

NONHARMONIC FOURIER SERIES IN THE SOBOLEV SPACES OF POSITIVE FRACTIONAL ORDERS

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Abstract. It is proved that if an exponential family $\mathcal{E} = \{e^{i\lambda_n t}\}$ forms a Riesz basis in $L^2(0, T)$, then the normalized family forms a Riesz basis in Sobolev space $H^s(0, T)$ on addition of $m - 1$ or m exponentials, where $m - 1 \leq s \leq m$ and $s \neq m - 1/2$.

Introduction

In the present paper it is proved that if an exponential family $\mathcal{E} = \{e^{i\lambda_n t}\}$ forms a Riesz basis in $L^2(0, T)$ then the normalized family forms a Riesz basis in the closure of its linear span in the Sobolev space $H^s(0, T)$ and this family forms a Riesz basis after addition of $m - 1$ or m exponentials, where $m - 1 \leq s \leq m$ and $s \neq m - 1/2$.

Notice that the Riesz basis theory of exponential families in $L^2(0, T)$ is well developed as in scalar (Pavlov 1979; Hrushchev, Nikol'skiĭ and Pavlov 1981) as well in the vector case (Ivanov 1983b; Avdonin and Ivanov 1989; Avdonin and Ivanov 1995b), while the properties of \mathcal{E} in the Sobolev spaces has not been adequately studied. The current status of the H^s -theory is that only results about connections of L^2 -basisness and H^s -basisness are known.

1.

Exponential families $\mathcal{E} = \{e^{i\lambda_n t}\}$ or $\mathcal{E} = \{\eta_n e^{i\lambda_n t}\}$, η_n being elements of an auxiliary Hilbert space, arise in diverse fields of mathematics. Let us mention the following ones.

(i) *Nonselfadjoint Model Operator Theory*

The inverse Fourier transform maps exponentials $e^{i\lambda_n t}$, $t > 0$, $\text{Im } \lambda > 0$, into simple fractions $1/(x + \lambda_n)$ belonging to the Hardy space H^2_+ of functions analytical in the upper halfplane. Simple fractions are the eigenfunctions of Sz.-Nagy – Foias dissipative model operator (the shift operator), see (Nicol'skiĭ 1980).

Some problems for vector exponentials $\eta_n e^{i\lambda_n t}$ connected with the model operator, are studied in (Ivanov 1985; Avdonin and Ivanov 1989; Avdonin and Ivanov 1995b).

(ii) *Resonance Scattering*

In (Pavlov 1971) it was established that completeness and basis properties of the resonance state family are equivalent to the same properties for joined system of the root functions of a model operator and its adjoint (joint completeness and joint basis property). Taking into account connection with exponentials mentioned above we reduce study of resonance state family to study of exponential family. For the Regge problem for multichannel system this approach was developed in (Ivanov and Pavlov 1978; Ivanov 1978; Ivanov 1983).

(iii) *Control Theory*

The Fourier method for control problem for hyperbolic equations leads to a moment problem for scalar or vector exponential family \mathcal{E} , where λ_n being eigenfrequencies of the hyperbolic system (Russell 1967; Avdonin and Ivanov 1984; Avdonin and Ivanov 1989a,b; Avdonin and Ivanov 1995a,b; Avdonin, Ivanov, and Ishmukhametov 1991). In some control problems it is essential to study exponential families not only in L^2 space but also in the Sobolev space H^s (Narukawa and Suzuki 1986; Avdonin and Ivanov 1989b; Avdonin and Ivanov 1995a,b).

2.

In this section we introduce concepts and notations and cite known results about basis property of exponential families.

Let Ξ be a family $\{\xi_n\}$ of elements of Hilbert space \mathcal{H} . By $\overline{\vee \Xi}$ (or $\overline{\vee_{\mathcal{H}} \Xi}$) we denote the closure of the linear span of Ξ in \mathcal{H} norm.

Definition. Family Ξ is called an \mathcal{L} -basis (Riesz basis in the closure of its linear span) if in \mathcal{H} there exists an operator \mathcal{V} (orthogonalizer) mapping family Ξ onto orthonormalized family and \mathcal{V} is an isomorphism onto its image.

Complete in \mathcal{H} \mathcal{L} -basis is called a *Riesz basis*.

Set $e_n(t) := c_n e^{i\lambda_n t}$, $n \in \mathbb{Z}$, where $c_n, c_n := \sqrt{(1 - e^{-2 \operatorname{Im} \lambda_n T})/2 \operatorname{Im} \lambda_n}$, are normalizing in $L^2(0, T)$ coefficients. The set $\{\lambda_n\}$ is called the spectrum of \mathcal{E} .

