A Digamma function based proof of the Riemann Hypothesis

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A nonharmonic Fourier series $L_2^{\#}(-\pi,\pi)$ based circle method to solve additive number theory problems

Klaus Braun

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Dedicated to my wife Vibhuta
on the occasion of her 60th birthday, August 25, 2021

Abstract:

Based on the negative real zeros of the Digamma function $\Psi(x)$ an alternative representation of the Riemann density function $J(x)$ is provided where the critical (oscillating) summand $\sum_{Im(\rho)>0} Li(x^{\rho}) + Li(x^{1-\rho})$ is replaced by two not oscillating sums, both enjoying the asymptotics $Li(x^{1/2})$, which proves the Riemann Hypothesis.

The real negative zeros $\{w_n\}_{n\in N}$ of the Digamma function $\Psi(x)$ and the imaginary part of the only complex valued zeros $\{z_n\}_{n\in N}$ of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ in the form $2\pi\omega_n := |Im(z_n)|$ enjoy the property $n-\frac{1}{2} < a_n := -w_n, \omega_n < n$. Correspondingly weighted „retarding" sequences $a_n^* := \frac{3a_n + a_{n+1}}{4}$ fulfill the Kadec condition $|n-a_n^*| \le L < \frac{1}{4}$, $n \in Z$ enabling the full power of non-harmonic Fourier series theory on $L_2^{\#}(-\pi,\pi)$ with its relation to the Paley-Wiener space. In line with the proof of the RH those sequences allow a split of the Riemann density function $J(x)$ into a sum of two number theoretical non-harmonic Fourier series, each of them governing a unit half circle. Correspondingly, each pair of primes (p,q) of binary number theoretical problems can be governed by those two different number theoretical distribution functions. This overcomes the current challenge of binary number problems analyzing problem specific relations between two prime numbers and their common behavior as x tends to infinite at the same time.

The number theoretical properties of the considered baseline functions are:

1. for the negative zeros of $\Psi(x)$ it holds $y_n := n + w_n \sim \frac{1}{logn}$
2. for the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ it holds

   i. ${}_1F_1\left(\frac{1}{2};\frac{3}{2},logn\right) \sim \frac{\mathrm{n}}{logn}$ for $x \to \infty$ (OlF) p. 257.

   ii. ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right) = e^{\frac{1}{3}z}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}}$, (BuH) p.184

   iii. $M\left[\,{}_1F_1\left(\frac{1}{2},\frac{3}{2},-x\right) + x\,{}_1F_1'\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right) = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s}\left(1-\frac{s}{2}\right)$, $0 < Re(s) < 1$.

In terms of ii) in the context of „convolution operators and zeros of entire functions", (CaD), based on Pólya's observation (PoG), Hilfssatz II, we note that all zeros of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi\mathrm{i}z\right)$ are real.

In terms of the negative zeros of the Digamma function we note their relation to the Weyl sums enabled by Landau's trigonometric sum (LaE).

All considered data from books, especially concerning gap and density theorems, non-harmonic complex Fourier analysis, non-harmonic Fourier series in the $L_2^{\#}(-\pi,\pi)$ framework and the related Paley-Wiener space are provided in Appendix II.

A Digamma function based
proof of the Riemann Hypothesis

The entire function

$$\xi(s) := (s-1)\frac{s}{2}\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}} = \xi(1-s)$$

has no zeros on the real axis. All zeros of $\xi(s)$ are situated in the strip $0 < \sigma < 1$, and lie symmetrically about the lines $t = 0$ and $\sigma = 1/2$. The zeros of $\zeta(s)$ are identical (in position and order of simplicity) with those of $\xi(s)$, except $\zeta(s)$ has simple zeros at each of the points $s = -2n$.

Putting $(A) := -\sum_{Im(\rho)>0}[Li(x^{\rho}) + Li(x^{1-\rho})]$ and $(B) := \int_x^{\infty}\frac{dt}{t(t^2-1)logt}$ the famous Riemann density formula is given by

$$J(x) - Li(x) + log2 = (A) + (B).$$

The formulae (A) and (B) result from the Fourier inverses of the $\log$ terms of the two canonical product representations

$$\xi(s) = \frac{1}{2}\prod_{\rho}\left(1-\frac{s}{\rho}\right), \frac{1}{\Gamma\left(1+\frac{s}{2}\right)} = \prod\left(1+\frac{s}{2n}\right)\left(1+\frac{1}{n}\right)^{-s/2}$$

In combination with the formulae (*)

A. $\quad -(A) = \frac{1}{2\pi i}\frac{1}{logx}\int_{a-i\infty}^{a+i\infty} log\xi(s)\, x^s \frac{ds}{s} = \int_0^{x^{\rho}}\frac{dt}{logt} + \int_0^{x^{1-\rho}}\frac{dt}{logt}$, where

1. $\int_0^{x^{\rho}}\frac{dt}{logt} = Li(x^{\rho}) - i\pi$ for $Im(\rho) > 0$
   resp.
2. $\int_0^{x^{1-\rho}}\frac{dt}{logt} = Li(x^{1-\rho}) + i\pi$ for $Im(1-\rho) < 0$

B. $\quad (B) = \frac{1}{2\pi i}\frac{1}{logx}\int_{a-i\infty}^{a+i\infty} log\Gamma(1+\frac{s}{2})\, x^s \frac{ds}{s} = \int_x^{\infty}\frac{dt}{t(1-t^2)\log t}$.

The essential difference between formulae (A) and (B) is the fact, that for the underlying non-trivial resp. trivial zeros in case (A) it holds, $Re(\rho_n) > 0$, while in case (B) it holds $Re(-2n) < 0$.

The „root of evil" to prove the RH is the fact, that the sum of the two terms A.a. and A.b. oscillates in sign as $x \to \infty$. Therefore, the series is only conditional convergent, i.e., the summation is understood by always pairing the associated non-trivial zeros of $\zeta(s)$. Technically speaking, the related system $\{e^{i\rho t}\}$ of the non-trivial zeta function zeros is minimal in $L_2(-\pi,\pi)$, i.e. there is no chance to build distribution function based on those sequences (Appendix I and II).

(*) The main tool to derive those formulae (A.) and (B.) is the function, (Appendix I)

$$H(\beta) := \frac{1}{2\pi i}\frac{1}{log\, x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log(1-\frac{s}{\beta})}{s}\right]x^s\, ds$$

where $a > Re(\beta)$ and where $log(1-\frac{s}{\beta})$ is defined for all complex numbers β other than real numbers $\beta \leq 0$ to be $log(1-\beta) - \log(-\beta)$. Let C^- denote the contour which goes over the lower semicircle from $1-\varepsilon$ to $1+\varepsilon$ and let C^+ denote the corresponding upper semicircle.

Lemma: For $Re(\beta) > 0$ when x is a fixed number with $x > 1$ it holds

i) $\quad H(\beta) = \int_{C^+}\frac{t^{\beta-1}}{logt}dt = \int_0^{x^{\beta}}\frac{du}{logu} = Li(x^{\beta}) - i\pi$, $Im(\beta) > 0$

ii) $\quad H(\beta) = \int_{C^-}\frac{t^{\beta-1}}{logt}dt = \int_0^{x^{\beta}}\frac{du}{logu} = Li(x^{\beta}) + i\pi$, $Im(\beta) < 0$.

Lemma: (PaR) p. 77, Appendix I and II: All the zeros z_n of the even Zeta function $\Xi(z):=\xi(1/2+iz)$ lie in the strip $|Im(z)|<1/2$. The corresponding product representation is given by, $(\xi(0)=-\xi(0)=1/2)$,

$$\Xi(z)=\Xi(0)\prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2}),$$

For $\lambda_n:=|z_n|$ with $\sum_1^{\infty}\lambda_n^{-2}<\infty$ and

$$H(z):=\Xi(0)\prod_{n=1}^{\infty}(1-\frac{z^2}{\lambda_n^2})$$

it holds $log\left|\frac{H(iy)}{\Xi(iy)}\right|=O(1)$. Thus, because of $log\Xi(iy)=O(y)+log\Gamma\left(\frac{y}{2}\right)\sim\frac{1}{2}ylogy$, $(y>0)$, it follows

$$logH(iy)\sim\frac{1}{2}ylogy \text{ resp. } -\frac{1}{logy}\int_{-y}^{y}log\left|\frac{\Xi(x)}{\Xi(0)}\right|\frac{dx}{x^2}\rightarrow\frac{\pi}{2} \text{ as } y\rightarrow\infty.$$

In order to prove the RH criteria

$$\pi(x)=Li(x)+O(\sqrt{x}\,log\,x) \text{ resp. } \pi(x)=Li(x)+O(x^{1/2+\varepsilon}),\ \varepsilon>0$$

we define a corresponding product representation to $H(z)$ in the form (*)

$$B(z):=\xi\left(\frac{1}{2}\right)\prod_n\left(1-\frac{z}{\gamma_n}\right)\left(1-\frac{z}{\bar{\gamma}_n}\right)=\xi\left(\frac{1}{2}\right)\prod_n\left(1-\frac{z^2}{|\gamma_n|^2}\right)$$

with $\gamma_n:=\frac{1}{2}+iw_n$ where w_n denotes the negative zeros of the Digamma function $\Psi(x)$. From the Lemma above it follows

$$log\left|\frac{\Xi(iy)}{B(iz)}\right|=O(1).$$

The crucial differentiator between the two product representations $H(z)$ and $B(z)$ is the change of sign of $Im(\gamma_n)<0$ and $Im(\bar{\gamma}_n)>0$, providing „pair" formulae related to the formulae A.1. and A.2 (**). The combination of those „pair" formulae results into

i) $$H(\rho_n)+H(\gamma_n)=Li(x^{\rho_n})+Li(x^{\gamma_n})\sim Li(x^{\frac{1}{2}})$$

ii) $$H(1-\rho_n)+H(\bar{\gamma}_n)=Li(x^{1-\rho_n})+Li(x^{\bar{\gamma}_n})\sim Li(x^{\frac{1}{2}})$$

from which it follows

$$|J(x)-Li(x)|=O(x^{\frac{1}{2}}).$$

(*) Considering the next section the related product representation is given by

$$B^*(z):=\xi\left(\frac{1}{2}\right)\prod_n\left(1-\frac{z}{\gamma_n}\right)\left(1-\frac{z}{\vartheta_n}\right) \text{ with } \vartheta_n:=\frac{1}{2}+i|\omega_n|.$$

(**)

$$H(\rho_n)=\frac{1}{2\pi i}\frac{1}{log\,x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log(1-\frac{s}{\rho_n})}{s}\right]x^s\,ds=\int_0^{x^{\rho_n}}\frac{du}{logu}=Li(x^{\rho_n})-i\pi\ ,\ Im(\rho_n)>0$$

$$H(1-\rho_n)=\frac{1}{2\pi i}\frac{1}{log\,x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log(1-\frac{s}{1-\rho_n})}{s}\right]x^s\,ds=\int_0^{x^{1-\rho_n}}\frac{du}{logu}=Li(x^{1-\rho_n})+i\pi\ ,\ Im(\rho_n)>0$$

$$H(\gamma_n)=\frac{1}{2\pi i}\frac{1}{log\,x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log(1-\frac{s}{\gamma_n})}{s}\right]x^s\,ds=\int_0^{x^{\gamma_n}}\frac{du}{logu}=Li(x^{\gamma_n})+i\pi$$

$$H(\bar{\gamma}_n)=\frac{1}{2\pi i}\frac{1}{log\,x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log(1-\frac{s}{\bar{\gamma}_n})}{s}\right]x^s\,ds=\int_0^{x^{\bar{\gamma}_n}}\frac{du}{logu}=Li(x^{\bar{\gamma}_n})-i\pi.$$

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The real negative zeros $\{w_n\}_{n\in N}$ of the Digamma function $\Psi(x)$ and the imaginary part of the only complex valued zeros $\{z_n\}_{n\in N}$ of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ in the form $2\pi\omega_n := |Im(z_n)|$ enjoy the same appreciated properties

$$(*)\ \ n-\tfrac{1}{2} < a_n := -w_n, \omega_n < n,$$

from which it follows

i) The sequences a_n have Snirelmann density ½ from a number theoretical perspective

ii) $|a_n - a_m| \geq \frac{1}{2}|n-m|$, $\lim_{n\to\infty}\frac{n}{a_n} = 1$ from an analytical perspective (*).

The Digamma zeros provide a distribution sequence for the horizontal line $\frac{1}{2} - i|w_n|$, while the corresponding Kummer function zeros provide a distribution sequence for the horizontal line $\frac{1}{2} + i|\omega_n|$; both can be mapped holomorphically to two unit half-circles.

