A proof of the Riemann hypothesis enabled by a new representation of the meromorphic symmetric Zeta function $\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi^*(1-s)$

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dedicated to my wife Vibhuta
on the occasion of her 62nd birthday
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Abstract

When $s \neq \nu$ and $\nu \in Z$, for Riemann's meromorphic Zeta function

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s) ,$$

a new representation is derived in the following form:

$$\xi^*(s) + \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} = \sum_{n=0}^\infty b_{2n}^*(s-\tfrac{1}{2})^{2n} + \frac{1}{\pi}\sum_{n=0}^\infty(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right]$$

with

$$b_{2n}^* := -2b_{2n} := -2\int_1^\infty \Phi(x)\left[\sum_{n=0}^\infty \frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} \text{ and } \Phi(x) := \sum_{n=1}^\infty (e^{-2\pi n x} - e^{-\pi n^2 x^2}) .$$

Accordingly, the non-trivial zeros $\left\{s_n = \frac{1}{2} + it_n\right\}$ of the zeta function are characterized by the identity of the following two convergent alternating power series representations:

$$\sum_{n=0}^\infty (-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^\infty(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2 + t_n^2}\right].$$

If a negative value $t_n^{2n} < 0$ exists, the affected term on the left side changes its sign, whereas the corresponding term on the right side does not. This proves the RH.

Keywords

Non-trivial zeros $\left\{s_n = \frac{1}{2} + it_n\right\}$, new Riemann's meromorphic Zeta function representation, two identical alternating power series representations do not allowing negative values t_n^{2n}.

1. Notations and Main Theorem

For the notations used herein, we refer to (Edwards, 2001). The baseline function for the zeta function theory is given by $\psi(x^2) := \sum_{n=1}^{\infty} e^{-\pi n^2 x^2}$, (Edwards, 2001) 1.7. It is related to Jacobi's functional equation, resulting in the symmetrical form of Riemann's functional equation (valid for all $s \neq 0,1$), in the form shown below, (Edwards, 2001) 1.7:

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s).$$

Riemann's entire related zeta function is given by $\xi(s) = \frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$, (Edwards, 2001) 1.8. Alternatively, for $\psi(x^2)$, we consider the function

$$\Phi(x) := \varphi(x) - \psi(x^2) := \sum_{n=1}^{\infty}\Phi_n(x) := \sum_{n=1}^{\infty}(e^{-2\pi n x} - e^{-\pi n^2 x^2})$$

with

$$\varphi(x) := \sum_{n=1}^{\infty} e^{-2\pi n x} = \frac{1}{1-e^{-2\pi x}} = \frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}.$$

The function $\Phi(x)$ resp. the related power series coefficients in the form

$$b_{2n} := \int_1^{\infty} \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}}$$

enables an alternative $\xi^*(s)$ –function representation with three $s \leftrightarrow (1-s)$ symmetric summands.

Main Theorem: For $s \neq \nu$ and $\nu \in Z$, it holds that

$$\xi^*(s) = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}\left(s-\frac{1}{2}\right)^{2n}.$$

When proving the main theorem (MT), the essential step is to prove the following lemma.

Lemma MT: For $s \neq \nu$, $\nu \in Z$, it holds that

$$-\frac{1}{2}\frac{1}{s(1-s)} = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - \int_1^{\infty}[x^s + x^{1-s}]\varphi(x)\frac{dx}{x}.$$

Corollary: The set of nontrivial zeroes $\left\{s_n = \frac{1}{2} + it_n\right\}$ of the zeta function is characterized by the following identity consisting of two convergent series representations:

$$\sum_{n=0}^{\infty} b_{2n}(s_n - \tfrac{1}{2})^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-z_n} + \frac{1}{(2n-1)+s_n}\right]$$

resp.

$$\sum_{n=0}^{\infty}(-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2+t_n^2}\right].$$

If a negative value $t_n^{2n} < 0$ exists, the affected term on the left side changes its sign, whereas the corresponding term on the right side does not. This proves the RH.

Remark (Gradshteyn and Ryzhik, 1965) 3.552: The function φ(x) is related to the Bernoulli polynomials in the form

$$\int_0^{\infty} x^{2m}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x} = \frac{|B_{2m}|}{2m},\ \int_0^{\infty} x^{2m}\frac{e^{-\pi x}}{\cosh(\pi x)}\frac{dx}{x} = (1-2^{1-2m})\frac{|B_{2m}|}{2m}.$$

Remark (Gradshteyn and Ryzhik, 1965) 3.552, 9.521: For Re(s) > 0, it holds that

$$\int_0^{\infty} x^s\varphi(x)\frac{dx}{x} = \frac{\Gamma(s)}{(2\pi)^s}\zeta(s,1) = \frac{1}{\sin(\pi s)}\left[\sin(\tfrac{\pi}{2}s)\sum_{n=1}^{\infty}\frac{\cos(2\pi n)}{n^{1-s}} + \cos(\tfrac{\pi}{2}s)\sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{n^{1-s}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\cos(\frac{\pi}{2}s)}\sum_{n=1}^{\infty}\frac{\cos(2\pi n)}{n^{1-s}} + \frac{1}{\sin(\frac{\pi}{2}s)}\sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{n^{1-s}}\right].$$

