



Prime pairs and the zeta function

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Received 29 August 2007; accepted 4 January 2008

Available online 30 September 2008

Communicated by C.K. Chui and H.N. Mhaskar

Dedicated to the memory of G.G. Lorentz

Abstract

Are there infinitely many prime pairs with given even difference? Most mathematicians think so. Using a strong arithmetic hypothesis, Goldston, Pintz and Yıldırım have recently shown that there are infinitely many pairs of primes differing by at most sixteen.

There is extensive numerical support for the prime-pair conjecture (PPC) of Hardy and Littlewood [G.H. Hardy, J.E. Littlewood, Some problems of ‘partitio numerorum’. III: On the expression of a number as a sum of primes, *Acta Math.* 44 (1923) 1–70 (sec. 3)] on the asymptotic behavior of $\pi_{2r}(x)$, the number of prime pairs $(p, p + 2r)$ with $p \leq x$. Assuming Riemann’s Hypothesis (RH), Montgomery and others have studied the pair-correlation of zeta’s complex zeros, indicating connections with the PPC. Using a Tauberian approach, the author shows that the PPC is equivalent to specific boundary behavior of a function involving zeta’s complex zeros. A certain hypothesis on equidistribution of prime pairs, or a speculative supplement to Montgomery’s work on pair-correlation, would imply that there is an abundance of prime pairs.

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Keywords: Tauberian theorem; Twin primes; Zeta’s complex zeros

Part I. Introductory survey

1. Prime pairs

The prime twins $(p, p + 2)$ with $p \leq 100$ are

(3, 5), (5, 7), (11, 13), (17, 19),

(29, 31), (41, 43), (59, 61), (71, 73).

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We also list the prime pairs $(p, p + 6)$ with $p \leq 100$:

(5, 11), (7, 13), (11, 17), (13, 19), (17, 23), (23, 29),
 (31, 37), (37, 43), (41, 47), (47, 53), (53, 59), (61, 67),
 (67, 73), (73, 79), (83, 89), (97, 103).

The reader should compare the totals, and may check that there are nine prime pairs $(p, p + 4)$ with $p \leq 100$.

Are there infinitely many prime twins $(p, p + 2)$, and infinitely many prime pairs $(p, p + 2r)$ for any given difference $2r \geq 2$? No proof is known, but one has found a prime twin for which p has more than 58 000 digits! See Twin Prime Search [48]. Denoting the number of prime twins $(p, p + 2)$ with $p \leq x$ by $\pi_2(x)$, one has a good idea how $\pi_2(x)$ should grow as $x \rightarrow \infty$; see Section 3. Similarly for the counting function $\pi_{2r}(x)$, the number of prime pairs $(p, p + 2r)$ with $p \leq x$.

Is it true that $\pi_6(x) \approx 2\pi_2(x)$ and that $\pi_4(x) \approx \pi_2(x)$ for all large x ? Assuming $\pi_2(x) \rightarrow \infty$, does

$$\pi_6(x)/\pi_2(x) \quad \text{tend to } 2 \text{ as } x \rightarrow \infty?$$

We would then say that $\pi_6(x)$ is asymptotic to $2\pi_2(x)$, in formula,

$$\pi_6(x) \sim 2\pi_2(x) \quad \text{as } x \rightarrow \infty.$$

2. Distribution of the primes

Legendre and Gauss already conjectured that $\pi(x)$, the number of primes $p \leq x$, is asymptotic to $x/\log x$, where $\log x$ denotes the natural logarithm. The so-called prime number theorem (PNT),

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \rightarrow \infty, \tag{2.1}$$

was proved only in 1896, more or less independently by the French mathematician Hadamard and the Belgian mathematician de la Vallée Poussin. (H's product representation of entire analytic functions played a big role in these first proofs of (2.1). A little later VP arrived at formula (2.4) with a good remainder; cf. Landau [37].)

Denoting the n -th prime by p_n , the formula $\pi(p_n) = n$ can be used to show that

$$p_n \sim n \log n \quad \text{as } n \rightarrow \infty,$$

a formula equivalent to the PNT. But how regular is the distribution of the primes? Suppose for a moment that the primes are very evenly distributed, say

$$|p_n - n \log n| < (1/3) \log n \quad \text{for all } n > n_0.$$

Then

$$|p_{n+1} - (n + 1) \log(n + 1)| < (1/3) \log(n + 1) \quad \text{for } n \geq n_0,$$

and it would follow that

$$p_{n+1} - p_n > (n + 2/3) \log(n + 1) - (n + 1/3) \log n > (1/3) \log n$$

for all $n \geq n_0$. But this would imply that there could be only a limited number of prime twins or prime pairs! Indeed, taking $n_0 > e^6$ as we may, one could not have $p_{n+1} - p_n = 2$ when $n \geq n_0$!

The conclusion is that the primes *cannot be very regularly* distributed. The PNT shows that in an average sense,

$$p_{n+1} - p_n \approx \log n.$$

However, it is known that the quotient

$$\frac{p_{n+1} - p_n}{\log n}$$

can become relatively small (and large) infinitely often. Erdős proved long ago that

$$c = \liminf_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{\log n} < 1.$$

Over the years, the best estimate for c came down to about $1/4$. Quite recently, Goldston, Pintz and Yildirim [23] could show that

$$c = \liminf_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{\log n} = 0. \tag{2.2}$$

In the other direction, it has been known for some time that

$$\limsup_{n \rightarrow \infty} \frac{p_{n+1} - p_n}{\log n} = \infty.$$

For (2.2) GPY used a result of Bombieri [4] and A.I. Vinogradov [50] on (weighted) equidistribution of primes in arithmetic progressions; see below. By assuming a strong hypothesis of Elliott and Halberstam [12] on this kind of equidistribution, Goldston, Pintz and Yildirim could actually prove that

$$\liminf_{n \rightarrow \infty} (p_{n+1} - p_n) \leq 16; \tag{2.3}$$

cf. also Goldston, Motohashi, Pintz and Yildirim [22] and the exposition by Soundararajan [45]. This conditional result would imply that there must be infinitely prime pairs $(p, p + 2r)$ for *some* difference $2r \leq 16!$

Remarks 2.1. A better approximation to $\pi(x)$ than (2.1) is given by the so-called logarithmic integral:

$$\pi(x) \sim \text{li}(x) = \int_2^x \frac{dt}{\log t} = \frac{x}{\log x} + \frac{x}{\log^2 x} + \dots; \tag{2.4}$$

cf. (2.5) and the conjectured estimate (4.6).

For primes in arithmetic progressions

$$a, a + d, a + 2d, \dots, (a, d) = 1, \quad a < d$$

the counting function $\pi(x, d, a)$ satisfies the asymptotic relation

$$\pi(x, d, a) \sim \frac{1}{\phi(d)} \text{li}(x),$$

where $\phi(d)$ is Euler’s function, the number of $n \leq d$ prime to d . A remainder estimate of de la Vallée Poussin implies that for every constant R ,

$$E(x, d) = \max_a \left| \pi(x, d, a) - \frac{\text{li}(x)}{\phi(d)} \right| \leq C(R, d) \frac{x}{(\log x)^R}. \tag{2.5}$$

Bombieri and Vinogradov proved a corresponding estimate that is uniform in d :

$$\sum_{d \leq x^\beta} \max_{y \leq x} E(y, d) \leq C(R, \beta) \frac{x}{(\log x)^R} \quad (2.6)$$

for any $\beta \in (0, 1/2)$. This result sufficed for (2.2). Elliott and Halberstam conjectured that (2.5) is valid for any number $\beta \in (0, 1)$. For (2.3) GPY had to assume this conjecture for a value of β close to 1.

3. The prime-pair conjecture of Hardy and Littlewood

How many prime twins $(p, p + 2)$ should one expect with $p \leq x$? Let us suppose for a moment that the primes are distributed more or less at random. Since $p_{n+1} - p_n \approx \log n$ on average, the ‘probability’ that a random n is prime would be about $1/\log n$. Now, for most $n \leq x$ one has $\log n \sim \log x$. Indeed, if $n > x^{1-\varepsilon}$ then $\log n > (1 - \varepsilon) \log x$. Thus, the chance that $n \leq x$ is prime would be about $1/\log x$. Similarly, the chance that $n + 2 \leq x$ is prime would be about $1/\log(n + 2) \approx 1/\log x$. Hence, the chance that a pair $(n, n + 2)$ with $n \leq x$ is a prime twin would be about

$$1/\log^2 x.$$

There is of course something wrong with this argument. For one thing, an even $n > 2$ cannot be prime. Also, for an odd n , the probabilities that n and $n + 2$ are both prime are not independent. However, one can correct for these facts, cf. Remark 3.2, and it remains a reasonable conjecture that $\pi_2(x)$, the number of prime twins $(p, p + 2)$ with $p \leq x$, is of the order $x/\log^2 x$. Around 1920 Viggo Brun developed a sieve method which gave an upper bound: $\pi_2(x) = \mathcal{O}(x/\log^2 x)$. Similar arguments apply to $\pi_{2r}(x)$, the number of prime pairs $(p, p + 2r)$ with $p \leq x$.

In 1923 Hardy and Littlewood published a long paper [25] on the Goldbach problems and prime pairs, prime triplets, etc. Using their new circle method and a strong hypothesis on the zeros of Dirichlet L -functions, they proved that every sufficiently large odd number n is the sum of three primes; later, I.M. Vinogradov [51] gave an unconditional proof. Using the circle method heuristically, H and L also formulated conjectures on the number of representations of an even number $2r$ as a sum of primes $p + q$ or a difference $p - q$. For prime pairs $(p, p + 2r)$, they arrived at the following precise conjecture:

Conjecture 3.1 (*Prime-Pair Conjecture, PPC*). *One has*

$$\pi_{2r}(x) \sim 2C_{2r} \frac{x}{\log^2 x} \quad \text{as } x \rightarrow \infty. \quad (3.1)$$

Here

$$C_2 = \prod_{p \text{ prime}, p > 2} \left\{ 1 - \frac{1}{(p-1)^2} \right\} \approx 0.6601618, \quad (3.2)$$

and

$$C_{2r} = C_2 \prod_{p|r, p > 2} \frac{p-1}{p-2}. \quad (3.3)$$

Thus, for example, $C_4 = C_8 = C_2$ and $C_6 = 2C_2$. There is a great deal of numerical support for the PPC. On the Internet one finds counts of twin primes for p up to 5×10^{15} by Nicely [42].

