Two-Dimensional Navier-Stokes Flow with Measures as Initial Vorticity

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Communicated by C. DAFERMOS

Introduction

This paper studies the nonstationary flow of a viscous incompressible fluid in R^2 when the initial vorticity is very singular. The governing equations of motion are the Navier-Stokes equations

$$\begin{gathered} u' - \nu \Delta u + (u \cdot \nabla) u + \nabla p = 0, \quad \nabla \cdot u = 0, \\ u \to 0 \quad \text{as } |x| \to \infty, \quad u(x, 0) = a(x), \quad \nabla \cdot a = 0, \end{gathered} \tag{1}$$

where u and p represent the unknown velocity and pressure, respectively, $\nu > 0$ is the kinematic viscosity, $(u \cdot \nabla) = \Sigma_i u^i \, \partial/\partial x_i$, $\nabla \cdot u = \Sigma_i \, \partial u^i/\partial x_i$ and $u' = \partial u/\partial t$. By normalization the density of the fluid is assumed to be 1.

We consider problem (1) in two dimensions, assuming that the initial vorticity

$$\nabla \times a = \partial a^2/\partial x_1 - \partial a^1/\partial x_2$$

is a finite Radon measure on R^2, and discuss its solvability. Velocity fields of this type include those with vortex sheets and point sources of vorticity. A rigorous relation between solutions of the Euler equations (system (1) with $\nu = 0$) and the classical theory of the motion of point vortices has been established only recently. See, *e.g.*, MARCHIORO & PULVIRENTI [19], [20] and TURKINGTON [31]. For the Navier-Stokes system (1), BENFATTO, ESPOSITO & PULVIRENTI [3] constructed a global smooth solution, assuming that the initial vorticity is a finite atomic measure whose variation is small compared with the viscosity, *i.e.*,

$$\nabla \times a = \sum_{j=1}^{m} \alpha_j \, \delta(x - z_j),$$

and $\nu/\Sigma_j |a_j|$ is sufficiently large; here $\delta(x - z_j)$ is the Dirac measure supported at $z_j \in R^2$. The results in [3] show that point-source vorticities can diffuse following the Navier-Stokes flow, provided ν is large. We note that this result does not follow from classical theories for the Navier-Stokes system, as developed by LERAY [17], LADYZHENSKAYA [16] or TÉMAM [30]. As pointed ou in [3], classical existence results for (1) fail to apply since the initial velocity a, with $\nabla \times a$ a measure, is not necessarily square-summable, even locally.

Our main goal in this paper is to show that there is a smooth global (in time) solution of (1), provided only that the initial vorticity $\nabla \times a$ is a finite measure on R^2. Evidently, this improves the result of [3] since no restriction is imposed on ν or on the size and the form of $\nabla \times a$.

To show existence, we follow a standard procedure. We first regularize the initial velocity a, consider the corresponding regular solutions of (1), and then take a subsequence converging to the desired solution of the original problem. As is well known, to carry out this process one needs good *a priori* estimates for regular solutions. For this purpose we study the vorticity equation for $v = \nabla \times u$:

$$v' - \nu \Delta v + (u \cdot \nabla) v = 0, \tag{2a}$$

$$u = K * v \tag{2b}$$

for *smooth* initial data $v(x, 0) = \nabla \times a$, where K is the vector function:

$$K(x_1, x_2) = (-x_2, x_1)/2\pi |x|^2, \quad x = (x_1, x_2),$$

and $*$ denotes convolution on R^2. These equations are derived formally by applying the operator $\nabla \times$ on (1) and using the condition $\nabla \cdot u = 0$. We note that there is no vorticity stretching term in (2a) since the space dimension is 2.

We regard (2a) as a linear parabolic equation for v with coefficients depending on u and write the corresponding fundamental solution as $\Gamma_u(x, t; y, s)$, $t \geqq s$. A bound for Γ_u established by OSADA [25] yields our key *a priori* estimates:

$$\begin{aligned} &C_1(t-s)^{-1} \exp[-C_2 |x-y|^2/(t-s)] \\ &\qquad \leqq \Gamma_u(x, t; y, s) \leqq C_3(t-s)^{-1} \exp[-C_4 |x-y|^2/(t-s)], \end{aligned} \tag{3}$$

where the positive constants C_j, $j = 1, 2, 3, 4$, depend only on ν and the L^1-norm of $\nabla \times a$. Estimate (3) makes it possible to control the behavior of v as $t \to 0$, uniformly in the approximation, so that the sequence of solutions with regularized initial data converges to a solution of the original problem (1), with $\nabla \times a$ a finite (Radon) measure on R^2. Estimates of the form (3), with C_j independent of the smoothness of coefficients, were first established by ARONSON [1] and ARONSON & SERRIN [2] for linear equations in divergence form. OSADA [25] extends the estimates in ARONSON [1] to a class of linear equations not in divergence form, which includes equation (2a) as a typical example.

The problem of existence of solutions for nonlinear evolution equations with measures as initial data has recently attracted the attention of many mathematicians. For example, MCKEAN [22], OSADA & KOTANI [24] and SZNITMAN [29] study the existence and uniqueness of solutions for the Burgers equation

$$u' + uu_x = \nu u_{xx}, \quad x \in R^1$$

with $u(x, 0) = c\,\delta(x)$, $c > 0$. For the problem

$$u' - \Delta u + u^p = 0, \quad x \in \mathbb{R}^n; \quad u(x, 0) = c\,\delta(x), \quad c > 0,$$

BRÉZIS & FRIEDMAN [5] prove that solutions exist when $0 < p < 1 + 2/n$ and do not exist when $p \geqq 1 + 2/n$; see also [35] for more general initial data. Their results on existence were extended to more general equations of the form $u' - \Delta u + f(u) = 0$ by NIWA [23]. For the problem

$$u' + f(u)_x = 0, \quad x \in R^1; \quad u(x, 0) = \delta(x),$$

LIU & PIERRE [18] discuss existence, (non-)uniqueness and asymptotic behavior of solutions satisfying the entropy admissibility criterion, under various assumptions on the form of the function f. Our main result may be understood as an example of existence of solutions in nonlinear parabolic equations with measures as initial data, namely global solutions to the problem (2a), (2b) when $v(x, 0)$ is an arbitrary finite measure on R^2.

In Section 1 we establish local existence of solutions for problem (1) in R^n, $n \geqq 2$, with initial velocity a in L^p, $p > n$, and show that this solution is regular for $t > 0$. For later use we discuss higher regularity up to $t = 0$. Since (1) is parabolic, these results are generally familiar. However, it is difficult to find the appropriate version in the literature, because here the initial velocity a is not necessarily square-summable, *i.e.*, the initial energy may be infinite.

From Section 2 onward we consider only two-dimensional flows. We extend the local solution obtained in Section 1 to global smooth solutions by appealing to the vorticity equation (2a), (2b). An argument of this type is found in MCGRATH [21]. Our results on global existence in Section 2 improve those in [19, 20] and [21] by relaxing assumptions on the initial data.

As a byproduct of our analysis, we prove in Section 2 that our solutions are persistent in the Sobolev spaces $W^{m,p}(R^2)$, $p > 2$, $m = 0, 1, 2, \ldots$ Namely, we show that if $a \in W^{m,p}(R^2)$ and $\nabla \times a \in L^q(R^2)$ with $1/q = 1/p + 1/2$, then the corresponding solution stays in $W^{m,p}(R^2)$ for all time and is bounded there, uniformly on each finite interval of time, independently of the viscosity ν. Such a uniform bound enables us to take a subsequence converging, as $\nu \to 0$, to a solution of the Euler equations. In fact we construct a global solution to the Euler equations under the same assumptions on a.

A property of persistence of this type is systematically studied by KATO [15] and PONCE [27] for the solutions of (1) with finite energy. Since our solution may have infinite energy, our results are not included in either [15] or [27]. After we completed this work, we learned that KATO & PONCE [34] extend their results to solutions which may have infinite energy. Their result covers our results for $m \geqq 2$. However, our results for $m = 0, 1$ are not contained even in [34]. In particular, our theorem of existence for the Euler equations seems new for initial data $a \in L^p(R^2)$, $\nabla \times a \in L^q(R^2)$, $1/q = 1/p + 1/2$. Recently, we have learned that DIPERNA & MAJDA [36] obtain a similar theorem on existence, assuming in addition $\nabla \times a \in L^1(\mathbb{R}^2)$. Their method seems different from ours.

Section 3 establishes our key *a priori* estimates for smooth solutions constructed in Section 2. It is crucial that our bound depends only on the L^1-norm of the initial vorticity $\nabla \times a$ and is otherwise independent of the regularity of a.