The property (*) enables the definition of corresponding weighted „retarding" sequences

$$w_n^* := \frac{3w_n + w_{n+1}}{4}\ \ ,\ \omega_n^* := \frac{3w_n + w_{n+1}}{4}$$

fulfilling the Kadec condition

$$|n - a_n^*| \leq L < \frac{1}{4},\ n\in Z.$$

Therefore, the „retarding" Digamma and Kummer function sequences above enable the full power of non-harmonic Fourier series theory on $L_2^{\#}(-\pi,\pi)$ with its relation to the Paley-Wiener space (YoR) (**).

Regarding the previous section this means that the Riemann density function $J(x)$ can be split into a sum of two sequences based non-harmonic Fourier series, each of them governing a unit half circle. Correspondingly, each pair of primes (p,q) of a binary number theoretical problem will be governed by those two different number theoretical distribution functions.

(*) The two properties (ii) are the standard assumptions of „gaps and density theorems" for entire function dealing with the determination of the rate of growth of those functions from their growth on sequences of points a_n, ((LeN), Appendix I and II).

(**) Kadec's Theorem (YoR) p. 36: If $\{\lambda_n\}_{n\in Z}$ is a sequence of real numbers for which $|\lambda_n - n| \leq L < 1/4$, then $\{e^{i\lambda_n t}\}_{n\in Z}$ satisfy the Paley-Wiener criterion and so forms a Riesz basis for $L_2\,(-\pi,\pi)$.

Regarding the stability of the class of Riesz bases $\{e^{i\lambda_n t}\}$ in $L_2\,(-\pi,\pi)$ Kadec's theorem can be dramatically improved, first under „small" displacements of the λ_n's and then under more general „vertical" displacements, (YoR) pp. 160 ff..

Corollary 1 (YoR) p. 164: Let $\{\lambda_1, \lambda_2, \lambda_3, ...\}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\{e^{iRe(\lambda_n)t}\}$ is a Riesz basis for $L_2\,(-\pi,\pi)$, then so is $\{e^{i\lambda_n t}\}$.

Corollary 2 (YoR) p. 164: if $\{\lambda_n\}_{n\in Z}$ be a sequence of scalars for which $sup_n|Re(\lambda_n) - n| < 1/4$ and $sup_n|Im(\lambda_n)| < \infty$, then the system $\{e^{i\lambda_n x}\}_{n\in Z}$ is a Riesz basis for $L_2^{\#}(-\pi,\pi)$.
(**) The totality of all entire functions of exponential type at most π (i.e., $|f(z)| \leq e^{\pi|z|}$) that are square integrable on the real axis is known as the Paley-Wiener (separable Hilbert) space PW (i.e. it holds $|f(x+iy)| \leq e^{\pi|y|}\|f\|$), equipped with the inner product $(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$. The Paley-Wiener space PW is isometrically isomorph to $L_2\,(-\pi,\pi)$. Every function $f \in PW$ can be recaptured from its values at the integers, which is achieved by the cardinal series representation of f, (YoR) p. 90.

Appendix I

„The roots of evil"
(LeB) p. 32

The challenge to prove the RH criteria

$$\pi(x) = Li(x) + O(\sqrt{x}\, log\, x) \text{ resp. } \pi(x) = Li(x) + O(x^{1/2+\varepsilon}),\ \varepsilon > 0$$

is the fact that the sum of the two terms in equation $\sum_{Im(\rho)>0}[Li(x^{\rho}) + Li(x^{1-\rho})]$ oscillates in sign as $x \to \infty$.

The connection between the growth of entire functions and the distribution of their integers is acompanied by corresponding gap and density theorems ((LeN). Worst case examples are the entire functions

$$\sin\left(\frac{\pi}{2}z\right) = \frac{\pi}{2}z\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(1-\frac{z}{2n})\,e^{z/2n} \quad , \quad \frac{1}{\Gamma(z/2)} = e^{\gamma z/2}\prod_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(1+\frac{z}{2n})\,e^{-z/2n},$$

as the canonical products are of maximal (infinite) type. „*The root of evil is the presence of a symmetry in the distribution of zeros of the first function and its absence for the second function*" (LeB) p. 32. For the second function it holds, (PaR) p. 39,

$$\frac{1}{\Gamma\left(\frac{1}{2}+iy\right)} \sim O(e^{\frac{\pi}{2}|y|}).$$

The „*root of evil*" concerning $\zeta(s)$ (with the symmetry in the distribution of its zeros) is the fact, that the related system $\{e^{i\rho t}\}$ is minimal in $L_2\,(-\pi,\pi)$, i.e. there is no chance to build any kind of zero sequences based distribution functions. This gap is due to the mean square modulus property of $\zeta(s)$, (EdH) 9.8, (LaE3b) §228, (WiN) p. 138 (*), (YoR) p. 109,

$$\frac{1}{2T}\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 dx \sim logT \text{ resp. } \int_{-\infty}^{\infty}\frac{\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2}{1+x^2}dx < \infty$$

which means that the functions $\zeta(s), \xi(s)$ are no PW functions.

(*) For the full analysis in the context of generalized harmonic analysis of Bohr's almost periodic functions we refer to (WiN) p. 150

Regarding the asymptotics of $\zeta(\sigma+it)$ for $0 \le \sigma \le 1$, we note (LaE3b) § 240,

$$\zeta(\sigma+it) = O\left(t^{\frac{1-\sigma}{2}}logt\right) = O\left(t^{\frac{1-\sigma}{2}+\varepsilon}\right).$$

The zeros of the Digamma function $\Psi(x)$ and the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$

The zeros $\{w_0, w_n\}_{n\in N}$ of the Digamma function are all real, where only w_0 is positive, which is appreciated in our case. The zeros $\{v_n\}_{n\in N}$ of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ are all complex with $Re(v_n) > 1/2$, with same distribution of the imaginary part as those of w_n:

i) $w_n \in (-n, 1/2 - n)$, $1 - w_0 \sim -\ 0{,}461 \in (-\frac{1}{2}, 0)$ (NiN) p. 99, (SeP)

ii) $\omega_n := \left|Im(\frac{v_n}{2\pi})\right| \in (n - 1/2, n)$ (SeA).

For more details especially regarding the not used asymptotics of the sequence $y_n := n + w_n = \frac{1}{\pi} arctan\left(\frac{\pi}{logn + \delta_n}\right) \sim \frac{1}{logn}$ we refer to appendix II.

From $w_n \in (-n, 1/2 - n)$ and $\omega_n :=\in (n - 1/2, n)$ it follows

$$|w_n - w_m| \geq \frac{1}{2}|n - m|,\ \lim_{n\to\infty}\frac{n}{w_n} = 1,\ |\omega_n - \omega_m| \geq \frac{1}{2}|n - m|,\ \lim_{n\to\infty}\frac{n}{\omega_n} = 1,$$

which is the standard prerequisite of „gaps and density theorems“ dealing with the determination of the rate of growth of analytic functions from their growth on sequences of points a_n, ((LeN).

In order to prove the RH we are only applying the zeros of the Digamma function. The combined usage of both sequences enables a truly unit circle method in a $L_2^{\#}(-\pi,\pi)$ framework applying the following properties

i) the sequences $\{-w_n, \omega_n\}$ have Snirelmann density ½

ii) the weighted „retarding“ sequences

$$w_n^* := \frac{3w_n + w_{n+1}}{4}\ w_{-n} := -w_n\ ,\ \omega_n^* := \frac{3w_n + w_{n+1}}{4}\ \omega_{-n} := -\omega_n$$

fulfills the Kadec condition:

$$|n - a_n^*| \leq L < \frac{1}{4},\ n \in Z$$

enabling the full power of non-harmonic Fourier series theory and its relation to the Paley-Wiener space (YoR) (*).

(*) The totality of all entire functions of exponential type at most π (i.e., $|f(z)| \leq e^{\pi|z|}$) that are square integrable on the real axis is known as the Paley-Wiener (separable Hilbert) space PW (i.e. it holds $|f(x+iy)| \leq e^{\pi|y|}\|f\|$), equipped with the inner product $(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$. The Paley-Wiener space PW is isometrically isomorph to $L_2(-\pi,\pi)$. Every function $f \in PW$ can be recaptured from its values at the integers, which is achieved by the cardinal series representation of f, (YoR) p. 90.

(**) Kadec's Theorem (YoR) p. 36: If $\{\lambda_n\}_{n\in Z}$ is a sequence of real numbers for which $|\lambda_n - n| \leq L < 1/4$, then $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ satisfy the Paley-Wiener criterion and so forms a Riesz basis for $L_2(-\pi,\pi)$.

Regarding the stability of the class of Riesz bases $\left\{e^{i\lambda_n t}\right\}$ in $L_2(-\pi,\pi)$ Kadec's theorem can be dramatically improved, first under „small“ displacements of the λ_n's and then under more general „vertical“ displacements, (YoR) pp. 160 ff..

Corollary 1 (YoR) p. 164: Let $\{\lambda_1, \lambda_2, \lambda_3, \ldots.\}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\left\{e^{iRe(\lambda_n)t}\right\}$ is a Riesz basis for $L_2(-\pi,\pi)$, then so is $\left\{e^{i\lambda_n t}\right\}$.

Corollary 2 (YoR) p. 164: if $\{\lambda_n\}_{n\in Z}$ be a sequence of scalars for which $sup_n|Re(\lambda_n) - n| < 1/4$ and $sup_n|Im(\lambda_n)| < \infty$, then the system $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$ is a Riesz basis for $L_2^{\#}(-\pi,\pi)$.

(IvS): On addition of one exponential the Riesz basis a $\left\{e^{2\pi i\lambda_n x}\right\}_{n\in Z}$ forms a Riesz basis of fractional Sobolev spaces $H_\beta^{\#}(0,1)$ of order β with $0 \leq \beta \leq 1$ and $\beta \neq 1/2$.

Über eine trigonometrische Summe

E. Landau, (LaE1)

Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen,
Mathematisch-Physikalische Klasse, Vol. 1928, p. 21-24, 1928

Theorem (LaE1): let $m > 1$, β_1 real, $0 < \vartheta \leq \beta_2 - \beta_1 \leq \beta_3 - \beta_2 \leq \cdots \beta_{m-1} - \beta_m \leq 1 - \vartheta$, $\sigma_n := e^{-2\pi i\beta_n}$,

$$S_m \coloneqq \left|\sum_{n=1}^{m} \sigma_n\right|.$$

Then it holds:

i) $S_m \leq \cot\left(\frac{\pi}{2}\vartheta\right)$;

ii) For $\vartheta = 1/2$ and every positive fraction $\vartheta < 1/2$ with odd nominator and odd denominator:

$$S_m = \cot\left(\frac{\pi}{2}\vartheta\right);$$

iii) For all other ϑ with $0 < \vartheta < 1/2$:

$$S_m < \cot\left(\frac{\pi}{2}\vartheta\right);$$

iv) For all ϑ with $0 < \vartheta \leq 1$ and every $\varepsilon > 0$:

$$S_m > \cot\left(\frac{\pi}{2}\vartheta\right) - \varepsilon.$$

Riesz Basis systems $\left\{e^{i\lambda_n t}\right\}_{n\in N}$ and the Paley-Wiener space PW

A complete sequence of vectors in a separable Hilbert space is a Riesz basis if and only if its moment space is equal to l^2 , (YoR) p. 142. A sequence of real or complex numbers $\{\lambda_n\}_{n\in N}$ is said to be an interpolating sequence for PW if the set of all sequences $\{f(\lambda_n)\}_{n\in N}$ where f ranges over PW, coincides with l^2 . If, in addition, the system $\left\{e^{i\lambda_n t}\right\}_{n\in N}$ is complete in $L_2^{\#}(\Gamma)$, then $f(\lambda_n) = c_n$ has exactly one solution, provided $c_n \in l^2$, and in this case we shall call $\{\lambda_n\}_{n\in N}$ a complete interpolating sequence. A complete interpolation sequence is „maximal“ in the sense that it is not contained in any larger interpolating sequence, and the converse is also true, (YoR) p. 142.

Putting

$$G(z) := z\prod_{k=0\to\infty}\left(1-\frac{z^2}{\lambda_n^2}\right) \quad \text{and } G_n(z) := \frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}$$

then $G_n(z)$ belongs to the Paley-Wiener space PW and $g_n(t)$ is the inverse Fourier transform of $G_n(z)$, i.e. for almost all $t \in [-\pi,\pi]$,

$$g_n(t) := \int_{-\infty}^{\infty} G_n(z)e^{ixt}dt.$$

The exponentials $e^{i\lambda_n t}$ are transformed into the reproducing functions $K_n(z) = \frac{\sin\,\pi(z-\lambda_n)}{\pi(z-\lambda_n)}$, $g_n(t)$ is transformed into $G_n(z)$, while the moment problem itself becomes

$$f(\lambda_n) = 0,\ n = 0, \pm 1, \pm 2, \pm 3, \ldots..,$$

since $f(\lambda_n) = (f, K_n)$. Here $c_n \in l^2$ and $f \in PW$ is to be found.

