A global bounded energy norm estimate of the 3D-NSE system enabled by complementary mechanical and turbulence energy spaces in the form $H_{1/2} = H_1 \otimes H_1^{\perp}$

Klaus Braun
unified-field-theory.de

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Global Existence and Uniqueness
of the Non-stationary 3D-Navier-Stokes
Initial-boundary Value Problem, (BrK1)

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Abstract

A global bounded energy norm estimate of a generalized 3D-NSE initial value problem is provided. Based on the Hilbert scale $H_\alpha,\ \alpha \in [-1,1]$, (SoH) p. 133, the standard variational (statistical) $L_2 = H_0$ framework is extended to the distributional Hilbert space $H_{-1/2} = H_0 \otimes H_0^{\perp}$. The corresponding generalized 3D Navier-Stokes initial value problem is given by ($\forall\ v \in H_{-1/2}$ and the time-dependency is not described)

$$(\dot{u}, v)_{-1/2} + (Au, v)_{-1/2} + (Bu, v)_{-1/2} = 0$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2}\ .$$

Accordingly the governing energy Hilbert space is given in the form $H_{1/2} = H_1 \otimes H_1^{\perp}$, (BrK). The corresponding energy norm inequality is given by

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right|$$

Applying the Sobolevskii-estimate, (GiY), resp. the lemma of Gronwall one gets

$$\left|(Bu, u)_{-1/2}\right| \le c \cdot \|u\|_{-1/2}\|u\|_1^2$$

resp.

$$\|u(t)\|_{-1/2} \le \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\}.$$

i.e. there is a global bounded energy norm inequality provided that $u_0 \in H_0$.

As it holds $(Bu, u)_0 = 0$, the non-linear term of the NSE system provides no contribution to the energy equality. Since the solution of the associated linearized equation is already as smooth as the data allow a solution of the non-linear NSE cannot be expected to be smoother than the corresponding linearized equations. Accordingly, the closed sub-space $H_1^{\perp}$ of $H_{1/2} = H_1 \otimes H_1^{\perp}$ may be interpreted as dynamic turbulence energy space providing an alternative model to Kolmogorov's statistical turbulence model. The corresponding "turbulence" energy operator with $H_1^{\perp}$ domain may be interpreted as compact disturbance of the self-adjoint Stokes operator, (BrK).

A $H_{1/2} = H_1 \otimes H_1^{\perp}$ energy Hilbert space based bounded energy norm estimate of the 3D-NSE system

Using the Stokes operator and its related Hilbert scale framework the Navier-Stokes equations can be represented as an evolution equation in H_0 . Since $P(gradp) = 0$ one gets

$$Au = Pf \text{ in } H_0 .$$

Putting $B(u) := P(u, grad)u$ and assuming $Pu_0 = u_0$ the NSE initial-boundary equation is given by

(*) $$\frac{du}{dt} + Au + Bu = Pf \text{ , } u(0) = u_0.$$

As u is divergence free and $u \cdot v$ identically vanishes on $\partial\Omega$ one gets

$$b(u, v, w) := ((u, grad)v, w) = \iint_{\Omega} (u, grad)v \cdot wdx = -b(u, w, v)$$

and especially $b(u, v, v) = (Bu, u) = 0$. This means that the non-linear term of the NSE system provides no contribution to the energy equality. Since the solution of the associated linearized equation is already as smooth as the data allow a solution of the nonlinear NSE cannot be expected to be smoother than the corresponding linearized equations.

Theorem: The generalized 3D Navier-Stokes initial value problem is governed by a bounded $H_{1/2}$ energy Hilbert space based energy norm estimate in the form

$$\|u(t)\|_{-1/2} \le \|u_0\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\} .$$

It ensures a global boundedness of a NSE system solution provided that $u_0 \in H_0$.

