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Notes on Magneto-Hydrodynamics, VIII
Nonlinear Wave Motion

K. O. Friedrichs and H. Kranzer

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AEC Computing and Applied Mathematics Center
Institute of Mathematical Sciences
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NOTES ON MAGNETO-HYDRODYNAMICS

VIII

NONLINEAR WAVE MOTION

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PREFACE

This note contains an expository analysis of the mathematical structure of the theory governing interaction of magnetic fields with conducting compressible fluids. One of its major aims is to emphasize the remarkably close parallelism between this theory and ordinary gas dynamics. It is shown that the basic equations governing magneto-hydrodynamics - or hydromagnetics for short - have essentially the same mathematical character as those governing gas dynamics and that, consequently, essentially the same mathematical methods that have proved successful in gas dynamics can be employed. This fact is illustrated by a detailed description of the hydromagnetic analogues of shocks, first discovered by Teller and de Hoffman, of sound waves, including the Alfvén waves, and of simple waves. In particular, a typical example is presented which serves to demonstrate that - as in gas dynamics - simple one-dimensional magneto-hydrodynamic flow problems can be solved with the aid of shocks and simple waves.

The work presented here originated in connection with a Seminar conducted in 1954 by H. Grad at New York University. A preliminary report appeared at the Los Alamos Scientific Laboratory in September 1954, reissued in March 1957 as Report LAMS 2105.

Some of the problems described in the preliminary report were subsequently treated extensively by Bazer and by Bazer and Ericson; their results are referred to and used in the present note. Also included is an appendix by K. von Hagenow, in which certain of the results on simple waves are shown to be deducible from the transformation properties of the differential equations.

The relativistic analogue of the problem treated here was formulated in a report by P. Reichel [12]; see also Zumino [17].

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Introduction. Physical assumptions.

The equations dealt with in this note represent a particular special case of the full set of magneto-hydrodynamic equations catalogued in MH-I. This specialization is achieved through a number of physical assumptions which serve to reduce the mathematical complexities inherent in these equations to almost manageable proportions.

Our first assumption is that there exists a scalar fluid pressure p which is a function of density and entropy. This function is assumed to have properties usually required in gas dynamics. Moreover, we shall assume in our discussion that heat conduction and viscosity may be disregarded. As a consequence various types of gases are excluded from treatment, such as gases in which the mean free path is not small compared with the significant dimensions of the problem.

We also shall assume that the flow velocity is small compared with the speed of light. Furthermore, we assume that the (mean) electric charge is negligible, so that the medium is essentially neutral, and that the displacement current may be neglected.*

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*It would not be necessary to make these assumptions. A strictly Lorentz-invariant counterpart of the treatment given in this report has been given by Reichel [12].

On the other hand, we do not assume that the flow velocity is small compared with the speed of sound. In other words, we assume the fluid to be compressible.

Moreover, we shall assume the electrical conductivity of the fluid to be infinite. This assumption enables us to express the electric field in terms of the magnetic field and the flow vector. Indeed, from equation (26) of MH-IV we find

$$E = B \times u. \tag{1}$$

It is not our intention to discuss in detail the significance of these various physical assumptions. Instead, we want to describe some of the mathematical consequences of the basic differential equations to be derived from these assumptions.

A remark might be made, though, about the assumption of compressibility. As is well known, compressibility need be taken into account only if the flow speed $|u|$ is comparable with the sound speed

$$a = c_{sound} = [dp/d\rho]^{1/2}. \tag{2}$$

Moreover, our approach will in general be worthwhile only if the Alfvén speed

$$c_{Alf} = \sqrt{\frac{B^2}{\mu\rho}}, \qquad \text{see Chapter II,} \tag{3}$$

is comparable with the sound speed, since otherwise hydromagnetic and compressibility effects could be separated. If c_{Alf} is much less than c_{sound}, which will frequently be the case, it might be possible to separate compressibility effects from hydromagnetic effects. We might then, for example, first treat compressibility effects in the absence of hydromagnetic effects, and then consider hydromagnetic effects under the assumption of a known fluid density.

1. Basic Equations

On the basis of all the assumptions described one can derive a system of differential equations which governs the flow of the fluid and the changes in the electromagnetic field. B is the magnetic field vector; the electric field vector is given by (1) in terms of the flow velocity u. In the absence of displacement current, the current per unit area, J, can be expressed in terms of the curl of B:

(1.1) $\mu J = \text{curl } B.$

The pressure $p = p(\rho, S)$ is a given function of density ρ and entropy S.

The two Maxwell equations which do not involve current and charge are retained; they may be written as

A_o $\text{div } B = 0$

and

A_1 $\dot{B} + \text{curl } (B \times u) = 0.$

The second term in A_1 is curl E by (1).

The force per unit volume which enters Newton's second law consists of the "Lorentz force" $-B \times J$ and the pressure gradient $-\nabla p$. Using (1.1), we may write this law in the form

A_2 $\rho\dot{u} + \rho(u\cdot\nabla)u + \nabla p + \mu^{-1}B \times \text{curl } B = 0.$

The continuity equation of fluid dynamics is retained:

A_3 $\dot{\rho} + \text{div } (\rho u) = 0.$

Finally, we add the law that the entropy per unit mass, S, is carried unchanged by the particles:

$A_4 \qquad S + (u\cdot\nabla)S = 0.$

These equations, essentially formulated by Lundquist, 1952, [5], and sometimes referred to as "Lundquist equations", will be the basis of our discussion. They include as a special case the two-dimensional problems treated in MH-VII.

It would be sufficient to require equation A_o to hold only at an initial time; as is well known, it then follows from A_1 that this equation is satisfied at all times.

Equations A_1 to A_4 are a system of eight non-linear partial differential equations of first order for the eight quantities B_x, B_y, B_z, u_x, u_y, u_z, ρ, and S. The first fact we emphasize is that <u>these equations are hyperbolic</u>. Specifically, they belong to the special class of <u>symmetric hyperbolic equations</u>, which is particularly well understood mathematically; see [14]. These equations share this property with Maxwell's equations, with the equations of elasticity, and with the equations of gas dynamics. Moreover, they share with the gas-dynamical equations the further property of being nonlinear with coefficients which involve the dependent but not the independent variables. It is because of the latter property, in addition to the hyperbolic character, that the same methods can be applied to the equations A_1 to A_4 that have proved successful in gas dynamics.

The fact that these equations of magneto-hydrodynamics belong to a class of equations which are most consistent in their mathematical character supports the confidence in the physical consistency of the various assumptions from which they were derived.

As is well known, physical processes governed by hyperbolic equations have the property that disturbances are propagated with finite speed. Thus, in ordinary compressible fluids, disturbances travel with the speed of sound relative to the motion of the fluid. Hence magneto-hydrodynamic disturbances also travel with finite speed. However, in contrast to gas dynamics, there are three "sound speeds." Accordingly, there are three modes of propagation in each direction. Moreover, these speeds depend on the direction; specifically, on the angle between the direction of propagation and the direction of the magnetic field. These remarkable facts were first discovered by Herlofson [3] and by van de Hulst [4], in connection with sinusoidal linearized wave motion. Our first task will be to find these speeds.

2. Characteristic Manifolds and Propagation of Disturbances

Characteristic manifolds - three dimensional manifolds in (x,y,z,t)-space - associated with a differential equation may be defined in many different ways; cf. [14,15]. Instead of giving a precise definition, it is sufficient in the present context to say that solutions of the differential equation may possess "small" discontinuities only on certain manifolds, and that such manifolds are called characteristic. We may consider such a manifold as being swept out by surfaces $\mathcal{S} = \mathcal{S}(t)$ in (x,y,z)-space; the motion of these surfaces will then also be called "characteristic". In a process described by a solution of the differential equation, therefore, a small discontinuity or "disturbance" present on a surface $\mathcal{S}(t_o)$ at an initial time t_o may at later times be present only on surfaces $\mathcal{S}(t)$ which move characteristically. Such a moving discontinuity will be called a "disturbance wave" or simply a "wave".

We introduce the normal vector n of unit length at each point of the surface $\mathcal{S}(t)$ and characterize the motion of the surface $\mathcal{S}(t)$ by its velocity c_{ch} in the normal direction at each of its points. It is convenient to introduce the normal component

$$(2.1) \qquad u_n = n \cdot u$$

of the flow velocity at this point and to write the characteristic velocity c_{ch} there as the sum

$$(2.2) \qquad c_{ch} = u_n \pm c;$$

by convention we always choose $c \geq 0$.

Thus $\pm c$ is the normal component of the characteristic velocity relative to the flow velocity.

In order to find the possible values of c one may first set up the relations between the possible discontinuities δB, δu, $\delta \rho$, δS of the quantities B, u, ρ, S on the surface $\mathcal{S}(t)$. Using the formalism of the theory of characteristics, cf. [14], the following relations are found:

B_1 $$\mp c\delta B + B\delta u_n - B_n \delta u = 0,$$

B_2 $$\mp \rho c\delta u + a^2 n\delta\rho + \mu^{-1} n(B\cdot\delta B) - \mu^{-1} B_n \delta B = 0,$$

B_3 $$\mp c\delta\rho + \rho\delta u_n = 0,$$

B_4 $$\mp c\delta S = 0.$$

Here a is the speed of sound, given by (2); furthermore

(2.3) $$\delta u_n = n\cdot\delta u,$$

and

(2.4) $$B_n = n\cdot B$$

is the normal component of the magnetic field.

The determinant of this system of eight homogeneous equations for the eight quantities δB, δu, $\delta\rho$, δS is found to be

(2.5) $$\det(B) = \rho c^2(\rho c^2 - B_n^2/\mu)\left\{\rho c^4 - (\rho a^2 + B^2/\mu)c^2 + a^2 B_n^2/\mu\right\}.$$

The characteristic velocities $\mp c$ are obviously the roots of the equation

(2.6) $$\det(B) = 0.$$

All eight roots of this equation are seen to be real, in accordance with the fact that the differential equations are hyperbolic.