Remark: Riemann's density function $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s}$ with $a > 1$, is zero for $0 \le x < 2$, (Edwards, 2001) 1.11. The functions $\Phi_n(x)$ enable a modified non-zero density function $J^*(x)$ for $1 \le x < 2$:

$\Phi_n(x) \ge 0$ for $n \ge 2$, $x \ge 1$; $\Phi_1(x) < 0$ for $1 \le x < 2$; $\Phi_1(2) = 0$; $\Phi_1(x) > 0$ for $x > 2$. Setting $\Phi^*_{(1,2)}(x) \coloneqq -\Phi_1(x)$, $\Phi^*_{(0,1)}(x) \coloneqq -\Phi_1\left(\frac{1}{x}\right)$ for $1 \le x < 2$; $\Phi^*_{(1,2)}(x) = \Phi^*_{(0,1)}(x) = 0$ for $x \ge 2$, $\Phi^*_{(2,\infty)}(x) \coloneqq J(x)$ for $x \ge 2$; $\Phi^*_{(2,\infty)}(x) = 0$ for $x < 2$; the three terms of the sum $\Phi^*_{(0,1)}(x) + \Phi^*_{(1,2)}(x) + \Phi^*_{(2,\infty)}(x) > 0$ have disjunct domains $0 < x < 1$, $1 \le x < 2$, $2 \le x < \infty$.

2. Proofs of the Lemma MT and the Main Theorem (MT)

Remark:

$$\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right]=\frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right)+\zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)}$$

Lemma of the MT: For $s \neq \nu$ and $\nu \in Z$, it holds that

$$-\frac{1}{2}\frac{1}{s(1-s)}=-\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right]+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s}+\frac{1}{(2n-1)+s}\right]-\int_1^{\infty}[x^s+x^{1-s}]\varphi(x)\frac{dx}{x}.$$

Proof: From (Milgram, 2013), we recall the combination of the two formulae (Milgram, 2013) in 4.6 and 4.8 :

$$\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}=\frac{1}{s-1}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s}+\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}\quad,$$

$$\frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}=\frac{1}{-s}-\frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{(2n-1)+s}+\int_1^{\infty}x^{s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

With $\frac{1}{s-1}+\frac{1}{-s}=\frac{1}{s(s-1)}$, one obtains

$$-\frac{1}{2}\frac{1}{s(s-1)}=-\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)}+\frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}\right]+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s}+\frac{1}{(2n-1)+s}\right]$$

$$-\int_1^{\infty}[x^{1-s}+x^s]\varphi(x)\frac{dx}{x}.$$

Main Theorem: For $s \neq \nu$ and $\nu \in Z$, it holds that

$$\xi^*(s) = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\frac{1}{2})^{2n}.$$

Proof: From the lemma of the MT, one obtain

$$\xi^*(s) = \int_1^{\infty} \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$$

$$= \int_1^{\infty}[\psi(x^2) - \varphi(x)][x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right]$$

$$= -\int_1^{\infty} \Phi(x)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right].$$

If the term $[x^s + x^{1-s}] = 2\sqrt{x}\left[\cosh\left(s-\frac{1}{2}\right)\log x\right]$ is expanded in the usual power series $\cosh(y) = \sum_{n=0}^{\infty}\frac{y^{2n}}{(2n)!}$ such that $y := \left(s-\frac{1}{2}\right)\log x$, then

$$\int_1^{\infty} \Phi(x)[x^s + x^{1-s}]\frac{dx}{x} = 2\int_1^{\infty} \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}(s-\frac{1}{2})^{2n}\right]\frac{dx}{\sqrt{x}}\ .$$

From this, it follows that

$$\xi^*(s) = -2\int_1^{\infty} \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}(s-\frac{1}{2})^{2n}\right]\frac{dx}{\sqrt{x}} - \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)}\right]$$

$$+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right]$$

$$= -2\sum_{n=0}^{\infty} b_{2n}(s-\frac{1}{2})^{2n} + \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)}$$

$$+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right].$$

3. Some Relations Involving Kummer Functions

The representation provided by the main theorem

$$\xi^*(s) = -\frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right)+\zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}$$

indicates an alternative entire zeta function in the following form:

$$\xi^{**}(s) = \xi^*(s)\sin(\pi s) = (1-s)\frac{\sin(\pi s)}{\pi s}\left[\pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s}\right]\zeta(s)\,.$$

In the critical stripe, the Mellin transform of the Kummer function ${}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)$ is given by (Gradshteyn and Ryzhik, 1965) 7.612:

$$M\left[\,{}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)\right](s) = \pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s} \quad,\quad 0 < \mathrm{Re}(s) < 1.$$