Table 1
Counting prime pairs

$2r$	x						C_{2r}/C_2
	10^3	10^4	10^5	10^6	10^7	10^8	
2	35	205	1224	8 169	58 980	440 312	1
4	41	203	1216	8 144	58 622	440 258	1
6	74	411	2447	16 386	117 207	879 908	2
8	38	208	1260	8 242	58 595	439 908	1
10	51	270	1624	10 934	78 211	586 811	4/3
12	70	404	2421	16 378	117 486	880 196	2
14	48	245	1488	9 878	70 463	528 095	6/5
16	39	200	1233	8 210	58 606	441 055	1
18	74	417	2477	16 451	117 463	880 444	2
20	48	269	1645	10 972	78 218	586 267	4/3
22	41	226	1351	9 171	65 320	489 085	10/9
24	79	404	2475	16 343	117 342	880 927	2
30	99	536	3329	21 990	156 517	1 173 934	8/3
210	107	641	3928	26 178	187 731	1 409 150	16/5
$L_2(x)$:	46	214	1249	8 248	58 754	440 368	

In Amsterdam, Fokko van de Bult [6] has recently counted the prime pairs $(p, p + 2r)$ with $2r \leq 10^3$ and $p \leq x = 10^3, 10^4, \dots, 10^8$. Table 1 is based on his work. The bottom line shows (rounded) values $L_2(x)$ of the comparison function $2C_2\text{li}_2(x)$, where

$$\text{li}_2(x) = \int_2^x \frac{dt}{\log^2 t} \sim \frac{x}{\log^2 x}. \tag{3.4}$$

Remarks 3.2. For the Hardy–Littlewood circle method, see Vaughan [49].

There are ‘Goldbach representations’ $2r = p + q$ for ‘most’ r ; cf. Hardy and Littlewood [26] (who used a strong hypothesis on zeros of L -functions), and van der Corput [11] (who used Vinogradov’s method and also considered $2r = p - q$). For later work on ‘Goldbach’, see Math. Reviews.

Bateman and Horn [2] formulated extensions of the Hardy–Littlewood conjectures and supported them by probabilistic arguments; see also the survey paper by Hindry and Rivoal [29] and cf. Soundararajan [45] for the special case of prime pairs.

4. Zeta function, Tauberian theory and PNT

Euler verified the infinity of primes by factoring the *zeta function*,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right) = \prod_{p \text{ prime}} \frac{1}{1 - 1/p^s},$$

and letting $s \searrow 1$. Comparison of $\zeta(s)$ with $\int_1^{\infty} t^{-s} dt$ shows that $\zeta(s) \sim 1/(s - 1)$ as $s \searrow 1$. Following Riemann, we will use $\zeta(s)$ also for complex $s = \sigma + i\tau$. Series and product are absolutely convergent for $\sigma > 1$, and define $\zeta(s)$ as an analytic function of s for $\sigma > 1$. A basic property is that the difference

$$\zeta(s) - \frac{1}{s - 1}$$

can be extended to an analytic function on the whole complex s -plane. One can use Euler’s product to derive that $\zeta(s) \neq 0$ for $\sigma \geq 1$. All nonreal zeros must lie in the strip $\{0 < \sigma < 1\}$ and there are infinitely many on the line $\{\sigma = 1/2\}$; cf. Titchmarsh [47]. According to Riemann’s (famous, unproved) Hypothesis (RH), all nonreal zeros of $\zeta(s)$ should lie on that line.

One of the simplest proofs of the prime number theorem (PNT) uses the zeta function and the *Tauberian theorem* of Wiener and Ikehara (1931–32). By a Tauberian theorem, one obtains information about an unknown quantity from the behavior of a function involving that quantity. The Wiener–Ikehara theorem reads as follows; see [30,53] and cf. [35]:

Theorem 4.1. *Consider any Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} \quad \text{with } a_n \geq 0 \tag{4.1}$$

that converges for $\sigma = \operatorname{Re} s > 1$ (so that the sum function is analytic there). Suppose that for some constant A , the difference

$$g(s) = f(s) - \frac{A}{s-1} \tag{4.2}$$

is analytic, or at least continuous, for $\sigma \geq 1$. Then

$$\sum_{n \leq x} a_n \sim Ax \quad \text{as } x \rightarrow \infty. \tag{4.3}$$

A simple example is provided by $f(s) = \zeta(s)$. Here $a_n = 1$, $A = 1$, and

$$\sum_{n \leq x} a_n = [x] \sim x.$$

To derive the PNT, observe first that the product $(s - 1)\zeta(s)$ is analytic and zero-free for $\sigma \geq 1$. From this it follows that $\log\{(s - 1)\zeta(s)\}$ is analytic for $\sigma \geq 1$. Hence, the derivative

$$\frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)}$$

is also analytic for $\sigma \geq 1$. Now differentiate the logarithm of Euler’s product for $\zeta(s)$. This gives

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p (\log p) \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right) = \sum_p \frac{\log p}{p^s} + g_1(s),$$

with a function $g_1(s)$ that is analytic for $\sigma > 1/2$. Combining results, one finds that

$$f(s) = \sum_p \frac{\log p}{p^s} = \frac{1}{s-1} + g_2(s), \tag{4.4}$$

where $g_2(s)$ is analytic for $\sigma \geq 1$. By Wiener–Ikehara, the conclusion is that

$$\theta(x) = \sum_{p \leq x} \log p \sim x. \tag{4.5}$$

This is equivalent to the PNT. Indeed, $\log p \sim \log x$ for ‘most’ $p \leq x$, so that

$$x \sim \sum_{p \leq x} \log p \sim (\log x) \sum_{p \leq x} 1 = (\log x)\pi(x).$$

Another simple proof of the PNT is due to Newman [41]; cf. [32,33]. One can show that the following error estimate is equivalent to Riemann’s Hypothesis:

$$\pi(x) - \text{li}(x) \ll x^{(1/2)+\varepsilon} \quad \text{for every } \varepsilon > 0. \tag{4.6}$$

Here the symbol \ll is shorthand for the \mathcal{O} -notation. Table 1 supports the analogous conjecture that for every r and $\varepsilon > 0$

$$\pi_{2r}(x) - 2C_{2r}\text{li}_2(x) \ll x^{(1/2)+\varepsilon}. \tag{4.7}$$

Starting with Montgomery’s work [39] one has realized that there is a deep connection between the prime-pair conjecture and the fine distribution of the complex zeros of the zeta function. Many authors have investigated the relation, notably Goldston, see [17] and Section 16; there are connections with random matrix theory. Following a lead of Arenstorf [1] we will use a Wiener–Ikehara Tauberian theorem to study prime pairs. This will show that the PPC has an equivalent formulation involving zeta’s complex zeros; see Sections 8–13. Furthermore, a certain randomness of the primes would imply that prime pairs with increasing values of $2r$ are equidistributed in some average sense. As a result there would be an abundance of prime pairs for some differences $2r$; see Sections 14 and 15.

Part II. Prime pairs and zeta’s zeros

5. Preliminary observations

As before, let

$$\pi_{2r}(x) = \{\# \text{ prime pairs } (p, p + 2r) \text{ with } p \leq x\}. \tag{5.1}$$

For Tauberian treatment of the PPC, one may introduce the sum

$$\theta_{2r}(x) = \sum_{\substack{p, p+2r \\ \text{prime}; p \leq x}} \log^2 p. \tag{5.2}$$

Since $\log p \sim \log x$ for most $p \leq x$, the PPC (3.1) is equivalent to the asymptotic relation

$$\theta_{2r}(x) \sim 2C_{2r}x. \tag{5.3}$$

By Wiener–Ikehara, this relation would follow if the function

$$\tilde{D}_{2r}(s) = \sum_{\substack{p, p+2r \\ \text{prime}}} \frac{\log^2 p}{p^s} \tag{5.4}$$

can be written as $2C_{2r}/(s - 1) + g_{2r}(s)$, where $g_{2r}(s)$ is continuous for $\sigma \geq 1$.

Following Brun’s early work, sieve methods have become an important part of prime-number theory. Using an advanced sieve, Jie Wu [54] has shown that $\pi_2(x) < 6.8 C_2 x / \log^2 x$ for all sufficiently large x . The best result in the other direction is Chen’s [10]: if $N(x)$ denotes the number of primes $p \leq x$ for which $p + 2$ has at most two prime factors, then $N(x) \geq cx / \log^2 x$ for some $c > 0$. There are related results for prime pairs $(p, p + 2r)$. In particular, there is a number x_0 independent of r such that

$$\pi_{2r}(x) \leq 9C_{2r} x / \log^2 x \leq C(\log \log 3r)x / \log^2 x \quad \text{for all } x \geq x_0; \tag{5.5}$$

see the book *Sieve Methods* by Halberstam and Richert [24].

We will also use the important fact that the prime-pair constants C_{2r} have *average* 1. Recently, Tenenbaum [46] found an elegant proof by using an appropriate Dirichlet series and the Wiener–Ikehara theorem. There is a strong estimate in the work of Bombieri and Davenport [5], which was improved further by Friedlander and Goldston [13] to

$$S_m = \sum_{r=1}^m C_{2r} = m - (1/2) \log m + \mathcal{O}\{\log^{2/3}(m + 1)\}. \tag{5.6}$$

The following sections lead up to a statement of the principal results of the paper in Section 8.

6. Refinement of the Tauberian approach

It will be convenient to use a two-way extension of the Wiener–Ikehara theorem due to the author [34]:

Theorem 6.1. *Let $\sum_{n=1}^\infty a_n/n^w$ with $a_n \geq 0$ converge to a sum function $f(w)$ for $w = u + iv$ with $u > 1$. Then*

$$\sum_{n \leq x} a_n \sim Ax \quad \text{as } x \rightarrow \infty \tag{6.1}$$

if and only if for $u \searrow 1$, the difference

$$f(u + iv) - \frac{A}{u + iv} = g(u + iv) \tag{6.2}$$

has a distributional limit $g(1 + iv)$, which on every finite interval $(-B, B)$ coincides with a pseudofunction (that may a priori depend on B).

The condition $\sum_{n \leq x} a_n = \mathcal{O}(x)$ would ensure that $f(u + iv)$ and $g(u + iv)$ have a distributional limit as $u \searrow 1$. A pseudofunction is the distributional Fourier transform of a bounded function which tends to zero at $\pm\infty$; locally, such a distribution is given by trigonometric series with coefficients that tend to zero. A pseudofunction cannot have pole-type singularities. In the case $a_n \geq 0$, local pseudofunction boundary behavior of $g(w)$ in (6.2) implies that

$$(w - w_0)g(w) \rightarrow 0 \tag{6.3}$$

for angular approach of w (from the right) to any point w_0 on the line $\{u = 1\}$; cf. [31], or [32], Theorem III.3.1.

