A geometric Hilbert scale based Mie electricity field theory accompanied by a complementary ground state energy space

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Maxwell‘s and Lorentz‘s encounter with Mie and Bohm

This paper is about the quantum energy Hilbert space $H_{1/2} = H_1^{(\cdot^{\cdot^{\cdot}})} \otimes H_1^{(\ddots)} \otimes H_1^{\perp}$ as proposed in (BrK), (BrK1), (BrK2) in the light of

(1) Mie’s tensor and manifolds based purely electricity field theory, (WeH)

(2) Bohm’s conception of explicate and implicate order based on an „undivided wholeness of modes of observation, instrumentation and theoretical understanding“ considering the difference between a lens and a hologram (a kind of mathematical lens (**)), (BoD).

All below relevant data from those two subject areas are taken from (WeH) (*), (BoD). For the following related subject areas we refer to (BrK2):

(a) Schrödinger’s vision of purely quanta waves governed by half-odd integers rather than integers, (ScE) p. 44

(b) Pauli’s exclusion principle accompanied with his spin conceptions

(c) Plemelj’s extended Green formulae accompanied with the conceptions of „flow“ and „mass element“, which are purely defined from the boundary layer w/o „a mathematical flux through the layer“ and enhancing the concept of „a mathematical particle density“, (***), (PlJ)

(d) Calderón’s wavelets interpreted as mathematical lenses.

For the convenience of interested readers we also provide some other related mathematical data in the appendix. We especially mention the theory of indefinite inner products and the hyperbolid generation by a self-adjoint operator on all of the Hilbert space.

(*) for whatever reason the german print version provides more physical rationals than the english versions

(**) (HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a-da$ with $da>0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify $b \leftrightarrow$ position; $(a\Delta)^{-1} \leftrightarrow$ enlargement; $g \leftrightarrow$ optics. “

(***) (PlJ) I, §8: "bisher war es ueblich fuer das Potential V(p) die Form $V(u)(s) = \oint \gamma(s-t)u(t)dt$ vorauszusetzen, wobei dann $u(t)dt$ die Massendichtigkeit der Belegung genannt wurde. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschraenkung, dass dadurch dem Potentials V(p) der groesste Teil seiner Leistungsfaehigkeit hinweg genommen wird." $V(u)(s) = \oint \gamma(s-t)du(t)$.” (PlJ) p. 11: “Vom Integral $\oint \frac{\partial U}{\partial n} ds$ auf einer nichtgeschlossenen Kurve ergibt sich aus der Gleichung (6) eine Eigenschaft von grosser Wichtigkeit. Das Integral hängt nämlich nur von den Endpunkten ab und nicht von der näheren Form der sie verbindenden Integrationskurve in der Weise, dass die Integrale alle gleich einander gleich sind, welche Integrationswege entsprechen, die durch stetige Deformation im Regularitätsgebiete auseinander hervorgehen. Sind also p und q zwei Punkte im Regularitätsgebiete und verbindet man sie durch irgendeine Kurve (die Tangenten hat), so ist $\int_p^q \frac{\partial U}{\partial n} ds$ wohl definiert und hat einen von der näheren Form der Kurve nicht abhängigen Wert.… Das Integral zwischen zwei Punkten p und q $\bar{U}(q) = -\int_p^q \frac{\partial U}{\partial n} ds$ ist, weil von der Kurve unabhängig, eine wohl definierte Funktion der Grenzen p und q und soll in seiner Abhängigkeit von q mit $\bar{U}$ bezeichnet werden.

Einstein relativity principle and the split up of the world into space and time

Einstein relativity principle tells us that the speed of light is independent from the motion state of the light source (it does not tell us, that clocks cannot run differently for observers in other galaxies). In other words, a translation in an ether cannot be distinguished from hibernation. The conclusion of the physicists was, „ether does not exist“ (WeH1) §22. The analysis of the Einstein relativity principle in the context of the Lorentz invariance lead to a decomposition of the world into space and time by projection. The related world-points constitute a four-dimensional manifold, i.e., in this world there exists four coordinates that the corresponding space-like zero cones translate into space-like zero cones, and the time-like vectors transform into time-like vectors (WeH1) §23. In terms of Bohm's explicate and implicate order conception, (*), the Einstein-world-model is related to the explicate order with three general transformations considered to be the essential determining features of a geometry in an Euclidean space of three dimensions: displacement operators, rotation operators and dilatation operators, which are the characterizing properties of the Riesz operators (**).

Regarding the NSE we note that the pressure p can be expressed in terms of the velocity by the formula $p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$ where (R_1, R_2, R_3) is the Riesz transform (***).

The fundamental principle of the SRT is the (Maxwell equations based) invariance principle building on the Lorentz transformation. The handicap of the Lorentz transformation is the fact, that no component of one component can be connected to another in another component. This results to three subgroups of L, which are the orthochronous Lorentz group, the proper Lorentz group, and the orthochorous Lorentz group. Associated with the restricted Lorentz group is the group of $2x2$ complex matrices of determinant one ($SL(2,C)$). The resolution of this handicap is the corresponding Lorentz group L has four disconnected components, where each of which is connected in the sense that any one point can be connected to any other (****).

The complex Lorentz group with its two connected components provides the central tool in the proof of the PCT theorem. In the context of the three characterizing one might rename the PCT theorem into DRD theorem (displacement, rotation, dilation).

(*) (BoD) A.2, p. 200: „What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. … The general transformations are considered to be the essential determining features of a geometry in an Euclidean space of three dimensions; those are displacement operators, rotation operators and the dilation operator.

