JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 70, 131–160 (1979)

Self-Adjoint Extensions of the Laplace Operator with Respect to Electric and Magnetic Boundary Conditions

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Submitted by C. L. Dolph

I. INTRODUCTION

The following considerations are motivated by the initial and boundary value problem of perfect reflection in the theory of electromagnetic wave propagation:

(EM) *Find a pair* (E, H) *of vector fields satisfying*

(i) *the Maxwell equations*

$$\nabla \times E + \mu \frac{\partial}{\partial t} H = 0, \tag{1.1}$$

$$\nabla \times H - \epsilon \frac{\partial}{\partial t} E - \sigma E = J \tag{1.2}$$

in the exterior Ω *of a finite collection of disjoint bounded bodies with smooth (say* C^3*) boundaries* $S_1, \ldots, S_n$;

(ii) *the boundary condition*

$$n \times E = 0 \qquad on \qquad \partial\Omega = S_1 + \cdots + S_n \tag{1.3}$$

where n *denotes the exterior normal unit vector on* $\partial\Omega$;

(iii) *the initial conditions*

$$E(x, 0) = E_0(x), \qquad H(x, 0) = H_0(x) \quad for\ x \in \bar{\Omega}. \tag{1.4}$$

$E(x, t)$ and $H(x, t)$ are the electric and the magnetic field produced by a given current distribution $J(x, t)$ in the presence of n perfectly conducting bodies with boundaries $S_1, \ldots, S_n$. We assume in this paper that Ω is filled by a homogeneous isotropic medium. In this case the dielectricity ϵ, the permeability μ and the electric conductivity σ are real numbers with $\epsilon > 0$, $\mu > 0$, $\sigma \geqslant 0$.

As we shall show in Section 2, problem (EM) can be reduced to each of the following two problems, by eliminating either the magnetic field H or the electric field E:

0022-247X/79/070131–30$02.00/0

(E) *Find a vector field E to given data J_1, γ_1, E_0, E_1 such that*

$$\Delta E = \epsilon\mu \frac{\partial^2}{\partial t^2} E + \mu\sigma \frac{\partial}{\partial t} E + J_1 \quad \text{in} \quad \Omega, \tag{1.5}$$

$$n \times E = 0, \qquad \nabla \cdot E = \gamma_1 \quad \text{on} \quad \partial\Omega, \tag{1.6}$$

$$E(x, 0) = E_0(x), \qquad \frac{\partial}{\partial t} E(x, 0) = E_1(x) \qquad \text{for } x \in \bar{\Omega}. \tag{1.7}$$

(H) *Find a vector field H to given data J_2, c, γ_2, H_0, H_1 such that*

$$\Delta H = \epsilon\mu \frac{\partial^2}{\partial t^2} H + \mu\sigma \frac{\partial}{\partial t} H + J_2 \quad \text{in} \quad \Omega, \tag{1.8}$$

$$n \times (\nabla \times H) = c, \qquad n \cdot H = \gamma_2 \quad \text{on} \quad \partial\Omega, \tag{1.9}$$

$$H(x, 0) = H_0(x), \qquad \frac{\partial}{\partial t} H(x, 0) = H_1(x) \qquad \text{for } x \in \bar{\Omega}. \tag{1.10}$$

Initial and boundary value problems of this type can be treated by extending the differential operator to a suitable self-adjoint operator and using the functional calculus for unbounded self-adjoint operators. This method has been applied with great success to boundary and initial value problems for the scalar wave equation with Dirichlet and Neumann boundary data by several authors, including Eidus [3], Shenk [7] and Wilcox [13]. The spectral-theoretical approach seems to be most appropriate to the discussion of the relations between the time-dependent theory and the corresponding time-independent problems arising in the investigation of time-harmonic wave fields and governed by the reduced wave equation (compare, for example, the proof of the limiting amplitude principle for the scalar wave equation by Eidus [3]). The purpose of this paper is to develop the foundations for a similar approach to the initial and boundary value problems (E) and (M) for the vector wave equation. The main step consists in the construction of concrete self-adjoint extensions A and A' of the vector Laplacian with respect to the electric boundary conditions

$$n \times E = 0, \qquad \nabla \cdot E = 0 \quad \text{on} \quad \partial\Omega \tag{1.11}$$

and to the magnetic boundary conditions

$$n \times (\nabla \times H) = 0, \qquad n \cdot H = 0 \quad \text{on} \quad \partial\Omega. \tag{1.12}$$

By applying the functional calculus for self-adjoint operators to A and A', we obtain weak solutions of the problems (E) and (M) which can be represented by spectral integrals employing the spectral sets of A and A'.

The plan of this paper is as follows: In Section 2 we discuss the relation between the classical solutions of the electromagnetic problem (EM) and the problems (E) and (M). In Section 3 we introduce the operators A and A'. The proof of their self-adjointness is based on a weak existence theory for the time-independent equation $(-\Delta + \lambda)\, G = F$ with respect to the boundary data (1.11) and (1.12) for sufficiently large real λ. Weak solutions of these boundary value problems are constructed in Sections 4 and 5, by giving equivalent formulations by means of bilinear forms and proving coerciveness properties. In the last section we discuss weak versions of the time-dependent problems (E) and (M) in the case of homogeneous boundary data.

It is planned to treat the following topics in subsequent papers:

(a) regularity properties of the vector Laplacian with respect to the boundary conditions (1.11) and (1.12);

(b) discussion of the spectral properties of the operators A and A';

(c) a more detailed investigation of the time-dependent problems (E), (M), (EM) including regularity properties of the solutions and asymptotic estimates as $t \to \infty$.

It can be shown that the spectrum of A and A' consists of an eigenvalue at $\lambda = 0$ which corresponds to electrostatics and to magnetostatics, respectively, and of a continuous part covering the half axis $[0, \infty)$. The dimensions of the null spaces of A and A' are n and $p = p_1 + \cdots + p_n$ where n is the number of reflecting bodies and p_j denotes the topological genus of the boundary S_j of the j-th reflector. For $\lambda > 0$, the projection operators P_λ and P'_λ in the spectral sets of A and A' can be expressed by outgoing and incoming solutions of the time-independent boundary value problems with frequency $\lambda^{1/2}$. In contrast to this, the spectrum in the scalar case studied in [3], [7] and [13] contains only a continuous part.

The presence of the eigenvalue $\lambda = 0$ leads to the consequence that, again in contrast to the scalar situation, the principle of limiting amplitude does not hold in general for the initial and boundary value problem (EM) with time-harmonic current distribution $J(x, t)$. By using the spectral-theoretical approach which will be developed here and in the following papers, it is possible to derive necessary and sufficient conditions for the validity of the principle of limiting amplitude. It turns out that a full discussion of the asymptotic behavior of the solution (E, H) of problem (EM) as $t \to \infty$ requires the investigation of the spectral properties of *both* operators A and A'. In fact, by applying the functional calculus for self-adjoint operators to A and A', we obtain two integral representations for (E, H) the first one of which employs the spectral set of A and the second one the spectral set of A'. The first representation allows a complete discussion of the asymptotic behavior of E while the second representation is appropriate for the discussion of H as $t \to \infty$. This remark shows that it is desirable to discuss *both* problems (E) and (M) and the corresponding self-adjoint operators A and A' in order to get sufficient insight into the asymptotic properties of the solution of problem (EM).

Some parts of this paper result from cooperation with A. Fetzer (compare [4]). The proof of Lemma 5.1 is related to investigations of R. Leis on the time-independent boundary value problem with electric boundary conditions for non-homogeneous anisotropic media [5]. Different approaches to a weak existence theory for problem (EM) have been discussed by C. H. Wilcox [12], G. Schmidt [8], G. Duvaut and J. L. Lions [2], and R. Picard [6].

The methods of this paper can be extended to non-homogeneous and anisotropic media and to interface problems (compare C. Weber [10]).

2. Relations Between the Problems (*EM*), (*E*) and (*M*)

In this section we study relations between the initial and boundary value problems (EM), (E) and (M) which were formulated in Section 1. We shall use the notation $\underline{C}^k := C^k \times C^k \times C^k$.

First we show:

Lemma 2.1. *Let* (E, H) *be a solution of* (EM) *where* $E, H, J \in \underline{C}^2(\bar{\Omega} \times [0, \infty))$ *and* $E_0, H_0 \in \underline{C}^2(\bar{\Omega})$. *Then* E *is a solution of* (E) *with*

$$\begin{aligned} J_1(x,t) &:= e^{-\sigma t/\epsilon}\nabla(\nabla \cdot E_0(x)) + \mu \frac{\partial}{\partial t} J(x,t) \\ &\quad - \frac{1}{\epsilon}\int_0^t e^{-\sigma(t-\tau)/\epsilon}\nabla(\nabla \cdot J(x,\tau))\,d\tau \qquad \text{for } x \in \bar{\Omega}, \\ \gamma_1(x,t) &:= e^{-\sigma t/\epsilon}\nabla \cdot E_0(x) - \frac{1}{\epsilon}\int_0^t e^{-\sigma(t-\tau)/\epsilon}\nabla \cdot J(x,\tau)\,d\tau \qquad \text{for } x \in \partial\Omega, \\ E_1(x) &:= \frac{1}{\epsilon}[\nabla \times H_0(x) - \sigma E_0(x) - J(x,0)] \qquad \text{for } x \in \bar{\Omega} \end{aligned}$$

and H *is a solution of* (H) *with*

$$J_2 := \nabla(\nabla \cdot H_0) - \nabla \times J, \qquad c := n \times J,$$

$$\gamma_2 := n \cdot H_0, \qquad H_1 := -\frac{1}{\mu}\nabla \times E_0.$$

Remarks 1. It is possible to weaken the regularity requirements for the solution and the data in Lemma 2.1 and in the remaining part of this section to a certain extent. It is not our intention to give "best" results in this respect.

2. The assumptions on E, H in Lemma 2.1 impose certain restrictions ("compatibility conditions") on the data E_0, H_0 and J, such as $n \times E_0 = 0$ and $n \times (\nabla \times H_0) = n \times J(., 0)$ on $\partial\Omega$. Here and in the following we do not state these restrictions explicitely.

Proof of Lemma 2.1. Set $y(t) = \nabla \cdot E(x, t)$ for fixed $x \in \bar{\Omega}$. Since $-\epsilon y' - \sigma y = \nabla \cdot J$ and $y(0) = \nabla \cdot E_0$, we obtain

$$\nabla \cdot E(x, t) = e^{-\sigma t/\epsilon} \nabla \cdot E_0(x) - \frac{1}{\epsilon} \int_0^t e^{-\sigma(t-\tau)/\epsilon} \nabla \cdot J(x, \tau)\, d\tau.$$

Furthermore, (1.1) and (1.2) imply that

$$\begin{aligned} \epsilon\mu \frac{\partial^2}{\partial t^2} E &= \mu \frac{\partial}{\partial t} (\nabla \times H - \sigma E - J) \\ &= -\nabla \times (\nabla \times E) - \mu\sigma \frac{\partial}{\partial t} E - \mu \frac{\partial}{\partial t} J \\ &= \Delta E - \nabla(\nabla \cdot E) - \mu\sigma \frac{\partial}{\partial t} E - \mu \frac{\partial}{\partial t} J. \end{aligned}$$

By inserting the above expression for $\nabla \cdot E$, it follows that E is a solution of (E) with the required data. The statement on H can be proved in the same way. The boundary condition $n \cdot H = n \cdot H_0$ follows from

$$n \cdot \frac{\partial}{\partial t} H = -\frac{1}{\mu} n \cdot (\nabla \times E) = -\frac{1}{\mu} \nabla_0 \cdot (n \times E) = 0$$

where $\nabla_0 \cdot a$ denotes the surface divergence of the tangential field a.