By taking the Fourier transform of $\left\{e^{int}\right\}_{n\in Z}$, we see that the set of functions

$$\left\{\frac{\sin\,\pi(z-n)}{\pi(z-n)}\right\}_{n\in Z}$$

forms an orthogonal basis for PW. Accordingly every function f in PW has an unique expansion of the form

$$f(z) = \sum_{-\infty}^{\infty} c_n \frac{\sin\,\pi(z-n)}{\pi(z-n)} \quad \text{with } \sum_{-\infty}^{\infty}|c_n|^2 < \infty.$$

The convergence of the series is understood to be in the metric of PW. But convergence in PW implies uniform convergence in each horizontal strip. This is an immediate consequence of the following useful estimate, (YoR) p. 90:

$$|f(x+iy)| \le e^{\pi|y|}\|f\|.$$

The core theorem of non-harmonic Fourier series theory is given by, (e.g. (LeN) p. 48, (PaR) p. 113, (YoR) p. 100),

> **Lemma**: If $\{\lambda_n\}$ is a sequence and L a constant such that $|\lambda_n - n| \le L < 1/4$, then the system $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$ is close in $L_2^{\#}(-\pi,\pi)$ (i.e. from $\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0$ it follows that $f(x)$ is identically zero) and possesses a unique biorthogonal set $\{h_n(x)\}$ such that for any $f \in L_2^{\#}(-\pi,\pi)$ the series
>
> $$\sum_{-\infty}^{\infty}\frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi - e^{i\lambda_n x}\int_{-\pi}^{\pi} f(\xi)h_n(\xi)d\xi$$
>
> converges uniformly to zero over the interval $[-\pi+\delta, \pi-\delta]$ for any $\delta > 0$. Moreover the difference of weighted sums (Riesz, Abel, and so on) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over $[-\pi+\delta, \pi-\delta]$.

Remark (IvS): On addition of one exponential the Riesz basis a $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis of fractional Sobolev spaces $H_\beta^\#(0,1)$ of order β with $0 \le \beta \le 1$ and $\beta \neq 1/2$. Let $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis for $L_2^\#(0,1)$. Then for each number μ, which do not belong to the spectrum $\{\lambda_n\}_{n\in Z}$, the exponential families

$$E_\mu^{(\beta)} = \left\{\frac{1}{(1+|2\pi\lambda_n|^\beta)} e^{2\pi i\lambda_n x}\right\}_{n\in Z} \cup \{e^{2\pi i\mu x}\}$$

form a Riesz basis for the Sobolev space $H_\beta\ (0,1)$.

Remark: We note that according to the Sobolev embedding theorem any $g \in H_\beta^\#(0,1)$ with $\beta < \frac{1}{2}$ is bounded, i.e. $|g(x)| \le c$.

Remark: Regarding the asymptotics of $\zeta(\sigma+it)$ we note

i) $\zeta(\sigma+it) = O(logt) = O(t^\varepsilon)$ for $\sigma \ge 1$, (LaE3a) § 46

ii) $\zeta(\sigma+it) = O(logt) = O(t^{\frac{1}{2}-\sigma+\varepsilon})$ for $\sigma \le 0$, (LaE3a) § 228

iii) $\zeta(\sigma+it) = O\left(t^{\frac{1-\sigma}{2}} logt\right) = O\left(t^{\frac{1-\sigma}{2}+\varepsilon}\right)$ for $0 \le \sigma \le 1$, (LaE3b) § 240.

Lemma: Let (YoR) p. 12: Let X denote the vector space of all finite linear combinations of functions of the form $e^{i\lambda x}$, $(-\infty < t < \infty)$, where the parameter λ is real. An inner product in X is defined by

$$((f,g)) := \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(t)\overline{g(t)}dt.$$

When X is closed by means of the metric generated by this inner product, we obtain a certain Hilbert space B^2 (B is for Besicovitch). The continuum of elements $e^{i\lambda x}$ forms a complete orthogonal subset of B^2. The Hilbert space B^2 contains the important class of Bohr almost periodic functions. Those functions are obtained by adding to X the limits of sequences of function in X that are uniformly convergent on the entire real line.

Relations to Dirichlet series

Lemma (LaE3b) §228: let $0 < \beta < 1$, and let $g(s) := \sum_1^\infty a_n e^{-s \log n}$ be absolute convergent for $Re(s) = \sigma = \gamma$. Then

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} \zeta(\beta+it)g(\gamma-it)dt = \sum_1^\infty a_n \frac{\log n}{n^{\beta+\gamma}},$$

i.e. it especially holds

$$(*)\ \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} \zeta(\beta+it)\zeta(\gamma-it)dt = \sum_1^\infty \frac{1}{n^{\beta+\gamma}} = \zeta(\beta+\gamma).$$

The latter formula (*) can be generalized by

Lemma (LaE3b) §228: let $-1 < \beta, \gamma$ with $\beta + \gamma = 1$, $\beta > 1, \gamma > 1$ or $\beta < 1, \gamma < 1$. Then

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} \zeta(\beta+it)g(\gamma-it)dt = \zeta(\beta+\gamma).$$

Lemma:

$$\frac{1}{2\pi}\int_0^\infty \left|\Gamma(\frac{1}{2}+it\right|^2 dx = \frac{1}{2}.$$

The most general results concerning the average values of $(|\zeta(s)|^2$ are provided in (LaE3b) §228:

Lemma: In the sense that the relative error approaches zero as $\omega \to \infty$

$$\frac{1}{2\omega}\int_{-\omega}^{\omega}(\left|\zeta(\frac{1}{2}+it)\right|^2 dt \sim \log\omega.$$

Regarding Dirichlet series we recall from (TiE) p. 138:

Lemma: Let $f(s) := \sum_1^\infty a_n e^{-s \log n}$, $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be absolute convergent for $Re(s) > 1/2$. Then for $\alpha > 1/2$

$$\langle f, g\rangle_{-\alpha} = \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\sigma+it)g(\sigma-it)dt = \sum_1^\infty \frac{1}{n^{2\alpha}} a_n b_n,$$

i.e. $f, g \in H_{-\alpha}^{\#} \cong l_2^{-\alpha}$ for $\alpha > 1/2$.

The generalization is provided by the "main theorem" from (LaE3b) §225:

Theorem 37: Let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$ and $\frac{\beta-\alpha_1}{\varepsilon_1} + \frac{\gamma-\alpha_2}{\varepsilon_2} > 1$ it holds

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta+it)g(\gamma-it)dt = \sum_1^\infty a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

Putting $\alpha = \alpha_1, = \alpha_2$ and $\varepsilon = \varepsilon_1 = \varepsilon_2$ Theorem 37 leads to

Theorem 38: Let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ and $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta, \gamma > \alpha$, $(\beta - \alpha) + (\gamma - \alpha) > \varepsilon$ it holds

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta+it)g(\gamma-it)dt = \sum_1^\infty a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

Putting $l := \limsup\left(\frac{\log n}{\lambda_n}\right)$ (chosing $\varepsilon_1 = \varepsilon_2 \coloneqq l$) Theorem 37 leads to

Theorem 39: Let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$, $(\beta - \alpha_1) + (\gamma - \alpha_2) > l$

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\gamma - it) dt = \sum_1^\infty a_n b_n \, e^{-\lambda_n(\beta + \gamma)}.$$

chosing $\varepsilon_1 = \varepsilon_2 \coloneqq \alpha, \beta = \gamma$ leads to

Theorem 40: Let $l := \limsup\left(\frac{\log n}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ and $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\beta - it) dt = \sum_1^\infty a_n b_n \, e^{-2\lambda_n \beta}.$$

Theorem 41: Let $l := \limsup\left(\frac{\log n}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta > \alpha + \varepsilon/2$

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f|(\beta + it)|^2 dt = \sum_1^\infty |a_n|^2 \, e^{-2\lambda_n \beta}$$

Theorem 42: Let $l := \limsup\left(\frac{\log n}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f|(\beta + it)|^2 \, dt = \sum_1^\infty |a_n|^2 \, e^{-2\lambda_n \beta}.$$

Lemma (ApT) p. 188: let $f(s) := \sum_1^\infty a_n e^{-s\lambda_n}$ absolute convergent for $\sigma > \sigma_a$ then

$$\frac{1}{2\omega} \int_{-\omega}^{\omega} e^{\lambda(\sigma + it)} f(\sigma + it) dt = \begin{cases} a_n & if \ \lambda = \lambda(n) \\ 0 & if \ \lambda \neq \lambda(n) \end{cases}.$$

Lemma (ApT) p. 188: let $\mu_n \coloneqq e^{\lambda_n}$, then $g(s) := \sum_1^\infty a_n e^{-s\mu_n}$ is absolute convergent for $\sigma > 0$; if $\sigma > \sigma_a$ then

$$\Gamma(s) f(s) = \int_0^\infty g(t) t^{s-1} dt$$

which is an extension of the classical formula

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{e^t - 1} t^{s-1} dt.$$

The entire function $\xi(s)$ is of order $\sigma = 1$ with finite order type

An entire function is of finite order, if $M_f(r) \coloneqq \max_{|z|=r}|f(z)| \overset{as}{<} e^{r^k}$ for some constant $k > 0$. The order of growth of an entire function is the greatest lower bound of these values of k. As entire functions with same order can grow differently, there are different types of orders, grouped into minimal, normal (mean), and infinite types. The entire function $f(z)$ is said to have a finite type, if for some $A > 0$ the inequality $M_f(r) \coloneqq \max_{|z|=r}|f(z)| \overset{as}{<} e^{Ar^k}$ is fulfilled. The greatest lower bound for those values of is called the type $\sigma = \sigma_f$ with respect to the order k. If $\sigma_f = 0$ the type is called minimal, if $0 < \sigma_f < \infty$ the type is called normal (or mean), if σ_f is infinite, the type is called maximal. Entire functions of order $k = 1$ and normal type $\sigma = \sigma_f$ are called entire functions of exponential type $\sigma = \sigma_f$. For an entire function $f(z)$ of minimal type with respect to an order k it holds $log|f(z)| = o(|z|^k)$, $|z| \to \infty$, (LeB) p 90.

Remark (YoR) p. 60: We note that entire functions $f(z)$ and $f'(z)$ are of the same order. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of order σ, then

$$\sigma = \lim_{n\to\infty} sup \frac{nlogn}{log\,(\frac{1}{|a_n|})}.$$

The exponential type" of is defined by the number $k = \lim_{r\to\infty} sup \frac{logM(r)}{r}$. The notion of „exponential type" is not to confused with that of „type"; An entire function of positive order ρ is said to be of type τ (relative to that order) if

$$\tau = \lim_{r\to\infty} sup \frac{logM(r)}{r^\rho}.$$

The entire function $\xi(s)$ is of order $\sigma = 1$ with finite order type (as on the circle $|s| = r$ for its maximum modulus function it holds $M(r) \sim \frac{1}{2} rlogr$. The function $\frac{1}{\Gamma(z)}$ has only zeros on the negative x-axis. It is the most prominent example where the density of the zero set is finite, while its canonical product is of maximal (i.e., infinite order) type as $\log(M(r)) \geq Crlogr$, (LeB) p. 32).

The entire function

$$\frac{1}{\Gamma\left(1+\frac{z}{2}\right)} = e^{\frac{\gamma z}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{z}{2n}\right) e^{-\frac{z}{2n}}$$

is most prominent example where the density of the zero set is finite, while the canonical product of order one is of maximal type, i.e. $logM(r) \geq Crlogr$.

Lemma (PaR) p. 41: $\Gamma(\frac{3}{2} + iv)\zeta\frac{3}{2} + iv) \in L_2(-\infty, \infty)$.