In (Russell 1982) D.L. Russell studied basis property of exponential families in the Sobolev spaces $H^m(0, T)$ with $m \in \mathbb{Z}$. Let us cite the result for $m > 0$.

Proposition 1. (Russell 1982) *Let family $\mathcal{E} = \{e_n\}$ be a Riesz basis in $L^2(0, T)$, points μ_1, \dots, μ_m be different and not belonging to the spectrum $\{\lambda_n\}$. Then the family*

$$\{c_n e^{i\lambda_n t} / (1 + |\lambda_n|^m)\} \cup \{e^{i\mu_j t}\}_{j=1}^m$$

forms a Riesz basis in the Sobolev space $H^m(0, T)$.

So the “original” basis family \mathcal{E} preserves \mathcal{L} -basis property in $H^m(0, T)$, but it does not preserve completeness. Let us notice, that the “effect” is characteristic for exponential family. For example, Legendre polynomial family $\{P_n\}$ is obviously complete in $H^m(-1, 1)$ but this family is not even uniformly minimal; it is easy to check that angles φ_n between polynomials P_n and P_{n+1} tend to zero:

$$\cos \varphi_n = \frac{|(P_n, P_{n+1})_{H^m}|}{\|P_n\|_{H^m} \|P_{n+1}\|_{H^m}} \xrightarrow{n \rightarrow \infty} 1.$$

In (Avdonin and Ivanov 1989b; Avdonin and Ivanov 1995b) the Russell’s theorem was generalized to subspaces generated by groups of exponentials and the obtained result was used for the controllability problem for a rectangular membrane (Avdonin and Ivanov 1989b; Avdonin and Ivanov 1995a,b).

In (Narukawa and Suzuki 1986) basis property of special exponential family was studied in $H^s(0, T)$ for noninteger s .

*The book (Avdonin and Ivanov 1989b) contains an incorrect result about vector exponential family in $H^s(0, T; \mathbb{C}^N)$. The same mistake was made in (Joð 1993).

Proposition 2. (Narukawa and Suzuki 1986) *Let λ_n^2 , $n = 1, 2, \dots$, be eigenvalues of a Sturm–Liouville operator*

$$-\frac{d^2}{dx^2} + p(x)$$

acting in $L^2(0, 1)$ with a smooth potential p and Dirichlet boundary conditions at $x = 0$ and $x = 1$. If $s - 1/2 \notin \mathbb{Z}$, and $s > 0$, then the family

$$\{e^{\pm i\lambda_n t}/(1 + |n|^s)\}_{n=1}^{\infty}$$

forms an \mathcal{L} -basis in $H^s(0, 2)$ and codimension of $\bigvee_{H^s} \mathcal{E}$ (dimension of the orthogonal complement to the linear span of \mathcal{E}) is equal to $\text{entier}(s + 1/2) + 1$.

Lemma 1. (Narukawa and Suzuki 1986) *Let $\Phi = \{\varphi_n\}_{n \in \mathbb{Z}}$, $\varphi_n(t) = e^{2\pi i n t/T}$, and $s > 0$, $s \notin \mathbb{Z} + 1/2$. Then the family $\Phi^{(s)} := \{\varphi_n/(1 + |n|^s)\}_{n \in \mathbb{Z}}$ forms an \mathcal{L} -basis in $H^s(0, T)$ and $\text{codim} \Phi^{(s)} = \text{entier}(s + 1/2)$.*

The proof of the lemma is rather direct and rests on the fact that harmonics are the eigenfunctions of the operator $A_p := -d^2/dx^2$ with periodic conditions. Therefore after normalisation Φ forms an \mathcal{L} -basis in H^s and the closure of its linear span has the form

$$\begin{aligned} \bigvee_{H^s} \Phi &= \mathcal{D} \left((1 + A_p)^{s/2} \right) \\ &= \{u \in H^s(0, T) \mid u(0) = u(T), u'(0) = u'(T), \dots, u^{(q)}(0) = u^{(q)}(T)\}, \\ & \qquad \qquad \qquad q \leq s - 1/2. \end{aligned}$$

So we have

$$\text{codim} \bigvee_{H^s} \Phi = \text{entier}(s + 1/2).$$

Lemma is proved.

3. Main Result

In the present paper, we do not assume that the spectrum of \mathcal{E} is the set of eigenfrequencies of a Sturm–Liouville operator, but we suppose that \mathcal{E} forms a Riesz basis in $L^2(0, T)$. In fact, the progress is that we change asymptotics

$$\lambda_n = 2\pi n/T + o(1)$$

by L^2 basis property of \mathcal{E} , which is more general than asymptotics, see (Avdonin 1974; Hrushchev, Nikol'skiĭ, and Pavlov 1981).