3. Fast and Slow Disturbance Waves

We shall first discuss the roots of the last factor of det (B). The condition that this factor vanishes can be written in the form

$$(3.1) \qquad c^2(\rho c^2 - B^2/\mu) = a^2(\rho c^2 - B_n^2/\mu)$$

or in the form

$$(3.2) \qquad (c^2 - a^2)(\rho c^2 - B_n^2/\mu) = c^2(B^2/\mu - B_n^2/\mu).$$

The larger and the smaller of the roots $c > 0$ of this equation will be denoted respectively by c_{fast} and c_{slow}.

From equation (3.2) one immediately deduces the inequalities

$$(3.3) \qquad c_{slow} \leq a \leq c_{fast}$$

and

$$(3.4) \qquad c_{slow} \leq b_n \leq c_{fast}.$$

Here

$$(3.5) \qquad b_n = [B_n^2/\mu\rho]^{1/2}$$

is the Alfven velocity, cf. (1), with the magnetic field vector replaced by its normal component B_n. The sound speed a is as given by (2). An equality sign can hold in relations (3.3) and (3.4) only if $B = B_n n$, so that the right member of (3.2) vanishes. In this case one of the two speeds equals a, the other equals b_n.

The possible disturbances, i.e., the solutions of equations B_1 to B_4 associated with $c = c_{slow}$ or $c = c_{fast}$, are found to be

(3.6) $$\delta B = k\rho c^2(B - B_n n),$$

$$\delta u = \mp kc(\mu^{-1}B_n B - \rho c^2 n),$$

$$\delta\rho = k\rho(\rho c^2 - B_n^2/\mu),$$

$$\delta S = 0.$$

Here k is any number $\neq 0$.

It is to be noted that δB has a tangential direction so that

(3.7) $$\delta B_n = n\cdot\delta B = 0.$$

The disturbance $\delta\rho$ vanishes when $c = b_n$. The disturbance δu may be written in the form

(3.8) $$\delta u = \mp(\mu\rho c)^{-1}B_n\delta B \pm \rho^{-1}c(\delta\rho)n,$$

which is on occasion useful. We also note down the relation

(3.9) $$\delta(p + B^2/2\mu) = a^2\delta\rho + \mu^{-1}B\cdot\delta B = k\rho c^2(\rho c^2 - B_n^2/\mu),$$

which follows from (3.6) and will prove useful later on.

Particular attention should be paid to the cases where the normal n is parallel or perpendicular to the magnetic field B.

In the first case, $B = B_n n$, one of the roots c agrees with the sound speed a. Unless $a = b_n = b$, formulas (3.6) remain valid for this root. They must in any case be modified for the other root $c = b$, since the latter root agrees with a root of another factor of det (B).

In the second case, $B_n = 0$, the fast speed is given by the noteworthy formula

$$(3.10) \qquad c_{fast} = \left[\frac{dp}{d\rho} + B^2/\rho\mu\right]^{1/2} = \left[a^2 + b^2\right]^{1/2}$$

Formulas (3.6) remain valid in this case. The other root

$$(3.11) \qquad c_{slow} = 0,$$

however, agrees with the root of another factor of det (B). Hence for this root formulas (3.6) must again be modified.

The needed modification of formulas (3.6) in these two cases will be described later (cf. (4.6) and (5.4)).

In order to illustrate disturbances of the type treated in this section, consider segments of plane wave fronts, fast and slow, which travel in the direction of the normal n after having passed through the origin at the time $t = 0$. At a time $t > 0$ these fronts pass through the points $c\,t\,n$ with $c = c_{fast}$ and $c = c_{slow}$. The locus of these points is shown in Figure 1. Also of significance is the envelope of these fronts, which is shown in Figure 1A. This envelope gives the position at the time $t > 0$ of the expanding wave front resulting from a point disturbance at the origin at $t = 0$. (That is, it represents the limiting form for large t of the wave fronts resulting from an initial disturbance in a region of finite extent--for small t such wave fronts may look quite different.) In these figures we have assumed $c_{Alf} < c_{sound}$, so that the slow speed c_{slow} agrees with the Alfvén speed for $B \parallel n$. This seems to be the more frequent situation. We also have drawn wave fronts of a third, intermediate, type which we are going to describe now.

4. Transverse Waves and Alfvén Waves

An intermediate wave belongs to the root

(4.1) $$c = b_n = [B_n^2/\mu\rho]^{1/2}$$

of equation (2.5). As seen from (3.4), the speed b_n lies between c_{slow} and c_{fast} except if $B \parallel n$ or $B \perp n$. The possible discontinuities associated with this wave, given as the solutions of equations (B), are

(4.2) $$\delta B = \mp k\rho c B \times n,$$

$$\delta u = k\mu^{-1} B_n B \times n,$$

$$\delta\rho = \delta S = 0,$$

with an arbitrary constant k. In this case, therefore, the disturbances are tangential to the wave front and perpendicular to the magnetic field. The relation

(4.3) $$\delta B^2 = 2B \cdot \delta B = 0,$$

implied by (4.2), shows that the magnetic field undergoes a rotation. This intermediate type of disturbance wave will also be referred to as a "transverse wave".

At this place we may interpose a remark about the Alfvén wave in an incompressible fluid. The conditions on the possible disturbances in this case are obtained from conditions (B) by setting $a^2\delta\rho = \delta p$ in B_2 and $\delta\rho = 0$ in B_3. In addition to the double root $c = 0$, one finds that $c = b_n$ is a double root. The corresponding disturbances are

$$(4.4) \qquad \delta B = kB_n n^*,$$

$$\delta u = \mp kb_n n^*$$

$$\delta p = -k\mu^{-1}B_n(B\cdot n^*),$$

$$\delta S = 0,$$

where n^* is an arbitrary unit vector perpendicular to n, and k is an arbitrary number. Evidently, these possible disturbances form a two-dimensional manifold, in accordance with the fact that $c = b_n$ is a double root. Relations (4.4) imply the important relation

$$(4.5) \qquad \delta(p + B^2/2\mu) = \delta p + \mu^{-1}B\cdot\delta B = 0,$$

which expresses the fact that the sum of fluid pressure p and "magnetic pressure" $B^2/2\mu$ is continuous across the surface $\mathcal{S}(t)$. (The notion of magnetic pressure will be discussed in Section 6.)

One might say that the Alfvén wave* results if the sound speed a and hence the fast disturbance speed c_{fast} become infinite while the slow wave speed c_{slow} coalesces with the Alfven wave speed $c_{Alf} = b_n$.

For a compressible fluid, we have already mentioned an analogous case in which the slow (or possibly the fast) disturbance speed coalesces with the Alfvén wave speed, namely, the case of a disturbance wave traveling parallel to the magnetic field:

$$B = B_n n.$$

*We use this term for disturbance waves in any direction in an incompressible medium although Alfvén described only waves traveling in the direction of B.

Again $c^2 = b_n^2 = b^2$ is a double root of equation (2.6) and there again exists a two-parameter family of disturbances, namely

$$(4.6) \qquad \delta B = kB_n n^*,$$

$$\delta u = \mp kbn^*,$$

$$\delta\rho = \delta S = 0.$$

Here n^* and k have the same meanings as in equation (4.4). Once more we have

$$(4.7) \qquad \delta(p + B^2/2\mu) = \mu^{-1}B \cdot \delta B = 0.$$

The disturbances (4.6) represent, in a real sense, the transition between a slow (or fast) wave and a transverse wave. Indeed, it is easy to see that they are obtained as the limit of equations (4.2) as B approaches the normal direction, and also as the limit of equations (3.6) as $B \to B_n n$ and $c^2 \to b_n^2 \to b^2$.

The fast, slow, and intermediate waves described in Sections 3 and 4 represent the three modes of wave propagation referred to at the end of Section 1. Since each type of wave may move in the direction of n or of -n, they correspond to six roots c. The propagation of disturbances associated with the remaining roots c = 0 will not be referred to as wave motion.

5. Contact Disturbances

About the remaining root $c = 0$ only a few remarks need be made. This root has the multiplicity 2 unless $B_n = 0$, in which case the multiplicity is 6. We confine ourselves for the present to the case $B_n \neq 0$. The relations (B) then possess only the solutions

$$(5.1) \qquad \delta B = kn$$

$$\delta u = 0, \qquad \delta \rho = 0,$$

$$\delta S = k_1,$$

with arbitrary k and k_1. We may just as well set $k = 0$, or, equivalently,

$$(5.2) \qquad \delta B_n = 0.$$

It is consistent to do so, for, as could be shown, this condition is satisfied on every surface $\mathcal{S}(t)$ if it is satisfied initially on $\mathcal{S}(t_o)$.

The remaining possibility of an entropy disturbance corresponds to a contact discontinuity. Since the discontinuity is small, we prefer to use the term "contact disturbance".

In striking contrast to the situation in gas dynamics a contact discontinuity in a hydromagnetic fluid does not permit a discontinuity in the tangential component of the velocity (provided $B_n \neq 0$). This remarkable fact will play a considerable role in the discussions of wave motions given in Section 13.

Suppose now that B *is* perpendicular to n:

$$B_n = 0.$$

Then we have

$$(5.3) \qquad c_{slow}^2 = b_n^2 = 0,$$

so that c = 0 is a sextuple root of (2.6). The corresponding six-parameter family of possible disturbances may be written as follows:

$$(5.4) \qquad \begin{aligned} \delta B &= K, \\ \delta u &= kn^*, \\ \delta\rho &= -\mu^{-1}a^{-2}K\cdot B, \\ \delta S &= k_1. \end{aligned}$$

Here K is an arbitrary vector, n^* an arbitrary unit vector perpendicular to n, and k and k_1 arbitrary numbers. In other words, all the disturbances are arbitrary, except that they must satisfy

$$(5.5) \qquad \delta(p + B^2/2\mu) = 0.$$

Equations (5.4) do not form as clear-cut a "transition case" as equations (4.6), since performing the appropriate limiting processes on equations (3.6), (4.2), or (5.1) will lead only to special cases of (5.4).