Remark: (PaR) p. 77: The even Zeta function $\Xi(z) := \xi(1/2 + iz)$ has all its zeros in the strip $|Im(z)| < 1/2$. Let z_n denote its zeros, then

$$\Xi(z) = \Xi(0) \prod_{n=1}^{\infty} (1 - \frac{z^2}{z_n^2}) \text{ with } \sum_1^{\infty} \lambda_n^{-2} < \infty \text{ where } \lambda_n \coloneqq |z_n|.$$

Putting

$$H(z) := \Xi(0) \prod_{n=1}^{\infty} (1 - \frac{z^2}{\lambda_n^2})$$

it holds $log\left|\frac{H(iy)}{\Xi(iy)}\right| = O(1)$. Thus, because of $log\Xi(iy) = O(y) + log\Gamma\left(\frac{y}{2}\right) \sim \frac{1}{2} ylogy$, $(y > 0)$, it follows

$$logH(iy) \sim \frac{1}{2} ylogy$$

and therefore

$$\frac{1}{logy} \int_{-y}^{y} log\left|\frac{\Xi(x)}{\Xi(0)}\right| \frac{dx}{x^2} \to -\frac{\pi}{2} \text{ as } y \to \infty.$$

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Appendix II

The Riemann density function

Regarding the distributions of the zeros of $\zeta(s)$ and $\xi(s)$ it holds, (InA) p. 48,

i) The zeros of $\xi(s)$ are all situated in the strip $0 < \sigma < 1$, and lie symmetrically about the lines $t = 0$ and $\sigma = 1/2$

ii) The zeros of $\zeta(s)$ are identical (in position and order of simplicity) with those of $\xi(s)$, except $\zeta(s)$has simple zero at each of the points $s = -2n$

iii) $\xi(s)$ has no zeros on the real axis.

We note the following equivalent critera for the RH:

i) $\pi(x) = Li(x) + O(\sqrt{x} \log x)$

ii) $\pi(x) = Li(x) + O(x^{1/2+\varepsilon})$, $\varepsilon > 0$, i.e. the relative error is $\frac{\pi(x)-Li(x)}{x} = O(x^{-\frac{1}{2}-\varepsilon})$ $\varepsilon > 0$

iii) $\psi(x) = x + O(\sqrt{x} \log^2 x)$

iv) The series $\sum_{n=1}^{\infty} \mu(n) n^{-s}$ is convergent for $Re(s) > 1/2$ and $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$.

The principal term in Riemann's formula for $J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s}$ derived from the Fourier inverse of $\frac{\log \zeta(s)}{s} = \int_0^{\infty} J(x) x^{-s-1} dx$, $Re(s) > 1$, is given by the $li_1(x)$ – function, (EdH) 1.14,

$$li_1(x) = \frac{1}{2} li(x + i0) + li(x - i0) = \lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log(s-1)}{s} \right] x^s ds.$$

Remark: The Riemann error term $\int_x^{\infty} \frac{dt}{t(1-t^2) \log t}$ reflects the trivial zeros of the Zeta function showing also up in the product formula, (EdH) 1.12, 1.16,

$$\Gamma\left(1 + \frac{s}{2}\right) = \Pi \left(1 - \frac{s}{-2n}\right)^{-1} \left(1 + \frac{1}{n}\right)^{s/2} \text{ resp. } \frac{1}{\Gamma\left(1+\frac{s}{2}\right)} = \Pi \left(1 + \frac{s}{2n}\right) \left(1 + \frac{1}{n}\right)^{-s/2} = e^{\gamma s/2} \Pi \left(1 + \frac{s}{2n}\right) e^{-s/(2n)}$$

where

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log\Gamma(1 + \frac{s}{2}) x^s \frac{ds}{s} = \int_x^{\infty} \frac{dt}{t(1-t^2) \log t}.$$

The main tool to derive this term and the terms involving the trivial and non-trivial roots of the zeta function is given by

$$H(\beta) := -\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log(1-\frac{s}{\beta})}{s} \right] x^s \, ds$$

where $a > Re(\beta)$ and where $\log(1 - \frac{s}{\beta})$ is defined for all complex numbers β other than real numbers $\beta \leq 0$ to be $\log(1 - \beta) - \log(-\beta)$. Let C^- denote the contour which goes over the lower semicircle from $1 - \varepsilon$ to $1 + \varepsilon$ and let C^+ denote the corresponding upper semicircle.

Lemma 1: For $Re(\beta) > 0$ when x is a fixed number with $x > 1$ it holds

iii) $\int_{C^+} \frac{t^{\beta-1}}{\log t} dt = \int_0^{x^\beta} \frac{du}{\log u} = Li(x^\beta) - i\pi$, $Im(\beta) > 0$

iv) $\int_{C^-} \frac{t^{\beta-1}}{\log t} dt = \int_0^{x^\beta} \frac{du}{\log u} = Li(x^\beta) + i\pi$, $Im(\beta) < 0$.

Lemma 2: For $Re(\beta) < 0$ it holds

$$H(\beta) = -\int_x^\infty \frac{t^{\beta-1}}{logt} dt = -\int_{x^\beta}^\infty \frac{du}{logu} .$$

Regarding the second part of this paper we note that except the single positive zero of the Digamma function, this lemma is also applicable to the sequence of all negative zeros of the Digamma function.

From lemma 1 it follows for the critical „oscillating“ term

$$-\sum H(\rho) = -\frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\sum log(1-\frac{s}{\rho})}{s} \right] x^s\, ds$$

involving the non-trivial zeta function roots $\{\rho\}$ the

Corollary 1:

$$-\sum H(\rho) = -\sum_{Im(\rho)>0} \left\{ \int_{C^+} \frac{t^{\rho-1}}{logt} dt + \int_{C^-} \frac{t^\rho}{logt} dt \right\} = -\sum_{Im(\rho)>0} \{Li(x^\rho) + Li(x^{1-\rho})\}.$$

From Lemma 2 it follows for the Riemann error function term

$$R(x) := \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{log\Gamma\left(1+\frac{s}{2}\right)}{s} \right] x^s\, ds$$

involving the trivial zeta function roots $\{-2n\}$ the

Corollary 2:

$$R(x) = -\sum_{n=1}^\infty H(-2n) = \sum_{n=1}^\infty \int_x^\infty \frac{t^{-2n-1} dt}{logt} = \int_x^\infty \frac{(\sum_{n=1}^\infty t^{-2n}) dt}{tlogt} = \int_x^\infty \frac{dt}{t(t^2-1)logt}.$$

The Riemann $\xi(s)$ function and the Gamma function $\Gamma(s)$

Lemma (EdH): The entire Zeta function $\xi(s) := (s-1)\frac{s}{2}\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}$ is of order 1 and it holds

i) $\xi(s) = \xi(1-s)$

ii) $\xi(s) = 4\int_1^\infty \frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx} x^{\frac{3}{4}} cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)logx\right] \frac{dx}{x}$

iii) $\xi\left(\frac{1}{2}+it\right) = 4\int_1^\infty \frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx} x^{\frac{3}{4}} \cos\left(\frac{t}{2}logx\right)\frac{dx}{x}$

iv) $\xi(s) = \sum_{n=0}^{\infty} a_{2n}(s-\frac{1}{2})^{2n}$, $\xi(\frac{1}{2}+it) = \sum_0^\infty (-1)^n a_{2n}\, t^{2n}$

where $a_{2n} := 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx} x^{3/4} \frac{(\frac{1}{2}logx)^{2n}}{(2n)!}\frac{dx}{x}$.

v) $\xi(s) = \xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right)$ where $\xi(0) = -\xi(0) = \frac{1}{2}$

vi) $\xi\left(\frac{1}{2}+it\right) = \xi\left(\frac{1}{2}\right)\prod_{Re(\alpha_n)>0}\left(1-\frac{t^2}{\alpha_n^2}\right)$ where $\rho_n = \frac{1}{2}+i\alpha_n$ (EdH) 1.16.

Lemma:

i) $\lim_{s\to 1}(s-1)\zeta(s) = \gamma$, (TiE) 2.1.16

ii) $\zeta(s) - \frac{1}{s-1} = 1 - \frac{1}{2}s\{\zeta(s+1)-1\} - \frac{s(s+1)}{2\cdot 3}\{\zeta(s+2)-1\} - \cdots$, (TiE) 2.14

iii) $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$, $\lim_{n\to\infty}\sum_{m=1}^{n}(\frac{1}{m+1} + 1 - logn) = \gamma$, (TiE) 2.1.16

iv) $\zeta(s) = \frac{s}{s-1} + s\int_1^\infty([t]-t)t^{-s-1}dt$ and therefore $log|(1-s)\zeta(s)| \ll log|s| + 1$.

Lemma:

i) $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, $-\frac{\zeta'(0)}{\zeta(0)} = -\log(2\pi)$, (TiE) 2.4.5

ii) $(s-1)\zeta(s) = e^{bs}\frac{1}{\Gamma\left(1+\frac{s}{2}\right)}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$, where $b := \log(2\pi) - 1 - \frac{\gamma}{2}$ (TiE) 2.12.6.

The trivial zeros of the Zeta function are derived from the functional equation, (ApT) p. 266,

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma\left(\frac{S}{2}\right)cos\,(\frac{\pi}{2}s)\zeta(s)$$

where the zeros of $\zeta(-2n) = 0$ are the zeros of the term $\cos(\frac{\pi}{2}s)$ with $s = 1+2n$.

Lemma (Backlund's estimate of $N(T)$, EdH) 6.7: let $N(T)$ denotes the number of roots between $0 < Im(s) < T$, then the relative error in the approximation

$$N(T) \sim \frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) - \frac{T}{2\pi}$$

is less than a constant times T^{-1} as $T\to\infty$.

Lemma:

i) $\frac{1}{\Gamma\left(1+\frac{z}{2}\right)} = e^{\frac{\gamma z}{2}} \prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}} = \prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)\left(1+\frac{1}{n}\right)^{-\frac{z}{2}}$, (GrI) 8.322

ii) $e^{\Psi(z)} = x\prod_{n=1}^{\infty}\left(1+\frac{1}{x+k}\right)e^{-\frac{1}{x+k}}$, (GrI) 8.364.

Lemma (PaR) p. 41: $\Gamma(\frac{3}{2}+iv)\zeta\frac{3}{2}+iv) \in L_2(-\infty,\infty)$.

A useful formula is, (InA) p. 57, $log\Gamma(z+\alpha) = \left(z+\alpha-\frac{1}{2}\right)logz + \frac{1}{2}log2\pi + O(\frac{1}{|z|})$ uniformly in any fixed angle $|\arg(z)| \leq \pi - \delta < \pi$ and any bounded range of α, as $|z| \to \infty$; and $log\Gamma'(z) = logz - \frac{1}{2z} + O(\frac{1}{|z|})$.

Lemma (PaR) pp. 31, 39:

$$\left|\Gamma\left(\frac{1}{2}+iy\right)\right| = \sqrt{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|\left|\Gamma\left(\frac{1}{2}-iy\right)\right|} = \sqrt{\frac{\pi}{\cosh(\pi y)}} \sim \sqrt{2\pi}e^{-\frac{\pi}{2}|y|} \sim \sqrt{\frac{\pi}{2}}\frac{1}{cosh\frac{\pi}{2}y}$$

i.e.

$$\Gamma\left(\frac{1}{2}+iy\right) = \int_{-\infty}^{\infty} e^{-e^u}e^{u(\frac{1}{2}+iy)}du \sim O(e^{-\frac{\pi}{2}|y|}).$$

A more general approximation formula is given

Lemma (GrI) 8.328,

$$\lim_{|y|\to\infty}|\Gamma(x+iy)|e^{\frac{\pi}{2}|y|}|y|^{\frac{1}{2}-x} = \sqrt{2\pi},\ x, y \text{ are real.}$$

Lemma (Theorem 20, (PaR) p. 128): Let $f(z)$ be a measureable function for which

$$\frac{1}{2T}\int_{-T}^{T}|f(x)|^2 dx$$

is bounded in T. Then

$$\int_{-\infty}^{\infty}\frac{|f(x)|^2}{1+x^2}dx < \infty.$$

As Hardy-Littlewood proved, (EdH) p. 201,

$$\frac{1}{2T}\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 dx \sim logT$$

it follows the

Corollary:

$$\int_{-\infty}^{\infty}\frac{\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2}{1+x^2}dx < \infty.$$

The Digamma function $\Psi(x) = log'\Gamma(x)$

For the Digamma function

$$\Psi(x) = log'\Gamma(x)$$

where $\frac{1}{\Gamma(s)} = \Psi(s)\frac{1}{\Gamma'(s)}$ resp. $-log\Gamma(s) = log\Psi(s) - log\Gamma'(s)$ it holds

Lemma (NiN) p. 99:

$$\Psi(x) = -\gamma - \sum_{k=0}^{\infty}(\frac{1}{x+k} - \frac{1}{1+k}).$$

Then for every $\varepsilon > 0$ there is a $R > 0$ that $|\Psi(x) - logx| < \varepsilon$ for $|x| \geq R > 0.$; it holds

$$\Psi(z) = logz + O\left(\frac{1}{|z|}\right) = log\Gamma(1+z) - log\Gamma(z) + O\left(\frac{1}{|z|}\right).$$

Lemma (NiN) p. 99, (SeP): Let $\Psi(x)$ denote the Digamma function and let $n \in N$;

i) all zeros of $\Psi(x)$ are real; there is only one positive zero $w_0 \sim 1{,}461$

ii) all negative zeros w_n of $\Psi(x)$ lie in the intervals $w_n \in (-n, 1/2 - n)$.