(**) The Riesz transforms are the generalization of the one-dimension Hilbert transform. The properties of the Riesz transforms have the following converse

Proposition 2 (StE) p. 58): Let $T = (T_1, T_2, \dots T_n)$ be an n-tuple of bounded transformations on $L_2(R^n)$. Suppose

- *i) each T_j commutes with the translation of R^n*
- *ii) each T_j commutes with the dilations of R^n*
- *iii) for every rotation of $\rho = (\rho_{jk})$ of R^n, $\rho T_j \rho^{-1} f = \sum_k \rho_{jk} T_k f$.*

Then the T_j are a constant multiple of the Riesz transforms, i.e. there exists a constant c, so that $T_j = cR_j$, $j = 1, \dots, n$.

(***) Regarding the NSE we note that the pressure p can be expressed in terms of the velocity by the formula $p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$ where (R_1, R_2, R_3) is the Riesz transform. The Leray-Hopf projector is the matrix valued Fourier multiplier given by $P(\xi) = Id - \frac{\xi \otimes \xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2})_{1 \le j,k \le n}$, $P = Id - R \otimes R =: Id - Q$, whereby Q is an orthogonal projector, i.e. it holds $Q := R \otimes R = (R_j R_k)_{1 \le j,k \le 1} = Q^2$. As a result the Leray-Hopf operator $P = Id - R \otimes R =: Id - Q = Id - \frac{D \otimes D}{D^2} Id - \Delta^{-1}(\nabla \times \nabla)$ is also an orthogonal projection.

(****) (StR): „The corresponding Lorentz group L has four disconnected components, where each of which is connected in the sense that any one point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component. This results to three subgroups of L, which are the orthochronous Lorentz group, the proper Lorentz group, and the orthochorous Lorentz group. Associated with the restricted Lorentz group is the group of 2x2 complex matrices of determinant one ($SL(2,C)$). ….. The alignment of the SRT with the proposed quantum field model is enabled by the complex Lorentz group $L(C)$, which is also essential in the proof of the PCT theorem. The central differentiator to the Lorentz group is the fact, that $L(C)$ has the (only two) connected components $L_{+/-}(C)$, where $L_+(C)$ denotes the proper complex Lorentz group. ….. For a general analysis of relativistic invariance it is reasonable that any relativistically invariant theory in which the states are spanned by the collision states of the elementary particles of the theory has, in a suitable basis, an essentially uniquely determined relativistic transformation law. This transformation law is identical to that of a theory of non-interacting elementary particles of the same masses and spins. Any relativistic theory of particles which does not have this transformation law will, in our opinion, require a novel physical interpretation. (as usual, in making this statement we are ignoring the special difficulties associated with zero mass particles.)

Maxwell‘s „a priori electron“ vs. Mie’s „electricity pressure“

The Mie electrodynamic field theory replaces ten unknown universal functions by only one accomplished by the principle of energy. Mie’s electrodynamics exists in a compressed one-world-function form governed by Hamiltonian‘s principle providing a single physical law, (WeH) §26.

The following text is basically a 1-2-1 copy from (WeH) §26. We just comment on the Maxwell/Lorentz concept of an a priori given electron. It is the root of evil for today’s large zoo of elementrary particles, when considering the two other „forces“ phenomena in the context of the SMEP.

In Maxwell’s phenomenogical theory of electricity, the concealed motions of the electrons are not taken into account as motions of matter, consequently electricity is not supposed attached to matter in his theory. The only way to explain how it is that a piece of matter carries a certain charge is to say this charge is that which simultaneously in the portion of space that is occupied by the matter at the moment under consideration. From this we see that the charge is not, as in the theory of electrons, an invariant determined by the portion of matter, but is dependent on the way the world has been split up into space and time.

The theory of Maxwell and Lorentz cannot hold for the interior of the electron; therefore, from the point of view of ordinary theory of electrons we must treat the electron as something given a priori, as a foreign body in the field.

A more general theory of electrodynamics has been proposed by Mie, by which it seems possible to derive the matter from the field.

In place of the ten unknown universal functions of the Maxwell theory there are only one (invariant scalar density), the Hamiltonian Function L; this is accomplished by the principle of energy. In Mie’s theory, the fundamental equation of electrical theory, suddenly acquires a much more vivid meaning by the appearance of potential as an electrical pressure; this is the required cohesive pressure that keeps the electron together (*).

Mie’s theory resolves the problem of matter into a determination of the expression of the Hamiltonian function in terms of the following four quantities and the laws for the field may be summarised in a very simple principle of variation, Hamilton’s principle:

$$\delta L = \frac{1}{2} H^{ik} \delta F_{ik} - s^i \delta \varphi_i .$$

The simplest invariants that may be formed from a vector having component φ_i and a linear tensor of the second order having component F_{ik} are the squares of the following expressions

1. $\varphi_i \varphi^i$
2. $2L^0 = \frac{1}{2} F_{ik} F^{ik}$
3. the linear tensor of the fourth order with components $\sum \pm F_{ik} F_{lm}$
4. $F_{ik} \varphi^k$.

(*) In Maxwell equations there is sometime the concept of „electromotive force“; it is not a force, but the work done by the electric field, i.e. it has the same unit of measure as „Volt“; the mathematical object is „circulation“ governed by the Stokes equation.