6. Conservation Laws

In this section we shall discuss the conservation form of the Lundquist equations, partly as preparation for the discussion of shocks.

It is customary to say that a system of partial differential equations has "conservation form" if each equation consists of the sum of derivatives of functions of the unknown quantities. The reason for this expression is that the laws of conservation of mass, momentum and energy are of this form. On the other hand the possibility of writing the equations in this form is the condition for the possibility of setting up shock relations.

Equation A_o (see Section 1) evidently has conservation form,

$$C_o \qquad \nabla \cdot B = 0.$$

This same is true of equation A_1, but we prefer to write it in the form

$$C_1 \qquad \dot{B} + \nabla \cdot uB - \nabla \cdot Bu = 0.$$

Here, and correspondingly in the following, the inner product in $\nabla \cdot uB$ involves only ∇ and u while the differentiation ∇ applies to the product uB. Thus, the x-component of $\nabla \cdot uB$ is

$$(\nabla \cdot uB)_x = \frac{\partial}{\partial x}(u_x B_x) + \frac{\partial}{\partial y}(u_y B_x) + \frac{\partial}{\partial z}(u_z B_x).$$

Similarly, the x-component of $\nabla \cdot Bu$ is

$$(\nabla \cdot Bu)_x = \frac{\partial}{\partial x}(B_x u_x) + \frac{\partial}{\partial y}(B_y u_x) + \frac{\partial}{\partial z}(B_z u_x).$$

Equation A_2 can be written in the form

$$C_2 \qquad \frac{\partial}{\partial t}(\rho u) + \nabla\circ u(\rho u) + \nabla p + \nabla(B^2/2\mu) - \nabla\circ B\mu^{-1}B = 0,$$

which expresses the law of conservation of momentum. Suppose the magnetic field B is such that the term $\nabla\circ B\mu^{-1}B$ vanishes; this will be the case under various symmetry conditions, cf. [9]. Then the conservation law C_2 has essentially the same form as in gas dynamics except that the expression $p + B^2/2\mu$ takes the place of the pressure. It is primarily because of this fact that the term $B^2/2\mu$ is referred to as "magnetic pressure".

The law of conservation of mass, A_3, is given in conservation form

$$C_3 \qquad \mathring{\rho} + \nabla\circ(\rho u) = 0.$$

Equation A_4, however, which describes the transport of entropy, is to be replaced by the law of conservation of energy, which assumes the form

$$C_4 \qquad \frac{\partial}{\partial t}(\tfrac{1}{2}\rho u^2 + \rho e + B^2/2\mu) + \nabla\circ u(\tfrac{1}{2}\rho u^2 + \rho e + p + B^2/\mu) - \nabla\circ\mu^{-1}B(B\circ u) = 0.$$

Here e is the internal energy per unit mass of the fluid, which may be considered as a function of density and entropy and is characterized by the relation

$$(6.1) \qquad de = TdS - pd(\rho^{-1}),$$

in which T is the temperature. It is to be noted that the expression $B^2/2\mu$ in the first term of C_4 plays the role of "magnetic energy per unit volume", while the expression B^2/μ in the second term of C_4 plays the role of "magnetic enthalpy per unit volume". It should also be noted that the expression

$$uB^2/\mu - \mu^{-1}B(B\cdot u)$$

occurring in C_4 may be written in the form

$$(6.2) \qquad \mu^{-1}[uB^2 - B(B\cdot u)] = \mu^{-1}[E \times B] = E \times H$$

by virtue of formula (1). This term is thus recognized as Poynting's energy flux per unit area.

7. Shocks

As is well known, continuous gas dynamical motions will break down at some time if they involve a compression. The same must be expected to happen in hydro-magnetic motions if the compressibility of the fluid cannot be neglected. Mathematically speaking, the solution of the differential equations ceases to exist beyond the time of breakdown. Physically speaking, the phenomenon is no longer governed by the differential equations from that time on. Actually, discontinuities, i.e., shocks, will appear.

As in gas dynamics, one assumes that the quantities on both sides of a shock front are governed by the laws of conservation. The shock conservation laws can then be derived from the conservation form (C) of the differential equations by a formalism which is quite analogous to the formalism by which the equations (B) for the disturbances are derived from the original differential equations (A). We denote by n the normal vector at any point of the shock front and by

$$[Q] = Q_1 - Q_o$$

the jump of any quantity Q across the shock front; here Q_1 is the value on the side toward which the normal n points and Q_o is the value on the other side. Further, we denote by U the velocity of the shock front in the normal direction. The recipe then requires one to replace the symbol $\frac{\partial}{\partial t}$ in equations (C) by $-U[\ldots]$ and the symbol ∇ by $n[\ldots]$. The result is the following set of equations.

$D_0 \qquad [B_n] = 0,$

$D_1 \qquad [(u_n - U)B - B_n u] = 0,$

$D_2 \qquad [(u_n - U)\rho u + (p + B^2/2\mu)n - \mu^{-1}B_n B] = 0,$

$D_3 \qquad [(u_n - U)\rho] = 0,$

$D_4 \qquad [(u_n - U)(\frac{1}{2}\rho u^2 + \rho e + B^2/2\mu) + u_n(p + B^2/2\mu) - \mu^{-1}B_n(B\cdot U)]=0.$

These are the relations which connect the values of the quantities B, u, ρ, S on one side of a shock front, or, more generally, of a discontinuity surface, with the values of these quantities on the other side and with the speed U of the surface.

These relations were derived by de Hoffmann and Teller [2] in 1950. These authors set up the conservation laws for shocks directly without relating them to differential equations. Also, they derived the equations in a Lorentz invariant form and only afterwards derived equations (D) as the non-relativistic approximation. The direct non-relativistic derivation was given by Lüst [7].

The jump conditions (D) are frequently supplemented by the statement that there is a "sheet current" flowing along the discontinuity surface and that the value J^* of this current per unit width is given by the relation

$(*) \qquad \mu J^* = n \times [B]$

in accordance with relation (1.1). Although this statement is of great significance for the description of the physical phenomena involved, it need not be taken into account in the mathematical analysis of the discontinuity condition.

We will speak of a shock if fluid crosses the front, $u_n - U \neq 0$. In that case we assume the direction of the normal vector n so chosen that fluid crosses in the direction of the normal:

$$(7.1) \qquad u_n - U > 0.$$

If $u_n - U = 0$, so that no fluid crosses, we speak of a contact discontinuity.

We shall present an analysis of the possible types of shocks and contact discontinuities which is completely analogous to the analysis of disturbance waves given in the preceding sections. Before doing this, however, we make an important observation of a general nature.

Gas dynamical shocks involve a rise in pressure and density, since the entropy increases across a shock. For hydromagnetic shocks we may state similarly: If the entropy increases across a shock front, pressure and density also increase. (The qualification is necessary in this case, for, as we shall see, there are shocks which do not involve changes in entropy, density, and pressure at all.) The proof of the statement follows from the identity

$$(7.2) \qquad [e + \tilde{p}\rho^{-1}] = -[\rho^{-1}][B]^2/4\mu^2,$$

in which $\tilde{p}$ is the mean value

$$(7.3) \qquad \tilde{p} = \tfrac{1}{2}(p_0 + p_1).$$

This remarkable identity, first given by Lüst, could be derived by forming a linear combination of relations (D) which is similar to the linear combination which expresses the entropy relation A_4 in terms of the relations (C), cf. Section 6. The left member of (7.2) is the "Hugoniot function", which vanishes for gas dynamical shocks. The right member involves the drop $-[\rho^{-1}]$ in specific volume ρ^{-1} and the square of the jump of the magnetic field. From the known properties of the Hugoniot function (cf. [12]), one can show that it has the same sign as $-[\rho^{-1}]$ only if it has the same sign as [S]. The statement made above then follows.

From relation (7.2) one can also derive the fact that <u>the increase of entropy across a shock is of the third order in</u> ρ <u>and</u> B.

In order to establish the analogy between the shock relations (D) and relations (B) characterizing disturbances it is convenient to introduce the notion of mean value

$$\tilde{Q} = \frac{1}{2}(Q_o + Q_1)$$

of any quantity Q and to use the formula

$$[PQ] = \tilde{P}[Q] + [P]\tilde{Q}.$$

It is also convenient to introduce the specific volume

$$(7.4) \qquad \tau = \rho^{-1}$$

instead of ρ, and the flux

$$(7.5) \qquad m = \rho(u_n - U)$$

instead of u_n. Note that, by D_3, the flux is the same on both sides of the shock front. Also, D_o permits us to take B_n as a constant. Relations D_1 to D_3 can now be written as

$$E_1 \qquad m\tilde{\tau}[B] + \tilde{B}[u_n] - B_n[u] = 0,$$

$$E_2 \qquad m[u] + [p]n + \mu^{-1}n\breve{B}\cdot[B] - \mu^{-1}B_n[B] = 0,$$

$$E_3 \qquad m[\tau] - [u_n] = 0.$$

These equations evidently correspond precisely to equations B_1 to B_3 if one lets

$$m, \tilde{\tau}, -[\tau]^{-1}[p], \tilde{B} \text{ correspond to } \mp\rho c, \rho^{-1}, \rho^2 a^2, B$$

and lets

$$[u], [\tau], [B] \text{ correspond to } \delta u, -\rho^{-2}\delta\rho, \delta B.$$

The analogue of relation B_4 is relation (7.2); it may be disregarded in the present context.