Now we prove the following converse of Lemma 2.1:

LEMMA 2.2. *Assume that* $J \in \underset{\sim}{C}^3(\bar{\Omega} \times [0, \infty))$ *and* E_0, $H_0 \in \underset{\sim}{C}^3(\bar{\Omega})$. *Let* $E \in \underset{\sim}{C}^3(\bar{\Omega} \times [0, \infty))$ *be a solution of problem* (E) *with data* J_1, γ_1 *and* E_1 *defined as in Lemma* 2.1, *and set, according to* (1.1),

$$H(x, t) = H_0(x) - \frac{1}{\mu} \int_0^t \nabla \times E(x, \tau)\, d\tau. \tag{2.1}$$

Then (E, H) *is a solution of* (EM).

LEMMA 2.3. *Assume that* $J \in \underset{\sim}{C}^3(\bar{\Omega} \times [0, \infty))$, $n \times E_0 = 0$ *on* $\partial\Omega$, *and* E_0, $H_0 \in \underset{\sim}{C}^3(\bar{\Omega})$. *Let* $H \in \underset{\sim}{C}^3(\bar{\Omega} \times [0, \infty))$ *be a solution of problem* (M) *with data* J_2, c, γ_2 *and* H_1 *defined as in Lemma* 2.1, *and set, according to* (1.2),

$$E(x, t) = e^{-\sigma t/\epsilon} E_0(x) + \frac{1}{\epsilon} \int_0^t e^{-\sigma(t-\tau)/\epsilon} [\nabla \times H(x, \tau) - J(x, \tau)]\, d\tau. \tag{2.2}$$

Then (E, H) *is a solution of* (EM).

The proofs of Lemma 2.2 and Lemma 2.3 are based on the following uniqueness statement for the scalar wave equation:

LEMMA 2.4. *Assume that* $\varphi \in C^2(\bar{\Omega} \times [0, \infty))$ *is a solution of the homogeneous problem*

$$\text{(i)} \qquad \Delta\varphi = \epsilon\mu \frac{\partial^2}{\partial t^2}\varphi + \mu\sigma \frac{\partial}{\partial t}\varphi \qquad \text{in} \qquad \bar{\Omega},$$

$$\text{(ii)} \qquad \varphi = 0 \left(\text{or } \frac{\partial}{\partial n}\varphi = 0\right) \qquad \text{on} \qquad \partial\Omega,$$

$$\text{(iii)} \quad \varphi(x, 0) = \frac{\partial}{\partial t}\varphi(x, 0) = 0 \qquad \text{for } x \in \bar{\Omega}.$$

Then we have $\varphi = 0$ *in* $\bar{\Omega} \times [0, \infty)$.

Note that no assumptions on the behavior of φ at infinity are required, according to the boundedness of the domains of dependence for hyperbolic problems. The proof of Lemma 2.4 is elementary and can be obtained by a slight modification of the uniqueness argument described in [14], Section 179 for 2-dimensional bounded domains Ω in the case $\sigma = 0$.

Proof of Lemma 2.2. The first Maxwell equation (1.1) and the initial conditions (1.4) are satisfied by the definition of H in (2.1). Hence it suffices to verify the second Maxwell equation (1.2). By (2.1) we have

$$\begin{aligned}
&\nabla \times H(x, t) - \epsilon \frac{\partial}{\partial t} E(x, t) - \sigma E(x, t) \\
&\quad = \nabla \times H_0(x) - \frac{1}{\mu}\int_0^t \nabla \times [\nabla \times E(x, \tau)]\, d\tau - \epsilon \frac{\partial}{\partial t} E(x, t) - \sigma E(x, t) \\
&\quad = \nabla \times H_0(x) + \frac{1}{\mu}\int_0^t \Big[\Delta E(x, \tau) - \nabla(\nabla \cdot E(x, \tau)) \\
&\qquad - \epsilon\mu \frac{\partial^2}{\partial t^2} E(x, \tau) - \sigma\mu \frac{\partial}{\partial t} E(x, \tau)\Big]\, d\tau - \epsilon \frac{\partial}{\partial t} E(x, 0) - \sigma E_0(x).
\end{aligned}$$

By using (1.5), (1.7) and the definitions of J_1 and E_1 in Lemma 2.1, we obtain

$$\begin{aligned}
&\nabla \times H(x, t) - \epsilon \frac{\partial}{\partial t} E(x, t) - \sigma E(x, t) \\
&\quad = \frac{1}{\mu}\int_0^t [J_1(x, \tau) - \nabla(\nabla \cdot E(x, \tau))]\, d\tau + J(x, 0) \\
&\quad = J(x, t) + \frac{1}{\mu}\nabla\varphi(x, t)
\end{aligned}$$

with

$$\varphi(x, t) = \int_0^t \left[e^{-\sigma\tau/\epsilon}\nabla \cdot E_0(x) - \nabla \cdot E(x, \tau) - \frac{1}{\epsilon}\int_0^\tau e^{-\sigma(\tau-\rho)/\epsilon}\nabla \cdot J(x, \rho)\, d\rho\right] d\tau. \tag{2.3}$$

We show that φ satisfies the assumptions of Lemma 2.4. The boundary condition $\varphi = 0$ on $\partial\Omega$ follows immediately from $\nabla \cdot E = \gamma_1$ on $\partial\Omega$ and the definition of γ_1 in Lemma 2.1. By differentiating (2.3), we get

$$\frac{\partial}{\partial t}\varphi(x, t) = e^{-\sigma t/\epsilon}\nabla \cdot E_0(x) - \nabla \cdot E(x, t) - \frac{1}{\epsilon}\int_0^t e^{-\sigma(t-\rho)/\epsilon}\nabla \cdot J(x, \rho)\, d\rho, \tag{2.4}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\varphi(x, t) &= -\frac{\sigma}{\epsilon}e^{-\sigma t/\epsilon}\nabla \cdot E_0(x) - \nabla \cdot \left[\frac{\partial}{\partial t}E(x, t)\right] \\ &\quad - \frac{1}{\epsilon}\nabla \cdot J(x, t) + \frac{\sigma}{\epsilon^2}\int_0^t e^{-\sigma(t-\tau)/\epsilon}\nabla \cdot J(x, \tau)\, d\tau.\end{aligned} \tag{2.5}$$

The initial conditions $\varphi(x, 0) = (\partial/\partial t)\,\varphi(x, 0) = 0$ follow from (2.3) and (2.4). By (2.4) and (2.5) we have

$$\begin{aligned}&\epsilon\frac{\partial^2}{\partial t^2}\varphi(x, t) + \sigma\frac{\partial}{\partial t}\varphi(x, t) \\ &\quad = -\nabla \cdot \left[\epsilon\frac{\partial}{\partial t}E(x, t) + \sigma E(x, t) + J(x, t)\right] \\ &\quad = -\int_0^t \nabla \cdot \left[\epsilon\frac{\partial^2}{\partial t^2}E(x, \tau) + \sigma\frac{\partial}{\partial t}E(x, \tau) + \frac{\partial}{\partial t}J(x, \tau)\right] d\tau\end{aligned}$$

since $\nabla \cdot [\epsilon(\partial/\partial t)\,E + \sigma E + J]$ vanishes for $t = 0$ by the definition of E_1 in Lemma 2.1. Hence we obtain by (1.5) and the definitions of J_1 and φ

$$\begin{aligned}&\epsilon\mu\frac{\partial^2}{\partial t^2}\varphi(x, t) + \mu\sigma\frac{\partial}{\partial t}\varphi(x, t) \\ &\quad = \int_0^t \nabla \cdot \left[J_1(x, \tau) - \Delta E(x, \tau) - \mu\frac{\partial}{\partial t}J(x, \tau)\right] d\tau \\ &\quad = \int_0^t \nabla \cdot \Big[e^{-\sigma\tau/\epsilon}\nabla(\nabla \cdot E_0(x)) - \Delta E(x, \tau) \\ &\qquad - \frac{1}{\epsilon}\int_0^\tau e^{-\sigma(\tau-\rho)/\epsilon}\nabla(\nabla \cdot J(x, \rho))\, d\rho\Big]\, d\tau \\ &\quad = \Delta\varphi(x, t).\end{aligned}$$

Here we have used that $\nabla \cdot (\Delta E) = \nabla \cdot [\nabla(\nabla \cdot E) - \nabla \times (\nabla \times E)] = \Delta(\nabla \cdot E)$. This shows that φ satisfies all assumptions of Lemma 2.4. Hence φ vanishes in $\bar{\Omega} \times [0, \infty)$ by Lemma 2.4, and the formula preceding (2.3) implies that (E, H) satisfies the second Maxwell equation (1.2). This concludes the proof of Lemma 2.2.

Proof of Lemma 2.3. Since $n \times (\nabla \times H) = c = n \times J$, it follows immediately from (2.2) and $n \times E_0 = 0$ on $\partial\Omega$ that (E, H) satisfies the second Maxwell equation (1.2), the boundary condition (1.3) and the initial conditions (1.4). Hence it suffices to verify the first Maxwell equation (1.1). First we show that

$$\psi = \nabla \cdot H - \nabla \cdot H_0 \tag{2.6}$$

satisfies the assumptions of Lemma 2.4 with $\partial\psi/\partial n = 0$ on $\partial\Omega$. By (2.6) we have $\psi(x, 0) = 0$. The second initial condition $(\partial/\partial t)\,\psi(x, 0) = 0$ follows from $(\partial/\partial t)\,\psi(x, 0) = \nabla \cdot H_1$ and $H_1 = -\mu^{-1}\nabla \times E_0$. The definition of J_2 in Lemma 2.1 implies that $\nabla \cdot J_2 = \Delta(\nabla \cdot H_0)$. Hence we obtain by (1.8)

$$\begin{aligned}\Delta\psi &= \Delta(\nabla \cdot H - \nabla \cdot H_0) = \nabla \cdot (\Delta H) - \Delta(\nabla \cdot H_0)\\ &= \nabla \cdot \left(\epsilon\mu \frac{\partial^2}{\partial t^2} H + \mu\sigma \frac{\partial}{\partial t} H + J_2\right) - \Delta(\nabla \cdot H_0)\\ &= \left(\epsilon\mu \frac{\partial^2}{\partial t^2} + \mu\sigma \frac{\partial}{\partial t}\right) \nabla \cdot H = \left(\epsilon\mu \frac{\partial^2}{\partial t^2} + \mu\sigma \frac{\partial}{\partial t}\right) \psi.\end{aligned}$$