Whereas in mechanics, a definite function L of action corresponds to every given mechanical system and has to be deduced from the constitution of the system, we are here concerned with a single system, the world. This is were the real problem of matter takes its beginning: we have to determine the „function of action", the world-function L, belonging to the world. For the present it leaves us in perplexity. If we choose an arbitrary L, we get a „possible" wolrd governed by this function of action, which will perfectly intelligible to us – more so than the actual world – provided that our mathematical analysis does not fail us. We are, of course, then concerned in discovering the only existing world, the real world of us. Judging from what we know of physical laws, we may expect that which belongs to it to bedistinguished by having mathematical properties. Physics, this time as a physics of fields, is again pursuing the object of reducing the totality of natural phenomena to a single physical law: it was believed that this goal was alomst within reach once before Newton's Principia, founded on the physics of mechanical point-masses was celebrating its triumph. For the present we do not know whether the phase-quantities on which Mie's theory is founded will suffice to describe matter or whether matter is purely „electrical" in nature.

Let us try the following hypothesis for L:

$$L = \frac{1}{2}|F|^2 + w(\sqrt{-\varphi_i \varphi^i})$$

(w is a symbol of a function with one variable); it suggests itself as being the simplest of those that go beyond Maxwell's Theory. We have no grounds for assuming that the world-function has actually this form. We have here the new circumstance that the density ρ is an universal function of the potential, the electrical pressure Φ. If $w(\Phi)$ is not an even function of Φ, the defining Poisson equation no longer holds after the transition from Φ to $-\Phi$; this would account for the difference between the natures of positive and negative electricity. Yet it certainly leads to a remarkable difficulty in the case of non-statical fields. If charges having opposite signs are to occur in the latter, the root $\sqrt{-\varphi_i \varphi^i}$ must have different signs at different points of the field.

A Hilbert space based Mie theory overcomes the current two Lorentz resp. Mie theory handicaps:

A. the Michelson-Morley experient, showing that a translation in an ether cannot be distinguished from hibernation; therefore, „ether" cannot not exist
B.
C. in the case of non-statical fields if charges have opposite signs there are different signs in the wolrd-function $L = \frac{1}{2}|F|^2 + w(\sqrt{-\varphi_i \varphi^i})$ at different points of the field.

A (quanta energy) Hilbert space decomposition $\boldsymbol{H_{1/2} = H_1^{(\cdot^{\cdot})} \otimes H_1^{(\cdot_{\cdot})} \otimes H_1^{\perp}}$
into an
explicate (world energy) Hilbert space $\boldsymbol{H_1 = H_1^{(\cdot^{\cdot})} \otimes H_1^{(\cdot_{\cdot})}}$
and into an
implicate (ground state energy) Hilbert space $\boldsymbol{H_1^{\perp}}$

The root of evil of the remarkable difficulties of different natures of positive an negative field electricity, i.e. there are different signs in $L = \frac{1}{2}|F|^2 + w(\sqrt{-\varphi_i\varphi^i})$ at different points of the field, if charges have opposite signs in the case of non-statical fields is the pure metrical structure of the underlying manifold concept. A geometric Hilbert space framework is equipped with the characterizing translation capabilities of the Euclidian space (displacement, dilatation, rotation) in line with the explicate order concept of D. Bohm. Mathematically speaking, the geometric structure of a Hilbert space determines the Hamiltonian function of a well defined energy minimanization problem.

Measurable actions in a specific physical situation where particle interaction happen are interpreted as the action of an underyling „*potential difference*" resp. „pressure". In the context of electrodynamics the potential difference is the „voltage", which is linked to „amperage" and „current resistence" by Ohm's law. In the context of the Mie theory this about the electric pressure.

The coarse-grained compactly embedded standard physical world Hilbert space H_1 (equipped with the Dirichlet integral inner product) into an overall mathematical $H_{1/2}$ Hilbert space world provides the energy model of the physical (explicative) world, while its complementary space $H_1^{\perp}$ builds the implicative purely mathematical (hologram) world.

The overall Hilbert space $H_{1/2}$ is governed by the conservation of energy principle.

The ground state energy space $H_1^{\perp}$ has a continuous spectrum, while H_1 has a discrete spectrum in line with Schrödinger's thermostatistics.

Handicap A: $H_1^{\perp}$ addresses the unfortune conclusion of the physicists („there exists no ether") from the results of the Michelson-Morley experiment, showing that a translation in an ether cannot be distinguished from hibernation.

Handicap B: The physical world is governed by the least action principle accompanied with a related self-adjoint (energy) operator defined on all of the Hilbert space H_1. It induces a decomposition of H_1 into a direct sum of two sub-spaces $H_1 = H_1^{(\cdot^{\cdot})} \otimes H_1^{(\cdot_{\cdot})}$ in the following way:

> let $\beta \coloneqq sup_{\|x\|=1}(Hx, x) < \infty$ and let E_t denotes the resolution of the identity corresponding to the hermitian operator H, then $P_1 \coloneqq E_\beta - E_0$ is a projector onto a subspace of H_1, (VaM) 11.2.

The decomposition $H_1 = H_1^{(\cdot^{\cdot})} \otimes H_1^{(\cdot_{\cdot})}$ provides the Hilbert space based model where an electric pressure keeps the electron together. The supporting mathematical theory is about linear operators in spaces with an indefinite metric accompanied with the Krein spaces (AzT).

For an overall vision about a simplification of the incompatible standard model of elementrary particles (SMEP) and the cosmology model (GRT) accompanied with Hamilton's quaternions and the conception of a S^3 unit sphere mathematical reality we refer to (UnA), (see also (BrK3)).