Lemma (NiN) p. 99: let $n \in N$ and zeros w_n the negative zeros of $\Psi(x)$ lying in the intervals $w_n \in (-n, 1/2 - n)$.

i) the sequence $y_n := n + w_n$ is characterized by the relations

$\Psi(1 - w_n) = \Psi(n + 1 - y_n) = \pi\cot(\pi w_n) = \pi\cot(\pi(-n + y_n)) = \pi\cot(\pi y_n)$, i.e.

$$y_n = \frac{1}{\pi} arctan\left(\frac{\pi}{\Psi(1-w_n)}\right)$$

ii) it holds $y_n, \frac{1}{2} - y_n \in (0, \frac{1}{2})$ and for large n

$$\pi\cot(\pi y_n) = logn + \delta_n \text{ with} \lim_{n\to\infty}\delta_n = 0,$$

resp.

$$y_n = \frac{1}{\pi} arctan\left(\frac{\pi}{logn + \delta_n}\right) = \frac{1}{logn} + \frac{\delta_n'}{logn}, \quad \lim_{n\to\infty}\delta_n' = 0$$

iii) $\frac{1}{2}\Psi(1 - w_n) = \frac{\pi}{2}\cot(\pi w_n) = \frac{\pi}{2}\cot(\pi y_n) = -\frac{1}{2w_n} - \frac{1}{1-w_n{}^2} + \sum_{k=0}^{\infty}(1 - s_{2k+1})w_n{}^{2k}.$

Proof: i), ii): with $\Psi(1 - x) - \Psi(\mathrm{x}) = \pi\cot(\pi\mathrm{x})$ it follows

$$\Psi(1 - w_n) = \Psi(n + 1 - y_n) = \pi\cot(\pi x_n).$$

On the other hand it holds, $(0 < y_n, 1 - y_n < 1)$, $\pi\cot(\pi w_n) = \pi\cot(\pi(-n + y_n)) = \pi\cot(\pi y_n)$ and, therefore, because of $\Psi(n) \sim logn$; with the lemma above it follows

$$\Psi(1 - w_n) = \pi\cot(\pi w_n) = \pi\cot(\pi y_n) = \frac{\pi}{2}\cot(\pi w_n) - \frac{1}{2w_n} - \frac{1}{1-w_n{}^2} + \sum_{k=0}^{\infty}(1 - s_{2k+1})w_n{}^{2k}$$

and therefore

$$\frac{\pi}{2}\cot(\pi w_n) = \frac{\pi}{2}\cot(\pi y_n) = -\frac{1}{2w_n} - \frac{1}{1-w_n{}^2} + \sum_{k=0}^{\infty}(1 - s_{2k+1})w_n{}^{2k}.$$

Corollary: $\frac{1}{2}\Psi(1+|w_n|)\sim\frac{1}{2|w_n|}$.

Lemma (Grl) 8.364:

$$log\Psi(z) = logx + \sum_{k=1}^{\infty} log\left(1+\frac{1}{x+k}\right)e^{-\frac{1}{x+k}}.$$

Lemma (NiN) p. 38: let $s_1 \coloneqq \gamma$ and $s_n \coloneqq \frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\cdots$ for $n \geq 2$, then

$$\Psi(1+z) = \frac{1}{2x} - \frac{\pi}{2}cot(\pi x) - \frac{1}{1-x^2} + \sum_{k=0}^{\infty}(1-s_{2k+1})x^{2k}.$$

Lemma (TiE) 2.15: for $0 < \sigma < 1$

$$\Psi(1+x) - logx = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\zeta(1-s)\frac{\pi}{\sin(\pi s)}x^{-s}ds.$$

Lemma (Grl) 6.469:

$$\int_0^1 \Psi(x)\sin(\pi x)\sin(n\pi x)\,dx = \begin{cases} 0 & n \; even \\ \frac{1}{2}log\frac{n-1}{n+1} & n \;\; odd \end{cases}.$$

The Kummer function $_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$

For the considered Kummer function $_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ it holds

Lemma:

i) $_1F_1\left(\frac{1}{2},\frac{3}{2};z\right) = e^{\frac{1}{3}z}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}}$, (BuH) p.184

ii) $_1F_1(\frac{1}{2};\frac{3}{2},x) \approx \frac{1}{\sqrt{\pi}}\frac{e^x}{x}$ for $x \to \infty$ (OlF) p. 257.

Let $M[h](s)$ denote the Mellin transform operator $M[h](s) = \int_0^\infty x^s h(x)\frac{dx}{x}$, where $M[-xh'](s) = sM[h](s)$ and $M[(xh)'](s) = (1-s)M[h](s)$, then it holds, (Grl) 7.612,

Lemma:

i) $M\left[\ _1F_1\left(\frac{1}{2},\frac{3}{2},-x\right)\right](\frac{s}{2}) = \int_0^\infty x^{s/2}\ _1F_1\left(\frac{1}{2},\frac{3}{2},-x\right)\frac{dx}{x} = \frac{\Gamma(s/2)}{1-s}$, $0 < Re(s) < 1$

ii) $M\left[x\ _1F_1'\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right) = -\frac{s}{2}\frac{\Gamma(s/2)}{1-s}$, $0 < Re(s) < 1$

iii) $M\left[\ _1F_1\left(\frac{1}{2},\frac{3}{2},-x\right) + x\ _1F_1'\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right) = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s}(1-\frac{s}{2})$, $0 < Re(s) < 1$

iv) $\Gamma\left(\frac{s}{2}\right) = \frac{1-s}{1-\frac{s}{2}}M\left[\ _1F_1\left(\frac{1}{2},\frac{3}{2},-x\right) + x\ _1F_1'\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right)$, $0 < Re(s) < 1$.

Lemma (SeA): all zeros z_n of $_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ are complex, and it holds

$$Re(z_n) > 1/2 \text{ and } (2n-1)\pi < |Im(z_n)| < 2n\pi\ ,\ n \in N.$$

The lemma about the zeros of the considered Kummer function is contained in the

Lemma (SeA): For the zeros of degenerate hypergeometric functions $_1F_1(a;c,z)$ it holds

1. Suppose that $1 \le a < c \le a+1$ and $c \neq 2$ if $a = 1$. Then all zeros of $_1F_1(a;c,z)$ lie in the half-plane
$$Re(z) < -\left[\sqrt{a-1} + \sqrt{1-(c-a)}\right]^2$$
2. Suppose that $0 < a \le 1$, $c \ge 1+a$, moreover $c \neq 2$ if $a = 1$. Then all zeros of $_1F_1(a;c,z)$ lie in the half-plane
$$Re(z) > \left[\sqrt{c-a-1} + \sqrt{1-a}\right]^2$$
3. Suppose that $0 < a \le 1$, $a < c \le 1+a$, moreover $c \neq 2$ if $a = 1$. Then all zeros of $_1F_1(a;c,z)$ lie in the horizontal strips $(2n-1)\pi < |Im(z)| < 2\pi n$.

Lemma (SeA): ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ is not of exponential type π and cannot expressed in the form

$$\int_{-\pi}^{\pi} e^{izt}\frac{k(t)dt}{(\pi^2-t^2)^{1/2}}\ ,\ vark(t)<\infty\ ,\ k(\pm\pi\mp 0\neq 0).$$

Lemma (SeA): for $-\pi<\arg(z)\,,\arg\,(-z)<\pi$ and real r it holds

i) $\quad {}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right)=\frac{\sqrt{\pi}}{2}(-z)^{-\frac{1}{2}}(1+O\left(\frac{1}{z^2}\right))+\frac{1}{2}z^{-1}e^z(1+\frac{1}{2z}+O\left(\frac{1}{z^2}\right))$

ii) $\quad z\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2},z\right)=\frac{\sqrt{\pi}}{4}(-z)^{-\frac{1}{2}}\left(1+O\left(\frac{1}{z^2}\right)\right)+\frac{1}{2}e^z\left(1-\frac{1}{2z}+O\left(\frac{1}{z^2}\right)\right).$

iii) $\quad z\ {}_1F_1''\left(\frac{1}{2},\frac{3}{2},z\right)=\frac{15\sqrt{\pi}}{8}(-z)^{-\frac{3}{2}}\left(1+O\left(\frac{1}{z^2}\right)\right)+\frac{5}{2}e^z\left(1-\frac{3}{2z}+O\left(\frac{1}{z^2}\right)\right).$

Lemma (OlF) p. 257: for real x it holds

i) $\quad {}_1F_1\left(\frac{1}{2},\frac{3}{2},x\right)\sim\frac{e^x}{x}$ as $x\to\infty$

ii) $\quad {}_1F_1\left(\frac{1}{2},\frac{3}{2},-x\right)\sim\frac{1}{\sqrt{x}}$ as $x\to\infty$.

Lemma (SeA): All zeros z_n^* of ${}_1F_1(\frac{1}{2},\frac{3}{2},-2\pi z)$ lie in the horizontal strips fulfilling the condition

$$n-1/2<y_n:=|Im(z_n^*)|<n \text{ and } |Re(z_n^*)|<\frac{1}{4\pi}$$

from which it follows for $k\geq 1$

$$\frac{1}{2}k\leq k-\frac{1}{2}<y_{n+k}-y_n<k+\frac{1}{2}.$$

The Hardy space $H^2(D)$ and the Besicovitch space

Lemma: Let (YoR) p. 12: Let X denote the vector space of all finite linear combinations of functions of the form $e^{i\lambda x}$, $(-\infty < t < \infty)$, where the parameter λ is real. An inner product in X is defined by

$$((f,g)) := \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(t)\overline{g(t)}dt.$$

When X is closed by means of the metric generated by this inner product, we obtain a certain Hilbert space B^2 (B is for Besicovitch). The continuum of elements $e^{i\lambda x}$ forms a complete orthogonal subset of B^2. The Hilbert space B^2 contains the important class of Bohr almost periodic functions. Those functions are obtained by adding to X the limits of sequences of function in X that are uniformly convergent on the entire real line.

Remark (PaJ) p. 4: The Hardy space $H^2(D)$ is defined as the space of all analytic functions on the disk $D = \{z \in C \big| |z| < 1\}$ for which the norm

$$|||f|||_2^2 := sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\varphi}\right|^2 d\varphi\right\}$$

is finite. The radial limit (function) $\tilde{f}(e^{i\varphi}) = \lim_{r\to 1} f(e^{i\varphi})$ on $\Gamma = S^1(R^2)$ exist almost everywhere with $|||f|||_2^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|f(e^{i\varphi})\right|^2 d\varphi$.

The Hardy space H^2 can also defined as the subspace of those $L_2(\Gamma)$ functions for which the negative Fourier coefficients vanish, that is

$$\hat{f}(n) = f_n = \frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(e^{i\varphi})e^{-in\varphi}d\varphi < 0,\ \nu < 0.$$

Then a function $\tilde{f}$ with $\tilde{f}(e^{i\varphi}) \sim \sum_{n=0}^{\infty} f_n e^{i\varphi}$ can be naturally identified with the power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$, defining an analytical function f in D.

In other words, the Hardy space H^2 is a Hilbert space, being a closed subspace of the Hilbert space $H := L_2^*(\Gamma)$ and the orthogonal projection $P_{H^2}: L_2^*(\Gamma) \to H^2$ is defined by

$$P_{H^2}:\ \sum_{n=-\infty}^{\infty} a_n e^{i\varphi} \ \to\ \sum_{n=0}^{\infty} a_n e^{i\varphi}.$$

As there is an isometric isomorphism between $L_2^*(\Gamma)$ and $l^2\,(Z)$, and as the sequence space $l^2\,(Z_+)$ maps to the Hardy space H^2, one may regard $l^2\,(Z_+)$ as embedding into $l^2\,(Z)$ as the subspace of all $(a_n)_{n=-\infty}^{\infty}$ with $a_n = 0$ for $n < 0$.

Remark: For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(D)$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in H^2(D)$ it holds $\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\varphi}\right|^2 d\varphi = \sum_{n=0}^{\infty}|a_n|^2 r^{2n}$ for $0 \le r < 1$ and

$$((f,g)) := \sum_{n=0}^{\infty} a_n \bar{b}_n$$

defines an inner product of $H^2(D)$ with $|||f|||_2^2 = (((f,f))$.