Before proving the theorem we prove the essential estimate of the nonlinear term of (*) in

Lemma:

$$\left\|A^{-1/4}P(u, grad)u\right\|_0 =\le c\left\|A^{1/2}u\right\|_0 \cdot \left\|A^{1/2}u\right\|_0 = c\|u\|_1 \cdot \|u\|_1 = c\|u\|_1^2.$$

Proof: In order to prove the lemma we will apply a more general lemma as provided in (GiY1), and below(*).

Choosing $p = 2$, $\delta = 1/4$, $\theta := \rho := 1/2$ which gives $\theta + \rho \ge \frac{1}{4}(n+1) = 1$. Then it follows

$$\left\|A^{-1/4}P(u, grad)u\right\|_0 = \left\|A^{-1/4}Bu\right\|_0 \le c\left\|A^{1/2}u\right\|_0 \cdot \left\|A^{1/2}u\right\|_0 = c\|u\|_1 \cdot \|u\|_1 = c\|u\|_1^2 \text{ .}$$

Proof of the theorem: Multiplying the homogeneous equation of (*) with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_{-1/2} + (Au, u)_{-1/2} + (Bu, u)_{-1/2} = 0 \text{ .}$$

Then, the corresponding generalized "energy" inequality is given by

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0 \le c\|u\|_1^2.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$. Applying the lemma of Gronwall gives

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\} .$$

(*) *Lemma 3.2 (GiY1); see also (FuH), (KaT) for $p = 2$): Let $0 \le \delta < \frac{1}{2} + \frac{n}{2}(1 - \frac{1}{p})$. We have*

$$\left\|A^{-\delta}P(u, grad)u\right\|_{L_p} \le M\left\|A^{\theta}u\right\|_{L_p} \cdot \|A^{\rho}u\|_{L_p}$$

with a constant $M = M(\delta, \theta, \rho, p)$ if $\delta + \theta + \rho \ge \frac{n}{2p} + \frac{1}{2}, \rho, \theta > 0, \ , \ \rho + \theta > \frac{1}{2}$. In particular, if $p = n$, we can choose $\delta = \theta = \frac{1}{4}, \rho = \frac{1}{2}$.
Lemma of Gronwall (general form): Let $\alpha(t)$ and $b(t)$ nonnegative functions in $[0, A)$ and $0 < \delta < 1$. Suppose a non-negative function $y(t)$ satisfies the differential inequality $y'(t) + b(t) \le \alpha(t)y^{\delta}(t)$ on $[0, A)$ and $y(0) = y_0$. Then for $0 \le t < A$

$$y(t) + \int_0^t b(\tau)d\tau \le (2^{\delta/(1-\delta)} + 1)y_0 + 2^{\delta/(1-\delta)}\left[\int_0^t \alpha(\tau)d\tau\right]^{\delta/(1-\delta)} .$$

The fluid intrinsic „pressure" artifact of the NSE system

In the current NSE system the pressure plays the key role to generate the „force" (resp. to provide the energy) to move a fluid particle forwards into the direction of the decreasing pressure. The „pressure" is a scalar quantity, however, its spacial shift generates a „pressure force", which acts on the fluid. Correspondingly, the negative pressure gradient represents the acceleration of the fluid which acts on the fluid. Its multiplication with the mass density of the fluid continuum it gives the fundamental force, which governs the movement of fluids orchestrated by the Newton law F = m * a.

In the current model it is the difference between the pressures of two fluid particles that generates the "pressure force", which moves the considered fluid forward. The negative gradient of this pressure (the relevant term in the NSE system) represents the acceleration of the considered fluid into the direction of the decreasing pressure. Conceptually spoken, the „pressure" force is governed by a fluid intrinsic energy, while the exterior force of the NSE system (the graviation force and the viscous forces) acting on the continuum are governed by mechanical energy. Correspondingly, regarding the boundary valus conditions the pressure field model should become an exterior Neumann problem, while the NSE system itself should become an interior dynamic fluid problem.