Ideal plasma and the decomposition $\boldsymbol{H_1 = H_1^{(-)} \otimes H_1^{(+)}}$

In case of a proper plasma heating model the above Hilbert space based (single particle focused) Mie theory needs to adapted in order to address the „two-nature-particles“ requirement of an ideal plasma. The key differentiator between plasma to neutral gas or neutral fluid is the fact that its electrically positively and negatively charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. The MHD model (*), describes an ideal plasma as a non-dissipative plasma particle flow of two incompressible charged particles with different natures.

For an overall vision about a simplification of the incompatible standard model of elementrary particles (SMEP) and the cosmology model (GRT) accompanied with Hamilton's quaternions and the conception of a S^3 unit sphere mathematical reality we refer to (UnA), (see also (BrK3)).

In case of plasma heating two types of electricity pressures generate the additional energy waves, i.e., it now requires two particle types with different nature and with the same cardinality, and the process is about frictional heating mathematically modelled as double layer potential governed by the extended Green formulae acoompanied with the concpets of „mass element“ and „potential difference“ defined on the boundary layer w/o the concept of a normal derivative.

The alternatively proposed adapted Hilbert space based Mie theory is accompanied by two types of (permanent) electric pressures generating the additional (energy) waves for the plasma heating phenomenon: the hermitian operaotr based decomposition $H_1 = H_1^{(\cdot^{\cdot^{\cdot}})} \otimes H_1^{(\ddots)}$ is replaced by a „balanced composition with respct to the cardinality of the concerned sub-space“ in the form

$$H_1 = H_1^{(-)} \otimes H_1^{(+)}.$$

We sketch the related building procedure for the 1D case:

The weighted Hermite polynomials

$$\varphi_n(x) := \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{with} \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \ , \ H_0(x) = 1, \ H_1(x) = x,$$

form a set of orthonormal functions in $L_2(-\infty, \infty)$.

The 1D counterpart of the Riesz operators is the Hilbert transform (appendix). From $\varphi_n, H\varphi_n \in L_2$, it follows $L_2 = \overline{span\{\varphi_n\}} = \overline{span\{H\varphi_n\}}$. Then, the orthogonality relation $(\varphi_n, H\varphi_n) = 0$ enables the following decomposition

$$H_1^{(-)} := \overline{span\{\varphi_{2n-1}\}},\ H_1^{(+)} := \overline{span\{H\varphi_{2n-1}\}}.$$

We note that

i) both sequences $\{\varphi_{2n-1}(x)\}_{n \in N}$, $\{H(\varphi_{2n-1}(x))\}_{n \in N}$ have Snirelmann density ½

ii) the decomposition addresses Schrödinger's vision of purely quanta waves governed by half-odd integers rather than integers, in line with Pauli`s exclusion principle, and also in line with an underlying ground state energy concept (ScE) p. 44.

(*) The MHD equations are derived from continuum theory of non-polar fluids with three kinds of balance laws,

(1) conservation of mass
(2) balance of linear momentum
(3) balance of angular momentum governed by the Ampere law and the Faraday law.The MHD equations consists of 10 equations with 10 parameters accompanied with appropriate boundary conditions from the underlying Maxwell equations (CaF).

Appendix

Extract (VaM) chapter IV

Let B be a self-djoint operator defined on all of the Hilbert space H. Then this operator induces a decomposition of H into a direct sum of two sub-spaces $H = H^1 \otimes H^2$. Both sub-spaces are no Hilbert spaces. However, the orthogonal projection operators P^1 and P^2 enable the definition of the indefinite metric

$$\varphi(x) \coloneqq \big((x)\big)^2 := \|P^1 x\|^2 - \|P^2 x\|^2$$

Thus, putting $x_1 \coloneqq P^1 x, x_2 \coloneqq P^2 x$ the operator B generates a hyperboloid and a related ellipsoid

i) Hyperboloid: $\varphi(x_1 + x_2) = \|x_1\|^2 - \|x_2\|^2 = c > 0$

ii) Ellipsoid: $\frac{\|x_1\|^2}{a_1^2} + \frac{\|x_2\|^2}{a_2^2} = 1$; elliptical region: $E_c \coloneqq \left\{x \in H | \frac{\|x_1\|^2}{a_1^2} + \frac{\|x_2\|^2}{a_2^2} \leq c, c > 0\right\}$.

The indefinite metric $\varphi(x)$ can be interpreted as a „potential“ accompanied with the gradient of the potential $\varphi(x)$ defined by

$$grad\varphi(x) = \mathrm{grad}\big((x)\big)^2 = 2P^1 x - 2P^2 x.$$

The corresponding potential operator is then given by

$$\boldsymbol{W}(x) := \frac{1}{2}\mathrm{grad}\big((x)\big)^2 = P^1 x - P^2 x.$$

The fundamental properties of the potential operator $\boldsymbol{W}(x)$ are completeness, invertibility, ($\boldsymbol{W} = \boldsymbol{W}^{-1}$) isometry, and symmetry. Thus, the bilinear form $(x, y)_W \coloneqq (\boldsymbol{W}(x), y)$ defines an inner product, (BoJ) p. 52.

From physical modelling perspective we note that the model enables the definition of a PDE specific potential criterion (e.g. the famous coupling constants) accompanied with related hyperbolic and conical region V_c and V_0, whose points satisfy the corresponding potential barrier conditions. Evidently V_c is a subspace of V_0.

We remark that if x is an exterior point of the conical region V_0, i.e.

$$\sqrt{\|P^1 x\|^2 - \|P^2 x\|^2} = \alpha > 0,$$

then those points of the ray $tx, t \in [0, \infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0, \infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\|P^1 x\| = \|P^2 x\|$ belong to K, and those points of S^1 for which $\|P^1 x\| > \|P^2 x\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by

$$t = \mathrm{c}\sqrt{\|P^1 x\|^2 - \|P^2 x\|^2}\,.$$

From this it is seen that $t \to \infty$ if $\|P^1 x\|^2 - \|P^2 x\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$.