Thus ψ satisfies the homogeneous wave equation. Since $n \times (\nabla \times H) = n \times J$, we have on $\partial\Omega$

$$n \cdot [\nabla \times (\nabla \times H)] = \nabla_0 \cdot [n \times (\nabla \times H)] = \nabla_0 \cdot (n \times J) = n \cdot (\nabla \times J)$$

and hence

$$\begin{aligned}\frac{\partial}{\partial n}\psi &= n \cdot \nabla\psi = n \cdot [\nabla(\nabla \cdot H) - \nabla(\nabla \cdot H_0)]\\ &= n \cdot [\nabla \times (\nabla \times H) + \Delta H - \nabla(\nabla \cdot H_0)]\\ &= n \cdot [\nabla \times J + \Delta H - \nabla(\nabla \cdot H_0)].\end{aligned}$$

Since $n \cdot H = \gamma_2 := n \cdot H_0$, we have $(\partial/\partial t)\, n \cdot H = 0$ on $\partial\Omega$ and hence by (1.8)

$$n \cdot \Delta H = n \cdot J_2 .$$

Thus we obtain

$$\frac{\partial}{\partial n}\psi = n \cdot [\nabla \times J + J_2 - \nabla(\nabla \cdot H_0)] = 0$$

by the definition of J_2. This proves that ψ satisfies all assumptions of Lemma 2.4. Therefore we have $\psi = 0$ and hence by (2.6)

$$\nabla \cdot H = \nabla \cdot H_0 \quad \text{in} \quad \bar{\Omega} \times [0, \infty). \tag{2.7}$$

It follows from (2.2) and (1.8) that

$$\begin{aligned}
&\nabla \times E(x, t) + \mu \frac{\partial}{\partial t} H(x, t) \\
&= e^{-\sigma t/\epsilon} \nabla \times E_0(x) + \mu \frac{\partial}{\partial t} H(x, t) \\
&\quad + \frac{1}{\epsilon} \int_0^t e^{-\sigma(t-\tau)/\epsilon} [\nabla(\nabla \cdot H(x, \tau)) - \Delta H(x, \tau) - \nabla \times J(x, \tau)]\, d\tau \\
&= e^{-\sigma t/\epsilon} \nabla \times E_0(x) + \mu \frac{\partial}{\partial t} H(x, t) + \frac{1}{\epsilon} \int_0^t e^{-\sigma(t-\tau)/\epsilon} \Big[\nabla(\nabla \cdot H(x, \tau)) \\
&\quad - \epsilon\mu \frac{\partial^2}{\partial t^2} H(x, \tau) - \mu\sigma \frac{\partial}{\partial t} H(x, \tau) - J_2(x, \tau) - \nabla \times J(x, \tau)\Big]\, d\tau.
\end{aligned}$$

Since $J_2 := \nabla(\nabla \cdot H_0) - \nabla \times J = \nabla(\nabla \cdot H) - \nabla \times J$ by (2.7), we obtain, by integrating by parts,

$$\begin{aligned}
&\nabla \times E(x, t) + \mu \frac{\partial}{\partial t} H(x, t) \\
&= e^{-\sigma t/\epsilon} \nabla \times E_0(x) + \mu \frac{\partial}{\partial t} H(x, t) \\
&\quad - \frac{1}{\epsilon} \int_0^t e^{-\sigma(t-\tau)/\epsilon} \left[\epsilon\mu \frac{\partial^2}{\partial t^2} H(x, \tau) + \mu\sigma \frac{\partial}{\partial t} H(x, \tau)\right] d\tau \\
&= e^{-\sigma t/\epsilon} \nabla \times E_0(x) + \mu \frac{\partial}{\partial t} H(x, t) - \mu \left[e^{-\sigma(t-\tau)/\epsilon} \frac{\partial}{\partial t} H(x, \tau)\right]_{\tau=0}^{t} \\
&= e^{-\sigma t/\epsilon} \nabla \times E_0(x) + \mu e^{-\sigma t/\epsilon} H_1(x) = 0
\end{aligned}$$

since $H_1 = -\mu^{-1} \nabla \times E_0$. Hence also the first Maxwell equation (1.1) holds. This concludes the proof of Lemma 2.3.

As a corollary of Lemma 2.1 and Lemma 2.4 we obtain the following uniqueness theorem for problem (EM):

LEMMA 2.5. *Assume that* (E, H) *is a solution of* (EM) *with* $J = 0$, $E_0 = 0$, $H_0 = 0$ *and that* $E, H \in \underline{C}^2(\bar{\Omega} \times [0, \infty))$. *Then we have* $E = 0$ *in* $\bar{\Omega} \times [0, \infty)$.

3. THE OPERATORS A AND A'

Now we turn to the main topic of this paper, the discussion of the vector Laplace operator with respect to the electric and magnetic boundary data (1.11) and (1.12). First we introduce the relevant linear spaces.

As in [11] we interpret the elements of Lebesgue and Sobolev spaces as distributions. In particular, we define $L_2 = L_2(\Omega)$ as the linear space consisting of all functionals F on $C_0^\infty = C_0^\infty(\Omega)$ such that

$$\|F\| := \operatorname{supr}\left\{|F\varphi| : \varphi \in C_0^\infty, \int |\varphi|^2\,dx = 1\right\} < \infty. \tag{3.1}$$

L_2 is a Hilbert space with norm (3.1). Every $u \in C(\Omega)$ with $\int |u|^2\,dx < \infty$ generates a functional $F_u \in L_2$ by

$$F_u\varphi = \int u\varphi\,dx \qquad \text{for } \varphi \in C_0^\infty, \tag{3.2}$$

and since $\partial\Omega$ is smooth, it can be shown that

$$\|u\| := \|F_u\| = \left[\int_\Omega |u|^2\,dx\right]^{1/2} \tag{3.3}$$

(compare [11], Lemma 2.5). In the following we identify u and F_u. In this sense C_0 is a dense subspace of L_2 (compare [11], Theorem 2.1). The k-th Sobolev space $H_k = H_k(\Omega)$ is defined by

$$H_k := \{F \in L_2 : D^pF \in L_2 \text{ if } |p| \leqslant k\} \tag{3.4}$$

where the derivative D^pF with $p = (p_1, p_2, p_3)$, $|p| = p_1 + p_2 + p_3$, $D^p = \partial_1^{p_1}\partial_2^{p_2}\partial_3^{p_3}$ are understood in the sense of the theory of distributions:

$$(D^pF)\,\varphi := (-1)^{|p|}\,F(D^p\varphi) \qquad \text{for } \varphi \in C_0^\infty. \tag{3.5}$$

H_k is a Hilbert space with the inner product

$$(F, G)_k := \sum_{0 \leqslant |p| \leqslant k} (D^pF, D^pG). \tag{3.6}$$

Furthermore, we shall use the space $\underline{L}_2 := L_2 \times L_2 \times L_2$ and $\underline{H}_k := H_k \times H_k \times H_k$ consisting of all triples $F = (F_1, F_2, F_3)$ with $F_i \in L_2$ or $F_i \in H_k$. $\underline{L}_2$ and $\underline{H}_k$ are Hilbert spaces with the inner products

$$(F, G) := (F_1, G_1) + (F_2, G_2) + (F_3, G_3) \tag{3.7}$$

and

$$(F, G)_k := (F_1, G_1)_k + (F_2, G_2)_k + (F_3, G_3)_k\,. \tag{3.8}$$

Note that we use the same notation $(\ ,\ )_k$ and $\|\ \|_k$ for the inner product and the corresponding norm in H_k and $\underline{H}_k$.

Now we introduce the following linear spaces:

$$\underline{S} := \Big\{E \in \underline{C}^2(\bar{\Omega}) \colon n \times E = 0 \text{ and } \nabla \cdot E = 0 \text{ on } \partial\Omega;$$

$$E, \partial_i E, \partial_i \partial_k E = O\left(\frac{1}{r^2}\right) \text{ for } i, k = 1, 2, 3 \text{ and } r = |x| \to \infty\Big\},$$

$$\underline{S}' := \Big\{H \in \underline{C}^2(\bar{\Omega}) \colon n \times (\nabla \times H) = 0 \text{ and } n \cdot H = 0 \text{ on } \partial\Omega;$$

$$H, \partial_i H, \partial_i \partial_k H = O\left(\frac{1}{r^2}\right) \text{ for } i, k = 1, 2, 3 \text{ and } r = |x| \to \infty\Big\}.$$

The spaces $\underline{S}$ and $\underline{S}'$ consist of classical vector fields which satisfy the electric or magnetic boundary conditions and which, together with their derivatives of order up to order 2, belong to $\underline{L}_2(\Omega)$. We denote the completions of $\underline{S}$ and $\underline{S}'$ in $\underline{H}_1$ with $\underline{V}$ and $\underline{V}'$. In particular, we have

$$\underline{V} = \{E \in \underline{H}_1 \colon \text{There exists a sequence } \{E_k\} \text{ in } \underline{S} \text{ such that } \| E - E_k \|_1 \to 0\}.$$

After these preparations we can define operators A and A' with domains of definition $D(A)$ and $D(A')$ by

$$\begin{aligned} D(A) &:= \{E \in \underline{V} \colon \Delta E \in \underline{L}_2 \text{ and } (\Delta E, F) = (E, \Delta F) \text{ for every } F \in \underline{S}\}, \\ AE &:= -\Delta E \qquad \text{if} \qquad E \in D(A) \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} D(A') &:= \{H \in \underline{V}' \colon \Delta H \in \underline{L}_2 \text{ and } (\Delta H, F) = (H, \Delta F) \text{ for every } F \in \underline{S}'\}, \\ A'H &:= -\Delta H \qquad \text{if} \qquad H \in D(A'). \end{aligned} \tag{3.10}$$

The Laplace operator in (3.9) and (3.10) has to be unterstood in the sense of distribution theory: $\Delta E := (\Delta E_1, \Delta E_2, \Delta E_3)$ with $(\Delta E_i)\,\varphi = E_i(\Delta\varphi)$ for every $\varphi \in C_0^\infty$.

The principal aim of the next sections is the proof of the following theorem:

THEOREM 3.1. *The operators A and A' defined by* (3.9) *and* (3.10) *are self-adjoint in the Hilbert space $\underline{L}_2$.*

As a first step we shall prove in this section that A and A' are symmetric and positive. For this purpose we introduce the bilinear form

$$B(F, G) := (\nabla \times F, \nabla \times G) + (\nabla \cdot F, \nabla \cdot G) \tag{3.11}$$

in $\underline{H}_1$ and show:

LEMMA 3.1. $-(\Delta F, G) = B(F, G)$ *if* $F, G \in \underline{S}$ *or* $F, G \in \underline{S}'$.

Corollary 1. *$(\Delta F, G) = (F, \Delta G)$ if $F, G \in \underset{\sim}{S}$ or $F, G \in \underset{\sim}{S}'$.*

Corollary 2. *$\underset{\sim}{S} \subset D(A)$ and $\underset{\sim}{S}' \subset D(A')$.*

Proof of Lemma 3.1. By the integral theorem of Gauss we obtain for $F, G \in \underset{\sim}{S}$ or $F, G \in \underset{\sim}{S}'$

$$
\begin{aligned}
(\Delta F, G) + B(F, G) &= \int_\Omega [\Delta F \cdot \bar{G} + (\nabla \times F) \cdot (\nabla \times \bar{G}) + (\nabla \cdot F)(\nabla \cdot \bar{G})]\, dx \\
&= \int_\Omega \{[\nabla(\nabla \cdot F) - \nabla \times (\nabla \times F)] \cdot \bar{G} + (\nabla \times F) \cdot (\nabla \times \bar{G}) \\
&\quad + (\nabla \cdot F)(\nabla \cdot \bar{G})\}\, dx \\
&= \int_\Omega \nabla \cdot [(\nabla \cdot F)\bar{G} - (\nabla \times F) \times \bar{G}]\, dx \\
&= -\int_{\partial\Omega} n \cdot [(\nabla \cdot F)\bar{G} - (\nabla \times F) \times \bar{G}]\, dS = 0
\end{aligned}
$$

since either $\nabla \cdot F = 0$ and $n \times G = 0$ or $n \cdot G = 0$ and $n \times (\nabla \times F) = 0$ on $\partial\Omega$. Corollary 1 follows from $(\Delta F, G) = -B(F, G) = -\overline{B(G, F)} = \overline{(\Delta G, F)} = (F, \Delta G)$. Corollary 2 is an immediate consequence of (3.9), (3.10) and Corollary 1.