To simplify the subsequent analysis, we replace the functions $\theta_{2r}(x)$ and $\tilde{D}_{2r}(s)$ of Section 5 by functions with the same behavior that involve von Mangoldt’s function $\Lambda(n)$. The latter is generated by the Dirichlet series

$$\sum_{n=1}^\infty \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} = \sum_{p \text{ prime}} (\log p) \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \dots \right);$$

cf. Section 4. One has $\Lambda(k) = \log p$ if $k = p^\alpha$ with p prime, and $\Lambda(k) = 0$ if k is not a prime power. Since there are only $\mathcal{O}(\sqrt{x})$ prime powers $p^\alpha \leq x$ with $\alpha \geq 2$, the difference between

$$\psi_{2r}(x) \stackrel{\text{def}}{=} \sum_{n \leq x} \Lambda(n)\Lambda(n + 2r) \tag{6.4}$$

and $\theta_{2r}(x)$ is not much larger than \sqrt{x} . Thus the PPC is also equivalent to the relation

$$\psi_{2r}(x) \sim 2C_{2r}x \quad \text{as } x \rightarrow \infty. \tag{6.5}$$

Similarly, the function

$$D_{2r}(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\Lambda(n)\Lambda(n+2r)}{n^s(n+2r)^s} \quad (s = \sigma + i\tau, \sigma > 1/2) \tag{6.6}$$

behaves in the same way as $\tilde{D}_{2r}(2s)$ when σ is close to $1/2$. Setting

$$D_{2r}(s) - \frac{C_{2r}}{s - 1/2} = G_{2r}(s), \tag{6.7}$$

Theorem 6.1 with $w = 2s$ shows that the PPC as formulated in (6.5) is equivalent to good boundary behavior of $G_{2r}(s)$ as $\sigma \searrow 1/2$.

Averages. The work on the Goldbach problems profited from knowledge of an average number of representations. Could one also introduce averages in the case of the PPC? One might try to study the average

$$\frac{2}{\lambda} \sum_{2r \leq \lambda} D_{2r}(s) \quad \text{for large } \lambda.$$

Under the PPC it should behave roughly like

$$\frac{2}{\lambda} \sum_{2r \leq \lambda} \frac{C_{2r}}{s - 1/2} \approx \frac{1}{s - 1/2}$$

for σ close to $1/2$, because the constants C_{2r} have average 1.

7. Auxiliary functions

Ordinary sums of functions $D_{2r}(s)$ do not handle well, but there are manageable combinations $V^\lambda(s)$ of functions $D_{2r}(s)$ with nonnegative coefficients. They are derived from a certain repeated complex integral $T^\lambda(s)$ which extends and modifies an integral of Arenstorf [1]; see Section 10. It involves a parameter $\lambda > 0$ and a parametric function E^λ ; the resulting formula for $V^\lambda(s)$ is

$$V^\lambda(s) = 2 \sum_{0 < 2r \leq \lambda} E^\lambda(2r) D_{2r}(s) = T^\lambda(s) - D_0(s) + H^\lambda(s). \tag{7.1}$$

Here, the function $D_{2r}(s)$ is given by (6.6), also when $r = 0$, and $H^\lambda(s)$ is holomorphic for $\sigma > 0$. The parametric function $E^\lambda(v) = E(v/\lambda)$ acts as a sieving device. The basic function $E(v)$ is taken even, of compact support, and decreasing on $[0, \infty)$ with derivative $E'(v)$ of bounded variation. For convenience, we normalize $E(v)$ so that its support is $[-1, 1]$ and $E(0) = 1$. A typical example is given by the Fourier transform of the Fejér kernel for \mathbb{R} ,

$$E_F^\lambda(v) = \frac{1}{\pi} \int_0^\infty \frac{\sin^2(\lambda t/2)}{\lambda(t/2)^2} \cos vt \, dt = \begin{cases} 1 - |v|/\lambda & \text{for } |v| \leq \lambda, \\ 0 & \text{for } |v| \geq \lambda. \end{cases} \tag{7.2}$$

One sometimes needs more smoothness, and then may use the Fourier transform of the Jackson kernel for \mathbb{R} ,

$$\begin{aligned}
 E_J^\lambda(v) &= \frac{3}{4\pi} \int_0^\infty \frac{\sin^4(\lambda t/4)}{\lambda^3(t/4)^4} \cos vt \, dt \\
 &= \begin{cases} 1 - 6(v/\lambda)^2 + 6(|v|/\lambda)^3 & \text{for } |v| \leq \lambda/2, \\ 2(1 - |v|/\lambda)^3 & \text{for } \lambda/2 \leq |v| \leq \lambda, \\ 0 & \text{for } |v| \geq \lambda. \end{cases} \tag{7.3}
 \end{aligned}$$

The PPC and the average 1 of the constants C_{2r} lead one to expect that $V^\lambda(s)$ has a first-order pole at $s = 1/2$ with residue

$$2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda) C_{2r} \sim \lambda \int_0^1 E(v) dv \stackrel{\text{def}}{=} A^E \lambda \tag{7.4}$$

as $\lambda \rightarrow \infty$; cf. (6.7) and (5.6).

For the following, we need a Mellin transform associated with the kernel $E^\lambda(v)$ via its Fourier transform:

$$\begin{aligned}
 M^\lambda(z) &\stackrel{\text{def}}{=} \frac{1}{\pi} \int_0^\infty \hat{E}^\lambda(t) t^{-z} dt = \frac{1}{\pi} \int_0^\infty t^{-z} dt \cdot 2 \int_0^\lambda E^\lambda(\omega) (\cos t\omega) d\omega \\
 &= \frac{2\lambda}{\pi} \int_0^1 E(v) dv \int_0^{\infty-} t^{-z} (\cos \lambda vt) dt \\
 &= \frac{2}{\pi} \lambda^z \Gamma(1 - z) \sin(\pi z/2) \int_0^1 E(v) v^{z-1} dv \\
 &= \frac{2}{\pi} \lambda^z \Gamma(-1 - z) \sin(\pi z/2) \int_0^{1+} v^{z+1} dE'(v); \tag{7.5}
 \end{aligned}$$

cf. (9.1). In the case of (7.2) this works out to the meromorphic function

$$M_F^\lambda(z) = \frac{2}{\pi} \lambda^z \Gamma(-z - 1) \sin(\pi z/2), \tag{7.6}$$

with (first-order) poles at $z = -1, 1, 3, \dots$ and residue $-\lambda/\pi$ at the pole $z = 1$. For the kernel in (7.3) one obtains

$$\begin{aligned}
 M_J^\lambda(z) &= \frac{3}{4\pi} \int_0^\infty \frac{\sin^4(\lambda t/4)}{\lambda^3(t/4)^4} t^{-z} dt \\
 &= \frac{24}{\pi} \lambda^z (1 - 2^{-z-1}) \Gamma(-z - 3) \sin(\pi z/2). \tag{7.7}
 \end{aligned}$$

In the general case $M^\lambda(z)$ turns out to be meromorphic for $x > -1$ with poles at $z = 1, 3, \dots$. The residue at $z = 1$ then is $-(2\lambda/\pi) \int_0^1 E(v) dv$. We need a good bound for $M^\lambda(z)$ on vertical lines: for fixed λ , one will have the majorization

$$M^\lambda(x + iy) \ll (|y| + 1)^{-x-3/2} \quad \text{when } -1 + \delta \leq x \leq C, |y| \geq 1. \tag{7.8}$$

This may be derived from the standard inequalities

$$\Gamma(z) \ll |y|^{x-1/2} e^{-\pi|y|/2}, \quad \sin(\pi z/2) \ll e^{\pi|y|/2} \tag{7.9}$$

that are valid for $|x| \leq C$ and $|y| \geq 1$. The inequality for $\Gamma(z)$ follows from asymptotic estimates in Whittaker and Watson [52].

The function $M_J^\lambda(z)$ of (7.7) is holomorphic for $x < 1$ and it satisfies an inequality (7.8) with $-7/2$ in the exponent instead of $-3/2$.

8. Results

Our results involve the complex zeros ρ of the zeta function. Taking multiplicities into account, the zeros above the real axis will be arranged according to non-decreasing imaginary part:

$$\rho = \rho_n = \beta_n + i\gamma_n, \quad 0 < \gamma_1 \approx 14 < \gamma_2 \approx 21 \leq \dots, n = 1, 2, \dots$$

(with $\beta_n = 1/2$ as far as zeros have been computed); we write $\bar{\rho}_n = \rho_{-n}$. In the theorem below the zeros appear in double sums

$$\begin{aligned} \Sigma_B^\lambda(s) &\stackrel{\text{def}}{=} \sum_{\rho, \rho'; |\text{Im } \rho| > B, |\text{Im } \rho'| > B} \Gamma(\rho - s)\Gamma(\rho' - s) \\ &\times M^\lambda(\rho + \rho' - 2s) \cos\{\pi(\rho - \rho')/2\}. \end{aligned} \tag{8.1}$$

Here, B may be any number ≥ 2 and $M^\lambda(\cdot)$ is given by (7.5). Under RH the double series is absolutely convergent for $1/2 < \sigma < 1$; cf. Lemma 9.2. Without RH the double sum may be interpreted as a limit of sums over the zeros ρ, ρ' with $|\text{Im } \rho|$ and $|\text{Im } \rho'|$ between B and R as $R \rightarrow \infty$; cf. [36]. Here, R should ‘stay away’ from the numbers γ_n ; cf. Section 12.

The formula for $V^\lambda(s)$ in (7.1) contains the function $T^\lambda(s)$ for which a repeated complex integral is introduced in Section 10. Moving the paths of integration in this integral and using the residue theorem one obtains

Theorem 8.1. *For any $\lambda > 0$, any sieving function E as described in Section 7, any $B > 2$ and for $s = \sigma + i\tau$ with $1/2 < \sigma < 1$, $|\tau| < B$ there are holomorphic representations*

$$\begin{aligned} V^\lambda(s) &= 2 \sum_{0 < 2r \leq \lambda} E\left(\frac{2r}{\lambda}\right) D_{2r}(s) \\ &= \frac{-1/4}{(s - 1/2)^2} + \frac{A^E \lambda}{s - 1/2} + \Sigma_B^\lambda(s) + H^\lambda(s, B) \\ &= \frac{A^E(\lambda - 1)}{s - 1/2} + \Sigma_B^\lambda(s) - \Sigma_B^1(s) + H^\lambda(s, B), \end{aligned} \tag{8.2}$$

where $A^E = \int_0^1 E(v)dv$ and $\Sigma_B^\lambda(s)$ is given by (8.1) (with proper interpretation of the double sum); the various functions $H^\lambda(s, B)$ are analytic for $1/2 \leq \sigma < 1$ (for $1/4 < \sigma < 1$ under RH) and $|\tau| < B$. On the real interval $\{1/2 \leq s \leq 3/4\}$ one has $H^\lambda(s, B) = \mathcal{O}(\lambda \log \lambda)$ as $\lambda \rightarrow \infty$.