From this analogy, or by direct computation, one finds that the determinant of the system E_1 to E_3 is

$$(7.6) \qquad \det(E) = \tilde{\tau}^2 m(\tilde{\tau} m^2 - B_n^2/\mu)(\tilde{\tau} m^4 + (\tilde{\tau}[\tau]^{-1}[p] - \tilde{B}^2/\mu)m^2 - [\tau]^{-1}[p] B_n^2/\mu).$$

The equation

$$(7.7) \qquad \det(E) = 0$$

is an equation for the flux m, but it may just as well be considered an equation for the shock velocity

$$(7.8) \qquad U = \tilde{u}_n - m\tilde{\tau},$$

cf. (7.5).

It is clear from (7.6) that there are fast, slow, and intermediate shocks and that the relationship between them is the same as that between the corresponding disturbance waves.

8. Fast and Slow Shocks

The flux m of a fast or a slow shock satisfies the equation

(8.1) $$m^2(\tilde{\tau} m^2 - \tilde{B}^2/\mu) = -[\tau]^{-1}[p](\tilde{\tau} m^2 - B_n^2/\mu),$$

which expresses the condition that the last factor of det (E) vanishes; cf. (3.1). In analogy with (3.3) and (3.4) we have

(8.2) $$m^2_{slow} \leq -[\tau]^{-1}[p] \leq m^2_{fast}$$

and

(8.3) $$m^2_{slow} \leq B_n^2/\mu\tilde{\tau} \leq m^2_{fast}.$$

In analogy with (3.6), the relations between the possible jumps across the shock front are

(8.4) $$[B] = k_1 m^2(\tilde{B} - B_n n),$$

(8.5) $$[u] = k_1 m(\mu^{-1} B_n \tilde{B} - \tilde{\tau} m^2 n),$$

(8.6) $$[\tau] = -k_1(\tilde{\tau} m^2 - B_n^2/\mu),$$

with an arbitrary constant k_1 (instead of $\tilde{\tau} k$).

The cases of a "parallel" shock, $\tilde{B} = B_n n$, and of a "perpendicular" shock, $B_n = 0$, were discussed in great detail by de Hoffmann and Teller [2]. Here we only mention that for the fast perpendicular shock, as well as for the "non-Alfvén" parallel shock, the jump of the velocity u is in the normal direction, as may be seen from (8.5).* Therefore, if in these cases the shock is observed from

*The slow perpendicular shock, for which m = 0, and the "Alfvén" parallel shock, for which $m^2 = B^2/\mu\tilde{\tau}$, are not governed by equations (8.4)-(8.6), but instead by equations analogous to (5.4) and (4.6) respectively.

an appropriate frame, the flow velocity will be normal on both sides.

For "oblique" shocks, those for which $\tilde{B} \neq B_n n \neq 0$, a frame can be so introduced that the flow velocity u is parallel to the magnetic field B on both sides of the front. This fact, emphasized by de Hoffmann and Teller, can be read off from relation D_1. One need only choose the frame such that $u = B_n^{-1}(u_n - U)B$ on one side; D_1 then implies that this relation also holds on the other side. In view of (1) the electric field E vanishes in this frame on both sides of the front.

Using formulas (8.4) to (8.6), we can easily describe the jump of the absolute value of the magnetic field across the shock front. From formula (8.4), we have

$$[B^2] = 2\tilde{B}\cdot[B] = k_1 m^2(\tilde{B}^2 - B_n^2).$$

Hence, using formula (8.6),

$$(8.7) \qquad [B^2] = -m^2[\tau](\tilde{\tau}m^2 - B_n^2/\mu)^{-1}(\tilde{B}^2 - B_n^2).$$

Since we have assumed the normal n to point in the direction in which the fluid crosses the shock front, the statement made preceding formula (7.2) implies that $[\tau] < 0$. In view of (8.3), we are therefore able to state: The magnetic field strength $|B|$ rises across a fast shock and drops across a slow shock. By virtue of D_0, the same is true of the magnitude of the tangential component of B. This fact will prove particularly significant in connection with specific flow problems.

Formula (8.4) implies that the tangential component of the magnetic field jumps in its own direction; i.e., the jump [B] is parallel to the tangential component of B_0 and has the same sense as the tangential component of $\tilde{B}$. Thus across a fast shock the tangential component of the magnetic field retains its direction and increases its magnitude, while across a slow shock this component may retain or reverse its direction and must decrease in magnitude.

The analysis of shocks as given here seems appropriate if one desires to obtain a quick survey of the possible types of shocks and to derive some of their simple properties. In this analysis we have extensively used mean values, involving values on both sides of the front. In a numerical problem, however, the more important questions are usually those which refer to the behavior of the various quantities on each side separately. To answer such questions, the analysis of Ericson and Bazer [11] may be more appropriate. We summarize below various results concerning such problems, the proofs of which may be found in [11].

One question that may arise is whether or not a tangential component of the magnetic field may be produced through a shock if it was absent ahead of it, or whether or not it may be wiped out if it was present ahead of it. Shocks through which this happens may be called "complete switch-on" or "switch-off" shocks. Clearly, a complete switch-on shock must be fast or a complete switch-off shock must be slow.

It can be shown that complete switch-on shocks exist only if the Alfvén speed is supersonic ahead of the shock; and even then only if the shock strength (e.g. measured by the pressure ratio p_1/p_0) lies below a critical value. If one lets the shock strength increase, the gain in tangential component of B first increases and then decreases, becoming zero when the critical strength is reached. On the other hand, complete switch-off shocks always exist if the Alfven speed behind the shock front is subsonic; if this speed is supersonic there, they exist only if the shock strength exceeds a critical value.

Gas dynamical shocks have the property that the normal component of the flow velocity relative to the shock velocity, $u_n - U$, is supersonic ahead of the shock front, i.e. on the side (0) from where the fluid comes, and subsonic behind it. For a hydromagnetic shock we may state: The normal flow velocity relative to a fast shock is greater than the fast disturbance speed ahead of the front and less than it but greater than (or equal to) the transverse disturbance speed behind it:

$$u_n - U > c_{fast} \qquad \text{on side (0), ahead,}$$

$$b_n \leq u_n - U < c_{fast\,(0)} \qquad \text{on side (1), behind.}$$

The equality sign holds only for complete switch-on shocks. The normal flow velocity relative to a slow shock is greater than the slow disturbance speed ahead of the front, but less than (or equal to) the transverse speed on both sides of the front:

$$u_n - U \leq b_n \qquad \text{on side (1), behind,}$$

$$b_n \geq u_n - U > c_{slow} \qquad \text{on side (0), ahead.}$$

The second equality sign holds only for complete switch-off shocks.

These facts may be illustrated in Figures 2 and 2A, in which the shock is assumed to be stationary, $U = 0$. Only those disturbance motions are shown that travel against the flow.

Just as in gas dynamics, a fast or a slow hydromagnetic shock is determined by prescribing all quantities ahead of it and the pressure $p > p^{(o)}$ or the relative velocity $v_n = u_n - U < v_n^{(o)}$ behind it. It is, however, not possible to prescribe arbitrarily the tangential component of the magnetic field behind the shock front, and where it is possible to prescribe it at all, the shock may not be uniquely determined. This fact is clearly indicated by the remarks about switch-on and switch-off shocks made above.

An important insight into the connections between the various types of shocks may be obtained by consideration of the family of slow shocks starting from a given state (with a non-zero tangential component of B) ahead of the front. As the shock strength increases, the tangential magnetic field component behind the front first decreases to zero, then increases with direction reversed, approaching in the limit the negative of its value ahead. In other words, a continuous transition may be made from a weak slow shock, such as is depicted in Figure 2A, through a complete switch-off shock to a special case of the "transverse" shock considered in the following section. This remarkable fact was first noticed by Ericson and Bazer [11].

9. Transverse Shocks. Contact Discontinuities

The root

(9.1) $\quad m_{tr} = (B_n^2/\mu\,\tilde{\tau})^{1/2}$

of equation det (E) = 0, cf. (7.7), is associated with a type of shock which will be called transverse.* Such transverse shocks correspond to the transverse waves discussed in Section 4; they are sometimes called "Alfvén shocks." In analogy to expressions (4.2) we find the expressions

(9.2) $\quad [B] = km\tilde{B} \times n,$

(9.3) $\quad [u] = k\mu^{-1}B_n\tilde{B} \times n,$

(9.4) $\quad [\tau] = 0,$

(9.5) $\quad [S] = 0.$

The last relation is derived from (7.2) and $[\tau] = 0$. Also, $[u_n] = 0$ holds, as seen from E_3. Thus, the only quantities that jump across a transverse shock are the tangential components of the magnetic field and of the velocity.

Relation (9.2) implies relation

(9.6) $\quad [B^2] = 0.$

In other words, the strength of the magnetic field is unchanged across a transverse shock. The magnetic field therefore rotates in the plane of the shock and the flow velocity undergoes a tangential change parallel to [B]. All other quantities remain continuous. The possibility of this occurrence necessitated the qualification made in the statement preceding formula (7.2).

*A transverse discontinuity is called a "shock" since fluid crosses the surface; in other respects it is closer to a contact discontinuity. See [16].

In general, the angle through which the tangential magnetic field $B_n = B - B_n n$ rotates in crossing a transverse shock is arbitrary. An interesting special case is that for which the rotation angle is 180°, so that the tangential magnetic field behind the shock is directed oppositely to that in front. As pointed out in Section 8, this special case also occurs as the limit of a slow shock when the shock strength approaches its maximum admissible value. Thus transverse shocks are continuously connected with the slow shocks considered earlier. This connection is more clearly shown in Figure 3.

Returning to the general transverse shock, it is obvious from (9.3) that a frame can be introduced such that the flow is parallel to the front on both sides. Therefore, the term "transverse" shock, introduced by de Hoffmann and Teller, would seem appropriate.

The possibility of such a transverse shock is, of course, to be understood as a mathematical possibility, referring to the existence of solutions of the equations (E) for the root (9.1). Whether or not such shocks are possible in nature is another question. In fact, it would seem that they are possible only under unusual circumstances; cf. Section 12.