In Section 4 we apply our *a priori* estimate derived in Section 3 and prove our main existence theorems. More precisely, we construct a global solution of (1) as well as of (2a), (2b) when the initial vorticity is a finite measure on R^2 and prove regularity for $t > 0$ as well as some decay estimates as $t \to \infty$. We clarify the meaning of the convergence to the initial velocity as $t \to 0$ by using Lorentz spaces. We further show that our solution is unique provided that the atomic part of $\nabla \times a$ is "small". We note that there is no restriction on the size of the continuous part of $\nabla \times a$. This conclusion covers the uniqueness theorem of [3] since it is assumed there that $\nabla \times a$ is a finite atomic measure with small total variation.

We are grateful to Professor MASAO YAMAZAKI for some suggestions on Lorentz spaces. We are also grateful to Professor NORIAKI SUZUKI for valuable suggestions on the proof of Lemma 4.4.

1. Local Solutions in R^n with Initial Data in L^p

This section establishes local existence of solutions in L^p for the Navier-Stokes system (1) in R^n, $n \geqq 2$, *without* assuming that the initial energy is finite. Although there are many references on the local existence in R^n, only a few results are known when the initial energy is not finite (see *e.g.*, [7, 12, 14, 16, 33, 34]). Consequently, we give here the details of our derivation for later use. The basic tool for constructing solutions is a standard successive approximation scheme which goes back to LERAY [17] and is systematically studied in [10, 11, 14, 32, 33, 34].

We shall also discuss higher regularity up to $t = 0$ to be used in the sequel. Since the equation is semilinear and parabolic, regularity for $t > 0$ and up to $t = 0$ is generally known (see, *e.g.*, [7, 10, 34]). However, we state and prove here our version of a regularity theorem which does not follow from a simple combination of known results.

Hereafter we use the following notation: BC denotes the class of bounded continuous functions. $L^p(R^n)$ represents the space of L^p-vector-valued or tensor-valued functions on R^n, as well as the space of L^p-scalar-valued functions on R^n; the norm of f in $L^p(R^n)$ is denoted by $\|f\|_p$. We denote $\mathrm{BC}([0, T); L^p(R^n))$ simply by $\mathrm{B}_{p,T}$. The norm of $u(x, t)$ in $\mathrm{B}_{p,T}$ is defined by

$$|u|_{p,T} = \sup_{0 \leqq t < T} \|u\|_p(t).$$

If $f = (f^1, \ldots, f^n)$ is a vector-valued function on R^n, ∇f denotes the tensor $\partial_i f^j$, $1 \leqq i,\ j \leqq n$, where $\partial_i = \partial/\partial x_i$. Similarly, for a nonnegative integer k, $\nabla^k f$ denotes the tensor $\partial_1^{\alpha_1} \ldots \partial_n^{\alpha_n} f^j$, $\alpha_1 + \ldots + \alpha_n = k$. The expression $\partial_t f$ denotes the time derivative of f.

Following the standard practice ([7, 10, 11, 12, 14, 32, 33, 34]), to solve (1) we transform it into its integral form:

$$u(t) = e^{\nu t \Delta} a + S[u](t), \qquad t \geqq 0, \tag{1.1}$$

where

$$S[u](t) = S[u, u](t); \qquad S[u, w](t) = -\int_0^t e^{\nu(t-s)\Delta} P(u \cdot \nabla) w(s)\, ds. \tag{1.2}$$

Here $e^{t\Delta}$ is the solution operator for the heat equation; P is a singular integral operator of convolution type (see [7]) namely the orthogonal projection onto the subspace of divergence-free vector fields of $L^2(R^n)$. A solution u of (1.1) is called a mild solution of the initial value problem (1) since (1) and (1.1) are equivalent, provided u and a are smoth and decay as $|x| \to \infty$. It turns out that the solutions treated in this paper are all smooth and satisfy the equations in the classical sense for $t > 0$. However, we should consider carefully the behavior of the solutions as $t \to 0$ in order to understand the meaning of the initial condition. We first derive basic estimates in L^p for the bilinear map $S[u, w]$. We observe that $\Sigma_j u^j \partial_j w = \Sigma_j \partial_j(u^j w)$ ($= \nabla \cdot (u \otimes w)$ for short) provided $\nabla \cdot u = 0$. This provides an alternative expression of S:

$$S[u, w](t) = -\int_0^t e^{\nu(t-s)\Delta} P \nabla \cdot (u \otimes w)\, ds = -\int_0^t \nabla \cdot e^{\nu(t-s)\Delta} P(u \otimes w)\, ds \tag{1.2'}$$

since P and ∇ commute with $e^{\nu t\Delta}$.

Lemma 1.1. *Let* $2 \leqq n < p < \infty$, $T > 0$ *and* $\sigma = 1/2 - n/2p$. *Then*

(i) $|S[u, w]|_{p,T} \leqq M(\nu T)^\sigma |u|_{p,T} |w|_{p,T}/\nu$ *provided that* $\nabla \cdot u = 0$;

(ii) $|(\nu t)^{\frac{1}{2}} \nabla S[u, w]|_{p,T} \leqq M(\nu T)^\sigma |u|_{p,T} |(\nu t)^{\frac{1}{2}} \nabla w|_{p,T}/\nu$;

(iii) $|\nabla S[u, w]|_{q,T} \leqq M(\nu T)^\sigma |u|_{p,T}^{2\sigma} |(\nu t)^{\frac{1}{2}} \nabla u|_{p,T}^{1-2\sigma} |\nabla w|_{q,T}/\nu$;

with $1/q = 1/p + 1/n$, *where* M *is a positive constant depending only on* n *and* p.

Proof. We estimate S and ∇S by applying the well known estimates:

$$\|\nabla e^{\nu t\Delta} f\|_r \leqq C(\nu t)^{-\frac{1}{2}-(n/s-n/r)/2} \|f\|_s, \quad 1 \leqq s \leqq r \leqq \infty, \tag{1.3}$$

$$\|Pf\|_r \leqq C \|f\|_r, \quad 1 < r < \infty, \tag{1.4}$$

(see [13, Chap. 9]) to (1.2) or (1.2'). Since the proofs of (i), (ii), (iii) are standard and similar to others, we give here only the proof of (iii). We take the gradient of $S[u, w]$ and use (1.3), (1.4) to get

$$\|\nabla S[u, w]\|_q(t) \leqq C \int_0^t [\nu(t-s)]^{-\frac{1}{2}} \|(u \cdot \nabla) w\|_q(s)\, ds \leqq C \int_0^t [\nu(t-s)]^{-\frac{1}{2}} \|u\|_\infty(s) \|\nabla w\|_q(s)\, ds$$

where C depends only on p and n. Since $p > n$, the Gagliardo-Nirenberg inequality [9, p. 24, Theorem 9.3] yields

$$\|u\|_\infty \leqq C \|u\|_p^{2\sigma} \|\nabla u\|_p^{1-2\sigma}.$$

We thus have

$$\|\nabla S[u, w]\|(t)$$
$$\leqq C \int_0^t [\nu(t-s)]^{-\frac{1}{2}} (\nu s)^{-\frac{1}{2}+\sigma} \|u\|_p^{2\sigma}(s) \|(\nu s)^{\frac{1}{2}} \nabla u\|_p^{1-2\sigma}(s) \|\nabla w\|_q(s)\, ds$$
$$\leqq M(\nu T)^\sigma |u|_{p,T}^{2\sigma} |(\nu t)^{\frac{1}{2}} \nabla u|_{p,T}^{1-2\sigma} |\nabla w|_{q,T}/\nu,$$

with M depending only on n and p. This yields (iii). □

We now construct a local solution in L^p, $p > n$.

Proposition 1.2. (i) *Suppose that the initial velocity a is in $L^p(R^n)$ for some $p > n$ and $\nabla \cdot a = 0$. Then there is a unique local solution u of* (1.1) *such that $u \in B_{p,T}$ for some $T > 0$ and*

$$|u|_{p,T} \leqq 2\|a\|_p. \tag{1.5}$$

(ii) *The time T can be selected so that*

$$T \geqq C\nu^{-1+1/\sigma}/\|a\|_p^{1/\sigma}, \quad \sigma = 1/2 - n/2p; \tag{1.6}$$

$$(\nu t)^{\frac{1}{2}} \nabla u \in B_{p,T} \quad \textit{with } |(\nu t)^{\frac{1}{2}} \nabla u|_{p,T} \leqq C\|a\|_p, \tag{1.7a}$$

where C depends only on n and p, and
(1.7b) *If $\nabla a \in L^q(R^n)$ with $1/q = 1/p + 1/n$, then $\nabla u \in B_{q,T}$ and $|\nabla u|_{q,T} \leqq 2\|\nabla a\|_q$.*

(iii) *Let m be a nonnegative integer and suppose that $\nabla^k a \in L^p(R^n)$ for $k = 0, \ldots, m$. Then the time T can be selected so that*

$$\nabla^k u \in B_{p,T} \quad \textit{and} \quad |\nabla^k u|_{p,T} \leqq C', \quad k = 0, \ldots, m; \tag{1.8a}$$

$$(\nu t)^{\frac{1}{2}} \nabla^{m+1} u \in B_{p,T} \quad \textit{and} \quad |(\nu t)^{\frac{1}{2}} \nabla^{m+1} u|_{p,T} \leqq C'; \tag{1.8b}$$

$$\nabla^k \partial_t^h u \in B_{p,T} \quad \textit{and} \quad |\nabla^k \partial_t^h u|_{p,T} \leqq C' \quad \textit{for } k + 2h \leqq m, \tag{1.8c}$$

where C' depends only on n, m, p and on bounds for ν and $\|\nabla^k a\|_p$, $k = 0, \ldots, m$.