The (extended) Wiener–Ikehara theorem will now show that the Hardy–Littlewood conjecture for prime pairs $(p, p + 2r)$ is true if and only if the sums $\Sigma_B^\lambda(s)$ exhibit certain specific boundary behavior as $\sigma \searrow 1/2$; cf. (6.7). For a precise result set

$$R(\lambda) = R^E(\lambda) = 2 \sum_{0 < 2r \leq \lambda} E\left(\frac{2r}{\lambda}\right) C_{2r} - A^E(\lambda - 1). \tag{8.3}$$

By induction, Theorems 8.1 and 6.1 imply

Corollary 8.2. *Suppose that the prime-pair conjecture for pairs $(p, p + 2r)$ is true for every $r < m$. Then the PPC for prime pairs $(p, p + 2m)$ is true if and only if, for some (or every) number $\lambda \in (2m, 2m + 2]$, some (or every) sieving function E and every $B > 2$, the function*

$$G^\lambda(s) = G_B^\lambda(s) \stackrel{\text{def}}{=} \Sigma_B^\lambda(s) - \Sigma_B^1(s) - \frac{R(\lambda)}{s - 1/2} \tag{8.4}$$

has good (local pseudofunction) boundary behavior for $\sigma \searrow 1/2$ when $|\tau| < B$.

Observe that the function $G^\lambda(s)$ does have good boundary behavior when $\lambda \leq 2$; under RH, it will even be analytic for $1/4 < \sigma < 1$ and $|\tau| < B$. Indeed, $V^\lambda(s) = 0$ for $\lambda \leq 2$. This supports the following

Conjecture 8.3. *For every $\lambda > 0$, every sieving function E and every $B > 2$, the function $G_B^\lambda(s)$ in (8.4) has an analytic continuation to the domain given by $1/4 < \sigma < 1$, $|\tau| < B$.*

If this is true, the counting functions $\pi_{2r}(x)$ all satisfy estimates of type (4.7).

Conditional abundance of prime pairs. We start with an important positivity property of certain double sums $\Sigma_B^\lambda(s)$ in (8.1):

Proposition 8.4. *For a sieving function E^λ (such as E_F^λ or E_J^λ) for which $\hat{E}^\lambda(t) \geq 0$, one has $\Sigma_B^\lambda(s) \geq 0$ when $1/2 < s < 1$.*

This positivity and a certain equidistribution hypothesis for prime pairs with increasing values of $2r$ would imply that there is an abundance of prime pairs for *some* difference $2r$:

Theorem 8.5. *Suppose that there are a positive integer m , a positive constant c and a sequence S of integers $\mu \rightarrow \infty$, such that for $\mu \in S$ and sufficiently large x , say $x \geq x_1 = x_1(\mu)$ with $\log x_1(\mu) = o(\mu)$, one has*

$$\frac{1}{m} \sum_{r=1}^m \pi_{2r}(x) \geq c \cdot \frac{1}{\mu} \sum_{r=1}^{\mu} \pi_{2r}(x). \tag{8.5}$$

Then

$$\limsup_{x \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \frac{\pi_{2r}(x)}{x / \log^2 x} \geq c. \tag{8.6}$$

There is both heuristic and numerical support for the hypothesis in the theorem; see Section 14.

Since the constants C_{2r} have average 1, the function $R(\lambda)$ in (8.3) is $o(\lambda)$ as $\lambda \rightarrow \infty$; cf. (7.4). (By (5.6) it will even be $\mathcal{O}(\log \lambda)$.) A speculative supplement to Montgomery’s work [39] on the pair-correlation of zeta’s complex zeros would imply that the differences $\Sigma_B^\lambda(s) - \Sigma_B^1(s)$ indeed have ‘upper residue’ $o(\lambda)$ as $s \searrow 1/2$. This too would imply that there is an abundance of prime pairs; see Section 16.

9. Complex representation for $E^\lambda(\alpha - \beta)$

For the discussion of $T^\lambda(s)$ in Section 10 we need a complex integral for the sieving function $E^\lambda(\alpha - \beta)$ in which α and β occur separately. It is obtained from the representation of $E^\lambda(\alpha - \beta)$

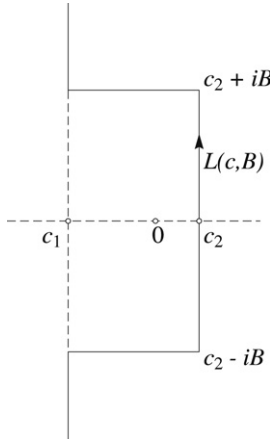


Fig. 1. The path $L(c_1, c_2, B)$.

as an inverse Fourier (cosine) transform:

$$E^\lambda(\alpha - \beta) = \frac{1}{\pi} \int_0^\infty \hat{E}^\lambda(t) \cos\{(\alpha - \beta)t\} dt$$

and a repeated complex integral for

$$\cos\{(\alpha - \beta)t\} = \cos \alpha t \cos \beta t + \sin \alpha t \sin \beta t.$$

To set the stage, we start with a complex representation for $\cos \alpha$ and $\sin \alpha$. Setting $z = x + iy$ we write $L(c)$ for the ‘vertical line’ $\{x = c\}$; the factor $1/(2\pi i)$ in complex integrals will be omitted. Thus

$$\int_{L(c)} f(z) dz \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(z) dz.$$

Mellin inversion of the improper Euler integral

$$\int_0^{\infty-} (\cos \alpha) \alpha^{z-1} d\alpha = \Gamma(z) \cos(\pi z/2) \quad (0 < x < 1) \tag{9.1}$$

now gives the improper complex integral

$$\cos \alpha = \int_{L(c)}^* \Gamma(z) \alpha^{-z} \cos(\pi z/2) dz = \lim_{A \rightarrow \infty} \frac{1}{2\pi i} \int_{c-iA}^{c+iA} \dots \quad (0 < c < 1/2);$$

there is a similar representation for $\sin \alpha$. It is important for us to have absolutely convergent integrals. We therefore replace the line $L(c)$ by a path $L(c, B) = L(c_1, c_2, B)$ with suitable $c_1 < c_2$ and $B > 0$ (cf. Fig. 1):

$$L(c, B) = \begin{cases} + & \text{the half-line} & \{x = c_1, -\infty < y \leq -B\} \\ + & \text{the segment} & \{c_1 \leq x \leq c_2, y = -B\} \\ + & \text{the segment} & \{x = c_2, -B \leq y \leq B\} \\ + & \text{the segment} & \{c_2 \geq x \geq c_1, y = B\} \\ + & \text{the half-line} & \{x = c_1, B \leq y < \infty\}. \end{cases} \tag{9.2}$$

Taking $c_1 < -1/2$ and $c_2 > 0$, one thus obtains the absolutely convergent repeated integral

$$\cos\{(\alpha - \beta)t\} = \int_{L(c,B)} \Gamma(z)\alpha^{-z}t^{-z}dz \int_{L(c,B)} \Gamma(w)\beta^{-w}t^{-w} \cos\{\pi(z - w)/2\}dw.$$

One now multiplies both sides by $\hat{E}^\lambda(t)$, integrates over $\{0 < t < \infty\}$, inverts the order of integration and uses (7.5). The result is

Proposition 9.1. *Let $\alpha, \beta > 0$, $-1/2 < c_1 < 0 < c_2 < 1/2$ and $B > 0$. Then for sieving functions $E^\lambda(\cdot)$ as in Section 7 one has*

$$E^\lambda(\alpha - \beta) = \int_{L(c,B)} \Gamma(z)\alpha^{-z}dz \int_{L(c,B)} \Gamma(w)\beta^{-w} \times M^\lambda(z + w) \cos\{\pi(z - w)/2\} dw, \tag{9.3}$$

where $M^\lambda(\cdot)$ is given by (7.5).

To justify the operations and verify the absolute convergence of the integral in (9.3) and subsequent repeated integrals, one may use a simple lemma:

Lemma 9.2. *For real constants a, b, c , the function*

$$\phi(y, v) = (|y| + 1)^{-a}(|v| + 1)^{-b}(|y + v| + 1)^{-c}$$

is integrable over \mathbb{R}^2 if and only if $a + b > 1$, $a + c > 1$, $b + c > 1$ and $a + b + c > 2$. For integrability over \mathbb{R}_+^2 the condition $a + b > 1$ may be dropped.

For verification let $A > 0$. On the subset of \mathbb{R}^2 where $|y| \leq A$, the condition $b + c > 1$ is necessary and sufficient for a finite v -integral. Similarly, for $|v| \leq A$ and the condition $a + c > 1$. When $|y + v| \leq A$ one needs the condition $a + b > 1$ in the case of \mathbb{R}^2 . For the fourth condition one looks at the set where $v \geq y \geq 1$. Setting $v = yr$ with new variable r ,

$$\phi(y, v)dydv \asymp y^{-a}(yr)^{-b}\{y(r + 1)\}^{-c}ydydr,$$

and the right-hand side is integrable over the set $\{1 < y < \infty, 1 < r < \infty\}$ if and only if $b + c > 1$ and $a + b + c > 2$. Similarly when y and v have opposite sign; one may of course assume that $|y + v| > 1$ then.

We turn to Proposition 9.1. Setting $z = x + iy$, $w = u + iv$, (7.9) and (7.8) give the following majorant for the integrand in (9.3) on the remote parts of the paths $L(c, B)$:

$$(|y| + 1)^{c_1-1/2}(|v| + 1)^{c_1-1/2}(|y + v| + 1)^{-2c_1-3/2}.$$

For integrability, one thus needs $-1 < c_1 < 0$. The more stringent requirements in Proposition 9.1 serve to keep the paths within the vertical strip $\{-1 < X < 1\}$ where $M^\lambda(Z)$ is known to be regular. In the case of a relatively smooth sieving function as in (7.3) the requirements can be relaxed.