The Riesz tranforms

We consider the pseudo-differential operator with symbol $|\xi|^{-1}$ ((EsG) (3.15'), (3.17')), defined by

$$\Lambda^{-1}u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}\int_{-\infty}^{\infty}\frac{u(y)dy}{|x-y|^{n-1}}\,, \quad n \geq 2.$$

We note that the function

$$\Lambda^{-2}u = \frac{\Gamma(\frac{n-2}{2})}{4\pi^{\frac{n}{2}}}\int_{-\infty}^{\infty}\frac{u(y)dy}{|x-y|^{n-2}}\,, \quad n \geq 3$$

is called the Newtonian potential. The inverse operator Λ (which is called the Calderón-Zygmund integrodifferential operator) with symbol $|\xi|^{1}$ is given by

$$(\Lambda u)(x) = -(\Delta\Lambda^{-1})u(x) = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\frac{\Delta_y u(y)}{|x-y|^{n-1}}dy.$$

The singular integral (Riesz) operators R_k

$$R_k u := -i\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\frac{x_k-y_k}{|x-y|^{n+1}}u(y)dy$$

enable an alternative representation of the Calderón-Zygmund operator Λ in the form, (EsG) (3.35),

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \sum_{k=1}^{n}\frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}p.v.\int_{-\infty}^{\infty}\sum_{k=1}^{n}\frac{x_k-y_k}{|x-y|^{n+1}}\frac{\partial u(y)}{\partial y_k}dy\ .$$

The Riesz transforms are the generalization of the one-dimension Hilbert transform. The properties of the Riesz transforms have the following converse (StE) p. 58,

> **Proposition 2**: Let $T = (T_1, T_2, \dots T_n)$ be an n-tuple of bounded transformations on $L_2(R^n)$. Suppose
>
> (a) Each T_j commutes with the translation of R^n
>
> (b) Each T_j commutes with the dillations of R^n
>
> (c) For every rotation of $\rho = (\rho_{jk})$ of R^n, $\rho T_j \rho^{-1} f = \sum_k \rho_{jk} T_k f$.
>
> Then the T_j are a constant multiple of the Riesz transforms, i.e. there exists a constant c, so that $T_j = cR_j$, $j = 1, \dots, n$.

The Hilbert transform

Some key properties of the Hilbert transform

$$(Hu)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x-y} d\, y = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy$$

are given in

Lemma:

i) The constant Fourier term vanishes, i.e., $(Hu)_0 = 0$

ii) $H\big(xu(x)\big) = xH\big(u(x)\big) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy$

iii) For odd functions it holds $H(xu(x)) = x(Hu)(x)$

iv) If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e., $\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = 0$

v) $\|H\| = 1$, $H^* = -H$, $H^2 = -I$, $H^{-1} = H^3$

vi) $H(f * g) = f * Hg = Hf * g \quad f * g = -Hf * Hg$

vii) If $(\phi_n)_{n \in N}$ is an orthogonal system, so it is for the system $(H(\phi_n))_{n \in N}$, i.e.

$$(H\phi_n, H\phi_n) = -(\phi_n, H^2\phi_n) = (\phi_n, \phi_n)$$

viii) $\|Hu\|^2 = \|u\|^2$, i.e. if $u \in L_2$, then $Hu \in L_2$.

Proof:

i) i) and v)-viii): (PeB), 2.9

ii) ii) The insertion of a new variable $z = x - y$ into the Hilbert transform of $xu(x)$, i.e., $H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{x-y} dy$ yields

$$H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z} dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xu(x-z)}{z} dz - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x-z) dz = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy.$$

iii) follows from i) and ii)

iv) $\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = \frac{i}{2\pi} \int_{-\infty}^{\infty} sign((\omega)|\hat{u}(\omega)|^2 d\omega$ whereby $|\hat{u}(\omega)|^2$ is even.

The Eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|\ldots\|$. We will consider mappings $K: H \to H$. Unless otherwise noticed the standard assumptions on K are:

i) K is symmetric, i.e., for all $x, y \in H$it holds $(x, Ky) = (x, Ky)$

ii) K is compact, i.e., for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent

iii) K is injective, i.e., $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} < \infty.$$

Lemma: Let K be bounded, and fulfill condition i) above, but not necessarily the two other conditions ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|}.$$

Theorem: There exists a countable sequence $\{\lambda_i, \phi_i\}$ of eigen-elements and eigenvalues $K\phi_i = \lambda_i \phi_i$ with the properties

i) the eigen-elements are pair-wise orthogonal, i.e. $(\phi_i, \phi_k) = \delta_{i,k}$

ii) the eigenvalues tend to zero, i.e., $\lim_{i \to \infty} \lambda_i$

iii) for the generalized Fourier sums it holds

$$S_n := \sum_{i=1}^{n} (x, \phi_i)\, \phi_i \to x \quad \text{with } n \to \infty \text{ for all } x \in H$$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \phi_i)^2$$

holds for all $x \in H$.