Proof. (i), (ii). Consider the following scheme of successive approximations for (1.1):

$$u_{j+1} = u_0 + S[u_j], \quad u_0 = e^{\nu t\Delta} a, \quad j = 0, 1, \ldots \tag{1.9}$$

Lemma 1.1 (i) and the estimate $\|e^{\nu t\Delta} a\|_p \leqq \|a\|_p$ together yield

$$|u_{j+1}|_{p,T} \leqq \|a\|_p + M(\nu T)^\sigma |u_j|_{p,T}^2/\nu.$$

This implies that, for all $j \geqq 0$

$$|u_j|_{p,T} \leqq K = 2r\theta^{-1}\|a\|_p, \quad r = 1 - (1-\theta)^{\frac{1}{2}} < 1, \tag{1.10a}$$

provided

$$0 < \theta = 4 \|a\|_p M(\nu T)^\sigma/\nu < 1. \tag{1.11}$$

For θ, $0 < \theta < 1$, we take $T > 0$ so that (1.11) holds.

Taking the gradient of (1.9) and then applying Lemma 1.1 (ii), together with (1.3) and (1.10a), yields

$$|(\nu t)^{\frac{1}{2}} \nabla u_{j+1}|_p \leqq C \|a\|_p + MK(\nu T)^\sigma |(\nu t)^{\frac{1}{2}} \nabla u_j|_p/\nu.$$

Here and hereafter we drop the subscript T to simplify the notation. By definition of K and (1.11), the second term of the right-hand side does not exceed $(r/2)\,|(\nu t)^{\frac{1}{2}} \nabla u_j|_p$. Hence

$$|(\nu t)^{\frac{1}{2}} \nabla u_j|_p \leqq 2C \|a\|_p, \quad \text{for all } j \geqq 0, \tag{1.10b}$$

with C depending only on p and n.

Similarly, we take the gradient of (1.9) and apply Lemma 1.1 (iii), (1.10a) and (1.10b) to get

$$|\nabla u_{j+1}|_q \leqq \|\nabla a\|_q + ML(\nu T)^\sigma \|a\|_p\, |\nabla u_j|_q/\nu$$

with L depending only on p and n. If θ is sufficiently small, say $0 < \theta < 2/L$, then the above estimate gives

$$|\nabla u_{j+1}|_q \leqq \|\nabla a\|_q + \tfrac{1}{2}\,|\nabla u_j|_q$$

which yields the bound

$$|\nabla u_j|_{q,T} \leqq 2\, \|\nabla a\|_q \quad \text{for all } j \geqq 0. \tag{1.10c}$$

Here and hereafter we fix T so that (1.11) holds with $\theta L < 2$.

We claim that u_j and $(\nu t)^{\frac{1}{2}} \nabla u_j$ (or ∇u_j) are Cauchy sequences in $B_{p,T}$ (or $B_{q,T}$). Indeed, we estimate the difference $w_j = u_{j+1} - u_j$, by applying Lemma 1.1 to (1.9). After a routine calculation, we see that (1.10a)–(1.10c) yield that $\Sigma_j\, |w_j|_p$, $\Sigma_j\, |(\nu t)^{\frac{1}{2}} \nabla w_j|_p$, $\Sigma_j\, |\nabla w_j|_q$ are finite. Since the idea of the proof is standard, the details may be omitted.

The estimate (1.6) is obvious from our choice of θ and T. Since u_j and $(\nu t)^{\frac{1}{2}} \nabla u_j$ (or ∇u_j) are Cauchy sequences in $B_{p,T}$ (or $B_{q,T}$), we see that the limit, $u = \lim u_j$, is a solution of (1.1) in $B_{p,T}$. Also u satisfies (1.5), (1.7a), (1.7b) by passing to the limit, $j \to \infty$, in (1.10a), (1.10b), (1.10c). The proof of uniqueness in $B_{p,T}$ is standard (see [12]), so (i), (ii) are proved.

(iii) The proof is routine and long so we just give an outline. We differentiate (1.9) with respect to the spatial variables and prove that for $k = 0 \dots m$

$$|\nabla^k u_j|_p \leqq C \qquad |(\nu t)^{\frac{1}{2}} \nabla^{m+1} u_j|_p \leqq C,$$

with C independent of j. This follows from our *a priori* bounds (1.10a), (1.10b) and Lemma 1.1. As in the proof of (i), (ii), we see that $\nabla^k u_j$ and $(\nu t)^{\frac{1}{2}} \nabla^{m+1} u_j$ are Cauchy sequences in $B_{p,T}$. This yields (1.8a), (1.8b). The estimate (1.8c) follows from (1.8a), (1.8b) and the equation

$$u' = \nu \Delta u - P(u \cdot \nabla)\, u. \quad \square$$

Remark. The basic idea of the above proof goes back to LERAY [17], who constructed a global regular solution, when $n = 2$, by a successive approximations scheme assuming that a is in $H^1 \cap L^\infty$. A proof of (i) is given in GIGA [12, Theorem 1 and Sect. 4].

The next theorem shows that the solutions of (1.1) in Proposition 1.2 are regular.

Proposition 1.3. (i) *Let* $a \in L^p(R^n)$, *for some* $p > n$ *and* $\nabla \cdot a = 0$. *Let* u *be the solution of* (1.1) *given in Proposition* 1.2. *Then,* $\nabla^k \partial_t^h u \in \mathrm{BC}([\varepsilon, T); L^p(R^n))$ *for all* $k, h \geqq 0$ *and* $0 < \varepsilon < T$. *Moreover,*

$$\sup_{[\varepsilon,T)} \|\nabla^k \partial_t^h u\|_p (t) \leqq C$$

where C *depends only on* ε, p, n, k, h *and on an upper bound for* $\|a\|_p$. *In particular,* u *is smooth for* $t > 0$ *and solves the Navier-Stokes system in the classical sense for* $t > 0$.

(ii) *Suppose further that* $\nabla^k a \in L^p(R^n)$ *for all* $k \geqq 0$. *Then* $\nabla_k \partial_t^h u$ *is bounded and continuous on* $R^n \times [0\ T)$ *for all* $k, h \geqq 0$. *Moreover,*

$$\sup_{[0,T)} \|\nabla^k \partial_t^h u\|_\infty (t) \leqq C,$$

where C *depends only on* p, n, k, h, ν *and on upper bounds for* $\max_{0 \leqq l \leqq k+2h+1} (\|\nabla^l a\|_p)$.

Proof. (i) By (1.7) we have $\|\nabla u\|_p (t_0) \leqq C$ for $0 < t_0 < T$ with C depending only on n, p, t_0 and $\|a\|_p$. We then solve the Navier-Stokes system for $t \geqq t_0$ with initial velocity $u(\cdot, t_0)$ and obtain $\|\nabla^2 u\|_p (2t_0) \leqq C$. Repeating this process, we infer that $\|\nabla^m u\|_p (mt_0)$ is bounded by the same constant C so long as $mt_0 < T$. Since t_0 can be taken arbitrarily small, this shows that $\nabla^m u$ is in $\mathrm{BC}([\varepsilon, T); L^p(R^n))$ for all $\varepsilon > 0$ and its norm is bounded by C depending only on p, n, m, ε and $\|a\|_p$. Combining this with (1.8c) yields the estimate in (i). The assertion on smoothness follows immediately from the Sobolev inequality.

(ii) This follows from (1.8c) by use of the Sobolev inequality. □

Remark. We note that Proposition 1.3 (ii) also follows from [7, Theorem 3.4] or [34]. However, apparently no estimate of the form (1.6) is given in [7] or [34] for the time T. Moreover, it seems that Proposition 1.3 (i) does not follow directly from the results of [7] or [34].