10. The complex integral for $T^\lambda(s)$

The function $T^\lambda(s)$ in (7.1) is defined by the integral below for $\sigma > 1 + |c_1|$, while for s with smaller real part it is defined by analytic continuation;

$$T^\lambda(s) = \int_{L(c,B)} \Gamma(z) \frac{\zeta'(z+s)}{\zeta(z+s)} dz \int_{L(c,B)} \Gamma(w) \frac{\zeta'(w+s)}{\zeta(w+s)} dw \times M^\lambda(z+w) \cos\{\pi(z-w)/2\} \tag{10.1}$$

Theorem 10.1. *Let $-1/2 < c_1 < 0 < c_2 < 1/2$. Then the integral (10.1) defines $T^\lambda(s)$ as a holomorphic function of $s = \sigma + i\tau$ for $\sigma > 1 - c_1$. Assuming RH, the integral gives $T^\lambda(s)$ as a holomorphic function for $\sigma > \max\{(1/2) - c_1, 1 - c_2\}$ and $|\tau| < B$.*

The integral has an analytic continuation to the half-plane $\{\sigma > 1/2\}$ given by the expansion

$$T^\lambda(s) = \sum_{k,l} \Lambda(k)\Lambda(l)k^{-s}l^{-s}E^\lambda(k-l). \tag{10.2}$$

Proof (Discussion). For $z \in L(c, B)$ and $\sigma > 1 - c_1$, the sum $z + s$ will stay away from the poles of ζ'/ζ . Under RH the same holds when $\sigma > \max\{(1/2) - c_1, 1 - c_2\}$ and $|\tau| < B$. Indeed, in that case $x + \sigma > 1/2$ and also $z + s \neq 1$: if $x + \sigma = 1$, then z must lie on the part of $L(c, B)$ where $|y| \geq B$, and then $y + \tau \neq 0$. Similarly, for $w = u + iv$. The absolute convergence of the repeated integral in (10.1) can be proved in the same way as that of (9.3). Indeed, the quotient $(\zeta'/\zeta)(Z)$ grows at most logarithmically in Y for $X \geq 1$, and for $X \geq (1/2) + \eta$ under RH; cf. Titchmarsh [47]. The holomorphy of the integral for $T^\lambda(s)$ now follows from locally uniform convergence in s .

For the second part we substitute the Dirichlet series for $(\zeta'/\zeta)(\cdot)$ into (10.1), initially taking $\sigma > 1 - c_1$. Integrating term by term and applying Proposition 9.1 one obtains the expansion (10.2). Because $E^\lambda(k - l) \neq 0$ only for finitely many values of $k - l$, the series represents a holomorphic function for $\sigma > 1/2$; cf. the proof of Theorem 10.2. The sum of the series provides an (the) analytic continuation of the integral to the half-plane $\{\sigma > 1/2\}$. \square

We will now derive (7.1).

Theorem 10.2. *For arbitrary $\lambda > 0$ and $\sigma > 1/2$,*

$$T^\lambda(s) = D_0(s) + 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda)D_{2r}(s) + H_1^\lambda(s), \tag{10.3}$$

where $H_1^\lambda(s)$ is holomorphic for $\sigma > 0$. On the interval $\{1/2 \leq s \leq 3/4\}$ one has $H_1^\lambda(s) = \mathcal{O}(\lambda)$ as $\lambda \rightarrow \infty$.

Proof. Taking $k = l$ in (10.2) one obtains the term $D_0(s)$ in (10.3). For $|k - l| = 2r$ one obtains a constant multiple of $D_{2r}(s)$. The coefficient is different from 0 only if $2r < \lambda$ and in fact equal to $2E(2r/\lambda)$. If $|k - l| = 2r - 1$, one can have $\Lambda(k)\Lambda(l) \neq 0$ only if either k or l is of the form 2^α for some $\alpha > 0$. The resulting functions, for which $2r - 1$ must be $< \lambda$, are holomorphic for $\sigma > 0$. Analyzing these functions one finds that for real $s \geq 1/2$, the sum $H_1^\lambda(s)$ of their values will be $\mathcal{O}(\lambda)$ as $\lambda \rightarrow \infty$. \square

It remains to determine the analytic character of $D_0(s)$:

Lemma 10.3. *One has*

$$D_0(s) = \sum_{k=1}^\infty \frac{\Lambda^2(k)}{k^{2s}} = \frac{1/4}{(s - 1/2)^2} + H_0(s), \tag{10.4}$$

where $H_0(s)$ is analytic for $\sigma \geq 1/2$, and for $\sigma > 1/4$ under RH.

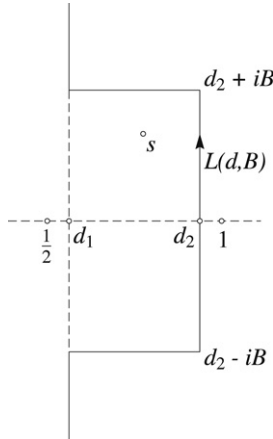


Fig. 2. The path $L(d_1, d_2, B)$.

Indeed, for $x = \text{Re } z > 1$

$$\begin{aligned} \sum_k \Lambda^2(k)k^{-z} &= \sum_p (\log^2 p)p^{-z} + H_1(z) = -\frac{d}{dz} \sum_p (\log p)p^{-z} + H_1(z) \\ &= -\frac{d}{dz} \sum_k \Lambda(k)k^{-z} + H_2(z) = \frac{d}{dz} \frac{\zeta'(z)}{\zeta(z)} + H_2(z) \\ &= \frac{1}{(z-1)^2} + H_3(z), \end{aligned}$$

where $H_1(z)$ and $H_2(z)$ define holomorphic functions for $x > 1/2$, while $H_3(z)$ is holomorphic for $x \geq 1$, and for $x > 1/2$ under RH. Finally take $z = 2s$.

11. From $T^\lambda(s)$ to a new integral $T_1^\lambda(s)$

In our transformations of the integral for $T^\lambda(s)$ it will be convenient to assume RH. With RH the process is simpler than without; cf. [36]. Another advantage is that a significant constituent of the function $T^\lambda(s)$ will appear right away.

Taking c_1, c_2 and s as in the first part of Theorem 10.1 we will move the paths of integration in the integral for $T^\lambda(s)$, but first change variables. Replacing z by $z' - s$ and w by $w' - s$ (and subsequently dropping the primes), one obtains

$$\begin{aligned} T^\lambda(s) &= \int_{L(c', B')} \Gamma(z-s) \frac{\zeta'(z)}{\zeta(z)} dz \int_{L(c', B')} \Gamma(w-s) \frac{\zeta'(w)}{\zeta(w)} \\ &\quad \times M^\lambda(z+w-2s) \cos\{\pi(z-w)/2\} dw. \end{aligned} \tag{11.1}$$

Here, the paths of integration initially depend on s , with horizontal segments that may be at different distances from the real axis. However, by our standard estimates and Cauchy’s theorem, one may choose $c'_1 = (1/2) + \eta$, $c'_2 = 1 + \eta$ (with $0 < \eta \leq 1/2$), use a constant B' , and take $(1/2) + \eta < \sigma < 1 + \eta$. Henceforth, the point s will be to the left of the paths.

We now move the new paths, one by one, across the poles $w = 1$ and $z = 1$ to new paths $L(d, B)$, where $d_1 = c'_1$ and $d_2 = 1 - \eta' > \sigma$; see Fig. 2. First, moving the w -path, the residue

theorem gives

$$T^\lambda(s) = \int_{L(c',B)} \cdots dz \int_{L(d,B)} \cdots dw + U_1^\lambda(s), \tag{11.2}$$

where

$$U_1^\lambda(s) = - \int_{L(c',B)} \Gamma(z-s) \frac{\zeta'(z)}{\zeta(z)} \Gamma(1-s) M^\lambda(z+1-2s) \sin(\pi z/2) dz. \tag{11.3}$$

In the latter integral, we move the path $L(c', B)$ across the pole $z = 1$ to the line $L(d_2) = \{x = d_2\}$. This provides the important residue

$$V_1^\lambda(s) \stackrel{\text{def}}{=} \Gamma^2(1-s) M^\lambda(2-2s). \tag{11.4}$$

By the definition of $M^\lambda(\cdot)$ in (7.5), the function $V_1^\lambda(s)$ is meromorphic for $0 < \sigma < 1$, with just one pole, a first-order pole at $s = 1/2$. Since $M^\lambda(z)$ has residue $-(2\lambda/\pi) \int_0^1 E(v) dv$ at $z = 1$, the function $V_1^\lambda(s)$ reveals the following *essential constituent* of $T^\lambda(s)$:

$$\frac{A^E \lambda}{s - 1/2}, \quad \text{with } A^E = \int_0^1 E(v) dv. \tag{11.5}$$

What remains of $V_1^\lambda(s)$ is holomorphic for $0 < \sigma < 1$. There is also a new integral along $L(d_2)$ corresponding to (11.3). Varying $d_2 > d_1$ and $d_1 > 1/2$ without letting $L(d_2)$ cross any singularities, this integral defines a function $H_2^\lambda(s)$ which is holomorphic for $1/4 < \sigma < 1$.

Returning to the repeated integral in (11.2), we move its z -path (after inverting order of integration) to $L(d, B)$ where $d_1 = (1/2) + \eta$, $d_2 = 1 - \eta'$. Besides a residue, this gives a new repeated integral which (after another inversion) takes the form

$$T_1^\lambda(s) \stackrel{\text{def}}{=} \int_{L(d,B)} \Gamma(z-s) \frac{\zeta'(z)}{\zeta(z)} dz \int_{L(d,B)} \Gamma(w-s) \frac{\zeta'(w)}{\zeta(w)} \times M^\lambda(z+w-2s) \cos\{\pi(z-w)/2\} dw. \tag{11.6}$$

Here, one has to take $(1/2) + \eta < \sigma < 1 - \eta'$ and $|\tau| < B$, but η and η' can be taken small and B large. The accompanying residue turns out to be equal to $H_2^\lambda(s)$. Summarizing we obtain

Proposition 11.1. *Assuming RH, there is a holomorphic decomposition*

$$T^\lambda(s) = T_1^\lambda(s) + \frac{A^E \lambda}{s - 1/2} + H_3^\lambda(s) \quad (1/2 < \sigma < 1), \tag{11.7}$$

where $H_3^\lambda(s)$ has an analytic continuation to the strip $\{1/4 < \sigma < 1\}$.

Analysis shows that for $1/2 \leq s \leq 3/4$, one has $H_3^\lambda(s) \ll \lambda \log \lambda$ as $\lambda \rightarrow \infty$. The factor $\log \lambda$ is due to the difference between (11.4) and (11.5).

12. From $T_1^\lambda(s)$ to sums $\Sigma_B^\lambda(s)$

We continue under RH, so that the complex zeros ρ of $\zeta(z)$ become

$$\rho = \rho_n = (1/2) + i\gamma_n, \quad 0 < \gamma_1 \approx 14 < \gamma_2 \approx 21 \leq \cdots, \gamma_{-n} = -\gamma_n.$$

The paths $L(d, B)$ of the integral in (11.6) will be moved one by one across the poles $z = \rho, w = \rho'$ of the quotients $(\zeta'/\zeta)(\cdot)$ on the line $L(1/2)$ with $|\gamma|, |\gamma'| > B$. Taking B different from all γ_n , the new paths of integration will have the form $L(d', B)$ with $d'_1 < 1/2$ and $d'_2 = d_2 < 1$.