The root m = 0 of equation (7.7) corresponds to a discontinuity of the normal component B_n of B. However, this possibility is excluded by condition D_o; cf. the arguments in Section 5, which could be carried over here.

A second root m = 0 would have occurred if we had not omitted relation D_4 in changing the system (D) over into the system (E).

The corresponding discontinuity would be a proper contact discontinuity involving no flow across the front, i.e.

$$u_n - U = 0.$$

In case $B_n \neq 0$ relations (D) then imply

$$[u] = 0, \quad [B] = 0, \quad [p] = 0.$$

The latter relation does not require $[\rho] = 0$, since $[S] \neq 0$ is compatible with D_4. Because of $[B] = 0$ we have a purely gas dynamical contact discontinuity, in fact, since $[u] = 0$, a special one.

In case $B_n = 0$, on the other hand, we can only conclude

$$[p + B^2/2\mu] = 0,$$

while the tangential components of u and of B may undergo any jumps. In fact, the contact discontinuities corresponding to both roots m = 0 coalesce in the case $B_n = 0$.

A contact discontinuity which involves a jump in the tangential flow velocity will be called a "shear flow discontinuity." It is remarkable that in a conducting fluid no shear flow discontinuity can be maintained if the magnetic field has a normal component $B_n \neq 0$ at the front. This fact will be the starting point for our discussion of special flow problems in Section 13.

10. Simple Waves*

Naturally, the hydro-magnetic flows most easily accessible to treatment are the one-dimensional flows. These are characterized by the condition that all quantities depend only on one space variable x say, in addition to the time, and hence are constant on each (y,z)-plane at each time. No restriction need be imposed as to the presence or absence of the y and z-components of the vectors B and u.

The problem of one-dimensional waves in a compressible conducting medium is certainly not the most urgent problem of magneto-hydrodynamics that needs to be solved; nevertheless, as in gas dynamics, the study of such problems contributes to an understanding of significant hydro-magnetic phenomena.

In gas dynamics, the simplest types of one-dimensional flows are the so-called "simple waves". Because a flow region adjacent to a region of constant state is always a simple wave, these waves may be used very effectively as building blocks in constructing solutions of flow problems. (A state of flow is referred to as constant in a region if all significant flow quantities are time- and space-independent in the region.)

Simple waves are also possible in magneto-hydrodynamics. They have essentially the same properties as those in gas dynamics. Moreover, it appears that they also can be effectively used as building blocks.

A simple wave may be characterized by describing, not the

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*For a theory of simple waves associated with general systems of differential equations whose coefficients do not involve the independent variables, see P. Lax [16].

motion of a particle, but the motion of a "phase", as determined by a set of values for the quantities B, u, ρ. In a simple wave each phase moves with constant velocity. That is, the values of B, u, ρ are constant on certain straight lines

$$(10.1) \qquad x = Ut + \xi$$

in the (x,t) plane. The number U, as well as B, u, and ρ, may be considered a function of the parameter ξ. Instead of U we introduce the quantity $c \geq 0$ defined by

$$(10.2) \qquad U = u_x \pm c.$$

This quantity c will turn out to be one of the characteristic disturbance speeds; i.e., one of the roots of equation (2.6).

From this characterization of simple waves one may derive the following recipe for setting up the equations governing them. In equations A (see Section 1), one should replace ∇ and $\frac{\partial}{\partial t}$ by $\{d/d\xi, 0, 0\}$ and $-(u_x \pm c)d/d\xi$, or, leaving open the choice of the parameter describing the phase, simply by $\{d, 0, 0\}$ and $-(u_x \pm c)d$.

The equations thus obtained are

$F_o \qquad dB_x = 0,$

$F_1 \qquad \mp c\, dB_y + B_y du_x - B_x du_y = 0,$

$\qquad \mp c\, dB_z + B_z du_x - B_x du_z = 0,$

F_2 $$\mp \rho c du_x + dp + d(B^2/2\mu) = 0,$$

$$\mp \rho c du_y + \mu^{-1} B_x dB_y = 0,$$

$$\mp \rho c du_z - \mu^{-1} B_x dB_z = 0,$$

F_3 $$\mp c d\rho + \rho du_x = 0,$$

F_4 $$\mp c dS = 0.$$

When the equations F_1 to F_4 are considered linear equations for the seven differentials $dB_y, \ldots, dS$ it is clear that the determinant of this system must vanish. One verifies - by direct computation or by comparison with equations (B) in Section 2 - that, except for a single factor of $-c$, this determinant is precisely det (B), cf. (2.5), with the normal direction n taken as the x-direction. Hence, the speed c must be one of the roots of the equation det (B) = 0, i.e., one of the characteristic disturbance speeds corresponding to the phase B, u, ρ.

According to which kind of speed c enters, the simple wave is fast, slow, or transverse; if c = 0 the wave is a "contact layer".

As we shall see in the subsequent sections, the differential equations (F) can be greatly simplified. In the case of a polytropic gas, they can even be solved explicitly.

11. Fast and Slow Simple Waves

Equation (3.2) for the fast and slow disturbance speeds may in the present case be written in the form

(11.1) $$\mu^{-1}(B_y^2 + B_z^2) = c^{-2}(c^2 - a^2)(\rho c^2 - B_x^2/\mu),$$

where, cf. (2),

$$a^2 = dp/d\rho.$$

It is more convenient for the present purpose to introduce the square ratio

(11.2) $$q = c^2/a^2$$

of disturbance speed to sound speed as independent variable and to try to express all other quantities in terms of q. It is also convenient to introduce the square ratio

(11.3) $$s = a^2/b_n^2 = \mu\rho a^2/B_x^2$$

of sound speed to "normal" Alfvén speed as dependent variable. Relation (11.1) then becomes

(11.4) $$B_y^2 + B_z^2 = (q - 1)(s - q^{-1})B_x^2.$$

From relations $F_{2,x}$ and F_3 one further derives the relation

(11.5) $$2(q - 1)ds = \gamma d(q - 1)(s - q^{-1}),$$

where

(11.6) $$\gamma = 1 + \rho \frac{d}{d\rho}(\log \frac{dp}{d\rho}).$$

(See the Appendix for a more systematic derivation of this relation.) For polytropic gases, γ is a constant, and relation (11.5) becomes a linear differential equation for s as a function of q, which can be solved explicitly. From relations F_1 and $F_{2,y}$, $F_{2,z}$ one infers that the ratio B_z/B_y is constant, so that, except for this constant, B_y and B_z can be determined explicitly in terms of the constant B_x by (11.4). Since ρ and a may be regarded as known functions of s, cf. (11.3), also $c = a\sqrt{q}$ may be considered known. The velocity u can now be determined by integrating the differentials

(11.7) $$du_x = \pm c\rho^{-1} d\rho,$$

(11.8) $$du_y = \mp(\mu\rho c)^{-1} B_x dB_y,$$

$$du_z = \mp(\mu\rho c)^{-1} B_x dB_z.$$

As implied by relation (3.4),

(11.9) $$\mu\rho c^2 \geq B_x^2 \quad \text{or} \quad s \geq q^{-1} \quad \text{in a fast wave,}$$

$$\mu\rho c^2 \leq B_x^2 \quad \text{or} \quad s \leq q^{-1} \quad \text{in a slow wave;}$$

thus the wave speed c may agree with the Alfvén speed. On the other hand, it could be shown that the wave speed c can never coalesce with the sound speed a; i.e.

(11.10) $$q > 1 \quad \text{in a fast wave,}$$

$$q < 1 \quad \text{in a slow wave.}$$

In a fast wave the compression may tend to infinity, and hence $s \to \infty$, while in a slow wave cavitation may be reached, $s \to 0$.

Of course, a fast (or slow) wave may be either a compression wave or a rarefaction wave. The tangential components of the magnetic field $|B_y|$ and $|B_z|$ increase across a fast, decrease across a slow compression wave, but they decrease across a fast, and increase across a slow rarefaction wave.

Finally, we mention that "centered rarefaction waves" exist, as in gas dynamics. They are characterized by the condition that at some initial time, say $t = 0$, all phases involved are concentrated at the same point. (One then must set ξ = const. in formula (10.1) and use a different parameter to characterize phases.) If we imagine that a centered wave separates two constant states of the fluid, these states will be adjacent at the time $t = 0$. In other words, the centered wave then resolves an initial discontinuity.

12. Transverse Waves and Contact Layers

While the ratio of H_z to H_y is constant across fast and slow waves, this is not the case for the transverse simple waves, which we are now going to describe. The speed c of these waves is the Alfven speed

$$(12.1) \qquad c = (B_x^2/\mu\rho)^{1/2}.$$

Furthermore, B_x, ρ, and S, and therefore p and u_x, are constant across these waves. The tangential magnetic field, however, rotates:

$$(12.2) \qquad B_y = G \cos \theta, \quad B_z = G \sin \theta,$$

with $\theta = \theta(x)$ being any function of the phase. Further,

$$(12.3) \qquad u_y = a_y \mp cB_x^{-1}B_y, \quad u_z = a_z \mp cB_x^{-1}B_z,$$

with any numbers a_y, a_z. Thus, <u>the flow in a transverse wave may be considered a shear flow.</u> All particles on the same (y,z)-plane move in the same straight line. This shear flow is evidently a steady flow if it is observed from a frame with respect to which $u_x \pm c = 0$. Thus, in contrast to the situation in gas dynamics, <u>there do exist non-constant steady flows in magneto-hydrodynamics.</u>

In order to maintain such a shear flow, it would be necessary to have at large (y,z)-distances a mechanism which supplies the velocity u in the proper directions there. It would seem that such a mechanism would not easily occur under natural circumstances and that it would have to be rather artificial.

A transverse wave may connect two constant states with different velocities u and magnetic fields B. One may imagine the layer covered by the wave to be arbitrarily thin; one then may approximate the transition by a discontinuity. This discontinuity would be exactly a transverse shock. Thus, a transverse shock may be considered the limit of transverse simple waves. The remark about the artificial nature of transverse waves, therefore, apply just as well to transverse shocks.