2. Global Existence and Persistency by Use of the Vorticity Equation

The goal of this section is to show global existence of solutions for the Navier-Stokes system (1) in R^2 without assuming that the initial energy is finite. As a byproduct we show that our solutions persist in the Sobolev spaces $W^{m,p}(R^2)$, $p > 2$, $m = 0, 1, 2, \ldots$ This leads to global existence of solutions for the Euler equations as $\nu \to 0$. It should be noted that we are dealing here with solutions with infinite energy, so our results are not included in either [15] or [27]. Since the standard energy method fails in our case, we are forced to appeal to the vorticity equation in order to get the desired results. Such an argument is found in McGRATH [21], under more stringent assumptions on the initial vorticity. Here we base our results on global existence for both the Navier-Stokes and the Euler equations on the results in Section 1. This relaxes the assumptions and simplifies the proofs of [21]. In what follows we always assume that the spatial dimension is 2, unless otherwise specified.

Suppose that the initial velocity a and all its derivatives are in $L^p(R^2)$ for some $p > 2$. Proposition 1.3 (ii) then says that there is a unique local solution of (1) which is smooth and bounded on $R^2 \times [0, T)$. We here apply $\nabla \times$ to (1) and obtain the vorticity equation for $v = \nabla \times u = \partial u^2/\partial x_1 - \partial u^1/\partial x_2$:

$$L_u v \equiv v' - \nu \Delta v + (u \cdot \nabla) v = 0, \quad t \in (0, T), \tag{V-1}$$

$$v(x, 0) = \nabla \times a.$$

Since u and all its derivatives are bounded on $R^2 \times [0, T)$, the linear parabolic operator L_u has a unique fundamental solution

$$\Gamma_u(x, t; y, s), \quad 0 \leqq s < t < T, \quad x, y \in R^2$$

such that $L_u \Gamma_u = 0$ as a function of (x, t) and

$$\lim_{t \downarrow s} \int_{R^2} \Gamma_u(x, t; y, s) f(y)\, dy = f(x)$$

for every $f \in \mathrm{BC}(R^2)$; see [8, Chapter 1].

Let us quickly review some properties of Γ_u which are needed later. It is well known that $\Gamma_u > 0$ and that the function

$$w(x, t) = \int_{R^2} \Gamma_u(x, t; y, s) f(y)\, dy \tag{2.1}$$

is a unique bounded classical solution of $L_u w = 0$ $(t > s)$, $w(x, s) = f \in \mathrm{BC}(R^2)$; see [8, Chapter 1, 2]. Since L_u has no zero$^{\text{th}}$-order term, $w \equiv 1$ is a unique bounded solution to $L_u w = 0$ $(t > s)$, $w(x, s) = 1$. By (2.1) this yields

$$\int_{R^2} \Gamma_u(x, t; y, s)\, dy = 1 \qquad 0 \leqq s < t < T. \tag{2.2}$$

The function $\Gamma_u^*(x, t; y, s) = \Gamma_u(y, s; x, t)$, $0 \leqq t < s < T$, is the fundamental solution of the adjoint problem

$$w' + \nu \Delta w - \nabla \cdot (uw) = 0, \quad 0 \leqq t < T,$$

which is the same as

$$w' + \nu \Delta w - (u \cdot \nabla) w = 0$$

since $\nabla \cdot u = 0$. In analogy to (2.2) we have

$$\int_{R^2} \Gamma_u(y, s; x, t)\, dy = 1, \quad 0 \leqq t < s < T. \tag{2.3}$$

The following result is immediately obtained from Propositions 1.2, 1.3 and the identities (2.2) and (2.3).

Proposition 2.1. (i) *Suppose that* $\nabla^k a \in L^p(R^2)$, $k = 0, 1, 2, \ldots,$ *for some* $p > 2$ *and that* $\nabla \cdot a = 0$. *Let* u *be the local solution of* (1) *given in Proposition* 1.2. *Then* $v = \nabla \times u$ *is given by*

$$v(x, t) = \int_{R^2} \Gamma_u(x, t; y, 0)\, (\nabla \times a)\, (y)\, dy, \quad 0 < t < T. \tag{2.4}$$

(ii) *Suppose further that* $\nabla \times a \in L^q(R^2)$ *for some* q *with* $1 \leqq q \leqq \infty$. *Then*

$$\|v\|_q\,(t) \leqq \|\nabla \times a\|_q, \quad 0 \leqq t < T. \tag{2.5}$$

We next consider how to recover the velocity field u from the solution v of the equation (V-1). Since $\nabla \cdot u = 0$, it is easily seen that

$$\Delta u = \nabla^\perp v, \quad \text{where } \nabla^\perp v = (-\partial v/\partial x_2, \partial v/\partial x_1).$$

It is thus to be expected that if u decays as $|x| \to \infty$, then

$$u = E * \nabla^\perp v = (\nabla^\perp E) * v$$

where $E = (2\pi)^{-1} \log |x|$ is a fundamental solution of Δ in R^2 and $*$ denotes the convolution in R^2. We shall now show that this is true in our setting. To this end we introduce certain function spaces. By $\mathcal{M}$ we denote the space of all finite Radon measures on R^2 with norm defined by the total variation. A measurable function f on R^2 is said to be in $L^{p,\infty}(R^2)$, $1 < p < \infty$, if

$$\|f\|_{p,\infty} = \sup_{\lambda > 0} \lambda\, [\text{mea}\, \{x;\, |f(x)| > \lambda\}]^{1/p} < \infty$$

where mea is Lebesgue measure in R^2. Although $\|f\|_{p,\infty}$ does not satisfy the usual triangle inequality, it is a pseudo-norm on the linear space $L^{p,\infty}$ and $L^{p,\infty}$ is a Banach space with a norm equivalent to $\|f\|_{p,\infty}$ (see [4]). $L^{p,\infty}$ is often called a Lorentz space.

In what follows we let

$$K(x) = \nabla^\perp E(x) = (-x_2, x_1)/2\pi\, |x|^2 \quad \text{for } x = (x_1, x_2) \in R^2$$

and consider the convolution operator $U = K * V = \int_{R^2} K(x - y)\, V(y)\, dy$. Note that $K \in L^{2,\infty}(R^2)$ and that K is not contained in any $L^p(R^2)$, $1 \leqq p \leqq \infty$.

Lemma 2.2. (i) *For* $U = K * V$ *we have the estimates*:

$$\|U\|_p \leqq C\, \|K\|_{2,\infty}\, \|V\|_q, \quad \text{if } 1 < q < 2, \quad V \in L^q(R^2) \quad \text{and} \quad 1/p = 1/q - 1/2; \tag{2.6a}$$

$$\|U\|_{2,\infty} \leqq C\, \|K\|_{2,\infty}\, \|V\|_{\mathcal{M}} \quad \text{for} \quad V \in \mathcal{M}; \tag{2.6b}$$

$$\|\nabla U\|_r \leqq C\, \|V\|_r \quad \text{for} \quad V \in L^r(R^2), \quad 1 < r < \infty, \tag{2.6c}$$

with C *independent of* V, *where* $\|V\|_{\mathcal{M}}$ *denotes the total variation of the Radon measure* V.

(ii) *Suppose that* $U \in L^p(R^2)$, $2 < p < \infty$, *with* $\nabla \cdot U = 0$ *and that* $\nabla \times U \in L^q(R^2)$ *with* $1/q = 1/p + 1/2$. *Then*

$$U = K * (\nabla \times U).$$

(iii) *Suppose that* $U \in L^{2,\infty}(R^2)$ *with* $\nabla \cdot U = 0$ *and that* $\nabla \times U \in \mathcal{M}$. *Then*

$$U = K * (\nabla \times U).$$

Proof. (i). (2.6a) is simply the generalized YOUNG's inequality (see [28, p. 32]). Since ∇K is a CALDERON-ZYGMUND kernel, (2.6c) follows from the standard theory of singular integral operators; see [13, Chapter 9]. To show (2.6b) consider the linear operator $Af = f * V$ for any fixed $V \in \mathcal{M}$. It is easily verified that A defines a bounded linear operator on each $L^p(R^2)$, $1 \leqq p \leqq \infty$, with norm $\leqq \|V\|_{\mathcal{M}}$. An interpolation theorem for Lorentz spaces ([4, Theorem 5.3.4]) now implies that A is bounded on $L^{2,\infty}(R^2)$ with norm $\leqq C\|V\|_{\mathcal{M}}$. This proves (2.6b).