Hilbert Scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm$\|\ldots\|$ and let A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x,Ax) \geq \|x\| \qquad \text{for all } x \in D(A).$$

The operator $K = A^{-1}$has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{\lambda_i, \phi_i\}$ with

$$A\phi_i = \lambda_i\phi_i \quad , \quad (\phi_i,\phi_k) = \delta_{i,k} \text{and } \lim_{i\to\infty} \lambda_i$$

ii) any $x \in H$ is represented by

$$(*) \qquad x = \sum_{i=1}^{\infty}(x,\phi_i)\,\phi_i \;\text{ and }\; \|x\|^2 = \sum_1^{\infty}(x,\phi_i)^2.$$

Lemma: Let $x \in D(A)$, then

$$(**) \;\; Ax = \sum_{i=1}^{\infty}\lambda_i(x,\phi_i)\,\phi_i \quad , \quad \|Ax\|^2 = \sum_{i=1}\lambda_i^2(x,\phi_i)^2,$$

$$(Ax,Ay) = \sum_{i=1}^{\infty}\lambda_i^2(x,\phi_i)\,(y,\phi_i).$$

Because of (*) there is a one-to-one mapping I of H to the space $\widehat{H}$ of infinite sequences of real numbers

$$\widehat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, \ldots)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x,\phi_i)\;.$$

If we equip $\widehat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_1^{\infty}(x,\phi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x},\hat{y})_\alpha = \sum_i^{\infty}\lambda_i^\alpha(x,\phi_i)\,(y,\phi_i) = \sum_i^{\infty}\lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x},\hat{x})_\alpha.$$

Let $\widehat{H}_\alpha$ denote the set of all sequences with finite α –norm. then $\widehat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space l_2.

Similarly one can define the spaces H_α: they consist of those elements $x \in H$ such that $Ix \in \widehat{H}_\alpha$ with scalar product

$$(x, y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x, \phi_i)\,(y, \phi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha.$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax, Ax)_0 \ , \ H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $lim\, \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, \dots)$ with

$$\|\hat{x}\|_2^2 < \infty \ , \ \|\hat{x}\|_0^2 = \infty \ .$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with

$$\mu = \frac{\gamma - \beta}{\gamma - \alpha} \text{ and } \nu = \frac{\beta - \alpha}{\gamma - \alpha}.$$

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$

ii) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

iii) $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Extension and generalizations

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i} t}(x, \phi_i)(y, \phi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 \ .$$

Now the factor has exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^α.

Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t)\|x\|_\alpha \text{ for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the (t)-norm is weaker than any α-norm. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (t)-norm.

It holds:

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the α-norm, which can be reformulated in the form ($\mu, \nu > 0, \mu + \nu > 1$)

$$\|x\|_\beta^2 \leq \nu\varepsilon\|x\|_\gamma^2 + \mu e^{-\nu/\mu}\|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \leq (\|x\|_\alpha^2)^\mu(\|x\|_\gamma^2)^\nu \ .$$

The counterpart of the fourth lemma above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\|x - y\| \leq \|x\|$

ii) $\|y\|_1 \leq \delta^{-1}\|x\|$

iii) $\|x - y\|_{(t)} \leq e^{-t/\delta}\|x\|$.

Eigen-functions and Eigen-differentials

Let H *be* a (infinite dimensional) Hilbert space with inner product $(.,.)$, the norm $\|\ldots\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1}compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$

$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \phi_\mu(\phi_\mu,*)d\mu \quad , \quad \mu \in [\lambda_0, \infty) \ ,$$

i.e.

$$dE_\lambda = \phi_\lambda(\phi_\lambda,*)d\lambda \ .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ. The set E_λ fulfills the following properties:

i) E_λ is a projection operator for all $\lambda \in R$

ii) for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

iii) $\lim_{\lambda \to -\infty} E_\lambda = 0$ and $\lim_{\lambda \to \infty} E_\lambda = Id$

iv) $\lim_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f:[a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) \ .$$

Symbolically one writes $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$. Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax,y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \qquad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2x,y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ , \quad \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function. It holds the following eigen-pair relations

$$A\phi_i = \lambda_i \phi_i \quad A\phi_\lambda = \lambda \phi_\lambda \quad \|\phi_\lambda\|^2 = \infty \ , (\phi_\lambda, \phi_\mu) = \delta(\phi_\lambda - \phi_\mu).$$

The ϕ_λ are not elements of the Hilbert space. The so-called eigen-differentials, which play a key role in quantum mechanics, are built as superposition of such eigen-functions.

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\phi_\lambda(x) = \lambda \phi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\phi_\lambda(x)$ it results into elements of H. The practical way to find eigen-differentials is looking for solutions of a distribution equation.

Non-linear minimization problems

Non-linear minimization problems can be analyzed as saddle point problems on convex manifolds in the following form (VeW):

$$(*) \quad J(u): a(u,u) - F(u) \to min \ , \quad u - u_0 \in U.$$

Let $a(\cdot,\cdot): \ V \times V \to R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V$ and $F(\cdot): V \to R$ a functional with the following properties:

$F(\cdot): V \to R$ is convex on the linear manifold $u_0 + U$, i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \leq (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

$F(u) \geq \alpha$ for every $u \in u_0 + U$

$F(\cdot): V \to R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \to R$ with

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = F_u(v).$$

Then the minimum problem (*) is equivalent to the variational equation

$$a(u,\phi) + F_u(\phi) = 0 \ \text{ for every } \ \phi \in U$$

and admits only an unique solution.

In case the sub-space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \to R$ is continuous with respect to convergence in the energy norm, then there exists a solution. We note that the energy functional is even strongly convex in whole V.