The dynamic turbulence energy sub-space $H_1^{\perp}$ of $H_{1/2} = H_1 \otimes H_1^{\perp}$

The $H_{1/2} = H_1 \otimes H_1^{\perp}$ energy Hilbert space decomposition may be interpreted as two complementary energy type spaces: the mechanical energy Hilbert space H_1, which is the domain of the Friedrichs extension of the Laplacian operator governed by Fourier waves. Its complementary closed sub-space of $H_{1/2}$ may be interpreted as dynamic turbulence energy space, where the corresponding "turbulence" energy operator with $H_1^{\perp}$ domain may be interpreted as compact disturbance of the related fractional self-adjoint Stokes operator, (BrK).

The Stokes operator is a self-adjoint positive definite operator with respect to the L_2 inner product. It has orthonormal eigenpairs and the inverse of the Stokes operator is bounded and compact. This means that the Stokes operator shows the same conceptual structure as the Laplacian operator, (TeR). Therefore, the non-stationary Stokes system shows the same structure as the heat operator. The related evolution equation of the nonstationary Stokes system shows solutions (in case of $u(0) = u_0 = 0$) in the form, (SoH) p. 203,

$$u(t) = \int_0^t S(t-\tau) f(\tau) d\tau \, , \, t \geq 0,$$

where $S(t) = e^{-tA}$ is defined by a spectral representation. In the appendix we provide a proof of an optimal shift theorem for the heat equation with respect to norms in the form

$$\|z\|_{L_2(0,T;H_k)}^2 = \int_0^T \|z(t)\|_{H_k}^2 dt \, .$$

Die Potentiale der einfachen und doppelten Scicht

Extract from

Plemelj's Potentialtheoretische Untersuchungen

The argument of Plemelj for his newly proposed potential is, (PU) S. 17:

"Bisher war es üblich für das Potential die Form $(V(p) = \oint log \frac{1}{r_{ps}} \cdot \mu'(s) ds$ vorauszusetzen, wobei dann $\mu'(s)$ die *Massendichtigkeit* der Belegung genannt wurde. *Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschränkung, daß dadurch dem Potentiale der größte Teil seiner Leistungsfähigkeit hinweg genommen wird*. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form $(V(p) = \oint log \frac{1}{r_{ps}} \cdot d\mu(s)$ verwendbar."

Neumann problem, hypersingular integral equations
and the Prandtl operator

Extracts from (LiI)

"*In order to find the characteristics of a flow past a body without separation, it is convenient to model the vortex layer next to the body by closed quadragles and vortex frames. The intensity of these vortex formations coincides with the density of the double layer potential on the surface of the body for which the values of the potential outside the body are the same as in the case of a purturbed flow. Thus we come to the problem of finding the potential outside a body in terms of its normal derivative and the density of the double layer potential. This chapter 4 is dedicated to the theory of the Neumann problem and the correponding integral equations with the double layer potential*", (LiI) p. 95.

In the Euclian space R^3, consider closed connected surfaces $S_1^0, \dots, S_k^0$ of class C^∞ (i.e., compact connected C^∞ manifolds of dimension 2). Suppose also that in R^3 there are non-closed connected surfaces surfaces $S_1^1, \dots, S_n^1$ of class C^∞ ((i.e., compact two-dimensional C^∞ manifolds with border) such that $\bar{S}_i^1 \cap \bar{S}_j^1 = \emptyset$ for $i \neq j$. We assume that there exist two-dimensional compact manifolds (without borders) of class $M_1, \dots M_{m_1}$ of class C^∞ such that every surface $S_j^1, j = 1, \dots, n$, belongs to one of the manifolds $M_1, \dots M_{m_1}$, and several surfaces S_i^1 may belong to one and the same manifold M_j. It is also assumed that the manifolds $M_1, \dots M_{m_1}$ are connected.

The surfaces S_i^0 coincide with the boundaries of the bounded domains Ω_i^0 and the surfaces M_i are the boundaries of bounded domains Ω_i^1. Moreover, $\overline{\Omega_i^0} \cap \overline{\Omega_j^0} = \emptyset$ $\overline{\Omega_i^1} \cap \overline{\Omega_j^1} = \emptyset$ for $i \neq j$, and $\overline{\Omega_i^0} \cap \overline{\Omega_j^1} = \emptyset$ for all i, j.

