The Prandtl (hyper singular) integral equation with double layer potential and the exterior Neumann problem

Klaus Braun

April, 2016

In [BrK] the global existence, uniqueness and a well-posed problem representation of the 3D Navier-Stokes equations is shown building on a $H_{1/2}$ (energy) Hilbert space framework. Following same conceptual idea, in [BrK1] optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data is proven. In [NaS] the elegant role of the $H_{1/2}$ space on the circle in the Teichmüller theory and the universal period mapping via quantum calculus is presented. In http://www.fuchs-braun.com/ we build on the the $H_{1/2}$ space on the circle and its relationship to the Bagchi reformulation of the Nyman-Beurling Riemann Hypothesis criterion ([BaB]) to provide one proof of the RH hypothesis.

The Prandtl airfoil uplift force model is based on potential theory requiring certain model adaptions, treatments and additional assumptions. The problem arises because lift on an airfoil in inviscid flow requires circulation in the flow around the airfoil, but a single potential function that is continuous throughout the domain around the airfoil cannot represent a flow with nonzero circulation. In space dimension $n=2$ the Cauchy-Riemann differential equations enable the definition of a complex function, by which the flow of an incompressible, vortex-free fluid can be modelled. In vector terminology this can be represented in the form ([RuC]):

$$\nabla \cdot \vec{v} = 0 \, , \; \nabla \times \vec{v} = 0 .$$

In [RuC] a generalization of the C-R differential equations for space dimension $n=3$ is proposed in the form

$$\nabla \cdot \vec{v} = 0 \quad , \quad (\nabla \times \vec{v}) \times \vec{v} = (\vec{v} \cdot \nabla)\vec{v} - \nabla(\frac{\vec{v} \cdot \vec{v}}{2}) = 0 \cdot$$

which allows also vortex flows with certain vortex line conditions. As for $n=2$ the vectors $\vec{w}$ and $\vec{v}$ are orthogonal it holds $\vec{w} \cdot \vec{v} := (\nabla \times \vec{v}) \cdot \vec{v} = 0$ it follows $|\vec{w} \times \vec{v}| = |\vec{w}| \cdot |\vec{v}|$. Therefore the vortex vanishes which leads to the CR differential equation above enabling harmonic analysis techniques.

We note that Runge's generalized CR differential equations are different from the standard generalized CR differential equations in the context of conjugate harmonic functions ([StE] III, 4), while its characterization by the Riesz transforms (whereby under rotation in R^n the Riesz operators transform in the same manner as the components of a vector) is still valid.

The additional vortex flows conditions are basically nothing else than fulfilling the Euler equation for stationary flows of an incompressible fluid under the condition that the external forces (if existing) do have a potential which is also compatible to the Bernoulli equation.

The relationship of the generalized CR differential equations to the Riesz operators, the Leray-Hopf orthogonal projection operator on the closed subspace of L_2 with zero divergence, the Calderon-Zygmund theory of singular integral operators and the confluent hypergeometric function of first kind is given in [LeN].

The application of the Cauchy integral theorem (analog to the case $n=2$) leads to Prandtl's hydrodynamic model of the fluid fuselage flow sticks along the surface of a body whereby in the boundary layer the velocity increases from zero to the velocity as modelled defined by the CR differential equations. Prandtl called the vortexes in this boundary "intermediate" layer as "bounded vortex".

In this note we provide the corresponding linkage of to the existing theory of hyper-singular integral equations and its application to the airfoil uplift force theory of L. Prandtl. The Prandtl operator enables also a $H_{1/2}$ (energy) Hilbert space, alternatively to the Laplacian operator with the standard H_1 (energy) Hilbert space.

For the following we restrict our self to the space dimension $n=3$ referring to [LiI]. For the case $n=1$ we refer to [KrR]. For a closed connected surface $S\subset R^3$ we consider the harmonic function ([LiI] 4)

$$(*)\quad u(x):=\frac{1}{4\pi}\oiint_S v(y)\frac{\cos\varphi_{xy}}{|x-y|^2}dS_y,$$

whereby φ_{xy} is the angle between the vector $|x-y|$ and the normal n_y to the surface at the point y and $v(y)$ is the density of the double layer potential. One can seek the solution of the Neumann boundary value problem

$$\begin{aligned}\Delta u&=0 &&\text{in } R^3-S\\ \frac{\partial u}{\partial n}&=f &&\text{on } S\end{aligned}$$

in the form (*), whereby the unknown function $v(y)$ is obtained by the equation

$$(\Pi u)(x):=\frac{1}{4\pi}\oiint_S v(y)\frac{\cos\varphi_{xy}}{|x-y|^2}dS_y=f(x).$$

The operator Π is called the Prandtl operator.