Therefore, formally, for $\omega(x) \coloneqq \sum_{n=1}^{\infty} {}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi n^2 x^2\right)$ may the zeta function $\zeta(s)$ be represented in the form shown below:

$$\frac{\zeta(s)}{1-s}\Gamma\left(1+\frac{s}{2}\right) = \int_0^{\infty} x^s\omega(x)\frac{dx}{x}\,.$$

Remark: Riemann built his famous power series representation of the entire zeta function

$$\xi(s) := \pi^{-\frac{s}{2}}\frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\zeta(s)$$

by multiplying $\xi^*(s) = \int_1^{\infty}\psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$ with $s(s-1)$ to govern the two poles of the last term at $s = 0$ and $s = 1$; multiplying this term by $\sin(\pi s)$ gives

$$\frac{1}{2}\frac{\sin(\pi s)}{s(1-s)} = \sin\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{2}s\right)\left[\frac{1}{s}+\frac{1}{1-s}\right] = \cos\left(\frac{\pi}{2}s\right)\left[\frac{\sin\left(\frac{\pi}{2}s\right)}{s}+\frac{\sin\left(\frac{\pi}{2}(1-s)\right)}{1-s}\right].$$

Remark: The term $\frac{\sin(\pi s)}{\pi s}$ is an entire function of order one and possesses an order type $\sigma = \pi/2$ with

$$\frac{\sin(\pi s)}{\pi s} = \prod_{n=1}^{\infty}\left(1-\frac{s^2}{n^2}\right) = \prod_{n=-\infty}^{\prime\, n=\infty}\left(1-\frac{s}{n}\right)e^{s/n},\ \text{(Levin, 1939, p. 32).}$$

Remark: The term $\frac{1}{1-s}$ resp. the term $\log(s-1)$ provides the principle term of the Riemann density function $J(x)$, i.e., $li_1(x) = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^s ds$ ($a > 1$), (Edwards, 2001) 1.14.

Remark Some properties of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ and the Digamma function Ψ(x) (Braun, 2021):

1. ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ is not of exponential type π.
2. If $Re(z) > 0, z \to \infty$, then ${}_1F_1\left(\frac{1}{2},\frac{3}{2};\mathrm{z}\right) = \frac{1}{2}\frac{e^z}{z}\left\{1 + O(\frac{1}{|z|})\right\}$.

   If $Im(z) = 0, x \to \infty$, then ${}_1F_1\left(\frac{1}{2};\frac{3}{2}, logx\right) \sim \frac{1}{2}\frac{\mathrm{x}}{logx}$.
3. All the zeros z_ν, $\nu \in Z - \{0\}$, of the function ${}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right)$ are simple and complex valued; they lie in the horizontal stripes $(2|\nu| - 1)\pi < |Im(z_\nu)| < 2\pi|\nu|$, and their real parts lie on the right side of the critical line $Re(z_\nu) > 1/2$ satisfying the asymptotics

$$Re(z_\nu) = \frac{1}{2}log2\pi|\nu| + log\sqrt{\pi} \pm \frac{1}{16\nu} + O\left(\frac{log|\nu|}{\nu^2}\right)), \nu \to \pm\infty.$$

4. The zeros z_ν of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};\mathrm{z}\right)$ are the zeros of ${}_1F_1\left(1;\frac{3}{2},-\mathrm{z}\right)$ due to the relationship

$${}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right) = e^z\ {}_1F_1\left(1,\frac{1}{2},-z\right).$$

5. All zeros of the Digamma function $\Psi(\mathrm{x}) = \log'\Gamma(\mathrm{x})$ are real; there is only one positive zero $w_0 \sim 1{,}461$; all negative zeros w_n of $\Psi(x)$ lie in the intervals $w_n \in (-n, 1/2 - n)$.

In summary:

a. The function ${}_1\mathrm{F}_1\left(\frac{1}{2},\frac{3}{2};\mathrm{z}\right)$ is of order one, i.e., it belongs to the subset LP^* of the Laguerre-Polya class LP consisting of all elements of LP of order < 2.

b. The function $\mathrm{G(z)} \coloneqq {}_1\mathrm{F}_1\left(\frac{1}{2},\frac{3}{2};2\pi\mathrm{iz}\right)$ has only real zeros z_ν, which lie in the intervals $|\nu| - 1/2 < |\mathrm{z}_\nu| < |\nu|$.

c. The holomorphic function $K(z) \coloneqq \prod_{n=1}^{\infty}(1 - \frac{z^2}{\kappa_n^2})$ with $\kappa_\nu \coloneqq \frac{1}{2} + \mathrm{i} \cdot \mathrm{Im}(\mathrm{z}_\nu)$ is proposed to govern the zeta function on the critical line.

d. The holomorphic function $D(z) \coloneqq \prod_{n=1}^{\infty}(1 - \frac{z^2}{|w_n|^2})$ is proposed to govern the zeta function on the real axis.

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