Fixing $s = \sigma + i\tau$ with $(1/2) + \eta < \sigma < 1 - \eta'$ and $|\tau| < B$, we first move the w -path in (11.6) and use the residue theorem to obtain

$$\begin{aligned} T_1^\lambda(s) &= \int_{L(d,B)} \Gamma(z-s) \frac{\zeta'(z)}{\zeta(z)} dz \int_{L(d',B)} \Gamma(w-s) \frac{\zeta'(w)}{\zeta(w)} \\ &\quad \times M^\lambda(z+w-2s) \cos\{\pi(z-w)/2\} dw + V_2^\lambda(s, B) \\ &= H_2^\lambda(s, B) + V_2^\lambda(s, B), \end{aligned} \tag{12.1}$$

say. Here

$$V_2^\lambda(s, B) = \int_{L(d,B)} \Gamma(z-s) \frac{\zeta'(z)}{\zeta(z)} \Sigma_B^\lambda(z, s) dz, \tag{12.2}$$

where

$$\Sigma_B^\lambda(z, s) = \sum_{|\operatorname{Im} \rho| > B} \Gamma(\rho-s) M^\lambda(z+\rho-2s) \cos\{\pi(z-\rho)/2\}. \tag{12.3}$$

We list the conditions that the paths and $s = \sigma + i\tau$ must satisfy so that singular points of the integrand in the integral for $H_2^\lambda(s, B)$ are avoided. For z and w on the paths of integration we require $z \neq s, w \neq s$ and

$$-1 < d_1 + d'_1 - 2\sigma \leq x + u - 2\sigma \leq d_2 + d'_2 - 2\sigma < 1, \quad |\tau| < B. \tag{12.4}$$

Additional conditions follow from the requirement that the repeated integral be absolutely convergent. For $|y|, |v| > B$, its integrand is majorized by

$$|y|^{d_1-\sigma-1/2} (\log |y|) |v|^{d'_1-\sigma-1/2} (\log |v|) (|y+v|+1)^{-d_1-d'_1+2\sigma-3/2}, \tag{12.5}$$

cf. (7.9) and (7.8) and the fact that $(\zeta'/\zeta)(W)$ grows at most logarithmically also for $0 \leq U \leq (1/2) - \eta$. By Lemma 9.2 the additional conditions are

$$-d_1 - d'_1 + 2\sigma + 1 > 1, \quad d'_1 - \sigma + 2 > 1, \quad d_1 - \sigma + 2 > 1. \tag{12.6}$$

From this we will derive

Proposition 12.1. *Formula (12.1) gives a holomorphic decomposition of $T_1^\lambda(s)$ which is valid for $s = \sigma + i\tau$ with $(1/2) + \eta < \sigma < 1 - \eta$ and $|\tau| < B$. Varying the paths one finds that $H_2^\lambda(s, B)$ is holomorphic for $1/4 < \sigma < 1$ (or $1/2 \leq \sigma < 1$ without RH) and $|\tau| < B$.*

Proof. Part (i), the function $H_2^\lambda(s, B)$. By the preceding $H_2^\lambda(s, B)$ is analytic for $d_1 + d'_1 < 2\sigma < d_1 + d'_1 + 1$. When one makes d'_1 small, so that σ can get close to $1/4$, one can no longer allow σ to get close to 1. However, one may take $d'_1 = 2\eta$ and gradually reduce η .

Part (ii). The integral for $V_2^\lambda(s, B)$ represents a holomorphic function for $(1 + \eta)/2 < \sigma < 1 - \eta$ and $|\tau| < B$. Indeed, under the conditions on s and the z -path, the general term in the series for $\Sigma_B^\lambda(z, s)$ is majorized, uniformly in ρ and z , by

$$|\rho|^{-\sigma} (|y + \rho| + 1)^{-x+2\sigma-2} e^{\pi(|y-\rho|-|\rho|)/2}. \tag{12.7}$$

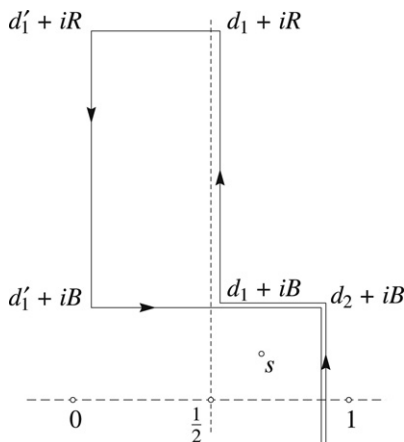


Fig. 3. Upper half of W_R .

Since $|\rho_n| \sim 2\pi n / \log n$ as $n \rightarrow \infty$, the series will be absolutely convergent for $z \in L(d, B)$. For $|y| > B$, the integrand in the integral for $V_2^\lambda(s, B)$ in (12.2) will be majorized by

$$\sum_{|\text{Im } \rho| > B} |y|^{\eta - \sigma} (\log |y|) |\rho|^{-\sigma} (|y + \rho| + 1)^{-\eta + 2\sigma - 5/2}.$$

Hence, by the analog of Lemma 9.2 for the integral of a sum, the integral is absolutely convergent.

Part (iii). The application of the residue theorem may be justified by starting with w -integrals over a sequence of closed contours W_R , $B < R = R_k \rightarrow \infty$, which are obtained from $L(d, B) - L(d', B)$ as follows. The parts where $|v| > R$ are deleted and replaced by the horizontal segments from $d_1 + iR$ to $d_1' + iR$ and $d_1' - iR$ to $d_1 - iR$. Observe that $L(d, B) - L(d', B)$ thus reduces to two rectangular paths, extending from the level $v = B$ to $v = R$, and the level $v = -R$ to $v = -B$, respectively; see Fig. 3. Here the numbers R are chosen ‘away from the numbers γ_n ’, in the sense that on the horizontal segments $\{v = \pm R\}$ one has $\zeta'(w)/\zeta(w) \ll \log^2 |v|$; cf. [47]. One can now use the standard estimates to verify that the double integrals associated with the segments $\{v = \pm R\}$ tend to zero when $\sigma > d_1$. \square

Next, the function $V_2^\lambda(s, B)$ of (12.2) is transformed by moving the z -path $L(d, B)$ to $L(d', B)$. Again using the residue theorem, one finds

$$\begin{aligned} V_2^\lambda(s, B) &= \int_{L(d', B)} \Gamma(z - s) \frac{\zeta'(z)}{\zeta(z)} \Sigma_B^\lambda(z, s) dz \\ &+ \sum_{|\text{Im } \rho'| > B} \Gamma(\rho' - s) \Sigma_B^\lambda(\rho', s) = H_3^\lambda(s, B) + \Sigma_B^\lambda(s), \end{aligned} \tag{12.8}$$

say. Here, the summation extends over the zeros ρ' of $\zeta(\cdot)$ with $|\text{Im } \rho'| > B$. The method used above for $H_2^\lambda(s, B)$ shows that the integral for $H_3^\lambda(s, B)$ defines a holomorphic function for $(1/4) + \eta < \sigma < 1$, $|\tau| < B$. Indeed, for $z \in L(d', B)$ and $|y| > B$, it follows from (12.7) that the integrand is majorized by

$$\sum_{|\text{Im } \rho| > B} |y|^{2\eta - \sigma - 1/2} (\log |y|) |\rho|^{-\sigma} (|y + \rho| + 1)^{-2\eta + 2\sigma - 2}.$$

The result now follows from an appropriate form of Lemma 9.2.

Using (12.3), $\Sigma_B^\lambda(s)$ may be written as the sum of a double series:

$$\begin{aligned} \Sigma_B^\lambda(s) &= \sum_{|\operatorname{Im} \rho| > B, |\operatorname{Im} \rho'| > B} \Gamma(\rho - s)\Gamma(\rho' - s) \\ &\times M^\lambda(\rho + \rho' - 2s) \cos\{\pi(\rho - \rho')/2\}. \end{aligned} \tag{12.9}$$

Here ρ and ρ' run over the complex zeros $(1/2) + i\gamma_n$ of $\zeta(\cdot)$ with $|\gamma_n| > B$. By the standard considerations, the sum represents a holomorphic function for $1/2 < \sigma < 1$ and $|\tau| < B$. Moreover, by the \mathbb{R}_+^2 part of Lemma 9.2, the corresponding sum in which $\gamma = \operatorname{Im} \rho$ and $\gamma' = \operatorname{Im} \rho'$ have the same sign represents an analytic function $H_4^\lambda(s, B)$ for $0 < \sigma < 1$ and $|\tau| < B$. Thus, the important part of $\Sigma^\lambda(s, B)$ is the part $\Sigma_-^\lambda(s, B)$ where $\operatorname{Im} \rho$ and $\operatorname{Im} \rho'$ have opposite sign:

$$\begin{aligned} \Sigma_-^\lambda(s, B) &\stackrel{\text{def}}{=} \sum_{|\operatorname{Im} \rho| > B, |\operatorname{Im} \rho'| > B; \operatorname{Im} \rho \cdot \operatorname{Im} \rho' < 0} \Gamma(\rho - s)\Gamma(\rho' - s) \\ &\times M^\lambda(\rho + \rho' - 2s) \cos\{\pi(\rho - \rho')/2\}. \end{aligned} \tag{12.10}$$

Combining the present results with Proposition 11.1 and checking the behavior of the functions $H_j^\lambda(s, B)$ for real s one obtains

Theorem 12.2. *Assume RH. Then for $s = \sigma + i\tau$ and any $B > 0$ there is a holomorphic decomposition*

$$T^\lambda(s) = \frac{A^E \lambda}{s - 1/2} + \Sigma_B^\lambda(s) + H_5^\lambda(s, B) \quad (1/2 < \sigma < 1), \tag{12.11}$$

where $\Sigma_B^\lambda(s)$ is given by (12.9) and $H_5^\lambda(s, B)$ has an analytic continuation to the domain given by $1/4 < \sigma < 1$, $|\tau| < B$. Here, $\Sigma_B^\lambda(s)$ may be replaced by $\Sigma_-^\lambda(s, B)$. On the interval $\{1/2 \leq s \leq 3/4\}$ one has $H_5^\lambda(s, B) = \mathcal{O}(\lambda \log \lambda)$ as $\lambda \rightarrow \infty$.

There is a corresponding result without RH; see [36]. In that case one has to define the sum of the double series in (12.9) as a suitable limit; one may use certain square partial sums. The function $H_5^\lambda(s, B)$ will still be analytic for $1/2 \leq \sigma < 1$, $|\tau| < B$, and $\mathcal{O}(\lambda \log \lambda)$ for real $s \searrow 1/2$.