(ii), (iii). The function $W = K * (\nabla \times U)$ is in $L^p(R^2)$ (or $L^{2,\infty}(R^2)$) by (2.6a) and (2.6b), and satisfies $\nabla \cdot W = 0$, $\nabla \times W = \nabla \times U$. Therefore, $Z = U - W$ is harmonic on R^2 and belongs to $L^p(R^2)$ (or $L^{2,\infty}(R^2)$). The mean-value theorem for harmonic functions yields, for every $x \in R^2$,

$$|Z(x)| \leqq \text{mea}\,(B)^{-1} \int_B |Z(y)|\,dy \leqq C\,\|Z\|_p \text{ (respectively } \leqq C\,\|Z\|_{2,\infty})$$

where B is the unit disc in R^2 with center x and C is independent of x. Liouville's theorem for harmonic functions now implies that Z is a constant, which must be equal to 0 since $Z \in L^p(R^2)$ (or $L^{2,\infty}(R^2)$). This proves (ii) and (iii). □

Proposition 2.3. *Let* $\nabla^k a \in L^p(R^2)$, $k = 0, 1, \ldots,$ *for some* $p > 2$. *Suppose further that* $\nabla \cdot a = 0$ *and* $\nabla \times a \in L^q(R^2)$ *with* $1/q = 1/p + 1/2$. *Then the local solution* u *given in Section* 1 *satisfies*

$$u(x, t) = K * (\nabla \times u) = \int_{R^2} K(x - y)\,(\nabla \times u)\,(y, t)\,dy, \qquad 0 \leqq t < T. \tag{V-2}$$

Moreover, the estimate

$$\|u\|_p\,(t) \leqq C\,\|\nabla \times u\|_q\,(t) \leqq C\,\|\nabla \times a\|_q, \qquad 0 \leqq t < T \tag{2.7}$$

holds with C *depending only on* p.

Proof. By Proposition 1.2 (i), $u(\cdot, t)$ is in $L^p(R^2)$. Thus (V-2) follows from Lemma 2.2 (ii). (2.7) is then immediately obtained from (V-2), (2.6a) and (2.5). □

We can now prove our result on global extension, using the estimate (2.7).

Theorem 2.4. *Suppose that* $\nabla^k a \in L^p(R^2)$, $k = 0, 1, \ldots,$ *for some* $p > 2$, *and that* $\nabla \cdot a = 0$. *Suppose further that* $\nabla \times a \in L^q(R^2)$ *with* $1/q = 1/p + 1/2$. *Then the local solution of* (1) *given in Proposition* 1.2 *may be extended uniquely to a global* (*in time*) *solution* u *such that* $u \in B_{p,\infty}$, $\nabla u \in B_{q,\infty}$ *and*

$$|u|_{p,\infty} \leqq C\,\|\nabla \times a\|_q, \qquad |\nabla u|_{q,\infty} \leqq C\,\|\nabla \times a\|_q$$

where C depends only on p. Moreover, the derivatives $\nabla^k \partial_t^h u$ *belong to* $B_{p,T}$ *for every finite* $T > 0$ *and satisfy*

$$|\nabla^k \partial_t^h u|_{p,T} \leqq C$$

with C depending only on p, k, h, T, ν, and on bounds for $\max_{0 \leqq l \leqq k+2h} (\|\nabla^l a\|_p)$ *and* $\|\nabla \times a\|_q$.

Proof. Take T as in the proof of Proposition 1.2 with $\|a\|_p$ replaced by $C\|\nabla \times a\|_q$, where C is the constant in (2.7). For any $t_0 \in (0, T)$, (2.7) shows that $\|u\|_p(t_0)$ has a bound depending only on p and $\|\nabla \times a\|_q$. Therefore, the argument in the proof of Proposition 1.2 ensures the existence of a unique solution on $[t_0, t_0 + T)$ with initial value $u(\cdot, t_0)$. Suppose now that u may be extended uniquely to some finite interval $[0\ T_1)$. Then (2.7) holds on $[0\ T_1)$ as seen from Propositions 2.2 and 2.3. Thus u may be extended uniquely to the interval $[0, T_1 + T)$. Since T is independent of T_1, we conclude that u may be extended in a unique way to the whole interval $[0, \infty)$. By (2.7) and (2.6c), we easily see that $u \in B_{p,\infty}$ and $\nabla u \in B_{q,\infty}$ and admit the required bounds. Bounds for $\nabla^k \partial_t^h u$ are obtained from Proposition 1.2 (iii). □

The assumption $\nabla^k a \in L^p(R^2)$, $k = 0, 1\ \ldots$, is made so that the local solution $u(x, t)$ be sufficiently regular up to $t = 0$. Since the equation (1) is parabolic, it is natural to expect global existence even if we drop the regularity assumptions on a.

Theorem 2.5. *Suppose that* $a \in L^p(R^2)$ *for some* $p > 2$ *with* $\nabla \cdot a = 0$ *and* $\nabla \times a \in L^q(R^2)$, $1/q = 1/p + 1/2$. *Then there is a unique global solution u of* (1) *such that* $u \in B_{p,\infty}$, $\nabla u \in B_{q,\infty}$ *and*

$$|u|_{p,\infty} \leqq C\|\nabla \times a\|_q, \qquad |\nabla u|_{q,\infty} \leqq C\|\nabla \times a\|_q$$

with C depending only on p. Moreover, all derivatives $\nabla^k \partial_t^h u$ *exist on* $R^2 \times [\varepsilon, \infty)$ *for any* $\varepsilon > 0$ *and satisfy*

$$\sup_{[\varepsilon, T]} \|\nabla^k \partial_t^h u\|_\infty (t) \leqq C$$

where C depends only on p, T, k, h, ε, ν, and on a bound for $\|\nabla \times a\|_q$.

Proof. Let u be the local solution in Proposition 1.3 (i). Since we have $\nabla a \in L^q(R^2)$ by Lemma 2.2 (i) (ii), (1.7b) now implies that ∇u is in $B_{q,T}$ for some T. For every t_0, $0 < t_0 < T$, we have

$$\|\nabla^k u(t_0)\|_p \leqq C, \quad k = 0, 1, 2, \ldots \tag{2.8}$$

by Proposition 1.3 (i), where $C = C(p, k, t_0, \nu, \|a\|_p)$. Applying Theorem 2.4 with initial data $u(t_0)$, we find that our solution can be extended globally in time. In particular we obtain $u \in B_{p,\infty}$ and $\nabla u \in B_{q,\infty}$ and

$$\|u\|_p(t), \quad \|\nabla u\|_q(t) \leqq A\|\nabla \times u\|_q(t_0) \quad t \geqq t_0$$

with A depending only on q.

Letting $t_0 \to 0$ we show that

$$\|u\|_p(t), \|\nabla u\|_q(t) \leqq A \|\nabla \times a\|_q, \quad t \geqq 0, \tag{2.9}$$

and this proves the first part of Theorem 2.5.

By (2.8) and (2.9), Theorem 2.4 yields

$$\sup_{[t_0,T]} \|\nabla^k \partial_t^h u\|_p(t) \leqq C \tag{2.10}$$

with $C = C(p, k, h, \nu, t_0, T, \|\nabla \times a\|_q)$. The proof is now completed by applying the Sobolev inequality to (2.10). □

Remark. MARCHIORO & PULVIRENTI [19] and OSADA [26] establish existence assuming that $\nabla \times a \in L^1 \cap L^\infty$. This assumption implies that $\nabla \times a \in L^q$, so one can apply Theorem 2.5 to get global existence.

We finally prove that our solution is persistent in the sense of KATO [15] and PONCE [27]. Although our argument is not original, we state our precise result since it concerns solutions with infinite energy and therefore is not contained in either [15] or [27]. In what follows, $W^{m,p}(R^2)$, $m = 0, 1, \ldots$, denotes the usual Sobolev space. The norm of $W^{m,p}(R^2)$ is written as $\|\cdot\|_{W^{m,p}}$.

Theorem 2.6. *Let $a \in W^{m,p}(R^2)$ for some $p > 2$ with $\nabla \cdot a = 0$ and $\nabla \times a \in L^q(R^2)$, where $1/q = 1/p + 1/2$. Then the solution u of* (1) *given in Theorem* 2.5 *is in* BC([0, T]; $W^{m,p}(R^2)$) *for all $T > 0$ and satisfies*

$$\sup_{[0,T]} \|u\|_{W^{m,p}}(t) \leqq C \textit{ uniformly for } \nu > 0. \tag{2.11}$$

Proof. It suffices to prove (2.11) since that $u \in$ BC([0, T], $W^{m,p}(R^2)$) follows directly from (1.8a) and (2.10). Since (2.11) for $m = 0$ follows from Theorem 2.5, we may assume $m \geqq 1$. First assume that $m = 1$ and consider the equation for the vorticity:

$$v' - \nu \Delta v + (u \cdot \nabla) v = 0 \quad (t > t_0), \tag{2.12}$$

$$v = \nabla \times u, \quad v(x, t_0) = \nabla \times u(x, t_0),$$

where $t_0 > 0$. By (2.9) and (2.10), applying (2.5) to (2.12) yields $\|v\|_p(t) \leqq \|\nabla \times u\|_p(t_0)$ for all $t \geqq t_0$ and therefore, by (2.6c), $\|\nabla u\|_p(t) \leqq C\|v\|_p(t) \leqq C\|\nabla \times u\|_p(t_0)$ for all $t \geqq t_0$ with C depending only on p. Since $\nabla u \in B_{p,T}$, by (1.8a) and $\nabla a \in L^p(R^2)$, letting $t_0 \to 0$ yields $\|\nabla u\|_p(t) \leqq C\|\nabla \times a\|_p$ for all $t \geqq 0$ and this establishes (2.11) for $m = 1$.