The proposed „energy" Hilbert space $H_{1/2}$ enables e.g. the method of Noble ((VeW) 6.2.4), (ArA) 4.2), which is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a "Hamiltonian" function W(·,·) which combines the pair of underlying operator equations (based on the "Gateaux derivative" concept)

$$Tu = \frac{\partial W(\acute{u},u)}{\partial \acute{u}} \ , \ T^* \acute{u} = \frac{\partial W(\acute{u},u)}{\partial u} \quad u \in E = H_{1/2} \ , \ \acute{u} \in \acute{E} = H_{-1/2}.$$

The $\boldsymbol{H_1 = H_1^{(\cdot\cdot)} \otimes H_1^{(\cdot\cdot)}}$ decomposition to model two spin-types of elementary mass particles

The standard energy Hilbert space H_1 is proposed to be interpreted as „ferminons mass/energy“ space; $H_1^{\perp}$ is proposed to be interpreted as the orthogonal „bosons / ether energy“ space. Both together build the newly proposed quantum energy space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$. The Hilbert (sub-) space H_1 is proposed as the model of the physical (fermions) reality of all affected quantum kinematical phenomena, in line with Einstein's idealized experiment considering the motion of a single electron moving in a field of force with a given potential (*).

A selfadjoint operator B defined on all of the Hilbert space H (e.g. $H = H_1$ and B the Friedrichs extension of the Laplacian operator) is bounded. Thus, the operator B induces a decomposition of H into the direct sum of the subspaces, enabling the definition of a potential and a corresponding „grad“ potential operator. Then a potential criterion defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions ((VaM) 11.2). The direct sum of the corresponding two subspaces of $H = H_1$ are proposed as a model to define a decomposition of the „fermions“ space H_1 into

$$H_1 = H_1^{(\cdot\cdot)} \otimes H_1^{(\cdot\cdot)}.$$

The potential criterion defines repulsive resp. attractive elementary mass particles. Then the corresponding proposed quantum energy Hilbert space is given by

$$H_{1/2} = H_1^{(\cdot\cdot)} \otimes H_1^{(\cdot\cdot)} \otimes H_1^{\perp}.$$

The theory of Hilbert spaces with an indefinite metric is provided in e.g. ((DrM), (AzT), (DrM), (VaM)). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK).

In case of a Hilbert space H, this is about a decomposition of H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 (see also the problem of S. L. Sobolev concerning Hermitean operators in spaces with indefinite metric, (VaM) IV). We note, that for a vector space H, the empty set, the space H, and any linear subspace of H are convex cones.

(*) ((HeW) p. 36, englisch version): „critique of the corpuscular theory“: „The motion and spreading of probability packets has been studied by various authors, … A simple consideration of Ehrenfest's may be mentioned, … considering the motion of a single electron moving in a field of force whose potential is $V(q)$. … If there were no spreading at all, it would be possible to make a Fourier analysis of the probability density into which only integral multiples of the fundamental frequency of the orbit enter. As a matter of fact, however, the „overtones“ of quantum theory are not exactly integral multiples of this fundamental frequency. The time in which the phase of the quantum theoretical overtones will be qualitatively the same as the time required for the spreading of the wave packet. Let J be the action variabe of classical theory, then this time will be $t = \frac{1}{h\frac{\partial \mathrm{v}}{\partial J}}$ and the number of revolutions performed in this time is $N \sim \frac{1}{h\frac{\partial \mathrm{v}}{\partial J}}$. In the special case of the harmonic oscillator, N becomes infinite – the wave packet remains small for all times. In general, however, N will be of the order of magnitude of the quantum number n “.

In relation to these considerations, one other idealized experiment (due to Einstein) may be considered. We imagine a photon which is represented by a wave packet built up out of Maxwell waves. (For a single photon the configuration space has only three dimensions; the Schrödinger equation of a photon can thus be regarded as formally identical with the Maxwell equations.) It will thus have a certain spatial extension and also a certain range of frequency. By reflection at a definite probability for finding the photon either in one part or in the other part oft he divided wave packet. After sufficient time the two parts will be sparated by any distance disired; now if any experiment yields the result that the photon is, say, in the reflected part of the packet, then the probability of finding the photon in the other part of the packet immediately becomes zero. The experiment at the position of the reflected packet thus exerts a kind of action (reduction of the wave packet) at the distant point occupied by the transmitted packet, and one sees that this action is propagated with velocity greater than that of light. However, it is also obvious that this kind of action can never be utilized for transmission of signals so that it is not in conflict with the postulates of the theory of relativity“.

The decomposition of the quantum state space $H_0 \otimes H_0^{\perp}$ resp. the quantum energy space $H_1 \otimes H_1^{\perp}$ is very much related to the „hidden variables in quantum theory" concept of D. Bohm (BoD) with the notions of implicate and explicate order:

(BoD) A.2, p. 200: „What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. … The general transformations are considered to be the essential determining features of a geometry in a Euclidean space of three dimensions; those are displacement operators, rotation operators and the dilatation operator.

In our case this puts the spot on the Riesz transformations. The Riesz operators fulfill certain properties with respect to commutation with translations homothesis and rotation ((PeB), (StE)). Let $SO(n)$ denote the rotation group. If $j \neq j$ then $R_j R_k$ is a singular convolution operator. On the other hand, it holds $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. The following identities are valid

$$\|R_j\| = 1 \ , R_j^* = -R_j \ , \ \sum R_j^2 = -I \ , \ \sum \|R_j u\|^2 = \|u\|^2 \ , u \in L_2 \ .$$

Let

$$m := m(x) := (m_1(x), \ldots m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and$\rho = \rho_{ik} \in SO(n)$, then

$$m(\rho(x)) = \rho(m(x)),$$

whereby

$$m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$$

and

$$m(\rho(x)) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(x\rho^{-1}(y)) + log \left|\frac{1}{x\rho^{-1}(y)}\right|) \frac{y}{|y|} d\sigma(y)$$

$$= c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(xy) + log \left|\frac{1}{xy}\right|) \frac{y}{|y|} d\sigma(y) \ .$$

(BoD), A3, p. 202: „Implicate order is generally to be described not in terms of simple geometric transformations, such as translations, rotations, and dilations, but rather in terms of a different kind of operations. … What happens in the broader context of implicate order we shall call a metamorphosis. … An example of such a metamorphosis metamorphosis M is determed by the Green's function relating amplitudes at the illuminated structure to those at the photographic plate".