Remark: The mapping $z \to 1 - \frac{1}{z}$ takes the right half plane $Re(z) > 1/2$ to the interior of the unit circle $D := \{z||z| < 1\}$ in the complex z-plane and maps the critical line $Re(z) = 1/2$ onto the unit circle

Proof:

$$\left|1 - \frac{1}{z}\right|^2 = \left|\frac{z-1}{z}\right|^2 = \frac{z-1}{z}\cdot\frac{\bar{z}-1}{\bar{z}} = \frac{|z|^2 - 2Re(z) + 1}{|z|^2} < \frac{|z|^2 - 1 + 1}{|z|^2} < 1.$$

Laguerre-Polya class LP of functions

Laguerre-Polya class LP of functions consists of the entire functions having only real zeros with a Weierstrass factorization of the form

$$az^q e^{\alpha z-\beta z^2}\prod_k(1-z/\alpha_k)e^{z/\alpha_k},$$

where a,α,α_k are real, $\beta\geq 0$ and q is a nonnegative integer, and α_k are the nonzero real numbers such that $\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}<\infty$. The subset LP^* consists of all elements of order <2. In this case β is neccessarily zero.

RH criterion (CaD): If the function

$$\Xi(t):=\xi(1/2+it)$$

can be realized as a convolution $\Xi(t)=(K*dF)(t)$ where $K(t)\in LP^*$, i.e., is an entire function from the Laguerre-Polya class of order <2, i.e.

$$cz^q e^{\alpha z}\prod_k(1-z/\alpha_k)e^{z/\alpha_k},$$

where c,α,α_k are real, and q is a nonnegative integer, this would prove the RH.

Some technical lemmata

In the neighborhood of $x\approx 1$ it holds

$$\frac{\pi}{2}tan\frac{\pi}{2}x=\frac{1}{1-x}+O(|1-x|)\,.$$

This indicates a circular density function on the half-circle in the form

$$log\frac{1}{x}\quad\rightarrow\quad log\frac{\pi}{2}(tan\frac{\pi}{2}(1-x))=log\frac{\pi}{2}cot(\frac{\pi}{2}x)$$

$$\frac{\pi}{2}log'\,(tan(\frac{\pi}{2}s))=\frac{\pi}{sin(\pi s)}=\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it).$$

Regarding the density $\frac{dx}{x}$ we also mention a complex relation to the Bessel functions, (WaG) p. 514,

$$dlogx=\frac{dx}{x}=\frac{\pi}{2}[J_0^2(x)+Y_0^2(x)]d\theta$$

with $\theta(x):=\arctan(\frac{Y_0(x)}{J_0(x)})$.

Entire functions and different types of same finite order

Extracts from (LeN) pp. 4, 32, (InA), (PaR)

An entire function is of finite order, if $M_f(r) \coloneqq \max_{|z|=r}|f(z)| \ ^{as}< e^{r^k}$ for some constant $k > 0$. The order of growth of an entire function is the greatest lower bound of these values of k. As entire functions with same order can grow differently, there are different types of orders, grouped into minimal, normal (mean), and infinite types. The entire function $f(z)$ is said to have a finite type, if for some $A > 0$ the inequality $M_f(r) \coloneqq \max_{|z|=r}|f(z)| \ ^{as}< e^{Ar^k}$ is fulfilled. The greatest lower bound for those values of is called the type $\sigma = \sigma_f$ with respect to the order k. If $\sigma_f = 0$ the type is called minimal, if $0 < \sigma_f < \infty$ the type is called normal (or mean), if σ_f is infinite, the type is called maximal. Entire functions of order $k = 1$ and normal type $\sigma = \sigma_f$ are called entire functions of exponential type $\sigma = \sigma_f$. For an entire function $f(z)$ of minimal type with respect to an order k it holds $log|f(z)| = o(|z|^k)$, $|z| \to \infty$, (LeB) p 90.

Examples:

i) $\sin(Az)$ is of order $k = 1$ and type $\sigma = |A|$, which mean that it is an entire function of exponential type $|A|$

ii) $\frac{\mathrm{Sin}(\sqrt{z})}{\sqrt{z}}$ is of order $k = \frac{1}{2}$ and type 1

iii) $\frac{1}{\Gamma(z)}$ is of maximal type, as $logM(r) \geq Crlogr$.

The function $\frac{1}{\Gamma(z)}$ is most prominent example where the density of the zero set is finite, while the canonical product is of maximal type (as $logM(r) \geq Crlogr$.

Remark: We note that the canonical product $\prod_{n=1}^{\infty}(1 - \frac{z}{a_n})$ is of exponential type zero, and so cannot be bounded along the real axis (applying Bernstein‘ inequality), (YoR) p. 118.

Remark: Sequences y_n fulfilling the Kadec condition $|n - y_n| \leq L < \frac{1}{4}$ play a central role in the theory of non-harmonic Fourier series and its isometric connection with the Paley-Wiener space. The corresponding Paley-Wiener function is given by

$$g(z) := \prod_{n=1}^{\infty}(1 - \frac{z^2}{y_n^2}).$$

By making a change of variables $z^2 = \omega$, one obtains an entire function of non-integer order $\rho = 1/2$, LeB) p. 89.

Theorem 17 (InA) p. 56: If $M(r)$ is the maximum of $|\xi(s)|$ on the circle $|s| = r$, then

$$M(r) \sim \frac{1}{2}rlogr.$$

Theorem XXIV (PaR) p. 77: If the y_n are real and positive, if the series $\sum \frac{1}{y_n^2}$ converges, and if $\varphi(z) := \prod_{n=1}^{\infty}(1 - \frac{z^2}{y_n^2})$, then the statements $log\varphi(iy) \sim \pi A|y|log|y|$, as $y \to \infty$ and $\int_{-\pi}^{\pi} \frac{log|\varphi(x)|}{x^2} dx \sim \pi^2 A \cdot log|y|$ are completely equivalent.

Remark: (PaR) p. 77: The even Zeta function $\Xi(z) := \xi(1/2+iz)$ has all its zeros in the strip $|Im(z)| < 1/2$. Let z_n denote its zeros, then

$$\Xi(z) = \Xi(0)\prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2}) \text{ with } \sum_1^{\infty}\lambda_n^{-2} < \infty \text{ where } \lambda_n := |z_n|.$$

Putting

$$H(z) := \Xi(0)\prod_{n=1}^{\infty}(1-\frac{z^2}{\lambda_n^2})$$

it holds $log\left|\frac{H(iy)}{\Xi(iy)}\right| = O(1)$ along the imaginary axis. Thus, because of $log\Xi(iy) = O(y) + log\Gamma\left(\frac{y}{2}\right) \sim \frac{1}{2}ylogy$, $(y>0)$, it follows

$$logH(iy) \sim \frac{1}{2}ylogy$$

and therefore

$$\frac{1}{logy}\int_{-y}^{y} log\left|\frac{\Xi(x)}{\Xi(0)}\right|\frac{dx}{x^2} \rightarrow -\frac{\pi}{2} \text{ as } y \rightarrow \infty.$$

Lemma ((PaR), p. 86: Let $\Phi(z) := \prod_{n=1}^{\infty}(1-\frac{z^2}{\pi^2\lambda_n^2})$ with $\lim_{n\to\infty}\frac{\lambda_n}{n} = 1$, then $\lim_{y\to\infty}\frac{log\Phi(iy)}{y} = 1$.

Lemma ((LeN), p. 92: Let $\Phi(z) := \prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2})$ with $\lim_{n\to\infty}\frac{n}{z_n} = D$, where D is real, and such that for some $d>0$, $|z_n - z_m| \geq d|n-m|$. Then for $\varepsilon > 0$, as $r \rightarrow \infty$

i) $\Phi(re^{i\theta}) = O(e^{\pi D|sin\theta|r+\varepsilon r})$

ii) $\frac{1}{\Phi(re^{i\theta})} = O(e^{-\pi D|sin\theta|r+\varepsilon r})$

iii) $\frac{1}{|\Phi'(z_n)|} = O(e^{\varepsilon|z_n|})$, $n \rightarrow \infty$.

Lemma: let a_n be a sequence fulfilling $n-\frac{1}{2} < a_n < n$. Then, because of $4n-1 < 3a_n + a_{n+1} < 4n+1$, $n \in N$ resp. $n-\frac{1}{4} < \frac{3a_n+a_{n+1}}{4} < n+\frac{1}{4}$, $n \in N$ it follows

$$\left|n-\frac{3a_n+a_{n+1}}{4}\right| \leq L < \frac{1}{4},\ n \in Z \text{ where } a_{-n} := -a_n \text{ and } a_0 \in \left(0,\frac{1}{2}\right).$$

Putting

$$\lambda_n := \frac{3|Im(z_n^*)|+|Im(z_{n+1}^*)|}{4},\ \lambda_0 \in \left(0,\frac{1}{2}\right)$$

one gets $|\lambda_n - n| \leq L < 1/4$ supporting the

Lemma (LeN) p. 48, (PaR) p. 113, (YoR) p. 100: If $\{\lambda_n\}$ is a sequence and L a constant such that

$$(*) \quad |\lambda_n - n| \leq L < 1/4,$$

then the system $\{e^{i\lambda_n x}\}_{n\in Z}$ is close in $L_2^{\#}(-\pi,\pi)$ (i.e. from $\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0$ it follows that $f(x)$ is identically zero) and possesses a unique biorthogonal set $\{h_n(x)\}$ such that for any $f \in L_2^{\#}(-\pi,\pi)$ the series

$$\sum_{-\infty}^{\infty}\frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi - e^{i\lambda_n x}\int_{-\pi}^{\pi} f(\xi)h_n(\xi)d\xi$$

converges uniformly to zero over the interval $[-\pi+\delta,\pi-\delta]$ for any $\delta > 0$. Moreover the difference of weighted sums (Riesz, Abel, and so on) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over $[-\pi+\delta,\pi-\delta]$.

Gap and Density Theorems

Lemma ((LeN), p. 89: For $\Phi(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2})$ with $\lim_{n\to\infty}\frac{n}{z_n} = D$, where D is real, and such that for some $d > 0$, $|re^{i\theta} - z_n| \geq \frac{1}{8}d$. Then for $\varepsilon > 0$, as $r \to \infty$

iv) $\Phi(re^{i\theta}) = O(e^{\pi D|sin\theta|r+\varepsilon r})$

v) $\frac{1}{\Phi(re^{i\theta})} = O(e^{-\pi D|sin\theta|r+\varepsilon r})$

vi) $\frac{1}{\Phi'(z_n)} = O(e^{\varepsilon|z_n|})$, $n \to \infty$.

This lemma is contained in

Lemma ((LeN), p. 92: Let $\Phi(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2})$ with $\lim_{n\to\infty}\frac{n}{z_n} = D$, where D is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n-m|$. Then for $\varepsilon > 0$, as $r \to \infty$

vii) $\Phi(re^{i\theta}) = O(e^{\pi D|sin\theta|r+\varepsilon r})$

viii) $\frac{1}{\Phi(re^{i\theta})} = O(e^{-\pi D|sin\theta|r+\varepsilon r})$

ix) $\frac{1}{|\Phi'(z_n)|} = O(e^{\varepsilon|z_n|})$, $n \to \infty$.

Lemma ((PaR), p. 86: Let $\Phi(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{\pi^2\lambda_n^2})$ with $\lim_{n\to\infty}\frac{\lambda_n}{n} = 1$, then $\lim_{y\to\infty}\frac{log\Phi(iy)}{y} = 1$.

Theorem XXIX ((PaR), p. 86: Let $\Phi(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{\pi^2\lambda_n^2})$ with $\lim_{n\to\infty}\frac{\lambda_n}{n} = 1$ belong to $L_2(-\infty,\infty)$. Then the set of functions $\{e^{\pm i\lambda_n x}\}$ cannot close $L_2(-\pi,\pi)$. Again, let $z\Phi(z)$ belong to $L_2(-\infty,\infty)$. Then the set $\{1, e^{\pm i\lambda_n x}\}$ cannot close $L_2(-\pi,\pi)$. In either case, a finite number of functions of the set may be replaced by other functions of the form $e^{i\lambda x}$ to the same number.