We next assume that $m = 2$. We apply ∇ to (2.12), multiply the resulting equality by $|\nabla v|^{p-2} \nabla v$ and integrate by parts, using $\nabla \cdot u = 0$, to get

$$\frac{d}{dt}\|\nabla v\|_p^p \leqq C\|\nabla u\|_\infty \|\nabla v\|_p^p, \quad t \geqq t_0 \tag{2.13}$$

with C depending only on p. To estimate $\|\nabla u\|_\infty$ we appeal to the following result of KATO [15, Lemma A3]:

$$\|\nabla u\|_\infty \leqq C(\|v\|_\infty + \|v\|_2 + \|v\|_\infty \log [1 + (\|\nabla v\|_p/\|v\|_\infty)]) \tag{2.14}$$

where C depends only on p. Using (2.5) and the Sobolev inequality, we have $\|v\|_\infty \leqq \|\nabla \times a\|_\infty \leqq C\|a\|_{W^{2,p}}$ and $\|v\|_2 \leqq \|v\|_q^{1-2/p}\|v\|_q^{2/p} \leqq \|\nabla \times a\|_q^{1-2/p}\|\nabla \times a\|_p^{2/p}$. Thus (2.14) gives

$$\|\nabla u\|_\infty \leqq C(1 + \log^+ \|\nabla v\|_p)$$

with C depending only on p, $\|\nabla \times a\|_q$ and $\|a\|_{W^{2,p}}$. Combining this with (2.13) and integrating with respect to t now yields

$$\|\nabla v\|_p(t) \leqq C \quad \text{for} \quad t \in [t_0, T] \quad , \tag{2.15}$$

where C depends also on T. Since $\nabla^2 u = \nabla K * (\nabla v)$ and since ∇K is a Calderon-Zygmund kernel (2.15) implies that $\|\nabla^2 u\|_p(t) \leqq C$ on $[0, T]$. This implies (2.11) for $m = 2$.

Suppose finally that $m \geqq 3$. We apply ∇^k to (2.12), multiply the resulting equality by $|\nabla^k v|^{p-2}\nabla^k v$ and integrate over R^2. Integrating by parts and using the condition $\nabla \cdot u = 0$ and the Sobolev inequality, we deduce, after summation over $k = 0, 1, \ldots, m-1$,

$$\frac{d}{dt}\|v\|^p_{W^{m-1,p}} \leqq C\|u\|_{W^{m-1,p}}\|v\|^p_{W^{m-1,p}}$$

where C depends only on m and p. Integrating this and then using the estimate $\|u\|_{W^{m,p}} \leqq C(\|v\|_{W^{m-1,p}} + \|\nabla \times a\|_q)$, which follows from (2.7) and the relation $\nabla^k u = \nabla K * (\nabla^{k-1} v)$, we arrive at (2.11) by induction on m. □

Theorem 2.6 suggests that we can obtain a solution of the Euler equations (system (1) with $\nu = 0$) by passing to the limit $\nu \to 0$. For $m \geqq 2$, this is carried out by KATO & PONCE [34] with no assumption on the vorticity $\nabla \times a$. For the cases $m = 0, 1$, which are excluded in [34], our Theorem 2.6 gives the following result.

Corollary 2.7. (i) *Let* $a \in L^p(R^2)$, $p > 2$, $\nabla \cdot a = 0$ *and* $\nabla \times a \in L^q(R^2)$ *with* $1/q = 1/p + 1/2$. *Then there is a function* u *such that*:
(a) $u: [0, \infty) \to L^p(R^2)$ *is bounded and continuous in the weak topology and* $u(\cdot, 0) = a$.
(b) $P\nabla \cdot (u \otimes u)$ *is defined as an element of* $L^\infty(0, \infty; W^{-1,p/2}(R^2))$.
(c) $u' + P\nabla \cdot (u \otimes u) = 0$ *for* $t > 0$.

(ii) *Let* $a \in W^{1,p}(R^2)$, $p > 2$, $\nabla \cdot a = 0$ *and* $\nabla \times a \in L^q(R^2)$ *with* $1/q = 1/p + 1/2$. *Then there is a function* u *such that*:
(d) $u: [0, \infty) \to W^{1,p}(R^2)$ *is bounded and continuous in the weak topology and* $u(\cdot, 0) = a$.
(e) $P(u \cdot \nabla)u$ *is defined as an element of* $L^\infty(0, \infty; L^p(R^2))$.
(f) $u' + P(u \cdot \nabla)u = 0$ *for* $t > 0$.

Proof. We fix a and denote by u_ν, $\nu > 0$, the corresponding solution of (1).

(i) From (2.6c) and (2.7) we see that $\|\nabla u_\nu\|_q$ and $\|u_\nu\|_p$ are bounded in $L^\infty(0,\infty)$. Since $q < p$, this implies that the u_ν are bounded in $L^\infty(0,\infty; W^{1,q}(D))$ for any fixed open disc D. Also, Δu_ν and $P(u_\nu \cdot \nabla) u_\nu = P \nabla \cdot (u_\nu \otimes u_\nu)$ are bounded in $L^\infty(0,\infty; W^{-1,q}(R^2))$ and $L^\infty(0,\infty; W^{-1,p/2}(R^2))$, respectively. Since $q < p/2$, $W^{-1,p/2}(D) \subset W^{-1,q}(D)$ with continuous injection. Thus the equation

$$u_\nu' - \nu \Delta u_\nu + P\nabla(u_\nu \otimes u_\nu) = 0, \quad t > 0,$$

implies that the u_ν' are bounded in $L^\infty(0,\infty; W^{-1,q}(D))$. Since D is arbitrary, Lemma 2.1 in [30, Chapter III] ensures the existence of a subsequence of u_ν (which we denote also by u_ν) so that $u_\nu \to u$ a.e. in $R^2 \times (0,\infty)$ as $\nu \to 0$. The preceeding observation shows that we may assume $u \in L^\infty(0,\infty; L^p(R^2))$ and $\nabla u \in L^\infty(0,\infty; L^q(R^2))$. Since $\nu \Delta u_\nu \to 0$ as $\nu \to 0$ in $L^\infty(0,\infty; W^{-1,q}(R^2))$, a simple limiting argument gives

$$\frac{d}{dt}(u, \phi) - (u \otimes u, \nabla\phi) = 0 \quad \text{in } t > 0$$

for every smooth and divergence-free vector field ϕ with compact support. We can thus apply de Rham's theorem [30, Chapter 1] to conclude that

$$u' + \nabla \cdot (u \otimes u) + \nabla \Pi = 0, \quad t > 0, \tag{2.16}$$

for some distribution Π on $R^2 \times (0,\infty)$. Taking the divergence of (2.16) gives

$$\Delta \Pi = -\Sigma_{j,k}\, \partial_j \partial_k (u^j u^k),$$

which shows that we may take $\Pi = \Sigma_{j,k} R_j R_k (u^j u^k)$, where R_j are the Riesz transforms. By the boundedness of the operators R_j in $L^r(R^2)$, $1 < r < \infty$, the function $\nabla \Pi$ is in $L^\infty(0,\infty; W^{-1,p/2}(R^2))$. Thus (2.16) implies $u' \in L^\infty(0,\infty; W^{-1,p/2}(R^2))$, so that (c) follows by applying P to (2.16). (b) follows from the boundedness of P in $W^{-1,p/2}(R^2)$. From (b) and (c) it follows that u is continuous from $[0,\infty)$ to $W^{-1,p/2}(R^2)$, and so from $[0,\infty)$ to $W^{-1,q}(D)$, for any D. Since $L^p(D) \subset L^q(D) \subset W^{-1,q}(D)$ with continuous injections, Lemma 1.4 in [30, Chapter III] implies that u is continuous from $[0,\infty)$ to $L^p(D)$ in the weak topology. Since D is arbitrary and $\|u\|_p(t)$ is bounded, the Banach-Steinhaus theorem implies (a).