Theorem. *Let the family $\mathcal{E} = \{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$ be a Riesz basis in $L^2(0, T)$, points μ_1, \dots, μ_m be different and not belonging to the spectrum $\{\lambda_n\}_{n \in \mathbb{Z}}$ of \mathcal{E} , and s be a positive number such that $m - 1 < s < m$, $s \neq m - 1/2$, for $m \in \mathbb{N}$.*

Then the family $\mathcal{E}^{(s)} := \{c_n e^{i\lambda_n t}/(1 + |\lambda_n|^s)\}_{n \in \mathbb{Z}}$ forms an \mathcal{L} -basis in $H^s(0, T)$ and, moreover, the family

$$\mathcal{E}^{(s)} \bigcup \{e^{i\mu_j t}\}_1^{m-1}, \tag{1}$$

or the family

$$\mathcal{E}^{(s)} \bigcup \{e^{i\mu_j t}\}_1^m \quad (2)$$

forms a Riesz basis in $H^s(0, T)$.

The proof of the theorem is based on “interpolation” between basis property of \mathcal{E} in L^2 space (assumption of the theorem) and basis property of $\mathcal{E}^{(m)}$ in H^m space (Proposition 1).

Proof. We assume without loss of generality that the spectrum $\{\lambda_n\}$ of \mathcal{E} is indexed as follows: $\Re \lambda_n \geq 0$ for $n \geq 0$, $\Re \lambda_n < 0$ for $n < 0$, and $\Re \lambda_n$ increases together with n .

Lemma 2. For the spectrum of \mathcal{E} we have

$$\frac{|\lambda_n|}{|n|} \xrightarrow{|n| \rightarrow \infty} \frac{2\pi}{T}.$$

Proof of the Lemma. It is known (Hrushchev, Nikol’skiĭ and Pavlov 1981; Minkin 1991; Avdonin and Ivanov 1995b) that the spectrum of \mathcal{E} is the set of zeros of a Cartright class function and

$$\frac{n_{\pm}(r)}{r} \xrightarrow{r \rightarrow \infty} \frac{T}{2\pi},$$

where

$$n_+(r) = \text{card}\{\lambda_n \mid |\lambda_n| \leq r, \Re \lambda_n \geq 0\}, \quad n_-(r) = \text{card}\{\lambda_n \mid |\lambda_n| \leq r, \Re \lambda_n < 0\}.$$

It follows from the definitions that for any $\varepsilon > 0$

$$\begin{aligned} n_+(|\lambda_n| + \varepsilon) &\geq n + 1, & n_+(|\lambda_n| - \varepsilon) &\leq n, & n > 0, \\ n_-(|\lambda_n| + \varepsilon) &\geq |n|, & n_-(|\lambda_n| - \varepsilon) &\leq |n| - 1, & n < 0. \end{aligned}$$

So we have for $n > 0$

$$\frac{n+1}{|\lambda_n| + \varepsilon} \leq \frac{n_+(|\lambda_n| + \varepsilon)}{|\lambda_n| + \varepsilon} \xrightarrow{n \rightarrow \infty} \frac{T}{2\pi}, \quad \frac{n}{|\lambda_n| - \varepsilon} \geq \frac{n_+(|\lambda_n| - \varepsilon)}{|\lambda_n| - \varepsilon} \xrightarrow{n \rightarrow \infty} \frac{T}{2\pi},$$

and similar relations for n_- . Lemma is proved.

Let us introduce operator \mathcal{V} on the linear span of \mathcal{E} by the formula

$$\mathcal{V}e_n = \varphi_n = e^{2\pi i n t / T}, \quad n \in \mathbb{Z}.$$

Since \mathcal{E} and Φ are both Riesz bases in $L^2(0, T)$, operator \mathcal{V} can be continued on the whole space L^2 and the continuation is an isomorphism.

We consider the restriction of \mathcal{V} on $\bigvee_{H^m} \mathcal{E}$. The \mathcal{V} maps family $\mathcal{E}^{(m)}$ – an \mathcal{L} –basis in H^m – into family $\Phi^{(m)} = \{e^{2\pi i n t / T} / (1 + |\lambda_n|^m)\}_{n \in \mathbb{Z}}$. The latter family is also an \mathcal{L} –basis in H^m . Indeed, by Lemma 1, family

$$\{e^{2\pi i n t / T} / (1 + |n|^m)\}_{n \in \mathbb{Z}}$$

forms an \mathcal{L} -basis in H^m and by Lemma 2 we have $|n| \asymp |\lambda_n|$, so elements of $\Phi^{(m)}$ are almost normed.