The Prandtl operator has the following properties ([LiI] (4.1.40), proposition 4.2.1, Theorem 4.2.2, proposition 4.3.1):

Theorem:
i) There is a representation

$$\prod = A + K$$

whereby

$$(Av)(x) := \frac{1}{4\pi} \oint\!\!\oint_S \frac{v(y)}{|x-y|^3} dS_y \quad \text{and} \quad (Kv)(x) := \frac{1}{4\pi} \oint\!\!\oint_S k(x,y) v(y) dS_y$$

with

$$\left| k(x,y) dS_y \right| \leq \left| \frac{|x-y|^2 \big((n_x, n_y) - 1\big) - 3(|x-y|, n_x)(|x-y|, n_y)}{|x-y|^5} \right| \leq \frac{c}{|x-y|} .$$

ii) The Prandtl operator $\prod : H_r \to H_{r-1}$ is bounded for $0 \leq r \leq 1$

iii) For $0 < r < 1$ the Prandtl operator is Noetherian, i.e. it has a right regularizer R with

$$R \prod = RL + RN ,$$

whereby RN is a compact operator in H_r , R is bounded from H_{r-1} to H_r and the operator N is bounded from H_r to H_0 ,The operators NR and LR are a compact operators in H_{r-1}.

iv) For $v \in H_r$, $r \geq 1/2$, the function

$$u(x) := \frac{1}{4\pi} \oint\!\!\oint_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y$$

is an element of $H_1(R^3 - S)$.

iv) For $1/2 \leq r < 1$ the exterior Neumann problem admits one and only on generalized solution.

Choosing $r := 1/2$ this leads to the

Corollary: The Prandtl operator $\prod : H_{1/2} \to H_{-1/2}$ is bounded, the function

$$u(x) := \frac{1}{4\pi} \oint\!\!\oint_S v(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_y$$

is an element of $H_1(R^3 - S)$ and the exterior Neumann problem admits one and only on generalized solution.

Relationship to the non-linear Navier-Stokes equations

The initial boundary value problem of the three dimensional Navier-Stokes equations is given by

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla) u = -\nabla p + f \quad & \text{in} \quad \Omega \times (0,T) \\ div(u) = 0 \quad & \text{in} \quad \Omega \times (0,T) \\ u(x,0) = u_0(x) \ , \quad & x \in \Omega \\ u(x,t) = u_1(x,t) \ , \quad & (x,t) \in \partial\Omega \times (0,T) \end{aligned}$$

The pressure p can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

where (R_1, R_2, R_3) is the Riesz transform.

Runge's generalized Cauchy-Riemann differential equations $div(u) = (\nabla \times u) \times u = 0$ (allowing potential and vortex flows) are related to the NSE by the formula

$$u \cdot \nabla u = (\nabla \times u) \times u + \nabla (\frac{u \cdot u}{2}) \ .$$

Applying formally the div-operator to the NSE the pressure filed must satisfy the following Neumann problem ([GaG])

$$\begin{aligned} \Delta p = (u \cdot \nabla) u - f \quad & \text{in} \quad \Omega \\ \frac{\partial p}{\partial n} = -\left[\Delta u - (u_1 \cdot \nabla) u - f \right] \cdot n \quad & \text{at} \quad \partial\Omega \end{aligned}$$

where n denotes the outward unit normal to $\partial\Omega$. From this follows that the prescription of the pressure at the boundary walls or at the initial time independently of u, could be incompatible with and, therefore, could retender the problem ill-posed.

References

[BaB] Bagchi B., On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann Hypothesis, Indian Statistical Institute, Bangalore Centre, (2005), www.isibang.ac.in

[BrK1] Braun K., Global existence and uniqueness of 3D Navier-Stokes equations, www. navier-stokes-equations.com, *not published*

[BrK1] Braun K., Optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data, www. navier-stokes-equations.com, *not published*

[GaG] Galdi G. P., The Navier-Stokes Equations: A Mathematical Analysis, Birkhäuser Verlag, Monographs in Mathematics, ISBN 978-3-0348-0484-4

[KrR] Kress R., Linear Integral Equations, Springer Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1989

[LeN] Lerner N., A Note on the Oseen Kernels, in *Advances in Phase Space Analysis of Partial Differential Equations,* Volume 78 of the series *Progress in Nonlinear Differential Equations and Their Applications*, 161-170

[LiI] Lifanov I. K., Poltavskii L. N., Vainikko G. M., Hypersingular Integral Equations and their Applications, Chapman & Hall/CRC, Boca Rato, London, NewYork, Washington, D. C., 2004

[NaS] Nag S., Sullivan, D., Teichmüller Theory and the Universal Period Mapping via Quantum Calculus and the $H_{1/2}$ Space on the Circle, Osaka J. Math. 32 (1995) 1-34

[RuC] Runge C., Über eine Analogie der Cauchy-Riemannschen Differentialgleichungen in drei Dimensionen, Nachrichten von der Königl. Gesellschaft der Wissenschaften, Mathematisch-Physikalische Klasse, Bd. 1922 (1922) 129-136

[StE] Stein E. M., Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, New Jersey, 1993