Lemma (LeN) p. 48, (YoR) p. 100: If $\{\lambda_n\}$ is a sequence and L a constant such that

$$(*) \quad |\lambda_n - n| \leq L < 1/4,$$

then the system $\{e^{i\lambda_n x}\}_{n\in Z}$ is close in $L_2^{\#}(-\pi,\pi)$ (i.e. from $\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0$ it follows that $f(x)$ is identically zero) and possesses a unique biorthogonal set $\{h_n(x)\}$ such that for any $f \in L_2^{\#}(-\pi,\pi)$ the series

$$\sum_{-\infty}^{\infty}\frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi - e^{i\lambda_n x}\int_{-\pi}^{\pi} f(\xi)h_n(\xi)d\xi$$

converges uniformly to zero over the interval $[-\pi+\delta, \pi-\delta]$ for any $\delta > 0$. Moreover the difference of weighted sums (Riesz, Abel, and so on) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over $[-\pi+\delta, \pi-\delta]$.

The following lemma show, roughly stated, that the rate of growth of an analytic function (in this case of order one) along a line can be determined by a sufficiently dense sequence of points on the line.

Lemma ((LeN), p. 100: Let $\Phi(z)$ be analytic in some sector $|\arg(z)| \leq \alpha$. Suppose

$$\lim_{n\to\infty} \sup \left(\frac{log|\Phi(re^{i\theta})|}{r}\right) \leq acos\theta + b|sin\theta|, \theta \leq \alpha < \frac{\pi}{2}.$$

Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} \frac{n}{z_n} = D$, where D is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n - m|$. If $b < \pi D$, then

$$\lim_{n\to\infty} \sup \left(\frac{log|\Phi(z_n)|}{|z_n|}\right) = \lim_{n\to\infty} \sup \left(\frac{log|\Phi(r)|}{r}\right).$$

A special case of this lemma is the

Lemma ((LeN), p. 101: Let $\Phi(z)$ be analytic and of finite exponential type in the half-plane $|\arg(z)| \leq \frac{\pi}{2}$. Let

$$\lim_{n\to\infty} \sup \left(\frac{log|\Phi(iy)|}{y}\right) = \pi L.$$

Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty} \frac{n}{z_n} = D$, where D is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n - m|$, then $L < D$ implies

$$(*) \quad \lim_{n\to\infty} \sup \left(\frac{log|\Phi(z_n)|}{|z_n|}\right) = \lim_{n\to\infty} \sup \left(\frac{log|\Phi(r)|}{r}\right).$$

Remark: If (*) does not hold there exists a c such that

$$\lim_{n\to\infty} \sup \left(\frac{log|\Phi(z_n)|}{|z_n|}\right) < c < a := \lim_{n\to\infty} \sup \left(\frac{log|\Phi(r)|}{r}\right).$$

Further extracts from the theory of nonharmonic Fourier series

A sequence of vectors in a separable Hilbert space is called complete, if its linear span is dense in the Hilbert space, resp. if the zero vector alone is perpendicular to every basis vector. A characterization of an orthogonal Schauder bases of a separable Hilbert space is that they are complete orthogonal sequences.

A complete sequence of vectors in a separable Hilbert space is a Riesz basis if and only if its moment space is equal to l^2 , (YoR) p. 142.

A sequence of real or complex numbers $\{\lambda_n\}_{n\in N}$ is said to be an interpolating sequence for PW if the set of all sequences $\{f(\lambda_n)\}_{n\in N}$ where f ranges over PW, coincides with l^2 . If, in addition, the system $\{e^{i\lambda_n t}\}_{n\in N}$ is complete in $L_2^{\#}(\Gamma)$, then $f(\lambda_n) = c_n$ has exactly one solution, provided $c_n \in l^2$, and in this case we shall call $\{\lambda_n\}_{n\in N}$ a complete interpolating sequence. A complete interpolation sequence is „maximal“ in the sense that it is not contained in any larger interpolating sequence, and the converse is also true, (YoR) p. 142.

… in more detail:

> **Lemma** (LeN) p. 48, (YoR) p. 100: If $\{\lambda_n\}$ is a sequence and L a constant such that
>
> $$(*) \quad |\lambda_n - n| \le L < 1/4,$$
>
> then the system $\{e^{i\lambda_n x}\}_{n\in Z}$ is close in $L_2^{\#}(-\pi,\pi)$ (i.e. from $\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0$ it follows that $f(x)$ is identically zero) and possesses a unique biorthogonal set $\{h_n(x)\}$ such that for any $f \in L_2^{\#}(-\pi,\pi)$ the series
>
> $$\sum_{-\infty}^{\infty} \frac{e^{inx}}{2\pi} \int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi - e^{i\lambda_n x} \int_{-\pi}^{\pi} f(\xi)h_n(\xi)d\xi$$
>
> converges uniformly to zero over the interval $[-\pi+\delta, \pi-\delta]$ for any $\delta > 0$. Moreover the difference of weighted sums (Riesz, Abel, and so on) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over $[-\pi+\delta, \pi-\delta]$.

Remark: (LeN) p. 48: if $|\lambda_n - n| \le L < 1/4$ is replace by $|\lambda_n - n| \le 1/4$, then the result of the lemma no longer holds.

Remark: Kadec's -theorem (YoR) p. 36, shows that the system $\{e^{i\lambda_n x}\}_{n\in Z}$ constitutes a Riesz basis for $L_2^{\#}(-\pi,\pi)$ whenever every μ_n is real and

$$|\lambda_n - n| \le L < 1/4,$$

but need not constitute a basis when $L = 1/4$.

On addition of one exponential the Riesz basis $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis of fractional Sobolev spaces $H_\beta^{\#}(0,1)$ of order β with $0 \le \beta \le 1$ and $\beta \ne 1/2$, (IvS),

> **Lemma**: Let $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis for $L_2^{\#}(0,1)$. Then for each number μ, which do not belong to the spectrum $\{\lambda_n\}_{n\in Z}$, the exponential families
>
> $$E_\mu^{(\beta)} = \left\{\frac{1}{(1+|2\pi\lambda_n|^\beta)} e^{2\pi i\lambda_n x}\right\}_{n\in Z} \cup \{e^{2\pi i\mu x}\}$$
>
> form a Riesz basis for the Sobolev space $H_\beta\ (0,1)$.

Remark: We note that according to the Sobolev embedding theorem any $g \in H_{\beta}^{\#}(0,1)$ with $\beta < \frac{1}{2}$ is bounded, i.e. $|g(x)| \leq c$.

Lemma, (YoR) p. 109: The system $\left\{e^{i\lambda_n t}\right\}$ is minimal in $L_2\,(-\pi,\pi)$ if and only if there exists a nontrival function $f(z)$ of exponential type at most π , zeros at every λ_n , and such that

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} dx < \infty.$$

The main ingredients to prove the Levinson lemma are the following two lemmata concerning the function

$$G(w) := G(u+iv) = (w-\lambda_0) \prod_{n=1}^{\infty} \left(1-\frac{w}{\lambda_n}\right)\left(1-\frac{w}{\lambda_{-n}}\right).$$

Lemma A (Levinson) (LeN) p. 55: If $\{\lambda_n\}$ is a sequence according to (*), then for different constants c it holds

i) $|G(w)| < c(|w|+1)e^{\pi|v|}$

ii) $|G(w)| > c|v|(|w|+1)^{-2} e^{\pi|v|}$

iii) $|G1/2+iv)| > c.$

Lemma B (LeN) p. 57: The functions $h_n(\xi)$ defined by

$$h_n(\xi) := \int_{-\infty}^{\infty} \frac{G(u)}{(u-\lambda_n)G'(u)} e^{-iu\xi} du$$

form a sequence of biorthogonal to $\left\{e^{i\lambda_n x}\right\}$ over $(-\pi,\pi)$.

Lemma (LeN) p. 89: Let $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ where $\lim_{n\to\infty} \frac{n}{\lambda_n} = D$ and for some $c > 0$

$$\lambda_{n+1} - \lambda_n \geq c.$$

Then on the abscissa of convergence there is at least one singularity in every interval of length exceeding $\lambda_n 2\pi D$.

Lemma (LeN) p. 89: If λ_n satisfies $\lim_{n\to\infty} \frac{n}{\lambda_n} = D$ and for some $c > 0$ $\lambda_{n+1} - \lambda_n \geq c$, and if

$$F(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

then as $n \to \infty$: $$\frac{1}{F'(\lambda_n)} = O(e^{\varepsilon\lambda_n}).$$

Lemma (LeN) p. 108: Let $\Phi(z)$ be analytic and of exponential type in the half-plane $|ph(z)| \leq \frac{\pi}{2}$ and let $\Phi(iy) = O(e^{\pi L|y|})$. If $\{\lambda_n\}$ be an increasing positive sequence such that $\lim_{n\to\to\infty} \frac{n}{\lambda_n} = D$, $\lambda_{n+1} - \lambda_n \geq d > 0$. Let $\Lambda(u)$ be the number of $\lambda_n < u$. If $\int_1^{\infty} \frac{\Lambda(y)-Ly}{y^2} dy = \infty$ and if $\Lambda(y) + t(y) > Ly$ for some positive $t(y)$ satisfying $\int_1^{\infty} \frac{t(y)}{y^2} dy < \infty$, then

$$\limsup_{n\to\infty} \frac{\log|\Phi(\lambda_n)|}{\lambda_n} = \limsup_{x\to\infty} \frac{\log|\Phi(x)|}{x}.$$

Theorem 13 (YoR) p. 160: If the system $\{e^{i\lambda_n x}\}_{n\in N}$ is a frame in $L_2^{\#}(-\pi,\pi)$ then there is a positive constant L with the property that $\{e^{i\mu_n x}\}_{n\in N}$ is also a frame in $L_2^{\#}(-\pi,\pi)$ whenever

$$|\lambda_n - \mu_n| \leq L \text{ for every } n.$$

Corollary (YoR) p. 161: If the system $\{e^{i\lambda_n x}\}_{n\in N}$ is a Riesz basis for $L_2^{\#}(-\pi,\pi)$ then there is a positive constant L with the property that $\{e^{i\mu_n x}\}_{n\in N}$ is also a Riesz basis for $L_2^{\#}(-\pi,\pi)$ whenever

$$|\lambda_n - \mu_n| \leq L \text{ for every } n.$$

Theorem 14 (YoR) p. 161: let the sequence $\{\lambda_n\}_{n\in N}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\{e^{iRe(\lambda_n)x}\}_{n\in N}$ is a frame in $L_2^{\#}(-\pi,\pi)$, then so is $\{e^{i\lambda_n x}\}_{n\in N}$.

Corollary 1 (YoR) p. 164: let the sequence $\{\lambda_n\}_{n\in N}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\{e^{iRe(\lambda_n)x}\}_{n\in N}$ is a Riesz basis for $L_2^{\#}(-\pi,\pi)$, then so is $\{e^{i\lambda_n x}\}_{n\in N}$.

Corollary 2 (YoR) p. 164: if $\{\lambda_n\}_{n\in Z}$ be a sequence of scalars for which

$$sup_n|Re(\lambda_n) - n| < 1/4 \text{ and } sup_n|Im(\lambda_n)| < \infty,$$

then the system $\{e^{i\lambda_n x}\}_{n\in Z}$ is a Riesz basis for $L_2^{\#}(-\pi,\pi)$.

Remark (YoR) p. 57: Let $f(z)$ be an entire function; then $f(z)$ and $f'(z)$ are of the same order. If $f(z)$ has at least one zero, but is not identically zero, then

$$\lambda = \lim_{n\to\infty} sup \frac{\log(n(r))}{logr}$$

where $n(r)$ is the number of zeros in the closed disk $|z| \leq r$ and λ denotes the exponent of convergence of the sequence of zeros. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of order σ, then

$$\rho = \lim_{n\to\infty} sup \frac{nlogn}{log\left(\frac{1}{|a_n|}\right)}.$$

Non-harmonic Fourier analysis and Kadec's theorem

Extracts from

„An Introduction to Nonharmonic Fourier Series", (YoR)

Robert M. Young

A sequence of vectors in a separable Hilbert space is called complete, if its linear span is dense in the Hilbert space, resp. if the zero vector alone is perpendicular to every basis vector. A characterization of an orthogonal Schauder bases of a separable Hilbert space is that they are complete orthogonal sequences.

Theorem (of Levinson) (YoR) p. 100:

The system $\{e^{i\mu_n x}\}_{n\in Z}$ built by the sequences $\{\mu_n\}_{n\in Z}$ of real or complex numbers for which $|\mu_n| \le |n| + \frac{1}{4}$ is complete in $L_2^{\#}(-\pi,\pi)$ where the constant $\frac{1}{4}$ is the best possible.

Kadec's -theorem (YoR) p. 36, shows that the system $\{e^{i\mu_n x}\}_{n\in Z}$ constitutes a basis for $L_2^{\#}(-\pi,\pi)$ whenever every μ_n is real and

$$|\mu_n - n| \le L < 1/4,$$

but need not constitute a basis when $L = 1/4$.