Now we prove the following generalisation of Lemma 3.1:

Lemma 3.2. *Assume that either $F \in D(A)$ and $G \in \underset{\sim}{V}$ or $F \in D(A')$ and $G \in \underset{\sim}{V}'$. Then we have*

$$-(\Delta F, G) = B(F, G). \tag{3.12}$$

Proof. Assume first that $F \in D(A)$ and $G \in \underset{\sim}{S}$. Consider a sequence $\{F_k\}$ in $\underset{\sim}{S}$ such that $\| F - F_k \|_1 \to 0$. By using the definition of $D(A)$ in (3.9) and Lemma 3.1, we obtain

$$-(\Delta F, G) = -(F, \Delta G) = -\lim_{k\to\infty}(F_k, \Delta G) = \lim_{k\to\infty} B(F_k, G) = B(F, G).$$

This proves (3.12) for $F \in D(A)$ and $G \in \underset{\sim}{S}$. Since $\underset{\sim}{S}$ is dense in $\underset{\sim}{V}$ with respect to the norm of H_1, (3.12) holds also for $F \in D(A)$ and $G \in \underset{\sim}{V}$. The case $F \in D(A')$, $G \in \underset{\sim}{V}'$ can be treated by a similar argument.

As an immediate consequence of Lemma 3.2, we obtain:

Lemma 3.3. *The operators A and A' are positive and symmetric.*

Proof. A and A' are positive since $-(\Delta F, F) = B(F, F) \geqslant 0$ if $F \in D(A) \cup D(A')$. The symmetry of A,

$$(AF, G) = (F, AG) \qquad \text{for } F, G \in D(A), \tag{3.13}$$

follows from

$$(AF, G) = -(\Delta F, G) = B(F, G) = \overline{B(G, F)} = -\overline{(\Delta G, F)}$$
$$= -(F, \Delta G) = (F, AG).$$

4. Weak Boundary Value Problems

Consider a complex number λ and a prescribed $F \in \underline{L}_2$. In order to prove that A is self-adjoint, we discuss the following two problems:

(D_λ) *Find $E \in D(A)$ such that*

$$AE - \lambda E = F. \tag{4.1}$$

(B_λ) *Find $E \in \underline{V}$ such that*

$$B(E, G) - \lambda(E, G) = (F, G) \tag{4.2}$$

for every $G \in \underline{V}$.

Recall that the bilinear form B is defined by (3.11). The aim of this section is the proof of the following equivalence theorem:

Lemma 4.1. *E is a solution of* (D_λ) *if and only if E is a solution of* (B_λ).

(D_λ) and (B_λ) can be considered as weak boundary value problems for the equation $-\Delta E - \lambda E = F$ with respect to the electric boundary conditions (1.11). Lemma 4.1 says that the differential operator version (D_λ) and the bilinear form version (B_λ) are equivalent.

Proof of Lemma 4.1. (a) Consider a solution E of (D_λ). Lemma 3.2 implies that $(AE, G) = B(E, G)$ for every $G \in \underline{V}$. Hence (4.2) holds for every $G \in \underline{V}$. Since $D(A) \subset \underline{V}$ it follows that E is also a solution of (B_λ).

(b) Now assume that E is a solution of (B_λ). Since $\underline{S} \subset D(A)$ by Corollary 2 of Lemma 3.1 and since $E \in \underline{V}$, Lemma 3.2 implies $-(E, \Delta G) = -\overline{(\Delta G, E)} = \overline{B(G, E)} = B(E, G)$ for every $G \in \underline{S}$ and hence by (4.2)

$$-(E, \Delta G) - \lambda(E, G) = (F, G) \qquad \text{for every } G \in \underline{S}. \tag{4.3}$$

Set $G = \bar{\varphi} e_i$ where $\varphi \in C_0$ and e_i denotes the i-th unit vector ($i = 1, 2, 3$). Then (4.3) implies

$$-(E_i, \Delta\bar{\varphi}) - \lambda(E_i, \bar{\varphi}) = (F_i, \bar{\varphi})$$

with $E = (E_1, E_2, E_3)$ and $F = (F_1, F_2, F_3)$. By using [11], Lemma 2.3 and the definition of derivatives in the sense of distribution theory (compare (3.5)), we obtain $(E_i, \Delta\bar{\varphi}) = E_i(\Delta\varphi) = (\Delta E_i)\,\varphi$ and hence $(-\Delta E_i - \lambda E_i)\,\varphi = F_i\varphi$ for every $\varphi \in C_0^\infty$. This yields

$$-\Delta E - \lambda E = F. \tag{4.4}$$

In particular, it follows that $\Delta E \in \underline{L}_2$. In order to prove that $E \in D(A)$, it remains to show that

$$(\Delta E, G) = (E, \Delta G) \qquad \text{for every } G \in \underline{S}. \tag{4.5}$$

Since $\underline{C}_0^\infty$ is dense in $\underline{L}_2$, there exists a sequence $\{G_k\}$ in $\underline{C}_0^\infty$ such that $\| G - G_k \| \to 0$. Since $G - G_k \in \underline{S}$, it follows from (4.3) that

$$-(E, \Delta(G - G_k)) = (\lambda E + F, G - G_k)$$

and hence

$$|(E, \Delta G) - (E, \Delta G_k)| \leqslant \| \lambda E + F \| \cdot \| G - G_k \|.$$

This implies

$$(E, \Delta G) = \lim_{k\to\infty}(E, \Delta G_k). \tag{4.6}$$

Set $G_k = (G_{k1}, G_{k2}, G_{k3})$. Since $G_{kj} \in C_0^\infty$ and $\Delta E \in \underline{L}_2$, we obtain by (3.5) and [11], Lemma 2.3

$$\begin{aligned}(E, \Delta G_k) &= \sum_{j=1}^{3} (E_j, \Delta G_{kj}) = \sum_{j=1}^{3} E_j(\Delta \bar{G}_{kj}) \\ &= \sum_{j=1}^{3} (\Delta E_j)\,\overline{G_{kj}} = \sum_{j=1}^{3} (\Delta E_j, G_{kj}) = (\Delta E, G_k)\end{aligned}$$

and hence by (4.6)

$$(E, \Delta G) = \lim_{k\to\infty}(E, \Delta G_k) = \lim_{k\to\infty}(\Delta E, G_k) = (\Delta E, G).$$

This completes the proof of (4.5). Hence we obtain $E \in D(A)$. By (4.4) we have $AE - \lambda E = F$ so that E is a solution of (D_λ). This concludes the proof of Lemma 4.1.

In a similar way we can consider a weak boundary value problem with respect to the magnetic boundary conditions (1.12). We use the following formulations:

(D'_λ) *Find $H \in D(A')$ such that*

$$A'H - \lambda H = F. \tag{4.7}$$

(B'_λ) *Find $H \in \underline{V}'$ such that*

$$B(H, G) - \lambda(H, G) = (F, G) \tag{4.8}$$

for every $G \in \underline{V}'$.

The argument used in the proof of Lemma 4.1 implies:

LEMMA 4.2. *H is a solution of* (D'_λ) *if and only if H is a solution of* (B'_λ).

5. Coerciveness Properties

In order to obtain existence theorems for the weak boundary value problems (D_λ) and (D'_λ), we study coerciveness properties of the bilinear form B. We show:

LEMMA 5.1. *The bilinear form*

$$B(E, G) = (\nabla \times E, \nabla \times G) + (\nabla \cdot E, \nabla \cdot G)$$

is coercive on $\underline{V}$ and $\underline{V}'$. This means that there exist constants $c_1 > 0$ and $c_2 \geqslant 0$ such that

$$B(G, G) \geqslant c_1 \| G \|_1^2 - c_2 \| G \|^2 \tag{5.1}$$

for every $G \in \underline{V} \cup \underline{V}'$.

Proof. Since $\underline{S}$ and $\underline{S}'$ are dense in $\underline{V}$ and $\underline{V}'$, respectively, with regard to the 1-norm, it suffices to prove (5.1) for every $G \in \underline{S} \cup \underline{S}'$. By using Lemma 3.1 and the integral theorem of Gauss, we obtain with $G = (G_1, G_2, G_3)$ and $n = (n_1, n_2, n_3)$

$$\begin{aligned} B(G, G) = -(\Delta G, G) &= - \sum_{i,k=1}^{3} \int_\Omega (\partial_k{}^2 G_i)\, \bar{G}_i \, dx \\ &= - \sum_{i,k=1}^{3} \int_\Omega [\partial_k(\bar{G}_i \partial_k G_i) - | \partial_k G_i |^2] \, dx \\ &= \sum_{i,k=1}^{3} \left[\left\| \frac{\partial}{\partial x_k} G_i \right\|^2 + \int_{\partial\Omega} n_k \bar{G}_i \partial_k G_i \, dS \right] \end{aligned}$$

and hence

$$B(G, G) = \| G \|_1^2 - \| G \|^2 + \int_{\partial\Omega} \bar{G} \cdot \frac{\partial}{\partial n} G \, dS \qquad \text{for } G \in \underline{S} \cup \underline{S}'. \tag{5.2}$$

Our next aim is to find a constant $a_1 > 0$ such that

$$\left| \int_{\partial\Omega} \bar{G} \cdot \frac{\partial}{\partial n} G \, dS \right| \leqslant a_1 \int_{\partial\Omega} | G |^2 \, dS \tag{5.3}$$

for every $G \in \underline{S} \cup \underline{S}'$. We decompose $\partial\Omega$ into a finite number of surface elements $S_1, \ldots, S_m$ with smooth parameter representations

$$z = z(u_1, u_2), \qquad (u_1, u_2) \in B_j, \qquad j = 1, \ldots, m$$

and choose a sufficiently small $\delta > 0$ such that

$$u = (u_1, u_2, u_3) \to x(u) := z(u_1, u_2) + u_3 n(z(u_1, u_2))$$

is a smooth injective mapping of the cylinder $Z_j = \bar{B}_j \times [0, \delta]$ into $\bar{\Omega}$ for every $j = 1, \ldots, m$. Set for $u \in Z_j$ and $i, k = 1, 2, 3$

$$t_i(u) := \frac{\partial}{\partial u_i} x(u), \qquad g_{ik} := t_i \cdot t_k, \qquad g := \det(g_{ik}),$$
$$(g^{ik}) := (g_{ik})^{-1}, \qquad (S_0 f)(u) := f(x(u)).$$