(ii) Theorem 2.6 shows that the u_ν are bounded in $L^\infty(0,\infty; W^{1,p}(R^2))$. Since $p > 2$, the Gagliardo-Nirenberg inequality: $\|f\|_\infty \leqq C\|f\|_p^{1-2/p} \|\nabla f\|_p^{2/p}$ yields the boundedness of $P(u_\nu \cdot \nabla) u_\nu$ in $L^\infty(0,\infty; L^p(R^2))$. This, together with the boundedness of Δu_ν in $L^\infty(0,\infty; W^{-1,p}(R^2))$, implies that the u_ν' are bounded in $L^\infty(0,\infty; W^{-1,p}(R^2))$. We can thus apply Lemma 2.1 in [30, Chapter III] to conclude that $u_\nu \to u$ as $\nu \to 0$, a.e. in $R^2 \times (0,\infty)$. As in the proof of (i), one can show that

$$u' + P\nabla \cdot (u \otimes u) = 0, \quad t > 0. \tag{2.17}$$

Since $u(\cdot, t) \in W^{1,p}(R^2)$ for almost all $t > 0$ and $\nabla \cdot u = 0$, we see that $\nabla \cdot (u \otimes u) = (u \cdot \nabla) u$. Thus (2.17) is rewritten in the form (f). (e) is obtained easily by applying the Gagliardo-Nirenberg inequality. (e) and (f) together imply that u is continuous from $[0,\infty)$ to $L^p(R^2)$. Since u lies in $L^\infty(0,\infty; W^{1,p}(R^2))$, Lemma 1.4 in [30, Chapter III] ensures the continuity of u as asserted in (d). □

Recently, KATO & PONCE [34] have extended their results in [15] and [27] to L^p spaces. They prove the persistency of solutions of (1) with $\nu \geqq 0$ in $H^{s,p}$, $s > 1 + 2/p$. However, our Theorem 2.6 and Corollary 2.7 are not covered by their results when m is 0 or 1.

3. New a priori Estimates

This section establishes new *a priori* estimates for solutions of (1) in R^2, which depend only on the norm of the measure $\nabla \times a$. These estimates allow us to take a subsequence of solutions for the regularized initial data which converges to the desired solution of the original problem. Our argument is based on a comparison theorem of OSADA [25] for the fundamental solution of the heat operator $\partial_t - \nu \Delta$ and also of the operator $L_b = \partial_t - \nu \Delta + (b \cdot \nabla)$ with $\nabla \cdot b = 0$. We note that [25] extends results in [1, 2] to operators in non-divergence form.

To be precise, we consider a parabolic operator in R^n $(n \geqq 2)$ of the form:

$$L_b = \partial_t - \nu \Delta + (b \cdot \nabla),$$

under the following assumptions:

The vector function $b = b(x, t)$ is bounded and continuous on $R^n \times [0, T)$, together with all its derivatives, and satisfies $\nabla \cdot b = 0$. (3.1)

There are functions $c^{ij}(x, t)$, $i, j = 1, \ldots, n$, such that (3.2)

$$\sup |c^{ij}(x, t)| \leqq \alpha, \quad i, j = 1, \ldots, n,$$

for some $\alpha > 0$ and

$$b^i = \Sigma_j \partial_j c^{ij}, \quad i = 1, \ldots, n, \quad \partial_j = \partial/\partial x_j$$

where b^i is the i^{th} component of b.

Since b is assumed to be smooth and bounded, L_b has a unique fundamental solution (see [8, Chapter 1, 2]), which we denote by $\Gamma_b(x, t; y, s)$, $x, y \in R^n$, $0 \leqq s < t < T$.

Theorem 3.1. ([25]). *Suppose that b satisfies* (3.1) *and* (3.2). *Then the following estimates hold for the fundamental solution Γ_b of L_b.*

(i) *There are positive constants C_j, $j = 1, 2, 3, 4$, depending only on n, α and ν such that*

$$C_1(t-s)^{-n/2} \exp[-C_2 |x-y|^2/(t-s)] \leqq \Gamma_b(x, t; y, s) \leqq C_3(t-s)^{-n/2} \exp[-C_4 |x-y|^2/(t-s)] \tag{3.3}$$

for all $x, y \in R^n$ *and* $0 \leqq s < t < T$.

(ii) *There is a* β, $0 < \beta < 1$, *depending only on* α *and* ν *such that*

$$|\Gamma_b(x, t; y, s) - \Gamma_b(x', t'; y', s')| \leqq C_5(|s-s'|^{\beta/2} + |y-y'|^{\beta} + |t-t'|^{\beta/2} + |x-x'|^{\beta}) \tag{3.4}$$

for all $\tau < t - s$, $t' - s' < \infty$ *and* $x, x', y, y' \in R^n$, *where* C_5 *depends only on* n, ν, α *and* $\tau > 0$.

The smoothness assumption on b is in fact unnecessary and is made here only in order to render the fundamental solution unique. For the full version of Theorem 3.1 and its proof, we refer the reader to [25].

Let us now consider the equation for the vorticity in R^2 for $v = \nabla \times u$:

$$L_u v \equiv v' - \nu \Delta v + (u \cdot \nabla) v = 0, \quad t > 0, \quad v(x, 0) = \nabla \times a; \tag{V-1}$$

$$u = K * v, \quad K(x) = (-x_2, x_1)/2\pi |x|^2, \quad x = (x_1, x_2). \tag{V-2}$$

The next two propositions show that Theorem 3.1 is applicable to L_u provided that the solution u of (1) is smooth on $R^2 \times [0, T)$ and $\nabla \times a$ is a finite measure on R^2.

Lemma 3.2 ([25]). *The function* $K = (K^1, K^2)$ *given in* (V-2) *is expressed as*

$$K^1 = \partial_1 A^3 + \partial_2 A^1, \quad K^2 = -\partial_1 A^1 - \partial_2 A^2,$$

where

$$A^1 = -x_1^2 x_2^2/\pi |x|^4, \quad A^2 = -3x_1x_2/2\pi |x|^2 + x_1^3 x_2/\pi |x|^4,$$
$$A^3 = -3x_1x_2/2\pi |x|^2 + x_1 x_2^3/\pi |x|^4.$$

Proof. The lemma is verified by direct calculation. □

Lemma 3.3. *Let* $U = K * V$ *with* $V \in \mathcal{M}$. *Then* U *may be expressed as*

$$U^i = \sum_{j=1}^{2} \partial_j c^{ij}, \quad i = 1, 2, \quad |c^{ij}(x)| \leqq M \quad \text{on } R^2$$

with M *depending only on an upper bound of* $\|V\|_{\mathcal{M}}$.

Proof. We define

$$c^{11} = A^3 * V, \quad c^{12} = A^1 * V, \quad c^{21} = -A^1 * V, \quad c^{22} = -A^2 * V,$$

where A^k, $k = 1, 2, 3$, are the functions introduced in Lemma 3.2. Since each A^k is in $L^\infty(R^2)$, we have $c^{ij} \in L^\infty(R^2)$ with $\|c^{ij}\|_\infty \leqq N \|V\|_{\mathcal{M}}$ where N depends only on $\|A^k\|_\infty$, $k = 1, 2, 3$. The expression for U follows immediately from Lemma 3.2 □

Using Theorem 3.1, Lemma 3.2 and 3.3, we now prove the main theorem of this section.

Proposition 3.4. *Let u be the unique global solution of* (1) *given in Theorem* 2.4. *Suppose further that* $v_0 \equiv \nabla \times a$ *is in* $L^1(R^2)$ *with* $\|v_0\|_1 \leqq m$, *and let* Γ_u *be the fundamental solution of the operator* L_u. *Then the following hold*:

$$\|v\|_1(t) \leqq \|v_0\|_1, \quad v = \nabla \times u, \quad \|u\|_{2,\infty}(t) \leqq C \|v_0\|_1 \quad \textit{for } t \geqq 0, \tag{3.5}$$

where $\|\cdot\|_{2,\infty}$ *is the norm of* $L^{2,\infty}(R^2)$ *and* C *depends only on* $\|K\|_{2,\infty}$.

$$C_1(t-s)^{-1}\exp\left[-C_2|x-y|^2/(t-s)\right]$$
$$\leqq \Gamma_u(x,t;y,s)\leqq C_3(t-s)^{-1}\exp\left[-C_4|x-y|^2/(t-s)\right],\quad t>s\geqq 0,\quad (3.6)$$

with C_j, $j=1,2,3,4$, *depending only on* ν *and* m.

$$\|v\|_r(t)\leqq Ct^{-1+1/r}\|v_0\|_1\quad \text{for } t>0 \quad \text{and } 1<r\leqq\infty, \qquad (3.7\text{a})$$

$$\|\nabla u\|_r(t)\leqq Ct^{-1+1/r}\|v_0\|_1\quad \text{for } t>0 \quad \text{and } 1<r<\infty, \qquad (3.7\text{b})$$

$$\|u\|_r(t)\leqq Ct^{1/r-1/2}\|v_0\|_1\quad \text{for } t>0 \quad \text{and } 2<r\leqq\infty, \qquad (3.7\text{c})$$

with C *depending only on* r, m *and* ν.

$$\sup_{[\varepsilon,T]}\|\nabla^k\partial_t^h u\|_\infty(t)\leqq C,\qquad \varepsilon>0 \qquad (3.8)$$

with C *depending only on* ε, h, k, ν, T *and* m.