In our case this relates to the closed sub-spaces $H_0^{\perp}$ and $H_1^{\perp}$.

Magnetohydrodynamics

One of the key differentiator between plasma to neutral gas of neutral fluid is the fact that its electrically positively and negatively charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not.

An ideal plasma is a non-dissipative flow of the incompressible charged particles (CaF).

The MHD equations are derived from continuum theory of non-polar fluids with three kinds of balance laws:

i) conservation of mass

ii) balance of linear momentum

iii) balance of angular momentum (Ampere law and Faraday law).

The MHD equations consists of 10 equations with 10 parameters accompanied with appropriate boundary conditions from the underlying Maxwell equations (CaF).

In (EyG) it is proven that smooth solutions of non-ideal (viscous and resistive) incompressible magneto-hydrodynamic (plasma fluid) equations satisfy a stochastic (conservation) law of flux. It is shown that the magnetic flux through the fixed Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons.

Hermite Polynomials

The weighted Hermite polynomials

$$\phi_n(x):=\frac{e^{-\frac{x^2}{2}}H_n(x)}{\sqrt{2^n n!\sqrt{\pi}}} \quad \text{with} \quad H_n(x):=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\ , \ \ H_0(x)=1,\ \ H_1(x)=x,$$

form a set of orthonormal functions in $L_2(-\infty,\infty)$, i.e., the Hermite polynomials have only real zeros. The relation to the Gaussian function is given by

$$f(x)=\pi^{1/4}\phi_0(\sqrt{2\pi}x)\ .$$

The Hermite polynomials $H_n(x)$ fulfill the recursion formula

$$H_n(\sqrt{2\pi}x)=2xH_{n-1}(\sqrt{2\pi}x)-(n-1)b_n\phi_{n-2}(x)-2(n-1)H_{n-2}(\sqrt{2\pi}x)\ .$$

Using the abbreviation

$$a_n:=\sqrt{\frac{2(n-1)!}{n!}} \qquad b_n:=\sqrt{\frac{(n-2)!}{n!}}$$

this gives the recursion formula

$$\phi_n(x):=a_n x\phi_{n-1}(x)-(n-1)b_n\phi_{n-2}(x)\ , \quad \phi_0(x):=\pi^{-1/4}e^{-\frac{x^2}{2}},\ \ \phi_1(x):=2^{-1/2}\pi^{-1/4}xe^{-\frac{x^2}{2}}\quad ,$$

from which the recursion formula for the corresponding Hilbert transforms can be calculated

$$\hat{\phi}_n(x):=a_n\left[x\hat{\phi}_{n-1}(x)-\frac{1}{\pi}\int_{-\infty}^{\infty}\phi_{n-1}(y)dy\right]-(n-1)b_n\hat{\phi}_{n-2}(x)$$

$$\hat{\phi}_0(x)=\pi^{1/4}\int_{-\infty}^{\infty}e^{-\frac{\omega^2}{2}}sin(\,\omega x)d\omega\ .$$

As $\phi_n, H\phi_n \in L_2$ is follows

$$L_2:=H:=span[\phi_n(x)]=span[H(\phi_n(x))].$$

Extract from

H. Weyl, Philosophy of Mathematics and Natural Science

The Physical Picture of the World
B. Matter and Fields. Ether

p. 171: „Just as the velocity of a water wave is not a substantial but a phase velocity, so the velocity with which an electron moves is only the velocity of an ideal „center of energy", constructed out of the field distribution. According to this view, there exists but one kind of natural laws, namely, field laws of the same transparent nature as Maxwell had established for the electromagnetic field. The obscure problem of laws of interaction between matter and field does not arise. This conception of the world can hardly be described as dynamical any more, since the field is neither generated nor acting upon an agent separate from the field, but following its own laws is in a quiet continuous flow. It is of the essence of the continuum. Even the atomic nuclei and the electrons are not ultimate unchangeable elements that are pushed back and forth by natural forces acting upon them, but they are themselves spread out continuously and are subject to fine fluent changes.

On the basis of rather convincing general considerations, G. Mie in 1912 pointed out a way of modifying the Maxwell equations in such manner that they might possibly solve the problem of matter, by explaining why the field possesses a „granular" structure and why the knots of energy remain intact in spite of the back and forth flux of energy and momentum. The Maxwell equations will not do because they imply that the negative charges compressed in an electron explode; to guarantee their coherence in spite of Coulomb's repulsive forces was the only service still required of substance by H. A. Lorentz's theory of electrons. The preservation of the energy knots must result from the fact that the modified field laws admit only of one state of field equilibrium – or of a few between which there is no continuous transition (static, spherically symmetry solutions of the field equations). The field laws should thus permit us to compute in advance charge and mass of the electron and the atomic weights of the various chemical elements in existence. And the same fact, rather than the contrast of substance and field, would be the reason why we may decompose the energy or inert mass of a compond body (approximately) into the non-resolvable energy or its last elementary constituents and the resolvable energy of their mutual bond."

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