Since operator $\mathcal{V}|_{\bigvee_{H^m} \mathcal{E}}$ maps one \mathcal{L} -basis $\mathcal{E}^{(m)}$ in H^m into another \mathcal{L} -basis $\Phi^{(m)}$, it is an isomorphism of subspaces $\bigvee_{H^m} \mathcal{E}$ and $\bigvee_{H^m} \Phi$.

It follows by interpolation theorem (see, for example, (Lions and Magenes 1968)) that the restriction of \mathcal{V} on interpolation space

$$\mathcal{H}_{\mathcal{E}}^s := [L^2, \bigvee_{H^m} \mathcal{E}]_{\theta},$$

with $s = \theta m$, is also an isomorphism onto

$$\mathcal{H}_{\Phi}^s := [L^2, \bigvee_{H^m} \Phi]_{\theta}.$$

From topological inclusions

$$H_0^m \subset \bigvee_{H^m} \mathcal{E} \subset H^m, \quad H_0^m \subset \bigvee_{H^m} \Phi \subset H^m$$

we deduce that metrics in interpolations spaces $\mathcal{H}_{\mathcal{E}}^s$ and \mathcal{H}_{Φ}^s are equivalent to H^s metric and these spaces may be considered as subspaces in $H^s(0, T)$.

Since $\mathcal{E}^{(s)}$ is the preimage under the isomorphism \mathcal{V} of family $\Phi^{(s)}$, and $\Phi^{(s)}$ is an \mathcal{L} -basis in H^s , the family $\mathcal{E}^{(s)}$ is also an \mathcal{L} -basis in H^s .

So, we have the following situation:

(i) The family

$$\{c_n e^{i\lambda_n t} / (1 + |\lambda_n|^{m-1})\}_{n \in \mathbb{Z}} \cup \{e^{i\mu_j t}\}_1^{m-1}$$

forms a Riesz basis in $H^{m-1}(0, T)$.

(ii) The family

$$\mathcal{E}^{(s)} = \{c_n e^{i\lambda_n t} / (1 + |\lambda_n|^s)\}_{n \in \mathbb{Z}}$$

forms an \mathcal{L} -basis in $H^s(0, T)$, $m - 1 < s < m$.

(ii) The family

$$\{c_n e^{i\lambda_n t} / (1 + |\lambda_n|^m)\}_{n \in \mathbb{Z}} \cup \{e^{i\mu_j t}\}_1^m \quad (3)$$

forms a Riesz basis in $H^m(0, T)$.

Since H^m is dense in H^s , family (3) is complete in $H^s(0, T)$. If family $\mathcal{E}^{(s)} \cup \{e^{i\mu_j t}\}_1^{m-2}$ is complete in H^s , then it is complete also in H^{m-1} what contradicts to (i).

So we have an alternative: one of the two families

$$\mathcal{E}^{(s)} \cup \{e^{i\mu_j t}\}_1^{m-1} \quad \text{or} \quad \mathcal{E}^{(s)} \cup \{e^{i\mu_j t}\}_1^m$$

forms a Riesz basis in $H^s(0, T)$. Theorem is proved.

4. Hypotheses

(i) In the main theorem we do not answer what family (family (1) or family (2)) forms a Riesz basis in H^s for given s . In the other words, we do not know the "defect" (codimension) of family $\mathcal{E}^{(s)}$ forming an \mathcal{L} -basis.

Hypotheses 1. For $m - 1 \leq s < m - 1/2$

$$\text{codim } \mathcal{E} = m - 1$$

and for $m - 1/2 < s \leq m$

$$\text{codim } \mathcal{E} = m.$$

(ii) By analogy with Proposition 2 and the results of D.L. Russell, and K. Narukawa and T. Suzuki (Russell 1982; Narukawa and Suzuki 1986) for exponential families in Sobolev spaces $H^s(0, T)$ of negative orders we may offer the hypotheses.

Hypothesis 2. (Generalization of the D.L. Russell result). If family \mathcal{E} forms a Riesz basis in $L^2(0, T)$ and $s < 0$, $s \notin \mathbb{Z} + 1/2$, then after elimination of entier $(|s| + 1/2)$ exponentials we obtain Riesz basis family in $H^s(0, T) = (H_0^{-s}(0, T))'$.

Hypothesis 3. (Generalization of the K. Narukawa and T. Suzuki results). If family \mathcal{E} forms a Riesz basis in $L^2(0, T)$ and $s < 0$, $s \notin \mathbb{Z} + 1/2$, then after addition of entier $(|s| + 1/2)$ exponentials we obtain Riesz basis family in $(H^{-s}(0, T))'$.

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