Only a short remark need be made about "contact layers". In accordance with $c = 0$, there is no flow across such a layer; i.e., u_x is constant across it. If $B_x = 0$, all other quantities may vary except that the relation

$$(12.4) \qquad p + (B_y^2 + B_z^2)/2\mu = \text{constant}$$

should hold. If $B_x \neq 0$, however, B, u, and p are constant, and only ρ and S may vary. Thus, in contrast to gas dynamics, a shear flow layer across which the tangential flow components u_y, u_z vary cannot be maintained in a conducting fluid if the magnetic field possesses a normal component $B_x \neq 0$.

13. The Resolution of a Shear Flow Discontinuity

As was explained at the end of Section 9, no shear flow discontinuity can remain unchanged in the presence of a magnetic field whose normal component is not zero. If such a discontinuity is present at an initial time, a wave motion must result which resolves it. The nature of this wave motion will now be described.

Specifically, we consider the following problem. At an initial time, $t = 0$, the fluid is at rest on one side, $x > 0$, of the plane $x = 0$, while on the other side, $x < 0$, it possesses a constant tangential velocity $(u_y, u_z) \neq 0$, but no normal velocity, so that $u_x = 0$. The density is constant and the same on both sides, and a constant magnetic field is present - the same on both sides - with a non-vanishing normal component $B_x \neq 0$.

The impossibility of maintaining a shear flow discontinuity may be visualized as follows. Instead of a discontinuity, consider a thin shear flow layer across which a tangential flow component, say u_y, varies smoothly from a positive value on the left hand side to the value zero on the right; for simplicity assume $u_z = 0$ throughout. Also assume $B_y = B_z = 0$ and $B_x > 0$. The basic assumption of magneto-hydrodynamics, embodied in formula (1), now implies that the electric field component E_z does not vanish on the left but does so on the right. Therefore, curl $E \neq 0$ in the layer. From A_1 we then conclude that the tangential magnetic field, specifically the component $-B_y$, will grow within the layer. Since this component at first remains zero outside of the layer, its curl, and hence the current J, will be different from zero in the layer. Specifically, the component J_z will vary from negative to positive values across the layer. Hence the fluid particles in the layer will experience a force whose x-component varies from negative to positive values across the layer and hence tends to push the particles away from the layer.

Before describing the details of the resulting wave motion we mention a concrete situation in which such initial shear flow discontinuity may occur; namely, if a jet of conducting fluid shoots into conducting fluid at rest so quickly that the hydro-magnetic adjustments just discussed have not yet developed. A similar case was described by Alfven [1] in explaining the possible origin of hydro-magnetic waves. Assuming the jet to proceed in the y-direction and to be much wider in the z-direction than in the x-direction, we may approximate it by a constant flow in the y-direction between two parallel planes x = constant. The waves which result from the two interfaces will interact only after some time. Up to this time, therefore, the situation may be described in terms of the wave motion resulting from a single interface.

We maintain that the resolution of the shear flow discontinuity is effected by two fast shocks followed by two slow centered rarefaction waves. After the waves have formed, the fluid has acquired the mean tangential velocity, while a tangential component of the magnetic field has increased (or developed if none was present originally).

A diagram of the resolving flow, in which only the x-component of the particle motion is indicated, is given in Figure 4.

It may be mentioned, incidentally, without giving supporting arguments, that after the waves coming from the two interfaces have interacted the fluid will come to rest in the jet region while the tangential magnetic field has a further increased value, at least for some time.

In attacking the problem in detail, we have, for simplicity, assumed that all z-components vanish and that, at time $t = 0$, B_x is a positive constant and $B_y = 0$. Further, we have assumed that ρ and p are constant, the same on both sides,

$$(13.1) \qquad \rho = \overset{o}{\rho}, \quad p = \overset{o}{p},$$

and that $u_x = 0$ on both sides. For convenience we have assumed that the flow is observed from a frame moving with the mean tangential velocity, so that

$$(13.2) \qquad u_y = \begin{cases} \overset{o}{u}_y, & x > 0, \\ -\overset{o}{u}_y, & x < 0, \end{cases}$$

$\overset{o}{u}_y$ being a positive constant.

The conditions to be satisfied after the passage of the waves (i.e., in the region of constant state containing $x = 0$) simplify because of the symmetry of the problem. They are

$$(13.3)_x \qquad u_x = 0,$$

$$(13.3)_y \qquad u_y = 0.$$

Since B_y is even in x, no condition need be imposed on B_y.

It was mentioned at the end of Section 8 that a shock would be determined if one quantity such as p or u_x is prescribed behind it, provided the state in front of it is known. Except for a qualification to be discussed below, the same is true for a simple wave. It is, therefore, natural to expect that two quantities could be prescribed behind a pair of waves when the state in front is known. It is not obvious, however, whether or not the two velocity components u_x, u_y may be prescribed without limitation behind a pair of waves. In the present problem this appears to be the case.

The qualification mentioned above is this: Suppose a piston at one end of a gas filled tube is withdrawn with speed greater than a certain "escape speed", cf. [15]. Then the resulting rarefaction wave will lead to "cavitation". The piston will separate from the gas and a vacuum zone will be formed. The gas at the edge of this zone will move with the escape velocity and not with the piston velocity. A similar phenomenon may occur here, so that condition $(13.3)_x$ must be modified. It should read

$$(13.4)_x \qquad \text{Either } u_x = 0 \quad \text{or} \quad \rho = 0,$$

thus permitting the presence behind the rarefaction waves of a vacuum zone which expands with an appropriate escape speed. When this vacuum zone is present, condition $(13.3)_y$ must be replaced by the condition that

$$(13.4)_y \qquad u_y B_x - u_x B_y = 0$$

at the edges of the vacuum zone. As noticed by Bazer [10], this is the correct condition to insure that the electric field $E = B \times u$ vanishes there and in the whole vacuum region.

For the description of the wave motion in detail it is necessary to solve a number of transcendental equations which are expressed in terms of explicitly given integrals derived from the considerations of Section 11.

Of particular interest is the magnitude of the tangential component B_y of the magnetic field that has developed at the center $x = 0$ after passage of the waves. For small values of the initial shear flow discontinuity $2\overset{o}{u}_y$, this component may be described by the formula

$$(13.5) \qquad B_y = (\varkappa \mu \overset{o}{\rho})^{1/2} \overset{o}{u}_y ,$$

in which $\varkappa$ in turn depends on $\overset{o}{u}_y$.

If $\overset{o}{u}_y$ approaches zero, $\varkappa$ approaches the value 1. In fact, this same value of $\varkappa$ would have been obtained if one had assumed the fluid to be incompressible and had described the resolution of the discontinuity with the aid of two Alfvén waves, one traveling in each direction.

If $\overset{o}{u}_y$ is sufficiently large, cavitation occurs. In such a case, the sound speed eventually becomes less than the Alfvén speed even if it was much larger originally. Thus the medium could certainly not be considered incompressible. The limiting form of the tangential component B_y as $\overset{o}{u}_y \to \infty$ is given by

$$(13.6) \qquad B_y \sim [(1+\gamma)\mu\rho_o]^{1/4} (B_x \overset{o}{u}_y)^{1/2},$$

where γ is as defined by (11.6).

The derivation of formulas (13.5) and (13.6), as well as complete details of the resolving flow, are given by Bazer [10].

APPENDIX: Simple Waves and Groups of Transformations

by K. von Hagenow

In the following it is demonstrated how the inherent symmetry of the equations of one-dimensional flow can be used to achieve the reduction of the equations of hydromagnetic simple waves (see Section 10) to a simple ordinary differential equation.

Simple waves are special solutions of the equations of one-dimensional flow which are obtained from equations A_1 through A_4 of Section 1 by assuming all quantities to depend only on the coordinate x and the time t. The divergence condition A_0 then reduces to B_x = const. The equations are non-relativistic, and, as physically no reference system can be preferred, they must be invariant under the 4-parameter group consisting of the following transformations: a translation with constant speed in an arbitrary direction and a rotation around the x-axis by an arbitrary angle. The direction of the x-axis has to remain unchanged, of course, for we have distinguished it from the other two coordinate axes by the assumption of one-dimensional flow. Polytropic gases, where the pressure p is given by

(14.1) $$p = \rho^{\gamma} e^{S}$$

allow the additional similarity transformation

(14.2) $$x' = \alpha x , \quad t' = t ,$$

with

(14.3) $$u' = \alpha u, \quad S' = S + 2\gamma \log \alpha$$

and pressure p and magnetic field B unchanged. If $B_x = 0$, the pressure p and field B can also be scaled by a factor:

$$(14.4)\qquad \begin{aligned} p' &= \beta p \\ B' &= \beta^{1/2} B. \end{aligned}$$

Now in simple waves, all variables u,B,p,S are constant on a family of straight characteristics (cf. eq. 10.1). As our transformations map straight lines into straight lines (in the x-t plane) and as invariance of the equation implies mapping of characteristics into characteristics, they also map simple waves into simple waves. That is, the system of ordinary differential equations F_1 through F_4 (section 10) is invariant under the corresponding transformation of the dependent variables.

Let us introduce new variables as follows: the pressure p, the square of the field, B^2, the angle ϕ between the y-z field component and the y-axis, the velocity components u_x in the x-direction, V in the direction of above field-component, W orthogonal to V in the y-z plane, and the entropy S. Then, if we write the system F_1 through F_4 of Section 10 symbolically:

$$(14.5)\qquad \sum a_{ik}\, dw_k = 0,$$

we see that we can add an arbitrary constant to each of the variables ϕ, u_x, V, W, i.e., except for a factor common to each row, the matrix a_{ik} can but depend on the remaining variables p, B^2, S.

S is constant according to F_4 so the system must be reducible to a single equation involving the quantities p and B^2 only, containing S as a parameter.