Let G^i be the (contravariant) components of $S_0 G$ in the curvilinear coordinate system defined by the vectors t_1, t_2, t_3:

$$S_0 G = G^i t_i \tag{5.4}$$

where we employ the usual summation convention and sum over i from 1 to 3. By forming inner products with t_k in (5.4), we obtain

$$G^i = g^{ik} t_k \cdot S_0 G. \tag{5.5}$$

In the following we use the sign $\stackrel{\circ}{=}$ to indicate that equality holds if $u_3 = 0$. Since $t_3 \stackrel{\circ}{=} S_0 n$, we have

$$g_{i3} \stackrel{\circ}{=} g_{3i} \stackrel{\circ}{=} 0 \qquad \text{if} \qquad i = 1, 2, \quad g_{33} \stackrel{\circ}{=} 1 \tag{5.6}$$

and hence

$$g^{i3} \stackrel{\circ}{=} g^{3i} \stackrel{\circ}{=} 0 \qquad \text{if} \qquad i = 1, 2, \quad g^{33} \stackrel{\circ}{=} 1. \tag{5.7}$$

We use the following elementary formulas:

$$S_0(\nabla \cdot G) = \frac{1}{g^{1/2}} \partial_i (g^{1/2} G^i), \tag{5.8}$$

$$S_0(\nabla \times G) = \frac{1}{g^{1/2}} \begin{vmatrix} t_1 & t_2 & t_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ t_1 \cdot S_0 G & t_2 \cdot S_0 G & t_3 \cdot S_0 G \end{vmatrix}. \tag{5.9}$$

Assume first that $G \in \underline{S}$. The boundary condition $n \times G = 0$ implies $t_1 \cdot S_0 G \mathrel{\overset{\circ}{=}} t_2 \cdot S_0 G \mathrel{\overset{\circ}{=}} 0$ and hence, by (5.5) and (5.7),

$$G^1 \mathrel{\overset{\circ}{=}} G^2 \mathrel{\overset{\circ}{=}} 0, \qquad G^3 \mathrel{\overset{\circ}{=}} t_3 \cdot S_0 G. \tag{5.10}$$

This implies by (5.4)

$$S_0 G \mathrel{\overset{\circ}{=}} (t_3 \cdot S_0 G)\, t_3\,. \tag{5.11}$$

Furthermore, we have

$$S_0\left(\frac{\partial}{\partial n} G\right) \mathrel{\overset{\circ}{=}} \frac{\partial}{\partial u_3} S_0 G \mathrel{\overset{\circ}{=}} \frac{\partial}{\partial u_3}(G^i t_i) \mathrel{\overset{\circ}{=}} G^i \frac{\partial}{\partial u_3} t_i + \left(\frac{\partial}{\partial u_3} G^i\right) t_i\,.$$

Note that $\partial t_3/\partial u_3 = 0$ since $t_3(u) = n(z(u_1, u_2))$. This, together with (5.10), yields

$$S_0\left(\frac{\partial}{\partial n} G\right) \mathrel{\overset{\circ}{=}} \left(\frac{\partial}{\partial u_3} G^i\right) t_i\,. \tag{5.12}$$

By (5.11) and (5.12) we obtain

$$S_0\left(\bar{G} \cdot \frac{\partial G}{\partial n}\right) \mathrel{\overset{\circ}{=}} S_0(\bar{G}) \cdot S_0\left(\frac{\partial}{\partial n} G\right) \mathrel{\overset{\circ}{=}} (t_3 \cdot S_0 G) \frac{\partial}{\partial u_3} G^3. \tag{5.13}$$

Now we apply the second boundary condition $\nabla \cdot G = 0$. Since (5.10) implies $\partial(g^{1/2} G^i)/\partial u_i \mathrel{\overset{\circ}{=}} 0$ for $i = 1, 2$, it follows from (5.8) that

$$S_0(\nabla \cdot G) \mathrel{\overset{\circ}{=}} \frac{1}{g^{1/2}} \frac{\partial}{\partial u_3}(g^{1/2} G^3) \mathrel{\overset{\circ}{=}} \frac{\partial}{\partial u_3} G^3 + \frac{1}{g^{1/2}}\left(\frac{\partial}{\partial u_3} g^{1/2}\right) G^3 \mathrel{\overset{\circ}{=}} 0.$$

Recall that $G^3 = t_3 \cdot S_0 G$. Hence (5.13) implies

$$S_0\left(\bar{G} \cdot \frac{\partial}{\partial n} G\right) \mathrel{\overset{\circ}{=}} -\frac{1}{g^{1/2}}\left(\frac{\partial}{\partial u_3} g^{1/2}\right)(t_3 \cdot S_0 G)^2. \tag{5.14}$$

By (5.14) we can find a constant $a > 0$ such that every $G \in \underline{S}$ satisfies

$$\left|\bar{G} \cdot \frac{\partial}{\partial n} G\right| \leqslant a\, |G|^2 \qquad \text{on} \qquad \partial\Omega. \tag{5.15}$$

Now we consider the case $G \in \underline{S}'$. The boundary condition $n \cdot G = 0$ implies

$$t_3 \cdot S_0 G = 0 \tag{5.16}$$

and hence $\partial(t_3 \cdot S_0 G)/\partial u_i = 0$ for $i = 1, 2$.

Therefore (5.9) yields

$$S_0(\nabla \times G) \overset{\circ}{=} \frac{1}{g^{1/2}}\Big[-\frac{\partial}{\partial u_3}(t_2 \cdot S_0G)\, t_1 + \frac{\partial}{\partial u_3}(t_1 \cdot S_0G)\, t_2 + \Big(\frac{\partial}{\partial u_1}(t_2 \cdot S_0G) - \frac{\partial}{\partial u_2}(t_1 \cdot S_0G)\Big)\, t_3\Big].$$

The second boundary condition $n \times (\nabla \times G) = 0$ implies

$$t_3 \times S_0(\nabla \times G) \overset{\circ}{=} \frac{1}{g^{1/2}}\Big[-\frac{\partial}{\partial u_3}(t_2 \cdot S_0G)\,(t_3 \times t_1) + \frac{\partial}{\partial u_3}(t_1 \cdot S_0G)\,(t_3 \times t_2)\Big] = 0$$

and hence

$$\frac{\partial}{\partial u_3}(t_1 \cdot S_0G) \overset{\circ}{=} \frac{\partial}{\partial u_3}(t_2 \cdot S_0G) = 0. \tag{5.17}$$

By (5.5), (5.7) and (5.16) we have $G^3 \overset{\circ}{=} 0$ so that (5.4) implies

$$S_0G \overset{\circ}{=} G^1 t_1 + G^2 t_2 . \tag{5.18}$$

Furthermore, we obtain by (5.4) and (5.5)

$$S_0\Big(\frac{\partial}{\partial n} G\Big) \overset{\circ}{=} \frac{\partial}{\partial u_3} S_0G \overset{\circ}{=} \frac{\partial}{\partial u_3}(G^i t_i) \overset{\circ}{=} \frac{\partial}{\partial u_3} g^{ik}(t_k \cdot S_0G)\, t_i$$

$$\overset{\circ}{=} (t_k \cdot S_0G)\frac{\partial}{\partial u_3}(g^{ik} t_i) + g^{ik} t_i \frac{\partial}{\partial u_3}(t_k \cdot S_0G)$$

and hence by (5.17) and (5.7)

$$S_0\Big(\frac{\partial}{\partial n} G\Big) \overset{\circ}{=} (t_k \cdot S_0G)\frac{\partial}{\partial u_3}(g^{ik} t_i) + t_3 \frac{\partial}{\partial u_3}(t_3 \cdot S_0G). \tag{5.19}$$

Since $t_1 \cdot t_3 \overset{\circ}{=} t_2 \cdot t_3 \overset{\circ}{=} 0$, (5.18) and (5.19) imply

$$S_0\Big(\bar{G} \cdot \frac{\partial}{\partial n} G\Big) \overset{\circ}{=} (\bar{G}^1 t_1 + \bar{G}^2 t_2) \cdot (t_k \cdot S_0G)\frac{\partial}{\partial u_3}(g^{ik} t_i).$$

This formula, together with (5.5), shows that a constant a' can be found such that every $G \in \underline{S}'$ satisfies

$$\Big|\, \bar{G} \cdot \frac{\partial}{\partial n} G \,\Big| \leqslant a' \mid G \mid^2 \qquad \text{on} \qquad \partial\Omega. \tag{5.20}$$

Thus (5.3) holds for every $G \in \underline{S} \cup \underline{S}'$ with $a_1 = \max(a, a')$.

In order to complete the proof of Lemma 5.1, we choose a vector field $b \in \underset{\sim}{C}^1(\bar{\Omega})$ with bounded support such that $b = n$ on $\partial\Omega$. Then we obtain

$$\int_{\partial\Omega} |G|^2 \, dS = \int_{\partial\Omega} (n \cdot b)(G \cdot \bar{G}) \, dS = -\sum_{i,k=1}^{3} \int_{\Omega} \partial_i(b_i G_k \bar{G}_k) \, dx$$

$$= -\int_{\Omega} (\nabla \cdot b) |G|^2 \, dx - 2 \sum_{i,k=1}^{3} \int_{\Omega} b_i \operatorname{Re}(G_k \partial_i \bar{G}_k) \, dx.$$

By applying Schwarz's inequality, we can find a constant a_2 such that

$$\int_{\partial\Omega} |G|^2 \, dS \leqslant a_2(\|G\|^2 + 2 \|G\| \cdot \|G\|_1)$$

for every $G \in \underset{\sim}{S} \cup \underset{\sim}{S}'$. By using

$$2\alpha\beta \leqslant \alpha^2 + \beta^2$$

with

$$\alpha = (2a_1a_2)^{1/2} \|G\| \qquad \text{and} \qquad \beta = (2a_1a_2)^{-1/2} \|G\|_1,$$

we obtain

$$\int_{\partial\Omega} |G|^2 \, dS \leqslant a_2(1 + 2a_1a_2) \|G\|^2 + \frac{1}{2a_1} \|G\|_1^2. \tag{5.21}$$

By (5.2), (5.3) and (5.21) we have

$$B(G, G) \geqslant \tfrac{1}{2} \|G\|_1^2 - [1 + a_1a_2(1 + 2a_1a_2)] \cdot \|G\|^2$$

for every $G \in \underset{\sim}{S} \cup \underset{\sim}{S}'$. This concludes the proof of Lemma 5.1.

6. The Self-Adjointness of A and A'

In this section we complete the proof of Theorem 3.1 on the selfadjointness of A and A'. Furthermore, we shall prove the following uniqueness and existence theorem for the weak boundary value problems (D_λ) and (D'_λ) introduced in Section 4:

Lemma 6.1. *The problems* (D_λ) *and* (D'_λ) *have a uniquely determined solution for every prescribed* $F \in \underset{\sim}{L}_2$ *and every complex number* λ *with* $\lambda \notin [0, \infty)$.