Consider the following harmonic functions

$$u_i^0(x) := \frac{1}{4\pi} \oiint_{S_i^0} \nu_i^0(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_{iy}^0$$
$$u_i^1(x) := \frac{1}{4\pi} \oiint_{S_i^0} \nu_i^1(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_{iy}^1 ,$$

where $x = (x_1, x_2, x_3) \in R^3$, $y = (y_1, y_2, y_3) \in S_i^0$ or S_i^1; φ_{xy} is the angle between the vector $|x - y|$ and the normal $\boldsymbol{n}_y$ to the surface S_i^0 or S_i^1 at the point y, $\nu_i^p(y) \in C^\infty(S_i^p)$ is the density of the double layer potential, $p = 0,1$. Consider the boundary value problem

$$\Delta u = 0 \quad x \in R^3 - (\cup S_i^p)$$
$$\left.\frac{\partial u}{\partial n}\right|_{S_i^p} = f_i^p ,$$
$$p = 0: \ i = 1, \dots, k; \ p = 1: \ i = 1, \dots, n,$$

where Δ is the Laplacian operator; $\partial u / \partial n|_{S_i^p}$ is the derivative along the normal to the surface S_i^p. We seek the solution of the boundary value problem as the double layer potential

$$u(x) := \frac{1}{4\pi} \sum_p \sum_i \oiint_{S_i^p} \nu_i^p(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_{iy}^p,$$

and for the unknown functions $\nu_i^p(y)$ we obtain the following system of equations:

$$\boldsymbol{Pr}[\nu_l^m](x) \coloneqq \frac{1}{4\pi} \sum_p \sum_i \frac{\partial}{\partial n_l^m} \oiint_{S_i^0} \nu_i^p(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_{iy}^p = f_l^m(x) ,$$

(*)

$$m = 0: \ l = 1, \dots, k; \ m = 1: \ l = 1, \dots, n.$$

Defintion 4.1.4: The operator $\boldsymbol{Pr}$ defined by (*) is called the *Prandtl operator*.

Note: $H_r(R^n)$ denotes the Sobolev-Slobodestkii space. For a closed connected surface $S \subset R^n$ it holds $\hat{H}_{r-1} = H_{r-1}$.

Proposition 2.3.3. The norm of the space H_λ with $0 < \lambda < 1$ is equivalent to

$$\|u\|_\lambda^2 = \int_{R^n} \int_{R^n} \frac{|u(x+y)|u(s)^2}{|y|^{n+2\lambda}} dxdy + \int_{R^n} |u(x)|^2 dx.$$

Denote by $H_r(v)$ the direct sum of all spaces $H_r(S_i^0)$ and $\dot{H}_r(S_j^1)$ and by $\hat{H}_r(v)$ the direct sum of all spaces $H_r(S_i^0)$ and $H_r(S_j^1)$.

The *Prandtl operator* has the following properties (LiI) pp. 108, 109, 111, 115:

Proposition 4.2.1. The Prandtl operator $\boldsymbol{Pr} : H_r \rightarrow \hat{H}_{r-1}$ is bounded for $0 \leq r \leq 1$.

Theeorem 4.2.2. For $0 < r < 1$, The Prandtl operator $\boldsymbol{Pr}$, which maps H_r into $\hat{H}_{r-1}$, is Noetherian.

Proposition 4.3.1. For $v \in H_r(v)$, $r \geq 1/2$, the function

$$u(x) := \frac{1}{4\pi} \sum_p \sum_i \oiint_{S_i^p} v_i^p(y) \frac{\cos \varphi_{xy}}{|x-y|^2} dS_{iy}^p$$

belongs to $H_1(\Omega)$, where $\Omega = R^3 - (\cup_p \cup_i S_i^p)$.

Theorem 4.3.2. For $1/2 \leq r < 1$, the exterior Neumann problem admits one and only on generalized solution.

