The Distribution of Zeros of Certain Entire Functions*

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1. INTRODUCTION

Linear difference and difference-differential equations with constant coefficients have long been of interest (cf. [1], [2], [3], [4]). More recently, generalizations to difference-integral and difference-differential-integral equations with the integrals of convolution type have been studied (cf. [5], [6], and (1.1) below). Much of the analysis concerns the homogeneous equations, which admit so-called fundamental solutions of the form \( q(t) = t^m e^{zt} \), where \( z \) is complex. Direct substitution shows that \( q(t) = t^m e^{zt} \) satisfies the equation iff \( z \) is a zero of multiplicity \( \geq m \) of a certain characteristic (entire) function \( \psi(z) \). The zeros of \( \psi(z) \) are also poles of the Laplace transform of solutions defined for \( t > 0 \). Residue evaluations of the inverse transform integrals yield expansions of more general solutions in terms of fundamental solutions. Thus, the distribution of the zeros of such entire functions \( \psi(z) \) is of some importance.

If the differences in the equation are commensurable, then a simple change of independent variable makes them integers. In the latter case, the difference terms and the difference-derivative terms contribute polynomials in \( e^z \) and in \( z \) and \( e^z \), respectively, to \( \psi(z) \). If the differences are incommensurable, then the characteristic function is more complicated and the problem of finding its zeros is accordingly more difficult.

We first consider the distribution of the zeros of the characteristic function of a difference-integral equation with integer differences.

Thus, consider the difference-integral equation

\[
\sum_{r=0}^{N} a_r q(t - r) = \int_0^N K(s) q(t - s) \, ds, \quad t \text{ real}
\]

(1.1)

where \( N \) is a fixed positive integer; \( K \) is an arbitrary function in complex \( L_1(0,N) \); the \( a_r \) are complex constants with \( a_0 \neq 0 \) and either \( a_N \neq 0 \) or

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Let (1.2) be written as

$$\psi(z) = e^{-Nz} P(e^z) - \int_0^N e^{-zs} K(s) \, ds$$

(2.1)
where
\[ P(\lambda) = \sum_{\nu=0}^{N} a_{\nu} \lambda^{N-\nu}. \] (2.2)

Suppose that \( \lambda \) is a nonzero zero of \( P(\lambda) \) of multiplicity \( \mu \). Define \( \xi \) so that
\[ e^{\xi} = \lambda, \quad \xi = \xi + i\eta, \quad 0 \leq \eta < 2\pi. \] (2.3)

Then \( e^{-Nz} P(e^{z}) \) and \( P(e^{z}) \) have common zeros of multiplicity \( \mu \) given by
\[ \xi + 2k\pi i \quad (k = 0, \pm 1, \ldots). \]

Anselone and Greenspan [5, p. 731] have shown that there is one and only one zero of \( \psi(z) \) near each zero of \( e^{-Nz} P(e^{z}) \); however, there is a slight solecism in their statement, which we rectify. There is a positive minimum distance \( \Delta \) between any two distinct zeros of \( e^{-Nz} P(e^{z}) \). For each \( \delta > 0 \) such that \( \delta < \frac{1}{2} \Delta \) and each \( k = 0, \pm 1, \ldots \) let
\[ \Gamma_{k}(\delta) = \{ z : |z - (\xi + 2k\pi i)| < \delta \}. \]

Then if \( \lambda \) is a nonzero zero of \( P(\lambda) \) of multiplicity \( \mu \), an application of Rouche’s theorem yields that there are \( \mu \) zeros (counting multiplicities) of \( \psi(z) \) in each \( \Gamma_{k}(\delta) \) for |\( k \)| sufficiently large.

Let \( \Omega \) denote the set of zeros of \( \psi(z) \) other than those near the zeros of \( e^{-Nz} P(e^{z}) \). Anselone and Greenspan have also shown that if \( a_{N} \neq 0 \), then \( \Omega \) is a finite set, and if \( \Omega \) is infinite, it consists of a sequence of complex numbers whose real parts converge to \( -\infty \).

The following theorem completes the description of the distribution of the zeros of \( \psi(z) \):

**Theorem 2.1.** If \( a_{N} = 0, a_{N-1} \neq 0 \), then \( \Omega \) is an infinite set if and only if \( K(s) \neq 0 \) on a subset of \((N - 1, N)\) of positive measure.

**Proof.** That \( \Omega \) is finite if \( K(s) = 0 \) a.e. on \((N - 1, N)\) follows from the work of Anselone and Greenspan, as indicated above.

For the remainder of the proof we will employ the following lemma of Cartwright [9, Theorem 55, p. 87]:

**Lemma 2.2.** If \( \psi(z) \) is an entire function of order one and mean type such that
\[ \int_{1}^{\infty} \{ \log^{+} |\psi(iy)| + \log^{+} |\psi(-iy)| \} \frac{dy}{y^{2}} < \infty \]

where \( \log^{+} a = \max (\log a, 0) \), then
(i) there exist $A_1$ and $A_2$ with $-\infty < A_1 \leq A_2 < \infty$ such that

$$h(\theta) = \max (A_1 \cos \theta, A_2 \cos \theta),$$

where $h(\theta)$ is the Phragmén-Lindelöf indicator function defined by

$$h(\theta) = \lim_{r \to \infty} r^{-1} \log |\psi(\text{re}^{i\theta})|.$$

(ii) $n(r) \sim (A_2 - A_1) r/\pi$ as $r \to \infty$,

where $n(r)$ is the number of zeros of $\psi(z)$ in the disk $|z| \leq r$.