Now that we know what equation to expect, we can obtain the reduction without actually introducing our new variables. Introducing the sound speed a with

$$(14.6) \qquad dp = a^2 d\rho,$$

and using equation F_3, we can write equation $F_{2,x}$

$$(14.7) \qquad (1-q)dp + d(B^2)/2\mu = 0$$

with

$$(14.8) \qquad q = c^2/a^2 \qquad \text{(cf. equation 11.2)}$$

and we know that q depends on B^2, p and S only, the latter being constant in a simple wave anyway. We can therefore express B^2 as a function of p and q, and get exactly equation (11.5), for the quantity s defined in equation (11.3) is related to p by

$$(14.9) \qquad \gamma dp = B_x{}^2/\mu ds$$

with γ defined by equation (11.6), while equation (11.4) allows one to express B^2 in terms of s and q for fast and slow waves. Transverse waves are trivial, because in them p and therefore B^2 are constants. Polytropic gases allow furthermore the transformation (14.3), i.e. the entropy S does not appear in (14.7). The special case of vanishing B_x allows the transformation (14.4), i.e. q must be homogeneous of degree 0 in p and B^2. It is noteworthy that we got our results without making explicit use of (11.4), which determines the disturbance speeds. If we actually compute B^2 in terms of p and q (for polytropic gases) we find that it is linear in p, i.e., equation (14.7) can be integrated explicitly. This fact cannot be deduced by looking at the symmetries of the one-dimensional equations, but we can make the following statement. If we treat also Δx as an unknown, then the equations A_o through A_4 allow (14.2,3,4) even for $B_x \neq 0$. But now, our reduction to (14.7) allows us only to conclude that

$$B^2 = B^2(p,q,B_x). \tag{14.10}$$

The wave formation leaves q fixed, for, according to (14.2), the characteristic direction, i.e. c, is multiplied by α, and from (14.1) we deduce that the sound speed a is transformed by the same factor. Then (14.4) shows that B^2 must be homogeneous of first degree in p and B_x^2 but linearity does not follow. If $B_x \neq 0$ it is only the simplest possibility compatible with the transformation properties of the equations. Equation (14.7) allows an interesting interpretation: With (14.6,8) we can write it

$$d(p + \frac{B^2}{2\mu}) = c^2 d\rho \tag{14.11}$$

i.e. the variation of the total "pressure"* with the density gives the square of the disturbance speed. This is of course true for any longitudinal wave in a compressible medium.

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*We should really talk about a pressure-tensor, see Lüst [7], for the magnetic force is anisotropic, but for one-dimensional flow only the 1-1 component gives a force in the x-direction.

Thus we can already conclude from this remark that the fast and slow waves are purely longitudinal, because the speed of transverse waves is in general not given by (14.11), but depends on restoring forces against torsion.

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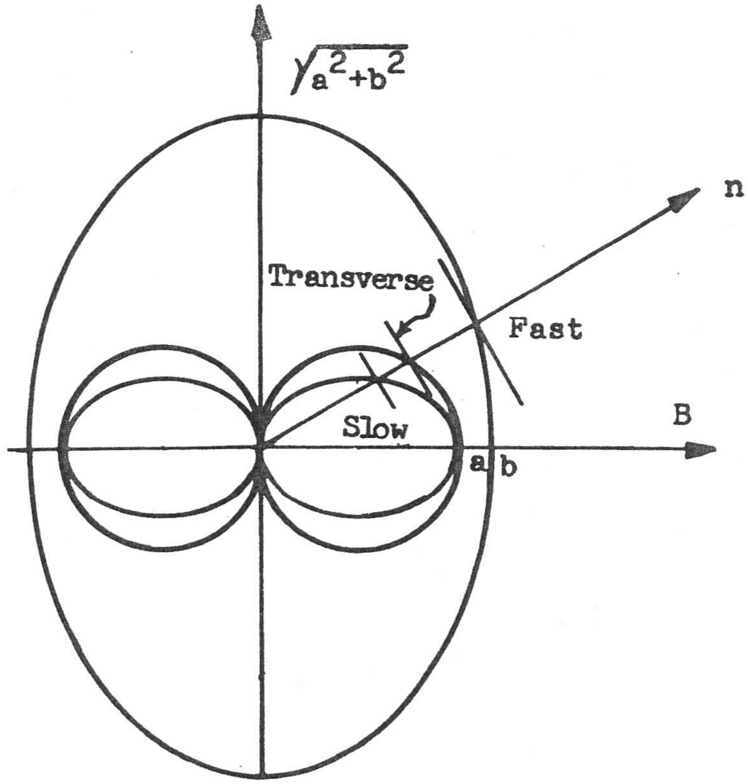


FIGURE 1

The Three Types of Disturbance Waves.

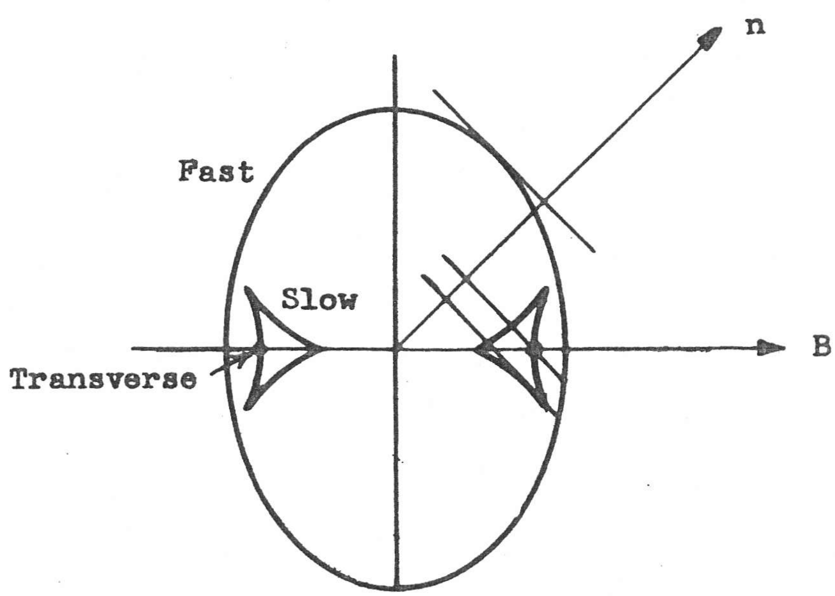


FIGURE 1A

Wave fronts at a time $t > 0$ originating from a point disturbance at the time $t = 0$. Their envelopes, also shown, are given by the intersection of the characteristic cone with t = constant and z = constant ($B_z = 0$).