In summary

For a closed connected surface $S \subset R^3$, the Prandtl operator $\boldsymbol{Pr} : H_{1/2} \rightarrow H_{-1/2}$ is Noetherian, is bounded, the function $u(x) := \frac{1}{4\pi} \oiint_S v(y) \frac{\cos \phi_{xy}}{|x-y|^2} dS_y$ is an element of $H_1(R^3 - S)$ and the exterior Neumann problem admits one and only on generalized solution.

The Garding type type inequality and compact operators

A variational representation of an operator in the form $B = A + K$, where A is a H_α - coercive operator with a compact disturbance K fullfills a coerciveness (Garding type type inequality) condition in the form, (AzA),

$$(Bu, v) \geq c \cdot \|u\|_\alpha \|v\|_\alpha - (Ku, v) \text{ or } (Bu, v) \geq c_1 \cdot \|u\|_\alpha^2 - c_2 \cdot \|u\|_\beta^2$$

with $H_\beta \subset H_\alpha$ compactly embedded.

The wavelet transform interpreted as a mathematical microscope

(HoM) 1.2: „*The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. ... Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane H of positions and details (where is which details generated?). ... Therefore, the parameter space H of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify*

$$b \leftrightarrow \textit{position}; \quad (a\Delta)^{-1} \leftrightarrow \textit{enlargement}; \quad g \leftrightarrow \textit{optics}“.$$

Optimal shift theorem for the heat equation

Extract from

L_∞-boundedness of the finite element
galerkin operator for parabolic problems

J. A. Nitsche, M. F. Wheeler, (NiJ)

The inverse of the fractional Stokes operators are compact, (TeR). Therefore, they have a discrete spectrum. This allow to omit the concept of a spectral measure dE_λ, when analyzing its mapping properties within Hilbert scales. In other words, In the respective Hilbert scale framework the heat and the Stokes operators show the same properties.

We introduce the Hilbert-scale $\{H_k | k \geq 0\}$ in the following way: Let $\{v_i, \lambda_i\}$ be the orthogonal set of eigen-pairs of the Laplacian, i.e.

$$-\Delta v_i = \lambda_i v_i \quad \text{in } \Omega$$
$$v_i = 0 \quad \text{in } \partial\Omega \; .$$

Any $z \in L_2(\Omega)$ admits the representation

$$z = \sum z_i v_i$$

with

$$z_i = (z, v_i) \; .$$

In addition Parseval's equation holds:

$$\|z\|^2 = \sum z_i^2 .$$

Now H_k is the subspace of functions such that $\|z\|_k^2 = \sum \lambda_i^k z_i^2$ is finite.

Remark: Since we have accepted only $z \in L_2$ the index $k$has to be non-negative.

For integers $k \leq 4$ – only these values will be relevant – the spaces H_k are connected with the unsual Sobolev-spaces W_2^m by:

$$H_0 = L_2, H_1 = \dot{W}_2^1, H_2 = \dot{W}_2^1 \cap W_2^2, H_3 = \{z | z \in H_2 \; and \; \Delta z \in H_1\}, H_4 = \{z | z \in H_2 \; and \; \Delta z \in H_2\} \, .$$

The H_k-norms are equivalent in these spaces to the corresponding W_2^k-norms. If $z = z(t)$ is an element of H_k for almost every $0 < t < T$ we will use the notation

$$\|z\|_{L_2(0,T;H_k)}^2 = \int_0^T \|z(t)\|_{H_k}^2 dt \; .$$

For the sake of completeness we will give the proof of the standard shift-theorem:

Theorem 3.1: Let the operator A be defined by

$$(*) \qquad Az = \dot{z} - \Delta u \quad \text{in } \Omega \times (0,T) \, ,$$
$$z = 0 \quad \text{on } \partial\Omega \times (0,T),$$
$$z_{t=0} = 0 \quad \text{in } \Omega \; .$$