First we prove Lemma 6.1 for $\lambda = -c_2$ where c_2 is chosen as in Lemma 5.1. Consider the bilinear form

$$B_0(G_1, G_2) := B(G_1, G_2) + c_2(G_1, G_2). \tag{6.1}$$

By Lemma 5.1 we have

$$B_0(G, G) \geqslant c_1 \| G \|_1^2 \tag{6.2}$$

for every $G \in \underline{V} \cup \underline{V}'$. Furthermore, by the definition of B in (3.11), there exists a constant $c_3 > 0$ such that

$$| B_0(G_1, G_2)| \leqslant c_3 \| G_1 \|_1 \cdot \| G_2 \|_1 \tag{6.3}$$

for all $G_1, G_2 \in \underline{V} \cup \underline{V}'$. Since $|(G, F)| \leqslant \| G \| \cdot \| F \| \leqslant \| G \|_1 \cdot \| F \|$ for every $G \in \underline{V}$, the mapping $G \to (G, F)$ is a bounded linear functional on the Hilbert space $\underline{V}$.

By (6.2) and (6.3) the bilinear form B_0 satisfies the assumptions of the representation theorem of Lax and Milgram. Hence there exists an $E \in \underline{V}$ such that

$$B_0(E, G) := B(E, G) + c_2(E, G) = (F, G)$$

for every $G \in \underline{V}$. By (4.2) E is a solution of (B_λ) and hence, by Lemma 4.1, also a solution of (D_λ) for $\lambda = -c_2$.

In order to show that E is uniquely determined, we consider a solution E of (D_λ) for $F = 0$ and $\lambda = -c_2$. By Lemma 4.1, E is also a solution of (B_λ) for $F = 0$ and $\lambda = -c_2$. Hence we have $B_0(E, G) = B(E, G) + c_2(E, G) = 0$ for every $G \in \underline{V}$. By setting $G = E$, we obtain $B_0(E, E) = 0$ and hence $E = 0$ by (6.2) since $c_1 > 0$. The same argument shows that also problem (D_λ') has a uniquely determined solution H for $\lambda = -c_2$. This concludes the proof of Lemma 6.1 in the case $\lambda = -c_2$.

Now we prove that A is self-adjoint. Denote the uniquely determined solution of (D_λ) with $\lambda = -c_2$ by KF. Then we have

$$KF \in D(A) \qquad \text{and} \qquad (A + c_2)\, KF = F \tag{6.4}$$

for every $F \in \underline{L}_2$. Recall that the adjoint operator A^* of A is defined by $D(A^*) := \{F \in \underline{L}_2\colon$ There exists a $F^* \in \underline{L}_2$ such that $(AG, F) = (G, F^*)$ for every $G \in D(A)\}$, $A^*F := F^*$. Note that F^* is uniquely determined by F since $D(A)$ is dense in $\underline{L}_2$ by Corollary 2 to Lemma 3.1. Since A is symmetric by Lemma 3.3, we have $D(A) \subseteq D(A^*)$ and $A^*F = AF$ for $F \in D(A)$. Hence it suffices to show that $D(A^*) \subseteq D(A)$. Assume that $F \in D(A^*)$. By (6.4) and the symmetry of A we have for every $G \in D(A)$

$$\begin{aligned}(AG + c_2G, F) &= (G, A^*F + c_2F)\\ &= (G, (A + c_2)\, K(A^*F + c_2F)) = (AG + c_2G, K(A^*F + c_2F))\end{aligned}$$

and hence $F = K(A^*F + c_2F) \in D(A)$ since $\{AG + c_2G\colon G \in D(A)\} = \underline{L}_2$ (by Lemma 6.1 with $\lambda = -c_2$). This shows that $D(A^*) \subseteq D(A)$ so that A is self-adjoint. A similar argument shows that A' is self-adjoint. This completes the proof of Theorem 3.1.

Since A and A' are self-adjoint and positive (compare Lemma 3.3), the inverse operators $(A - \lambda)^{-1}$ and $(A' - \lambda)^{-1}$ are defined on the whole Hilbert space $\underline{L}_2$ if $\lambda \notin [0, \infty)$. Hence problems (D_λ) and (D'_λ) are uniquely solvable if $\lambda \notin [0, \infty)$. This concludes the proof of Lemma 6.1.

7. Weak Initial and Boundary Value Problems

In this section we apply Theorem 3.1 to weak versions of the initial and boundary value problems (E) and (M) (see Section 1) and prove uniqueness and existence statements. We shall consider vector-valued functions $t \to F(t)$ defined on the real half line $R_0{}^+ := [0, \infty)$ with values in the Hilbert space $\underline{L}_2$. The basic concepts of calculus, such as continuity, derivatives and Riemann integration, can be easily extended to this situation (compare, for example, [1], Section 2). We consider the following weak variant of problem (E):

(E*) *Find a vector-valued function* $E: R_0{}^+ \to \underline{L}_2$ *to given data* E_0, $E_1 \in \underline{L}_2$ *such that*

$$E \in C^2(R_0{}^+, \underline{L}_2), \tag{7.1}$$

$$\Delta E = \epsilon\mu E'' + \mu\sigma E' + J_1 \qquad \text{for } t \geqslant 0, \tag{7.2}$$

$$E(t) \in D(A) \qquad \text{for } t \geqslant 0, \tag{7.3}$$

$$E(0) = E_0, \qquad E'(0) = E_1. \tag{7.4}$$

Here E' and E'' denote the first two derivatives of the $\underline{L}_2$-valued function E while the Laplacian $\Delta E(t)$, for fixed t, is understood in the sense of the theory of distributions as in the previous sections. Note that (7.3) and (7.4) impose the compatibility condition $E_0 \in D(A)$. The property (7.3) can be interpreted as a weak formulation of the electric boundary conditions (1.11).

First we discuss uniqueness and show:

Lemma 7.1. *Assume that E is a solution of* (E*) *with $E_0 = E_1 = 0$ and $J_1 = 0$. Then we have $E = 0$ in $R_0{}^+$.*

Proof. Let B be the bilinear form introduced in (3.11). Lemma 3.2 and (7.3) imply for t, $t + h \in R_0{}^+$

$$\begin{aligned}
&\frac{1}{h}\,[B(E(t+h), E(t+h)) - B(E(t), E(t))] \\
&\quad = \frac{1}{h}\,[B(E(t+h), E(t+h) - E(t)) + B(E(t+h) - E(t), E(t)) \\
&\quad = -\left(\Delta E(t+h), \frac{1}{h}\,[E(t+h) - E(t)]\right) - \left(\frac{1}{h}\,[E(t+h) - E(t)], \Delta E(t)\right).
\end{aligned}$$

Since $\Delta E \in C(R_0^+, \underline{L}_2)$ by (7.1) and (7.2), we can perform the limit $h \to 0$ and obtain

$$\frac{d}{dt} B(E, E) = -(\Delta E, E') - (E', \Delta E) = -2\mathrm{Re}(\Delta E, E')$$

and hence by (7.2) (with $J_1 = 0$)

$$\frac{d}{dt} B(E, E) = -2\epsilon\mu \,\mathrm{Re}(E'', E') - 2\mu\sigma \| E' \|^2. \tag{7.5}$$

In particular, the real valued function $t \to B(E(t), E(t))$ is continuously differentiable in R_0^+. Since

$$\frac{d}{dt}(E', E') = (E', E'') + (E'', E') = 2\mathrm{Re}(E'', E'),$$

(7.5) can be rewritten as

$$\rho' = -\mu\sigma \| E' \|^2 \tag{7.6}$$

where ρ is the real-valued function defined by

$$\begin{aligned} \rho(t) &:= \tfrac{1}{2}[B(E(t), E(t)) + \epsilon\mu \| E'(t)\|^2] \\ &= \tfrac{1}{2}[\| \nabla \times E(t)\|^2 + \| \nabla \cdot E(t)\|^2 + \epsilon\mu \| E'(t)\|^2]. \end{aligned} \tag{7.7}$$

In particular, we have $\rho(t) \geqslant 0$ for $t \geqslant 0$. On the other hand, since $E_0 = E_1 = 0$, (7.4) implies $\rho(0) = 0$ and hence, by (7.6), $\rho(t) \leqslant 0$ for $t \geqslant 0$. Thus we obtain $\rho = 0$ and hence $E' = 0$ in R_0^+ by (7.7). Since the fundamental theorem of calculus can be extended to vector-valued functions (see [1], Section 2), we conclude that

$$E(t) = E_0 + \int_0^t E'(\tau)\, d\tau = E_0 = 0$$

for $t \geqslant 0$. This completes the proof of Lemma 7.1.

The construction of a solution E of problem (E*) will be based on the functional calculus for unbounded self-adjoint operators. For convenience, we collect the main facts which are used in the following (see, for example, [9], Sections 19–20 for detailed proofs). Since A is self-adjoint and positive, there exists a uniquely determined set of projection operators P_λ, $\lambda \geqslant 0$ in $\underline{L}_2$ such that $P_0 = 0$, $P_\lambda P_\mu = P_\mu P_\lambda = P_{\min(\lambda,\mu)}$, $P_\lambda G \to P_\mu G$ for every $G \in \underline{L}_2$ as $\lambda \uparrow \mu$, $P_\lambda G \to G$ for every $G \in \underline{L}_2$ as $\lambda \to \infty$, and

$$\begin{aligned} D(A) &= \left\{ G \in \underline{L}_2 \colon \int_0^\infty \lambda^2 \, d(\| P_\lambda G \|^2) < \infty \right\}, \\ AG &= \int_0^\infty \lambda \, d(P_\lambda G) \qquad \text{for every } G \in D(A). \end{aligned} \tag{7.8}$$

The set $\{P_\lambda \colon \lambda \geqslant 0\}$ is called the spectral set of A. Consider a complex-valued function $f \in C[0, \infty)$. The operator $f(A)$ is defined by

$$\begin{aligned} D(f(A)) &:= \left\{ G \in \underline{L}_2 \colon \int_0^\infty |f(\lambda)|^2 \, d(\| P_\lambda G \|^2) < \infty \right\}, \\ f(A)\, G &:= \int_0^\infty f(\lambda)\, d(P_\lambda G) \qquad \text{for } G \in D(A). \end{aligned} \tag{7.9}$$

The properties of the operators P_λ mentioned above imply that $\lambda \to \| P_\lambda G \|^2$ is an increasing function with $\| P_\lambda G \|^2 \to \| G \|^2$ as $\lambda \to \infty$ and that

$$\left\| \int_a^b f(\lambda)\, d(P_\lambda G) \right\|^2 = \int_a^b |f(\lambda)|^2 \, d(\| P_\lambda G \|^2) \tag{7.10}$$

for every $G \in \underline{L}_2$ and every pair a, b of real numbers with $0 \leqslant a < b < \infty$. Hence the convergence of the first integral in (7.9) is equivalent to the convergence of the second integral. Formulas (7.8) and (7.9) imply $A = id(A)$ where id denotes the identity: $id(\lambda) = \lambda$ for $\lambda \geqslant 0$. Set $f_1(\lambda) := \lambda f(\lambda)$ for $\lambda \geqslant 0$. Then $G \in f_1(A)$ implies $f(A)\, G \in D(A)$ and

$$A[f(A)\, G] = f_1(A)\, G = \int_0^\infty \lambda f(\lambda)\, d(P_\lambda G) \qquad \text{for } G \in D(f_1(A)). \tag{7.11}$$

After these preparations we continue the discussion of problem (E*).