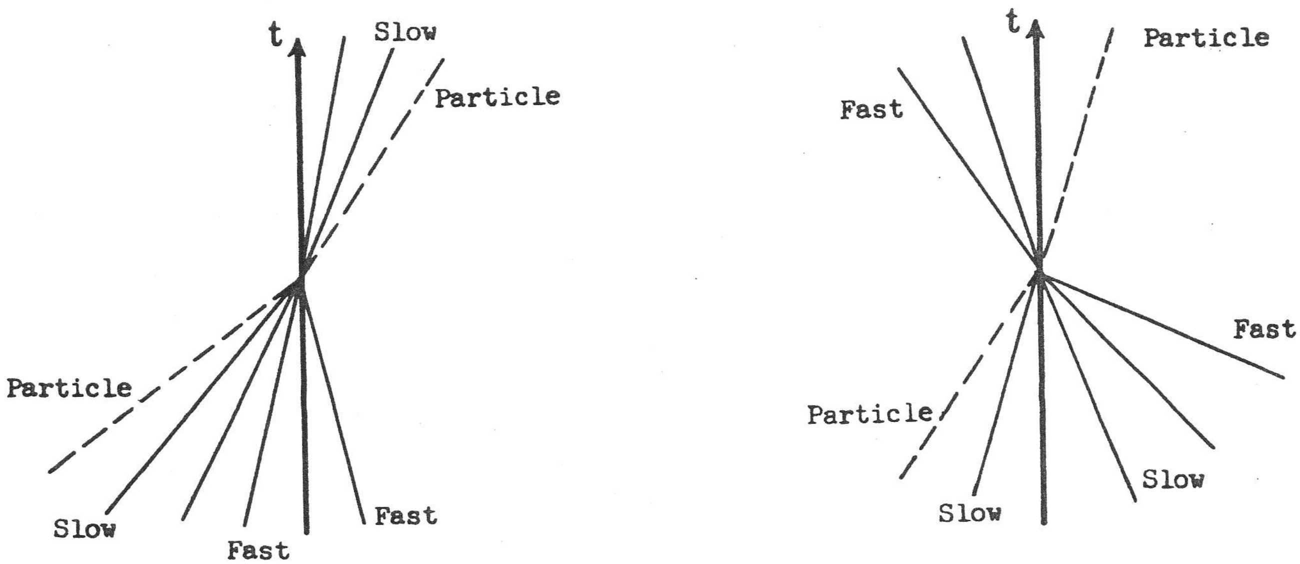


FIGURE 2

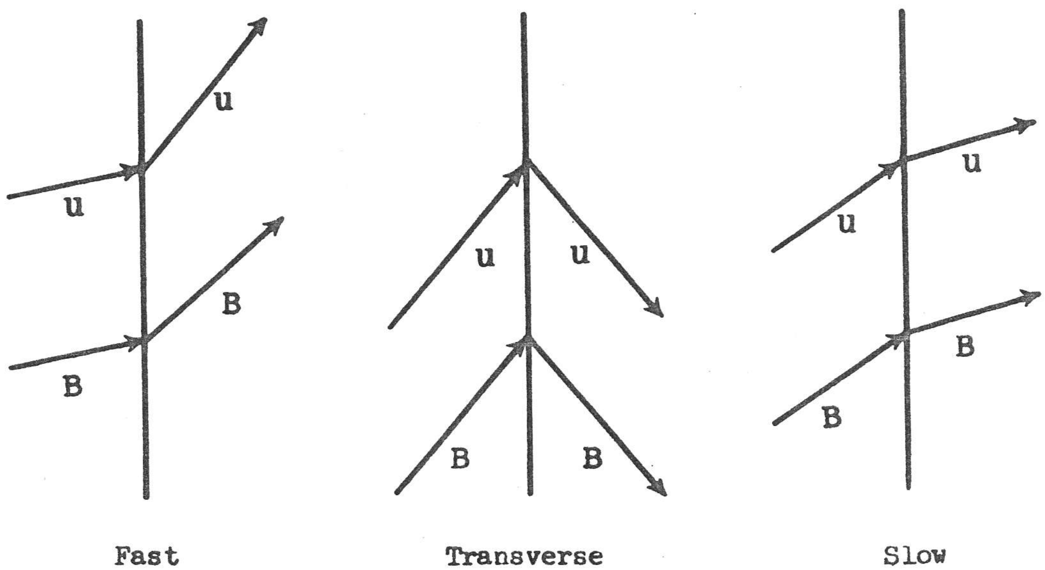


FIGURE 2A

Transitions Through Stationary Shock Fronts

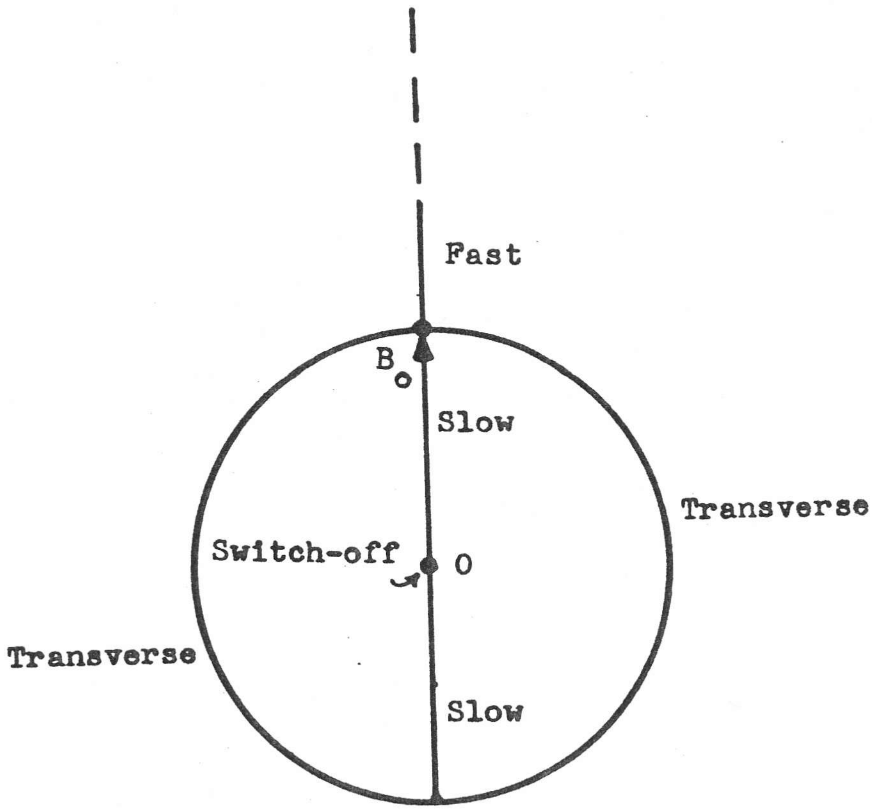
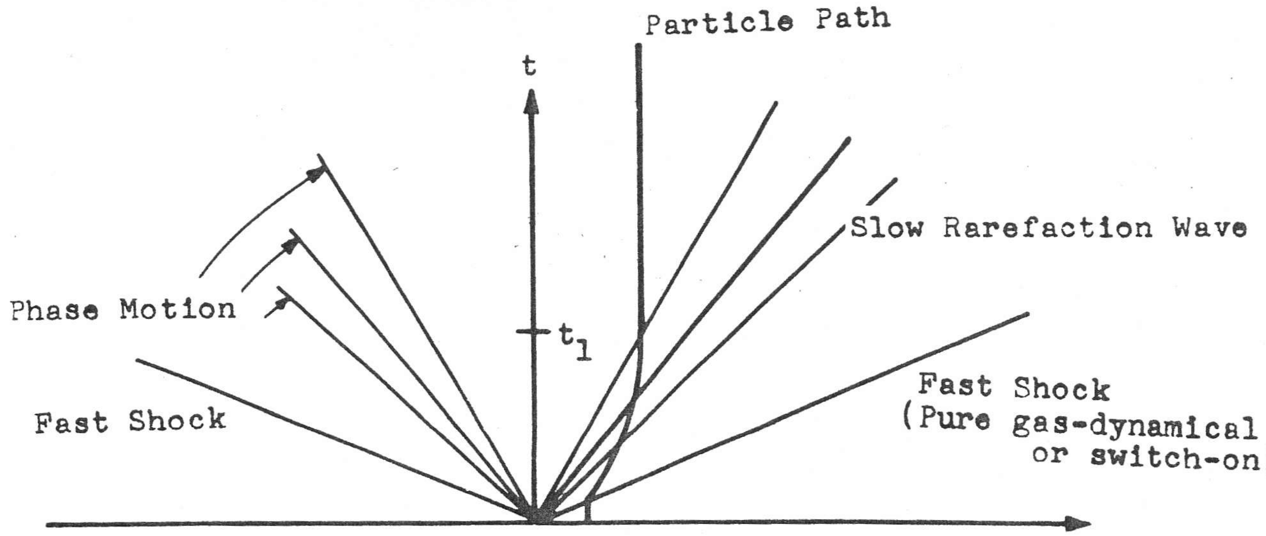
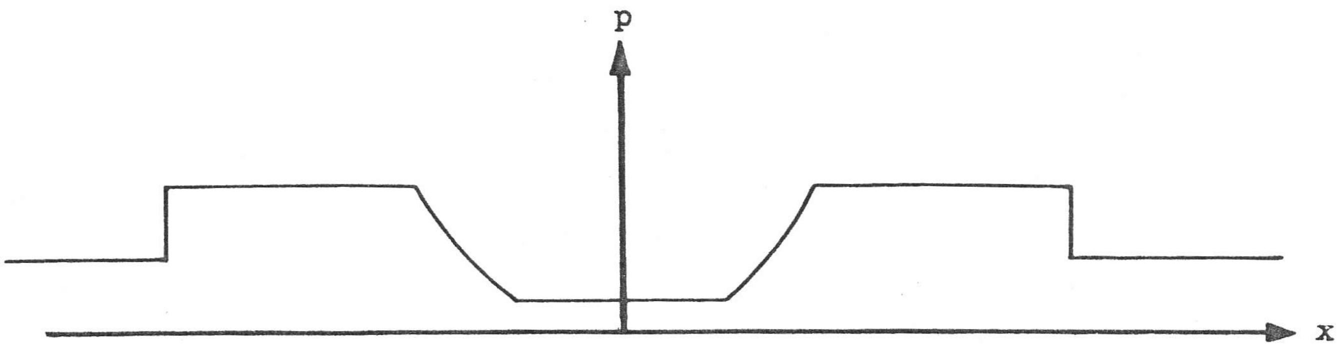


FIGURE 3

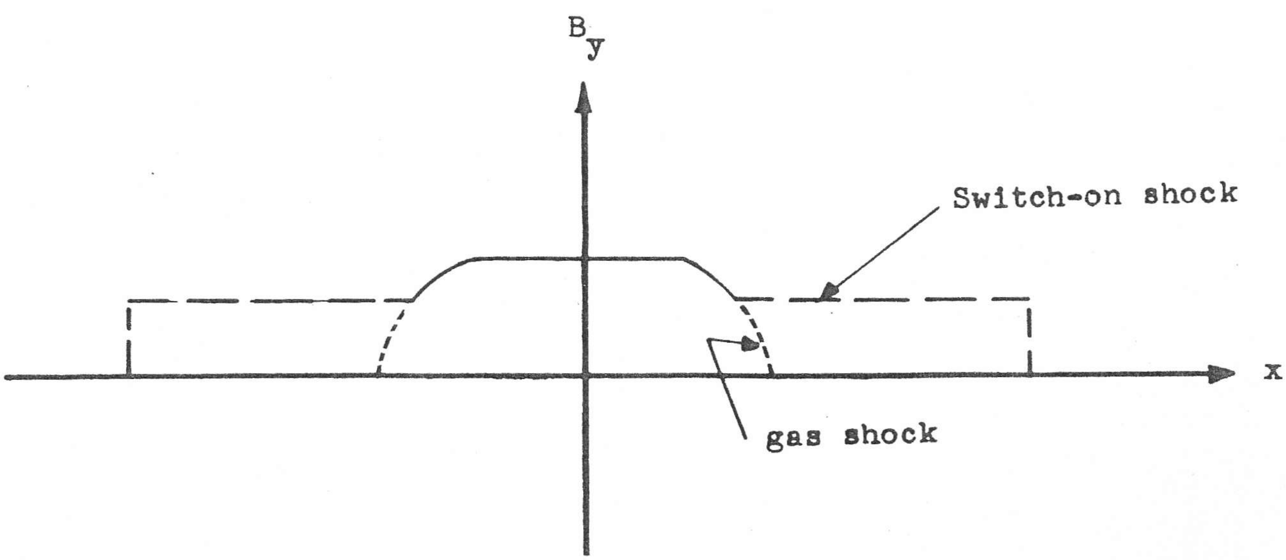
Locus of the possible tangential components of magnetic field behind the various types of shock corresponding to the fixed tangential component B_o ahead of the shock.



Resolution of a Shear Flow Discontinuity



Typical Distribution of Pressure at Time $t_1 > 0$



Typical Distribution of Tangential Magnetic Field at Time $t_1 > 0$

FIGURE 4