13. Proof of Theorem 8.1 and Corollary 8.2

Proof of Theorem 8.1. The proof is obtained by combining Theorem 10.2, Lemma 10.3 and Theorem 12.2. For $1/2 < \sigma < 1$ these results give the following holomorphic representations for the sum $V^\lambda(s)$ in (7.1):

$$\begin{aligned} V^\lambda(s) &= 2 \sum_{0 < 2r \leq \lambda} E(2r/\lambda) D_{2r}(s) = T^\lambda(s) - D_0(s) - H_1^\lambda(s) \\ &= T^\lambda(s) - \frac{1/4}{(s - 1/2)^2} + H_6^\lambda(s) \\ &= -\frac{1/4}{(s - 1/2)^2} + \frac{A^E \lambda}{s - 1/2} + \Sigma_B^\lambda(s) + H_7^\lambda(s, B). \end{aligned} \tag{13.1}$$

Here, $A^E = \int_0^1 E(v)dv$ and $\Sigma_B^\lambda(s)$ is given by (8.1) with suitable interpretation of the double sum. The functions $H_j^\lambda(\cdot)$ are holomorphic for $1/2 \leq \sigma < 1$ (for $1/4 < \sigma < 1$ under RH) and $|\tau| < B$. For the final line of (8.2) one applies (13.1) to $V^1(s) = 0$ and subtracts the result from (13.1) for $V^\lambda(s)$.

Results for H_1^λ in Theorem 10.2 and H_5^λ in Theorem 12.2 show that on the interval $\{1/2 \leq s \leq 3/4\}$ one has $H_7^\lambda(s, B) = \mathcal{O}(\lambda \log \lambda)$ as $\lambda \rightarrow \infty$. \square

Proof of Corollary 8.2. The proof uses induction with respect to $m \geq 1$. The induction hypothesis is that the PPC for pairs $(p, p + 2r)$ is known to hold for every $r < m$.

(i) Suppose that the PPC is also true for $r = m$. Then, if λ is any number in $(2m, 2m + 2]$, the PPC is true for every $r < \lambda/2$. Hence for any E , cf. (6.7), the function

$$W^\lambda(s) \stackrel{\text{def}}{=} 2 \sum_{0 < 2r < \lambda} E(2r/\lambda) \left(D_{2r}(s) - \frac{C_{2r}}{s - 1/2} \right) \tag{13.2}$$

has good (local pseudofunction) boundary behavior as $\sigma \searrow 1/2$. Now by (8.2)–(8.4), for any B ,

$$G_B^\lambda(s) = W^\lambda(s) - H^\lambda(s, B),$$

where $H^\lambda(s, B)$ is holomorphic for $1/2 \leq \sigma < 1$ and $|\tau| < B$. Hence $G_B^\lambda(s)$ will also have good boundary behavior for $|\tau| < B$.

(ii) Conversely, suppose that for some E , some $\lambda \in (2m, 2m + 2]$ and every B , the function $G_B^\lambda(s)$ has good boundary behavior as $\sigma \searrow 1/2$ when $|\tau| < B$. Then, with this λ , the sum $W^\lambda(s)$ in (13.2) has good boundary behavior. But we know from the induction hypothesis that

$$2 \sum_{0 < 2r < 2m} E(2r/\lambda) \left(D_{2r}(s) - \frac{C_{2r}}{s - 1/2} \right)$$

has good boundary behavior, hence, so does the difference

$$2E(2m/\lambda) \left(D_{2m}(s) - \frac{C_{2m}}{s - 1/2} \right).$$

Since $E(2m/\lambda) \neq 0$ this implies the PPC for pairs $(p, p + 2m)$. \square

14. Introduction to Theorem 8.5

We first prove the positivity result in Proposition 8.4, taking $B = 2$.

Proof. Considering the double sum $\Sigma_2^\lambda(s)$ of (8.1) with $1/2 < s < 1$ it will be convenient to replace ρ' by $\bar{\rho}'$. Set $\Omega_R(t, s) = \Omega_R'(t, s) + \Omega_R''(t, s)$, where

$$\Omega_R'(t, s) = \sum_{|\text{Im } \rho|, |\text{Im } \rho'| < R} \Gamma(\rho - s)\Gamma(\bar{\rho}' - s)t^{2s - \rho - \bar{\rho}'} \cos(\pi\rho/2) \cos(\pi\bar{\rho}'/2),$$

and $\Omega_R''(t, s)$ is the corresponding function with sin instead of cos. Then

$$\Omega_R'(t, s) = \left| \sum_{|\text{Im } \rho| < R} \Gamma(\rho - s)t^{s - \rho} \cos(\pi\rho/2) \right|^2 \geq 0,$$

and similarly for $\Omega_R''(t, s)$. Hence, $\Omega_R(t, s) \geq 0$, and by (8.1) and (7.5)

$$\Sigma_2^\lambda(s) = \frac{1}{\pi} \lim_{R \rightarrow \infty} \int_0^\infty \hat{E}^\lambda(t) \Omega_R(t, s) dt \geq 0. \quad \square$$

For the discussion of **Theorem 8.5** it is convenient to consider the function

$$\tilde{D}_{2r}(s) = \sum_{p, p+2r \text{ prime}} \frac{\log^2 p}{p^{2s}} = \int_2^\infty \frac{d\theta_{2r}(x)}{x^{2s}} = \int_2^\infty \frac{\log^2 x}{x^{2s}} d\pi_{2r}(x), \tag{14.1}$$

which is obtained from (5.4) when we replace s by $2s$. It differs from $D_{2r}(s)$ in (6.6) only by a function that is analytic for $\sigma > 1/4$. We also have to deal with the difference on the interval $\{1/2 < s < 3/4\}$:

Lemma 14.1. *For $1/2 < s < 3/4$, one has the estimate*

$$\tilde{D}_{2r}(s) - D_{2r}(s) = \mathcal{O}(\log^2 2r),$$

which holds uniformly in r .

Proof. We may restrict ourselves to the main part of $D_{2r}(s)$,

$$D_{2r}^*(s) = \sum_{p, p+2r \text{ prime}} \frac{\log p \log(p + 2r)}{p^s (p + 2r)^s}. \tag{14.2}$$

Indeed, analysis of the other terms $\Lambda(n)\Lambda(n + 2r)n^{-s}(n + 2r)^{-s}$ in the series for $D_{2r}(s)$ shows that for $s \geq 1/2$, their sum is $\mathcal{O}(\log 2r)$ uniformly in r . The critical part consists of the terms in which n is a prime and $n + 2r$ the square of a prime.

For the comparison of $D_{2r}^*(s)$ with $\tilde{D}_{2r}(s)$ we use the majorizations

$$\pi_{2r}(x) \leq \pi(x) \ll \frac{x}{\log x}, \quad \pi_{2r}(x) \leq \frac{9C_{2r}x}{\log^2 x} \ll \frac{(\log 2r)x}{\log^2 x}. \tag{14.3}$$

The second inequality holds uniformly in r when $x \geq x_0$; cf. (5.5). Our comparison may be carried out in two steps, first considering

$$\int_2^\infty \frac{\log x \log(x + 2r) - \log^2 x}{x^s(x + 2r)^s} d\pi_{2r}(x).$$

Here, one may estimate as follows:

$$\begin{aligned} \int_2^{2r} \dots &\ll \int_e^{2r} \frac{\log x \log(1 + r)}{x} d\pi(x) \ll \log^2 2r, \\ \int_{2r}^\infty \dots &\ll \int_{2r}^\infty \frac{(\log x)(2r/x)}{x} d\pi(x) = \mathcal{O}(1). \end{aligned}$$

In the second step one has to deal with

$$\int_2^\infty \frac{\log^2 x}{x^s} \left(\frac{1}{x^s} - \frac{1}{(x + 2r)^s} \right) d\pi_{2r}(x).$$

If $2r \geq x_0$ one will split at $2r$ and use the second majorant for $\pi_{2r}(x)$ on $(2r, \infty)$; in the case $2r < x_0$ one splits at x_0 . \square

By (14.1) and (5.5) the ‘upper residue’ of $\tilde{D}_{2r}(s)$ for $s \searrow 1/2$,

$$\omega_{2r} = \limsup_{\delta \searrow 0} \delta \tilde{D}_{2r}\{(1/2) + \delta\}, \tag{14.4}$$

is finite. But how can we prove that $\omega_{2r} > 0$ for some r ?

Take $E^\lambda(v) = E_F^\lambda(v)$ as in (7.2), so that $\hat{E}^\lambda(t) \geq 0$ and $A^E = 1/2$. Then **Theorem 8.1** with $B = 2$, **Lemma 14.1** and **Proposition 8.4** imply the following inequality for $1/2 < s < 3/4$ and $\lambda = 2\mu \geq 2$:

$$2 \sum_{r \leq \mu} E(r/\mu) \tilde{D}_{2r}(s) \geq -\frac{1/4}{(s - 1/2)^2} + \frac{\mu}{s - 1/2} - \mathcal{O}(\mu \log^2 2\mu).$$

Here $0 \leq E(r/\mu) \leq 1$. Setting $s - 1/2 = \delta$ and multiplying by $\delta/(2\mu)$, it follows that

$$\frac{1}{\mu} \sum_{r \leq \mu} \delta \tilde{D}_{2r}\{(1/2) + \delta\} \geq \frac{1}{2} - \frac{1}{8\mu\delta} - \mathcal{O}(\delta \log^2 2\mu). \tag{14.5}$$

This, however, is not enough to conclude that $\omega_{2r} > 0$ for some fixed r , since we have to take $\mu\delta \geq \gamma > 1/4$ to ensure positivity of the right-hand side of (14.5) as $\delta \searrow 0$.

A convenient hypothesis on the distribution of prime pairs will enable us to overcome the difficulty. For $\mu \in \mathbb{N}$ set

$$A(x, \mu) = \frac{1}{\mu} \sum_{r=1}^{\mu} \pi_{2r}(x). \tag{14.6}$$

If the Prime-Pair Conjecture is true, the averages $A(x, \mu)$ with not too small μ will all grow at about the same rate as $x \rightarrow \infty$; indeed, the constants C_{2r} have average 1. However, leaving aside the PPC, there is both heuristic and numerical support for such a growth hypothesis.