Then A is a bijective mapping of $L_2(H_{k+2}) \cap \{z | \dot{z} \in L_2(H_k)\}$ to $L_2(H_k)$ and

$$(**) \qquad \|z\|_{L_2(H_{k+2})} \leq \|Az\|_{L_2(H_k)} \, .$$

Proof: Let z_i resp. f_i denote the "Fourier"-coefficients of z resp. Az with respect to $\{v_i\}$. Multiplication of (*) with v_i and integration over Ω leads to the uncoupled first order system

$$\dot{z}_i + \lambda_i z_i = f_i \quad \text{for } 0 < t < T$$
$$z_i(0) = 0$$

The solution of which is $z_i(t) = \int_0^t e^{-\lambda_i(t-s)} f_i(s)ds$. Application of Schwartz' inequality in the proper way gives

$$|z_i|^2 \leq \left\{\int_0^t e^{-\lambda_i(t-s)} f_i^2(s)ds\right\}\left\{\int_0^t e^{-\lambda_i(t-s)} ds\right\} \leq \lambda_i^{-1} \int_0^t e^{-\lambda_i(t-s)} f_i^2(s)ds$$

and further by interchanging the order of integration

$$\int_0^T |z_i|^2 dt \leq \lambda_i^{-1} \left[\int_0^T f_i^2(s)ds\right] \cdot \left[\int_s^T e^{-\lambda_i(t-s)}(s)dt\right] \leq \lambda_i^{-2} \int_0^T f_i^2(s)ds.$$

Because of the defintion of the H_k– resp. $L_2(H_k)$- norms (**) is proven.

The Riesz, the Calderón-Zygmund and the Schrödinger 2.0 operators

The Riesz transformations are the n-dimensional generalizations of the 1-dimensional Hilbert transformation. They arise when study the Neumann problem in upper half-plane. The Riesz transforms

$$R_k u = -ic_n p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y)dy \, , \quad c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$$

commutes with translations and homothesis, having nice properties relative to rotation, (PeB), (StE) (*).
The "rotation property" plays a key role in the context of the rotation group $SO(n)$ (*):

> let $m := m(x) := (m_1(x), \dots m_n(x))$ be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then it holds $m(\rho(x)) = \rho(m(x))$,
> i.e. $m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$.

The Calderón-Zygmund operators Λ with symbol $|\nu|$ and its inverse operator Λ^{-1} may be represented in the following forms, (EsG) 3.15, 3.17, 3.35, (LiI) p. 58 ff., (**)

$$(\Lambda u)(x) = (\sum_{k=1}^n R_k D_k u)(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^n p.v. \int_{-\infty}^{\infty} \sum_{k=1}^n \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy$$

$$= -\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta\Lambda^{-1})u(x)$$

$$(\Lambda^{-1}u)(x) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{u(y)}{|x-y|^{n-1}} dy \, , n \geq 2 \, .$$

Note: For space dimension $n = 1$ this is about $\Lambda = DH = PH$, where H denotes the Hilbert transformation and $D = P$ the Schrödinger momentum operator $P = -i\frac{d}{dx}$, (MeY) p. 5. In (BrK2) the Calderón-Zygmund operators Λ is proposed as alternative Schrödinger 2.0 momentum operator.

(*) If $j \neq j$ then $R_j R_k$ is a singular convolution operator. On the other hand, it holds $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. The following identities are valid

$$\|R_j\| = 1 \, , R_j^* = -R_j \, , \sum R_j^2 = -I \, , \sum \|R_j u\|^2 = \|u\|^2 \, , u \in L_2 \, .$$

Let $m := m(x) := (m_1(x), \dots m_n(x))$ be the vector of the Mikhlin multipliers of the Riesz operators and$\rho = \rho_{ik} \in SO(n)$, then

$m(\rho(x)) = \rho(m(x))$, whereby $m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$

and

$$m(\rho(x)) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(x\rho^{-1}(y)) + log\left|\frac{1}{x\rho^{-1}(y)}\right|) \frac{y}{|y|} d\sigma(y)$$

$$= c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(xy) + log\left|\frac{1}{xy}\right|) \frac{y}{|y|} d\sigma(y) \, .$$

(**) They are special Calderón-Zygmund (Pseudo Differential-, convolution-) operators $T(f) = S * F$ with a distribution S defined by symbols $m(\omega) \in C^\infty(R^n - \{0\})$ with the following properties, (MeY)

i) $m(\mu\omega) = m(\omega), \mu > 0$
ii) the mean of $m(\omega)$ on the unit sphere is zero
iii) it holds $m(\omega) = \frac{\omega_j}{|\omega|}$.