First we study the special case $J_1 = 0$. We try to determine real-valued functions f and g such that

$$E(t) = \int_0^\infty f(\lambda, t)\, d(P_\lambda E_0) + \int_0^\infty g(\lambda, t)\, d(P_\lambda E_1) \tag{7.12}$$

is a solution of (E*) with $J_1 = 0$. According to (7.11), we consider the initial value problems

$$\begin{aligned} &\left(\lambda + \epsilon\mu \frac{\partial^2}{\partial t^2} + \mu\sigma \frac{\partial}{\partial t}\right) f(\lambda, t) = 0, \\ &f(\lambda, 0) = 1, \qquad \frac{\partial}{\partial t} f(\lambda, 0) = 0 \end{aligned} \tag{7.13}$$

and

$$\begin{aligned} &\left(\lambda + \epsilon\mu \frac{\partial^2}{\partial t^2} + \mu\sigma \frac{\partial}{\partial t}\right) g(\lambda, t) = 0, \\ &g(\lambda, 0) = 0, \qquad \frac{\partial}{\partial t} g(\lambda, 0) = 1. \end{aligned} \tag{7.14}$$

Note that the required initial conditions for f and g ensure that the vector-valued function E defined by (7.12) satisfies (7.4). The solutions f and g of (7.13) and (7.14) are given by

$$\begin{aligned}
f(\lambda, t) &= e^{-\sigma t/2\epsilon} \cosh\left(t\left(\frac{\sigma^2}{4\epsilon^2} - \frac{\lambda}{\epsilon\mu}\right)^{1/2}\right) \\
&\quad + \frac{\sigma}{2\epsilon}\frac{1}{(\sigma^2/4\epsilon^2 - \lambda/\epsilon\mu)^{1/2}} \sinh\left(t\left(\frac{\sigma^2}{4\epsilon^2} - \frac{\lambda}{\epsilon\mu}\right)^{1/2}\right) && \text{if} \quad \lambda < \frac{\sigma^2\mu}{4\epsilon}, \\
&= e^{-\sigma t/2\epsilon}\left(1 + \frac{\sigma}{2\epsilon}t\right) && \text{if} \quad \lambda = \frac{\sigma^2\mu}{4\epsilon}, \\
&= e^{-\sigma t/2} \cos\left(t\left(\frac{\lambda}{\epsilon\mu} - \frac{\sigma^2}{4\epsilon^2}\right)^{1/2}\right) \\
&\quad + \frac{\sigma}{2\epsilon}\frac{1}{(\lambda/\epsilon\mu - \sigma^2/4\epsilon^2)^{1/2}} \sin\left(t\left(\frac{\lambda}{\epsilon\mu} - \frac{\sigma^2}{4\epsilon^2}\right)^{1/2}\right) && \text{if} \quad \lambda > \frac{\sigma^2\mu}{4\epsilon}
\end{aligned}$$

and

$$\begin{aligned}
g(\lambda, t) &= e^{-\sigma t/2\epsilon}\frac{1}{(\sigma^2/4\epsilon^2 - \lambda/\epsilon\mu)^{1/2}} \sinh\left(t\left(\frac{\sigma^2}{4\epsilon^2} - \frac{\lambda}{\epsilon\mu}\right)^{1/2}\right) && \text{if} \quad \lambda < \frac{\sigma^2\mu}{4\epsilon}, \\
&= te^{-\sigma t/2\epsilon} && \text{if} \quad \lambda = \frac{\sigma^2\mu}{4\epsilon}, \\
&= e^{-\sigma t/2\epsilon}\frac{1}{(\lambda/\epsilon\mu - \sigma^2/4\epsilon^2)^{1/2}} \sin\left(t\left(\frac{\lambda}{\epsilon\mu} - \frac{\sigma^2}{4\epsilon^2}\right)^{1/2}\right) && \text{if} \quad \lambda > \frac{\sigma^2\mu}{4\epsilon}.
\end{aligned}$$

These formulas imply that

$$\left(\frac{\partial}{\partial t}\right)^k f(\lambda, t) = O(\lambda^{k/2}) \qquad \text{and} \qquad \left(\frac{\partial}{\partial t}\right)^k g(\lambda, t) = O(\lambda^{(k-1)/2}) \tag{7.15}$$

as $\lambda \to \infty$ uniformly with regard to t. By (7.12) and (7.15) we have $E \in C^k(R_0{}^+, \underline{L}_2)$ and

$$E^{(j)}(t) = \int_0^\infty \left(\frac{\partial}{\partial t}\right)^j f(\lambda, t)\, d(P_\lambda E_0) + \int_0^\infty \left(\frac{\partial}{\partial t}\right)^j g(\lambda, t)\, d(P_\lambda E_1) \tag{7.16}$$

for $j \leqslant k$ if $E_0 \in D(A^{k/2})$ and $E_1 \in D(A^{(k-1)/2})$. In fact, the integral

$$\int_0^\infty [\lambda^k\, d(\| P_\lambda E_0 \|^2) + \lambda^{k-1}\, d(\| P_\lambda E_1 \|^2)]$$

exists under these assumptions by (7.9) so that we conclude from (7.10), (7.15) and the monotonicity of the functions $\lambda \to \| P_\lambda E_0 \|^2$ and $\lambda \to \| P_\lambda E_1 \|^2$ that the integrals in (7.16) converge uniformly with regard to t. Hence the Stieltjes integration and the differentiation with respect to t can be interchanged.

Now assume that $E_0 \in D(A)$ and $E_1 \in D(A^{1/2})$. Then we have $E \in C^2(R_0^+, \underline{L}_2)$. Since $f(\lambda, t) = O(1)$ and $g(\lambda, t) = O(\lambda^{-1/2})$ as $\lambda \to \infty$, it follows as above that the integral

$$\int_0^\infty [\lambda^2 f(\lambda, t)^2 \, d(\| P_\lambda E_0 \|^2) + \lambda^2 g(\lambda, t)^2 \, d(\| P_\lambda E_1 \|^2)]$$

converges. Hence (7.11) and (7.12) imply $E(t) \in D(A)$ for every $t \geqslant 0$ and

$$AE(t) = \int_0^\infty \lambda f(\lambda, t) \, d(P_\lambda E_0) + \int_0^\infty \lambda g(\lambda, t) \, d(P_\lambda E_1). \tag{7.17}$$

This, together with (7.13), (7.14) and (7.16), shows that

$$AE + \epsilon\mu \frac{\partial^2}{\partial t^2} E + \mu\sigma \frac{\partial}{\partial t} E = 0.$$

Hence E is a solution of problem (E*) in the case $J_1 = 0$. We collect the results in the following theorem:

Lemma 7.2. *Assume that $E_0 \in D(A)$, $E_1 \in D(A^{1/2})$ and $J_1 = 0$. Then problem* (E*) ***has a uniquely determined solution E. E is given by** (7.12) **where f and g are the solutions of the initial value problems** (7.13) **and** (7.14). If, in addition, $E_0 \in D(A^{k/2})$ and $E_1 \in D(A^{(k-1)/2})$ then $E \in C^k(R_0^+, \underline{L}_2)$.*

Note that the assumptions $E_0 \in D(A^{k/2})$ and $E_1 \in D(A^{(k-1)/2})$ are satisfied if E_0, $E_1 \in C_0{}^k(\Omega)$ for even k and $E_0 \in C_0^{k+1}(\Omega)$, $E_1 \in C_0^{k-1}(\Omega)$ for odd k. If $\sigma = 0$ then we have

$$f(\lambda, t) = \cos\left(t\left(\frac{\lambda}{\epsilon\mu}\right)^{1/2}\right), \qquad g(\lambda, t) = \left(\frac{\epsilon\mu}{\lambda}\right)^{1/2} \sin\left(t\left(\frac{\lambda}{\epsilon\mu}\right)^{1/2}\right)$$

and hence

$$E(t) = \int_0^\infty \left[\cos\left(\left(\frac{\lambda}{\epsilon\mu}\right)^{1/2} t\right) d(P_\lambda E_0) + \left(\frac{\epsilon\mu}{\lambda}\right)^{1/2} \sin\left(\left(\frac{\lambda}{\epsilon\mu}\right)^{1/2} t\right) d(P_\lambda E_1)\right]. \tag{7.18}$$

Now we reduce the general case to the case $J_1 = 0$ by applying Duhamel's principle. We need the following preparations.

Lemma 7.3. *Assume that $h \in C[0, \infty)$, $F \in D(A^\gamma)$ and*

$$\| h \|_\gamma^* := \max_{0 \leqslant \lambda \leqslant 1} | h(\lambda)| + \operatorname{supr}_{\lambda \geqslant 1} \lambda^{-\gamma} | h(\lambda)| < \infty$$

with real $\gamma > 0$. Then we have

$$\left\| \int_0^\infty h(\lambda) \, d(P_\lambda F) \right\| \leqslant \| h \|_\gamma^* (\| F \| + \| A^\gamma F \|).$$

The proof follows from the estimates

$$\left\| \int_0^1 h(\lambda)\, d(P_\lambda F) \right\|^2 = \int_0^1 | h(\lambda)|^2\, d(\| P_\lambda F \|^2) \leqslant (\| h \|_\gamma^*)^2 \int_0^1 d(\| P_\lambda F \|^2)$$

$$= (\| h \|_\gamma^*)^2 \| P_1 F \|^2 \leqslant (\| h \|_\gamma^*)^2 \| F \|^2$$

and

$$\left\| \int_1^\infty h(\lambda)\, d(P_\lambda F) \right\|^2 = \int_1^\infty | h(\lambda)|^2\, d(\| P_\lambda F \|^2) \leqslant (\| h \|_\gamma^*)^2 \int_0^\infty \lambda^{2\gamma}\, d(\| P_\lambda F \|^2)$$

$$= (\| h \|_\gamma^*)^2 \left\| \int_0^\infty \lambda^\gamma\, d(P_\lambda F) \right\|^2 = (\| h \|_\gamma^*)^2 \| A^\gamma F \|^2.$$

LEMMA 7.4. *Suppose that* $p \in C(R_0^+ \times R_0^+)$ *and that*

$$\lambda^{-\gamma} p(\lambda, t) \to 0 \qquad as \qquad \lambda \to \infty \tag{7.19}$$

uniformly in $[0, T]$ *for every* $T > 0$ *(with real* $\gamma > 0$*). Furthermore, assume that* $J \in C(R_0^+, \underline{L}_2)$, $J(t) \in D(A)$ *for every* $t \geqslant 0$ *and* $A^\gamma J \in C(R_0^+, \underline{L}_2)$. *Then the* $\underline{L}_2$-*valued function*

$$Q(t, \tau) := \int_0^\infty p(\lambda, t)\, d[P_\lambda J(\tau)]$$

is continuous in $R_0^+ \times R_0^+ : Q \in C(R_0^+ \times R_0^+, \underline{L}_2)$.