Heuristics. Let $\pi(x)$ as usual denote the number of primes $\leq x$. For any odd prime p , the number of prime pairs $(p, p + 2r)$ with $r \leq \mu$ is equal to $\pi(p + 2\mu) - \pi(p)$. For the sum $\sum_{r \leq \mu} \pi_{2r}(x)$ we let p run over the odd primes $\leq x$, hence

$$\sum_{r=1}^{\mu} \pi_{2r}(x) = \sum_{3 \leq p \leq x} \{\pi(p + 2\mu) - \pi(p)\}.$$

Now, consider the corresponding sum over all integers $n \in (0, x]$. It may be telescoped as follows:

$$\sum_{n \leq x} \{\pi(n + 2\mu) - \pi(n)\} = \pi(x + 1) + \dots + \pi(x + 2\mu) \approx \frac{2\mu x}{\log x}$$

as $x \rightarrow \infty$. Hence, if the primes are ‘randomly distributed’ among the positive integers $n \leq x$, with ‘density’ approximately $1/\log x$, one would expect that for not too small μ and $x \rightarrow \infty$,

$$A(x, \mu) = \frac{1}{\mu} \sum_{3 \leq p \leq x} \{\pi(p + 2\mu) - \pi(p)\} \approx \frac{2x}{\log^2 x}.$$

Numerics. **Table 2** is based on computations by Fokko van de Bult [6]. It lists the sums

$$\sum_{km < r \leq (k+1)m} \pi_{2r}(x) \quad \text{and} \quad \sum_{0 < r \leq 10m} \pi_{2r}(x)$$

for $m = 50, k = 0, \dots, 9$ and $x = 10^3, \dots, 10^6$. For large x the final totals are comparable to ten times the preceding sums.

In view of all this, it appears reasonable to propose

Table 2
Adding numbers of prime pairs

k	x			
	10^3	10^4	10^5	10^6
0	2 698	14 885	90 550	605 087
1	2 621	15 078	93 010	621 580
2	2 550	15 090	93 501	624 766
3	2 453	14 754	91 679	614 148
4	2 424	14 848	92 735	621 114
5	2 409	14 917	93 268	627 333
6	2 354	14 545	91 490	614 479
7	2 334	14 666	92 820	623 366
8	2 318	14 615	92 547	622 323
9	2 244	14 366	91 216	614 764
Sum	24 405	147 764	922 816	6 188 960

Hypothesis 14.2. There are a positive integer m , a positive constant c and a sequence S of integers $\mu \rightarrow \infty$, such that for $\mu \in S$ and sufficiently large x , say $x \geq x_1 = x_1(\mu)$ with $\log x_1(\mu) = o(\mu)$, one has

$$A(x, m) = \frac{1}{m} \sum_{r=1}^m \pi_{2r}(x) \geq c \cdot \frac{1}{\mu} \sum_{r=1}^{\mu} \pi_{2r}(x). \tag{14.7}$$

15. Conditional abundance of prime pairs

We restate [Theorem 8.5](#):

Theorem 15.1. *Let m and c be as in [Hypothesis 14.2](#). Then*

$$\limsup_{x \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \frac{\theta_{2r}(x)}{x} \geq c. \tag{15.1}$$

As a result, there must be an abundance of prime pairs:

$$\limsup_{x \rightarrow \infty} \frac{1}{m} \sum_{r=1}^m \frac{\pi_{2r}(x)}{x/\log^2 x} \geq c. \tag{15.2}$$

Proof. The function

$$\Phi(x, \delta) = \frac{\log^2 x}{x^{2s}} \quad \text{with } s = s_\delta = (1/2) + \delta, \delta \in (0, 1/4) \tag{15.3}$$

is decreasing for $x \geq e^2$. We may assume that $x_1 = x_1(\mu) \geq e^2$. By [\(14.1\)](#), using integration by parts in the integral over $[x_1, \infty)$,

$$\tilde{D}_{2r}(s_\delta) \geq -\Phi(x_1, \delta)\pi_{2r}(x_1) - \int_{x_1}^{\infty} \pi_{2r}(x)d\Phi(x, \delta).$$

To bound the first term on the right one may use the inequality $\pi_{2r}(x) \leq 9C_{2r}x/\log^2 x$ which by [\(5.5\)](#) is valid for every r when $x \geq x_0$; we may of course assume $x_1 \geq x_0 \geq e^2$. Thus

$$-\Phi(x_1, \delta)\pi_{2r}(x_1) \geq -9C_{2r}.$$

Hence, by (14.1) and (14.7), taking μ in S ,

$$\begin{aligned} \frac{\delta}{m} \sum_{r \leq m} \left\{ \tilde{D}_{2r}(s_\delta) + 9C_{2r} \right\} &\geq - \int_{x_1}^\infty \frac{\delta}{m} \sum_{r \leq m} \pi_{2r}(x) d\Phi(x, \delta) \\ &\geq c \frac{\delta}{\mu} \sum_{r \leq \mu} \left[\tilde{D}_{2r}(s_\delta) + \int_2^{x_1} \pi_{2r}(x) d\Phi(x, \delta) \right]. \end{aligned} \tag{15.4}$$

Using a separate estimate for $\int_2^{x_0}$ based on the inequality $\pi_{2r}(x) \leq \pi(x)$ and recalling that the numbers C_{2r} have average 1, the final integral leads to an error term on the right-hand side that is

$$\geq -C\delta \{ \log^2 x_0 + \log x_1(\mu) \}.$$

Combining (15.4) and (14.5) one now finds that

$$\frac{\delta}{m} \sum_{r \leq m} \tilde{D}_{2r}(s_\delta) \geq c \left(\frac{1}{2} - \frac{1}{8\mu\delta} \right) - \mathcal{O}[\delta \{ \log^2 2\mu + \log x_1(\mu) \}]. \tag{15.5}$$

For given $\varepsilon \in (0, 1/2)$ we choose $\mu \rightarrow \infty$ in S and $\delta \searrow 0$ in $(0, 1/4)$ such that $1/(8\mu\delta) = \varepsilon/2$. Since $\log x_1(\mu) = o(\mu)$ we may then conclude that

$$\limsup_{\delta \searrow 0} \frac{\delta}{m} \sum_{r \leq m} \tilde{D}_{2r}(s_\delta) \geq (1 - \varepsilon)c/2. \tag{15.6}$$

We will show that this implies (15.1). Suppose to the contrary that one would have $(xm)^{-1} \sum_{r \leq m} \theta_{2r}(x) \leq c_1 < c$ for all $x \geq x_2 \geq 2$. Majorizing $\theta_{2r}(x)$ by $(\log^2 x)\pi(x)$ on the interval $2 \leq x \leq x_2$, we then find

$$\begin{aligned} \frac{1}{m} \sum_{r \leq m} \tilde{D}_{2r}\{(1/2) + \delta\} &= \frac{1}{m} \sum_{r \leq m} \int_{x_2}^\infty x^{-1-2\delta} d\theta_{2r}(x) + \mathcal{O}(1) \\ &\leq (1 + 2\delta) \int_{x_2}^\infty x^{-2-2\delta} \frac{1}{m} \sum_{r \leq m} \theta_{2r}(x) dx + \mathcal{O}(1) \leq (1 + 2\delta)c_1/(2\delta) + \mathcal{O}(1). \end{aligned}$$

As a result the left-hand side of (15.6) would be $\leq c_1/2 < c/2$. For small ε this gives a contradiction, which proves (15.1).

For (15.2) one may still use integration by parts:

$$\theta_{2r}(x) = \int_2^x (\log^2 t) d\pi_{2r}(t) \leq (\log^2 x)\pi_{2r}(x). \quad \square$$

16. Pair-correlation of zeta’s zeros and conditional abundance of prime pairs

Details on the results in this section may be found in the manuscript [36].

The analog of Lemma 9.2 for series can be used to show the following about the double sum $\Sigma_2^\lambda(s)$ in (8.1). The part in which $\text{Im } \rho$ and $\text{Im } \rho'$ have the same sign defines a meromorphic function for $1/2 \leq \sigma < 1$ whose only poles occur at complex zeros of $\zeta(\cdot)$. Thus, for a study of its pole-type behavior near the point $s = 1/2$, the double sum $\Sigma_2^\lambda(s)$ may be reduced to the sum $\Sigma_2^\lambda(s, 2)$ in which $\text{Im } \rho$ and $\text{Im } \rho'$ have opposite sign. Hence in the study of the PPC under RH, the differences of zeta’s zeros on the same side of the real axis play a key role. Careful asymptotic analysis gives

Theorem 16.1. Assume RH. Then the pole-type behavior of $\Sigma_{-}^{\lambda}(s, 2)$ and $\Sigma_{2}^{\lambda}(s)$ as $s \searrow 1/2$ is the same as that of the reduced sum

$$\Sigma_{*}^{\lambda}(s) = 2\pi \sum_{\gamma, \gamma'; |\gamma' - \gamma| < \gamma^{1/2}} \gamma^{-2s + i(\gamma - \gamma')} M^{\lambda}\{1 - 2s + i(\gamma - \gamma')\}, \quad (16.1)$$

where γ and γ' run over the imaginary parts of the zeros of $\zeta(\cdot)$ in the upper half-plane and $M^{\lambda}(\cdot)$ is given by (7.5).

The expression in (16.1) is reminiscent of the pair-correlation function of zeta's complex zeros which was first studied by Montgomery [39]. See also Gallagher and Mueller [15], Heath-Brown [27], Gallagher [14], Goldston and Montgomery [21], Goldston [16,17], Goldston and Gonek [18,19], Hejhal [28], Rudnick and Sarnak [43,44], Goldston, Gonek, Özlük and Snyder [20], Bogomolny and Keating [3], Chan [7–9], Montgomery and Soundararajan [40], and LMS Lecture Notes vol. 322 [38].

Since the constants C_{2r} have average 1, the function $R(\lambda)$ in (8.3) is $o(\lambda)$ as $\lambda \rightarrow \infty$; cf. (7.4). The corresponding hypothesis below regarding $\Sigma_{2}^{\lambda}(s) - \Sigma_{2}^{1}(s)$ would follow from a plausible counterpart to the pair-correlation work.

Hypothesis 16.2. For smooth E the ‘upper residue’

$$\omega(\lambda) = \omega^E(\lambda) = \limsup_{s \searrow 1/2} (s - 1/2) \{ \Sigma_{2}^{\lambda}(s) - \Sigma_{2}^{1}(s) \} \quad (16.2)$$

is $o(\lambda)$ as $\lambda \rightarrow \infty$.

If Hypothesis 16.2 is true, there will be an abundance of prime pairs:

Theorem 16.3. Assume Hypothesis 16.2. Then for every $\varepsilon > 0$, there is a positive integer m , depending on $\omega(\cdot)$ and ε , such that

$$\limsup_{x \rightarrow \infty} \frac{1}{m} \sum_{r \leq m} \frac{\pi_{2r}(x)}{x / \log^2 x} > 2 - \varepsilon. \quad (16.3)$$

Here the constant 2 would be optimal.

Acknowledgments

The author wishes to thank Fokko van de Bult for numerical support and Jan van de Craats for the drawings.

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