The Leray-Hopf and the Landau collision operators

Extract from (LeN), (LiP)

The Leray-Hopf projector (that projector is also called the Helmholtz-Weyl projector by some authors) is the following matrix valued Fourier multiplier, given by

$$\boldsymbol{P}(\xi) = Id - \frac{\xi \otimes \xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2})_{1 \le j,k \le n} \quad , \boldsymbol{P} = Id - R \otimes R =: Id - \boldsymbol{Q}.$$

We can also consider the $n \times n$ matrix of operators given by $\boldsymbol{Q} := R \otimes R = (R_j R_k)_{1 \le j,k \le n}$ sending the vector space of $L^2(R^n)$ vector fields into itself. The operator $\boldsymbol{Q}$ is selfadjoint and is a projection since $\sum_l R_l^2 = Id$ so that $\boldsymbol{Q}^2 = \sum_l (R_j R_l R_l R_k)_{j,k} = \boldsymbol{Q}$. As a result the (Leray-Hopf or Helmholtz-Weyl) operator

$$\boldsymbol{P} = Id - R \otimes R =: Id - \boldsymbol{Q} = Id - \frac{D \otimes D}{D^2} Id - \Delta^{-1}(\nabla \times \nabla)$$

is also an orthogonal projection; the operator is in fact the orthogonal projection onto the closed subspace of L^2 vecor fields with null divergence. When almost all collisions are grazing, the Landau (or Fokker-Planck) collision operator is defined by

$$Q(f,f) = \frac{\partial}{\partial v_i} \left\{ \int_{R^N} a_{ij}(v-w) \left[f(w) \frac{\partial f(v)}{\partial v_j} - f(v) \frac{\partial f(w)}{\partial w_j} \right] dw \right\}$$

The matrix $a_{jk}(z)$ is symmetric, non-negative, even z and is typically of the following form if $N = 3$,

$$a_{jk}(z) := \frac{a(z)}{|z|} \left\{ \delta_{jk} - \frac{z_j z_k}{|z|^2} \right\},$$

were a is even, smooth (for instance) and positive on R^n. The unknown function f corresponds at each time t to the density of particle at the point x with velocity v. The matrix and therefore the collision operator can be approximated by the linear Pseudo Differential Operator (PDO) of order zero with symbol

$$\frac{z}{|z|} \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} = \frac{z}{|z|} \boldsymbol{P}(z) := \frac{z}{|z|} [Id - \boldsymbol{Q}](z).$$

On Boltzmann and Landau equations

Some extracts from (LiP), (LiP1)

In (LiP) some properties of the solutions of the following kinetic equations are studied

$$(*) \qquad \frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f) \text{ for } t \ge 0, x \in R^N, v \in R^N,$$

where $N \ge 1$, f is a non-negative function and $Q(f,f)$ is a non-local, quadratic operator. Physically, such equations provide a mathematical model for statistical evolution of large number of particles interacting through "collisions". They are used for the description of a moderately rarefied gas or of plasma. The unkonwn function f corresponds at each time t to the density of particles at the point x with velocity v. If the operator Q were 0, (*) would simply mean that the particles do not interact and f would be constant along particles paths ($\dot{x} = v,\ \dot{v} = 0$). This conservation no longer holds if collisions occurs, in which case the rate of changes of f has to be specified.

All "compactness-stability" and existence results are shown under a certain conditions on V_0. This condition is satisfied in the case the Vlasov-Poisson system where $N = 3, V_0 = \frac{1}{|x|}$, as the condition holds in view of classical results on Riesz transforms, (LiP1).

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