Proof. Let $\epsilon > 0$ and $T > 0$ be given. Set

$$M_1 := \max_{0 \leqslant \tau \leqslant T} \| J(\tau) \|,$$

$$M_2 := \max_{0 \leqslant \tau \leqslant T} \| A^\gamma J(\tau) \|,$$

$$M_3 := \max\{| p(\lambda, t)| : 0 \leqslant \lambda \leqslant 1, 0 \leqslant t \leqslant T\},$$

$$M_4 := \operatorname{supr}\{\lambda^{-\gamma} \,| p(\lambda, t)| : 1 \leqslant \lambda < \infty, 0 \leqslant t \leqslant T\},$$

$$M := \max(M_1, M_2, M_3, M_4).$$

Note that $M_4 < \infty$ since (7.19) holds uniformly in $[0, T]$. Furthermore, choose $\delta_1, \delta_2, \delta_3, \delta_4 > 0$ such that

(i) $\| J(\tau_1) - J(\tau_2) \| \leqslant \epsilon/8M$ if $\tau_1, \tau_2 \in [0, T]$ and $| \tau_1 - \tau_2 | \leqslant \delta_1$,

(ii) $\| A^\gamma J(\tau_1) - A^\gamma J(\tau_2) \| \leqslant \epsilon/8M$ if $\tau_1, \tau_2 \in [0, T]$ and $| \tau_1 - \tau_2 | \leqslant \delta_2$,

(iii) $| p(\lambda, t_1) - p(\lambda, t_2)| \leqslant \epsilon/8M$ if $t_1, t_2 \in [0, T]$, $\lambda \in [0, 1]$ and $| t_1 - t_2 | \leqslant \delta_3$,

(iv) $| \lambda^{-\gamma} p(\lambda, t_1) - \lambda^{-\gamma} p(\lambda, t_2)| \leqslant \epsilon/8M$ if $t_1, t_2 \in [0, T]$, $\lambda \in [1, \infty)$ and $| t_1 - t_2 | \leqslant \delta_4$.

The choice of δ_4 is possible since (7.19) holds uniformly in $[0, T]$. Set

$$\delta := \min(\delta_1, \delta_2, \delta_3, \delta_4)$$

and consider two points $(t_1, \tau_1), (t_2, \tau_2) \in [0, T] \times [0, T]$ such that $(t_1 - t_2)^2 + (\tau_1 - \tau_2)^2 < \delta^2$. Then it follows from Lemma 7.3 that

$$\begin{aligned}
&\| Q(t_1, \tau_1) - Q(t_2, \tau_2)\| \\
&\quad \leqslant \| Q(t_1, \tau_1) - Q(t_2, \tau_1)\| + \| Q(t_2, \tau_1) - Q(t_2, \tau_2)\| \\
&\quad = \left\| \int_0^\infty [p(\lambda, t_1) - p(\lambda, t_2)]\, d[P_\lambda J(\tau_1)] \right\| + \left\| \int_0^\infty p(\lambda, t_2)\, d[P_\lambda(J(\tau_1) - J(\tau_2))] \right\| \\
&\quad \leqslant \| p(\cdot, t_1) - p(\cdot, t_2)\|_\gamma^* [\| J(\tau_1)\| + \| A^\gamma J(\tau_1)\|] \\
&\qquad + \| p(\cdot, t_2)\|_\gamma^* [\| J(\tau_1) - J(\tau_2)\| + \| A^\gamma J(\tau_1) - A^\gamma J(\tau_2)\|] \\
&\quad \leqslant 2 \frac{\epsilon}{8M} \cdot 2M + 2M \cdot 2 \frac{\epsilon}{8M} = \epsilon.
\end{aligned}$$

This estimate shows that Q is continuous in every square $[0, T] \times [0, T]$ and hence in $R_0^+ \times R_0^+$. This concludes the proof of Lemma 7.4.

Now we discuss problem (E*) under the assumptions $J_1 \in C(R_0^+, \underline{L}_2)$, $J_1(t) \in D(A)$ for every $t \geqslant 0$ and $AJ_1 \in C(R_0^+, \underline{L}_2)$. By Lemma 7.2 there exists for every $\tau \geqslant 0$ a uniquely determined $\underline{L}_2$-valued function G_τ with the properties

$$\begin{aligned}
G_\tau &\in C^2(R_0^+, \underline{L}_2), \\
G_\tau(t) &\in D(A) \qquad \text{for } t \geqslant 0, \\
-\Delta G_\tau + \epsilon\mu G_\tau'' + \mu\sigma G_\tau' &= 0 \qquad \text{for } t \geqslant 0, \\
G_\tau(0) = 0, \qquad G_\tau'(0) &= \frac{1}{\epsilon\mu} J_1(\tau).
\end{aligned} \tag{7.20}$$

By (7.12) we have

$$G_\tau(t) = \frac{1}{\epsilon\mu} \int_0^\infty g(\lambda, t)\, d[P_\lambda J_1(\tau)]. \tag{7.21}$$

Since $J_1(\tau) \in D(A) \subset D(A^{1/2})$, it follows from (7.16) that

$$\left(\frac{d}{dt}\right)^j G_\tau(t) = \frac{1}{\epsilon\mu} \int_0^\infty \left(\frac{\partial}{\partial t}\right)^j g(\lambda, t)\, d[P_\lambda J_1(\tau)] \tag{7.22}$$

for $j = 1, 2$. The formula preceding (7.15) shows that $p(\lambda, t) := (\partial/\partial t)^j g(\lambda, t)$ satisfies the assumptions of Lemma 7.4 with $\gamma = 1$ if $j \leqslant 2$. Hence Lemma 7.4, with $\gamma = 1$, implies that $G_\tau(t)$, $G_\tau'(t)$ and $G_\tau''(t)$, as $\underline{L}_2$-valued functions of t and

τ, are continuous in $R_0^+ \times R_0^+$. By (7.20), also $\Delta G_\tau(t)$ is continuous in $R_0^+ \times R_0^+$.

According to Duhamel's principle, we set

$$E(t) = \int_0^t G_\tau(t - \tau)\, d\tau. \tag{7.23}$$

Since $G_\tau(t - \tau)$, $G_\tau'(t - \tau)$ and $G_\tau''(t - \tau)$ are continuous in the set $\{(t, \tau): t \geqslant 0, 0 \leqslant \tau \leqslant t\}$ by the consideration above, we can form E' and E'' by the usual rules which are valid also for vector-valued functions (see [1], Section 2):

$$\begin{aligned} E'(t) &= G_\tau(0) + \int_0^t G_\tau'(t - \tau)\, d\tau, \\ E''(t) &= G_\tau'(0) + \int_0^t G_\tau''(t - \tau)\, d\tau. \end{aligned} \tag{7.24}$$

We show that $E(t) \in D(A)$ for $t \geqslant 0$ and that

$$\Delta E(t) = \int_0^t \Delta G_\tau(t - \tau)\, d\tau. \tag{7.25}$$

Since $G_\tau(t - \tau)$ and $\Delta G_\tau(t - \tau)$ depend continuously on τ in $[0, t]$ and since $G_\tau(t - \tau) \in D(A)$ for $0 \leqslant \tau \leqslant t$, we obtain by (3.9) for every $F \in \underline{S}$

$$\begin{aligned} (E(t), \Delta F) &= \left(\int_0^t G_\tau(t - \tau)\, d\tau, \Delta F\right) \\ &= \int_0^t (G_\tau(t - \tau), \Delta F)\, d\tau \\ &= \int_0^t (\Delta G_\tau(t - \tau), F)\, d\tau \end{aligned}$$

and hence

$$(E(t), \Delta F) = \left(\int_0^t \Delta G_\tau(t - \tau)\, d\tau, F\right) \qquad \text{for } F \in \underline{S}. \tag{7.26}$$

By setting $F = \bar{\varphi} e_i$ with $\varphi \in C_0^\infty$ and $E = (E_1, E_2, E_3)$, $G_\tau = (G_{\tau 1}, G_{\tau 2}, G_{\tau 3})$, we conclude from (7.26) that

$$\begin{aligned} \Delta E_i(t)\, \varphi &= E_i(t)\,(\Delta\varphi) = (E_i(t), \Delta\bar{\varphi}) = (E(t), \Delta(\bar{\varphi} e_i)) \\ &= \left(\int_0^t \Delta G_\tau(t - \tau)\, d\tau, \bar{\varphi} e_i\right) = \left[\int_0^t \Delta G_{\tau i}(t - \tau)\, d\tau\right] \varphi \end{aligned}$$

for $i = 1, 2, 3$ and every $\varphi \in C_0^\infty$. Hence (7.25) holds. Formulas (7.25) and (7.26) imply $\Delta E(t) \in \underline{L}_2$ and

$$(\Delta E(t), F) = (E(t), \Delta F) \qquad \text{for every } F \in \underline{S}. \tag{7.27}$$

This, together with (3.9), yields

$$E(t) \in D(A) \qquad \text{for } t \geqslant 0. \tag{7.28}$$

It follows from (7.20), (7.23), (7.24), (7.25) and (7.28) that E has the following properties:

$$\begin{aligned} E &\in C^2(R_0{}^+, \underline{L}_2), \\ -\Delta E + \epsilon\mu E'' + \mu\sigma E' &= J_1 \qquad \text{for } t \geqslant 0, \\ E(t) &\in D(A) \qquad \text{for } t \geqslant 0, \\ E(0) = E'(0) &= 0. \end{aligned} \tag{7.29}$$

Now assume in addition that $J_1(t) \in D(A^{k/2})$ for every $t \geqslant 0$ and $A^{k/2} J_1 \in C(R^+, \underline{L}_2)$ where k is an integer with $k > 2$. Then $p(\lambda, t) := (\partial/\partial t)^j g(\lambda, t)$ satisfies the assumptions of Lemma 7.4 with $\gamma = k/2$ if $j \leqslant k$. Hence it follows from Lemma 7.4 that the $\underline{L}_2$-valued functions $G_\tau^{(j)}(t), j = 0, 1,..., k$ are continuous in $R_0{}^+ \times R_0{}^+$. Thus we obtain by (7.23) $E \in C^k(R_0{}^+, \underline{L}_2)$. These considerations, together with Lemma 7.2, imply:

THEOREM 7.1. *Assume that* $E_0 \in D(A)$, $E_1 \in D(A^{1/2})$, $J_1 \in C(R_0{}^+, \underline{L}_2)$, $J_1(t) \in D(A)$ *for every* $t \geqslant 0$, *and* $AJ \in C(R_0{}^+, \underline{L}_2)$. *Then problem* (E*) *has a uniquely determined solution which can be obtained as the sum of the righthand-sides in formulas* (7.12) *and* (7.23) *where G is defined by* (7.21) *and f and g are the solutions of* (7.13) *and* (7.14). *If, in addition,* $E_0 \in D(A^{k/2})$, $E_1 \in D(A^{(k-1)/2})$, $J_1(t) \in D(A^{k/2})$ *for every* $t \geqslant 0$, *and* $A^{k/2} J_1 \in C(R_0{}^+, \underline{L}_2)$, *then we have* $E \in C^k(R_0{}^+, \underline{L}_2)$.

In the same way we can discuss the corresponding weak version (H*) of problem (H). In this case we have to replace the generalized boundary condition (7.3) by $H \in D(A')$ and P_λ by the spectral set P'_λ of A'. The solution is given by

$$H(t) = \int_0^\infty f(\lambda, t)\, d(P'_\lambda H_0) + \int_0^\infty g(\lambda, t)\, d(P'_\lambda H_1) + \int_0^t G_\tau^*(t - \tau)\, d\tau \tag{7.30}$$

with

$$G_\tau^*(t) = \frac{1}{\epsilon\mu} \int_0^\infty g(\lambda, t)\, d[P_\lambda J_2(\tau)] \tag{7.31}$$

where f and g are defined as above.

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