A complete sequence of vectors in a separable Hilbert space is a Riesz basis if and only if its moment space is equal to l^2 , (YoR) p. 142.

A sequence of real or complex numbers $\{\lambda_n\}_{n\in N}$ is said to be an interpolating sequence for PW if the set of all sequences $\{f(\lambda_n)\}_{n\in N}$ where f ranges over PW, coincides with l^2 . If, in addition, the system $\{e^{i\lambda_n t}\}_{n\in N}$ is complete in $L_2^{\#}(\Gamma)$, then $f(\lambda_n) = c_n$ has exactly one solution, provided $c_n \in l^2$, and in this case we shall call $\{\lambda_n\}_{n\in N}$ a complete interpolating sequence. A complete interpolation sequence is „maximal" in the sense that it is not contained in any larger interpolating sequence, and the converse is also true, (YoR) p. 142.

A set of vectors in a normed vector space is said to be minimal if each vector in the set lies outside the closed subspace spanned by the others. A set of vectors in a normed vector space is said to be linked if each vector in the set lies within the closed subspace spanned by the others.

We note that a system $\{e^{i\mu_n x}\}_{n\in Z}$ is minimal in $L_2^{\#}(\Gamma)$ (i.e. each vector in the set lies outside the closed subspace spanned by the others) if and only if there exists a nontrivial entire function $f(z)$ of exponential type at most π, zeros at every $f(\mu_n) = 0$, and such that $\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} < \infty$, (YoR) p. 110.

A complete system $\left\{e^{i\lambda_n x}\right\}$ is unaffected when one of the terms $\left\{e^{i\lambda x}\right\}$ is replaced by a different term $\{e^{i\mu x}\}$:

For if $f(z)$ belongs to the Paley-Wiener space PW, if $f(\lambda_n) = 0$, whenever $\lambda_n \ne \mu$ and if $f(\mu) = 0$, then the function $g(z) = \frac{z-\lambda}{z-\mu} f(z)$ also belongs to PW and satisfies $g(\lambda_n) = 0$ for every λ_n .

If $\left\{e^{i\lambda_n x}\right\}$ is complete, then λ_n is a set of uniqueness for $g(z)$, and so $g(z)$ must vanish identically. It follows that $f(z)$ must also vanish identically, and hence the new system (with $e^{i\lambda x}$ replaced by $e^{i\mu x}$) is also complete.

The trigonometric moment problem in $L_2\,(-\pi,\pi)$ is given by, (YoR) p.124,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)e^{int}dt = c_n.$$

A basis for a Hilbert space is a Riesz basis, if it is equivalent to an orthogonal basis, that is, if it is obtained from an orthogonal basis by means of a bounded invertable operator.

The „generalized“ trigonometric moment problem in $L_2\,(-\pi,\pi)$ is given by, (YoR) p.124,

$$(*)\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)e^{i\lambda_n t}dt = c_n.$$

As $\left\{e^{i\lambda_n t}\right\}$ is a Riesz basis its moment space is l^2 and (*) admits a solution whenever c_n is square-summable, and for these sequences only. If $\{g_n\}$ is the unique sequence in $L_2\,(-\pi,\pi)$ biorthogonal to $\left\{e^{i\lambda_n t}\right\}$, then the unique solution to (*) is given by the norm-convergent series

$$\varphi(t) = \sum_{-\infty}^{\infty} c_n g_n(t).$$

Lemma (YoR) p. 119: Let $n(r)$ denote the counting function of a real increasing sequence λ_n. Then

$$\limsup_{r\to\infty}\frac{n(r)}{r} = \limsup_{r\to\infty}\frac{n}{\lambda_n}.$$

Lemma (YoR) p. 117: Let λ_n be a sequence of real numbers such that $C = \sum_{n=1}^{\infty}\frac{1}{|\lambda_n|} < \infty$. If $f(z) = \prod_{n=1}^{\infty}\frac{\sin\left(\frac{\pi z}{\lambda_n}\right)}{\frac{\pi z}{\lambda_n}}$ then $f(z)$ is a nontrivial entire function of exponential type at most πC, bounded by 1 on the real axis, and zero at every λ_n.

Lemma (YoR) p. 106: The completeness of a system $\{e^{i\lambda_n t}\}$ (accompanied by an entire function $f(z)$ with $f(\lambda_n) = 0$) is unaffected when one of the terms $e^{i\lambda t}$ is replaced by a different term $e^{i\mu t}$ (different from those already existing, i.e. if additionally it holds $f(\mu) = 0$, then $g(z) := \frac{z-\lambda}{z-\mu}f(z)$ must vanish identically).

The totality of all entire functions of exponential type at most π (i.e., $|f(z)| \le e^{\pi|z|}$) that are square integrable on the real axis is known as the Paley-Wiener (separable Hilbert) space $\boldsymbol{P}$, equipped with the inner product $(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$, which is isometrically isomorph to $L_2\,(-\pi,\pi)$.

Putting

$$G(z) := z\prod_{k=0\to\infty}\left(1-\frac{z^2}{\lambda_n^2}\right) \quad \text{and} \quad G_n(z) := \frac{G(z)}{G'(\lambda_{\boldsymbol{n}})(z-\lambda_{\boldsymbol{n}})}$$

then $G_n(z)$ belongs to the Paley-Wiener space $\boldsymbol{P}$ and $g_n(t)$ is the inverse Fourier transform of $G_n(z)$, i.e. for almost all $t \in [-\pi,\pi]$,

$$g_n(t) := \int_{-\infty}^{\infty} G_n(z)e^{ixt}dt.$$

The exponentials $e^{i\lambda_n t}$ are transformed into the reproducing functions $K_n(z) = \frac{\sin\,\pi(z-\lambda_n)}{\pi(z-\lambda_n)}$, $g_n(t)$ is transformed into $G_n(z)$, while the moment problem itself becomes

$$f(\lambda_n) = 0,\ n = 0, \pm 1, \pm 2, \pm 3, \dots..,$$

since $f(\lambda_n) = (f, K_n)$. Here $c_n \in l^2$ and $f \in \boldsymbol{P}$ is to be found.

The solution of the moment problem is given by

$$(**)\quad f(z) = \sum_{-\infty}^{\infty} c_n \frac{G_n(z)}{G_n'(\lambda_n)(z-\lambda_n)}.$$

Moreover, since the expansion $f = \sum_{-\infty}^{\infty}(f, K_n)G_n$ is valid for every function f belonging to $\boldsymbol{P}$ and $\{G_n\}$ is a Riesz basis for $\boldsymbol{P}$, $\sum_{-\infty}^{\infty}|f(\lambda_n)|^2 < \infty$ for all f. Thus (**) with $c_n \in l^2$ represents the most general function in $\boldsymbol{P}$. If f belongs to $\boldsymbol{P}$ and has the representation

$$f(z) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(t)e^{izt}dt \quad \text{with } \varphi \in L_2(-\pi, \pi),$$

then Planchel's theorem shows that

$$\|f\|^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi(x)|^2 dx = \|\varphi\|^2.$$

By taking the Fourier transform of $\{e^{int}\}_{n\in Z}$, we see that the set of functions

$$\left\{\frac{\sin\, \pi(z-n)}{\pi(z-n)}\right\}_{n\in Z}$$

forms an orthogonal basis for $\boldsymbol{P}$. Accordingly every function f in $\boldsymbol{P}$ has an unique expansion of the form

$$f(z) = \sum_{-\infty}^{\infty} c_n \frac{\sin\, \pi(z-n)}{\pi(z-n)} \quad \text{with} \quad \sum_{-\infty}^{\infty}|c_n|^2 < \infty.$$

Every function $f \in \boldsymbol{P}$ can be recaptured from its values at the integers, which is achieved by the cardinal series representation of f, (YoR) p. 90.

> **Lemma**, (YoR) p. 109: The system $\{e^{i\lambda_n t}\}$ is minimal in $L_2(-\pi, \pi)$ if and only if there exists a nontrival function $f(z)$ of exponential type at most π, zeros at every λ_n, and such that
>
> $$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2} dx < \infty.$$

A number of important characteristic properties of Riesz bases for a separable Hilbert space H are the following equivalent statements, (YoR), appendix, e.g.,

- The sequence $\{f_n\}$ forms a Riesz basis
- There is an equivalent inner product on H, with respect to which the sequence $\{f_n\}$ becomes an orthogonal basis for H
- The sequence $\{f_n\}$ is complete in H, and there exist positive constants A and B such that for an arbitrary positive integer n and arbitrary scalars $c_1, c_2, \ldots c_n$ one has

$$A\sum_{i=1}^{n} c_i^2 \le \|\sum_{i=1}^{n} c_i f_i\|^2 \le B\sum_{i=1}^{n} c_i^2$$

- The sequence $\{f_n\}$ is complete in H, and its Gram matrix $((f_i, f_l))_{i,j=1}^{\infty}$ generates a bounded invertable operator on $\{l^2\}$
- The sequence sequence $\{f_n\}$ is complete in H and possesses a complete biorthogonal sequence sequence $\{g_n\}$ such that

$$\sum_{n=1}^{\infty}|(f, f_n)|^2 < \infty \text{ and } \sum_{n=1}^{\infty}|(f, g_n)|^2 < \infty \text{ for every } f \text{ in } H$$

- The sequence $\{f_n\}$ is both, a Bessel basis and a Hilbert basis
- The sequence $\{f_n\}$ is an exact frame, (YoR).

The trigonometric system is stable $L_2(-\pi,\pi)$ under „sufficiently small“ perturbances of the integers. This means that if λ_n is a sequence of real or complex numbers for which $\{\lambda_n - n\}$ is in some sense „small“, then the system $\{e^{i\lambda_n t}\}$ will form a basis for $L_2(-\pi,\pi)$, in fact, a Riesz basis. Accordingly, every function $f \in L_2(-\pi,\pi)$ will have an unique nonharmonic Fourier series expansion $f(t) = \sum_{-\infty}^{\infty} c_n e^{i\lambda_n t}$ (in the mean) with $\sum_{-\infty}^{\infty}|c_n|^2 < \infty$. The possibility of such nonharmonic expansions was discovered by Paley and Wiener. …. In the present setting that criterion takes the form

(*) $\left\|\sum_{-\infty}^{\infty} c_n \left(e^{int} - e^{i\lambda_n t}\right)\right\| \leq \lambda < 1$, whenever $\sum_{-\infty}^{\infty}|c_n|^2 \leq 1$.

The Paley-Wiener criterion for Hilbert spaces: Let $\{e_n\}_{n\in N}$ be an orthonormal basis for a Hilbert space H and let $\{f_n\}_{n\in N}$ be close“ in the sense that $\|\sum c_i(e_i - f_i)\| \leq \lambda\sqrt{\sum|c_i|^2}$ for some constant $0 \leq \lambda < 1$, and arbitrary scalars $c_1, c_2, \dots c_n$ $(n = 1,2. \dots)$. Then $\{f_n\}_{n\in N}$ is a Riesz basis for H.

Kadec's Theorem (YoR) p. 36: If $\{\lambda_n\}_{n\in Z}$ is a sequence of real numbers for which $|\lambda_n - n| \leq L < 1/4$, then $\{e^{i\lambda_n t}\}_{n\in Z}$ satisfy the Paley-Wiener criterion and so forms a Riesz basis for $L_2(-\pi,\pi)$.

Regarding the stability of the class of Riesz bases $\{e^{i\lambda_n t}\}$ in $L_2(-\pi,\pi)$ Kadec's theorem can be dramatically improved, first under „small“ displacements of the λ_n ‘s and then under more general „vertical“ displacements, (YoR) pp. 160 ff.

Corrollary. If the system $\{e^{i\lambda_n t}\}$ is a Riesz basis for $L_2(-\pi,\pi)$ then there is a positive constant L with the property that $\{e^{i\mu_n t}\}$ is also a Riesz basis for $L_2(-\pi,\pi)$ whenever $|\lambda_n - \mu_n| \leq L$ for every n.

Corollary 1 (YoR) p. 164: Let $\{\lambda_1, \lambda_2, \lambda_3, \dots.\}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\{e^{iRe(\lambda_n)t}\}$ is a Riesz basis for $L_2(-\pi,\pi)$, then so is $\{e^{i\lambda_n t}\}$.

Corollary 2 (YoR) p. 164 If $\{\lambda_n\}_{n=-\infty}^{n=\infty}$ is a sequence of scalars for which $|Re(\lambda_n) - n| < \frac{1}{4}$, then the system $\{e^{i\lambda_n t}\}_{n=-\infty}^{n=\infty}$ is a Riesz basis for $L_2(-\pi,\pi)$.

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