Clearly $\psi(z)$ is of order one and mean type. By the Riemann-Lebesgue lemma [10, p. 11]

$$\int_0^N K(s) e^{-(x_0 + iy)s} ds \to 0 \quad \text{as} \quad |y| \to \infty$$

for each fixed $x_0$. Also

$$\left| \int_0^N K(s) e^{-z\bar{s}} ds \right| \leq \int_0^N |K(s)| ds \quad \text{for} \quad \text{Re} \ z \geq x_0$$

and so, by the Phragmén-Lindelöf theorem [11, p. 47]

$$\int_0^N K(s) e^{-z\bar{s}} ds \to 0 \quad \text{as} \quad |y| \to \infty$$

uniformly on each finite $x$-interval. Therefore, on each finite $x$-interval, the asymptotic behavior of $\psi(x)$ as $|y| \to \infty$ is that of $e^{-Nz} P(e^z)$, i.e.,

$$\psi(x + iy) = O(1) \quad \text{as} \quad |y| \to \infty,$$

uniformly on each finite $x$-interval. Hence

$$\int_1^\infty \left\{ \log^+ |\psi(iy)| + \log^+ |\psi(-iy)| \right\} \frac{dy}{y^2} < \infty.$$
Since \( K(s) \neq 0 \) on a subset of \((N - 1, N)\) of positive measure,

\[
\psi(re^{it}) \sim \int_{N-1}^{N} K(s) e^{rs} ds.
\]

By a lemma of Titchmarsh [12, Lemma 2.3], the assumptions on \( K \) insure that there exists a sequence \( \{R_n\} \) of real numbers with \( R_n \to \infty \) such that

\[
\left| \int_{N-1}^{N} K(s) e^{R_n s} ds \right| > A e^{(N-1+\varepsilon)R_n}
\]

for some \( A > 0 \) and some \( \varepsilon > 0 \). Consequently

\[
h(\pi) = \lim_{r \to \infty} r^{-1} \log |\psi(re^{it})| \geq N - 1 + \varepsilon.
\]

If \( \cos \theta > 0, |\psi(re^{it})| \sim |a_0| \neq 0 \) and hence

\[
h(\theta) = \lim_{r \to \infty} r^{-1} \log |a_0| = 0.
\]

Now \( h(\theta) \) is the support function of a convex set (the indicator diagram of \( \psi \)) which we have just shown contains the points 0 and \(- (N - 1 + \varepsilon)\), cf. Levin [11, p. 77]. Hence

\[
h(\theta) \geq - (N - 1 + \varepsilon) \cos \theta \quad \text{for} \quad \frac{\pi}{2} < \theta < \frac{3\pi}{2}.
\]

By (i) of Lemma 2.2,

\[
h(\theta) = \max (A_1 \cos \theta, A_2 \cos \theta),
\]

where \( A_1 \leq - (N - 1 + \varepsilon) \) and \( A_2 = 0 \). Consequently there exists \( p(r) \) with \( n(r) \sim p(r) \) and

\[
p(r) \geq (N - 1 + \varepsilon) \frac{r}{\pi}
\]

for \( r \) sufficiently large. Let \( n_1(r) \) be the number of zeros of

\[
\sum_{r=0}^{N-1} a_r e^{-r^2}.
\]

Since

\[
\sum_{r=0}^{N-1} a_r \lambda^r
\]

has at most \( N - 1 \) distinct zeros, there exists \( q(r) \) such that \( n_1(r) \sim q(r) \) and

\[
q(r) \leq (N - 1) \frac{r}{\pi}
\]
for $r$ sufficiently large. Thus the number of points in $\Omega$ is

$$n(r) - n_1(r) \sim p(r) - q(r) \geq \frac{er}{\pi}$$

for $r$ sufficiently large, i.e., $\Omega$ is an infinite set.

3. Zeros of a More General Entire Function

The methods of this paper are also applicable to the more general function

$$\psi(z) = \sum_{\nu=0}^{m} a_{\nu}[1 + \epsilon_{\nu}(z)] e^{-\beta_{\nu} z} - \int_{\gamma}^{0} K(s) e^{-s \nu} ds$$

(3.1)

where the $\beta_{\nu}$ are commensurable real numbers such that

$$0 = \beta_0 < \beta_1 < \cdots < \beta_m \leq \beta; \quad a_{\nu} \neq 0, \quad v = 0, 1, \cdots, m,$$

and

$$\lim_{|z| \to \infty} \epsilon_{\nu}(z) = 0, \quad v = 0, 1, \cdots, m.$$  

We again assume that $K \in L_2(0, \beta)$.

By Rouché's theorem $\psi(z)$ has one and only one zero near each zero of

$$g(z) = \sum_{r=0}^{m} a_{r}[1 + \epsilon_{r}(z)] e^{-\beta_{r} z}.$$  

(3.2)

Let $\Omega$ be the set of zeros of $\psi(z)$ other than those near the zeros of $g(z)$. If $\beta_m = \beta$, we again have that $\Omega$ is finite.

It has been shown by Bellman and Cooke [1, p. 405] that the number $n_1(r)$ of zeros of $g(z)$ in the disk $|z| \leq r$ is asymptotic to $q(r)$ where

$$q(r) \leq \frac{r \beta_m}{\pi}$$

Applying the methods of Section 2 we obtain

**Theorem 3.1.** If $\beta_m < \beta$, $\Omega$ is infinite if and only if $K(s) \neq 0$ on a subset of $(\beta_m, \beta)$ of positive measure.

**References**