Proof. By the assumption, u together with its derivatives on each slab $R^2\times[0,T]$ are smooth and bounded. Therefore, the fundamental solution Γ_u exists and is unique. The estimates (3.5) follow from (2.5) and (2.6b). The estimate (3.6) is obtained from Theorem 3.1, since Lemma 3.3 applies to $u=K*v$ in view of the estimate (3.5) for v.

The estimate (3.7a) follows from (3.6). Lemma 2.2 together with (3.7a) yield (3.7b) and (3.7c) for $2<r<\infty$. The remaining case (3.7c) for $r=\infty$, namely,

$$\|u\|_\infty\leqq Ct^{-\frac{1}{2}}\|v_0\|_1,$$

is deduced by applying the Gagliardo-Nirenberg inequality: $\|u\|_\infty\leqq C\|u\|_r^{1-2/r}\|\nabla u\|_r^{2/r}$, $r>2$ (see [9, p. 24, Theorem 9.3]) to (3.7b) and (3.7c) for finite r.

It remains to prove (3.8). Taking $t_0=\varepsilon/2$, we see by (3.7c) that

$$\|u\|_r(t_0)\leqq C,\qquad r>2$$

with C depending only on t_0, r, ν and m, where $t_0=\varepsilon/2$. Applying Proposition 1.3 (i) with initial data $u(t_0)$ and $p=r$ yields (3.8) by uniqueness. □

Our next theorem concerns the continuity of the function $v(\cdot,t)=(\nabla\times u)(\cdot,t)$ when $\nabla\times a$ is a measure, and enables us to give a precise meaning to the initial condition $u(\cdot,0)=a$.

Proposition 3.5. *Let* u *and* a *be as in Proposition* 3.4, *and let* $v=\nabla\times u$, $v_0=\nabla\times a$. *Then for each* $m>0$ *and* $T>0$ *the functions* $v(\cdot,t)$, $\|v_0\|_1\leqq m$, *are equicontinuous from* $[0,T]$ *to* $\mathcal{M}$ *under the topology of weak convergence of measures. In other words, the pairing* $(v(\cdot,t),\phi)$ *of* $\phi\in \mathrm{BC}(R^2)$ *with the measure* $v(\cdot,t)$ *satisfies*

$$(v(\cdot,t),\phi)\to(v(\cdot,s)\,\phi)\qquad \text{as } t\to s$$

for all $s\in[0,T]$, *and the convergence is uniform in* v *for* $\|v_0\|_1\leqq m$.

Proof. On $\mathcal{M}_m^+ = \{\mu \in \mathcal{M}; \mu \geqq 0, \|\mu\|_{\mathcal{M}} \leqq m\}$ consider the function

$$R(\mu_1, \mu_2) = \inf \int_{R^2 \times R^2} (|x-y| \wedge 1)\, d\lambda(x, y), \qquad \mu_1, \mu_2 \in \mathcal{M}_m^+$$

where the infimum is taken over all measures $\lambda \geqq 0$ on $R^2 \times R^2$ such that $\Pi_1 \lambda = \mu_1$ and $\Pi_2 \lambda = \mu_2$. Here Π_1 (or Π_2) is the projection from $R^2 \times R^2$ onto the first (or second) factor, and $\Pi_i \lambda$, $i = 1, 2$, is the image of the image λ under Π_i. For arbitrary measures μ_1 and μ_2 on R^2 with $\|\mu_i\|_{\mathcal{M}} \leqq m$, $i = 1, 2$, we define

$$R(\mu_1, \mu_2) = R(\mu_1^+, \mu_2^+) + R(\mu_1^-, \mu_2^-)$$

where μ_i^+ and μ_2^- denote the positive and negative part of μ_i, respectively. It is known (see [6]) that the function R is a distance function on $\{\mu \in \mathcal{M}; \|\mu\|_{\mathcal{M}} \leqq m\}$ which defines a topology equivalent to that of weak convergence. We shall use the function R in showing equicontinuity. Without loss of generality we may assume that $v_0 \geqq 0$ and therefore $v(\cdot, t) \geqq 0$ for all $t \geqq 0$. Consider the measures $\mu(t) = v(x, t)\, dx$ on R^2 and $\lambda(t, s) = \Gamma_u(x, t; y, s)\, v(y, s)\, dx\, dy$ on $R_x^2 \times R_y^2$. Then we have $\mu(t) \geqq 0$, $\lambda(t, s) \geqq 0$ and

$$\Pi_1 \lambda(t, s) = \left[\int_{R^2} \Gamma_u(x, t; y, s)\, v(y, s)\, dy\right] dx = v(x, t)\, dx = \mu(t);$$

$$\Pi_2 \lambda(t, s) = \left[\int_{R^2} \Gamma_u(x, t; y, s)\, dx\right] v(y, s)\, dy = v(y, s)\, dy = \mu(s).$$

Note that here we have used the positivity of Γ_u, the identity (2.3), the integral representation (2.4) for v and the Chapman-Kolmogorov equality:

$$\Gamma_u(x, t; y, s) = \int_{R^2} \Gamma_u(x, t; z, t')\, \Gamma_u(z, t'; y, s)\, dz, \qquad 0 \leqq s < t' < t. \tag{3.9}$$

By (3.6) and the definition of R we see that

$$\begin{aligned} R(\mu(t), \mu(s)) &\leqq \iint_{R^2 \times R^2} |x-y|\, \Gamma_u(x, t; y, s)\, v(y, s)\, dx\, dy \\ &\leqq C_1 (t-s)^{-1} \iint_{R^2 \times R^2} |x-y| \exp\left[-C_2 |x-y|^2/(t-s)\right] v(y, s)\, dx\, dy \\ &= C(t-s)^{\frac{1}{2}} \|v\|_1 (s) \leqq C \|v_0\|_1 (t-s)^{\frac{1}{2}} \leqq mC(t-s)^{\frac{1}{2}} \end{aligned}$$

for $0 \leqq s < t \leqq T$, where C depends only on m and ν. This shows the desired equicontinuity. □

Remark. Proposition 3.5 can be proved directly without introducing R. In fact, since $v(x, t) = \int_{R^2} \Gamma_u(x, t; y, s)\, v(y, s)\, dy$, by using (3.5) and the upper estimate for Γ_u in (3.6) one can prove, by a standard argument, that $(v(\cdot, t)\, \phi)$ converges to $(v(\cdot, s), \phi)$ uniformly in $s \geqq 0$ and $\|v_0\|_1 \leqq m$ as $t \downarrow s$. Clearly this implies the equicontinuity of $(v(\cdot, t)\;\; \phi)$ on $[0, T]$. However, the proof using R seems conceptually simpler. The function R is used in [3, 19] and [20] in a similar context.

The results obtained in this section are applied in Section 4 to construct a global solution of the problem (1) when $\nabla \times a$ is a measure.

4. Main Theorems

In this section, we apply the *a priori* estimates derived in Section 3 to construct a global solution of (1) as well as of (2a), (2b) when the initial vorticity $\nabla \times a$ is a general finite Radon measure on R^2. It turns out that our solution is smooth for $t > 0$ and decays as $t \to \infty$. We also study how the velocity converges to a as $t \to 0$. We further show that our solution is unique provided that the atomic part of the measure $\nabla \times a$ is sufficiently small.

We begin by selecting a reasonable function space for a when $\nabla \times a$ is a finite Radon measure on R^2 and $\nabla \cdot a = 0$. By (2.6b) and Lemma 2.2 (iii), a is expressed as the sum of $K * (\nabla \times a) \in L^{2,\infty}(R^2)$ and a harmonic vector field. Since our initial velocity a is supposed to decay as $|x| \to \infty$ it is natural to assume that a is in $L^{2,\infty}(R^2)$ with $\nabla \cdot a = 0$ and $\nabla \times a \in \mathcal{M}$ so that $a = K * (\nabla \times a)$.

To study convergence to the initial velocity, we give a sufficient condition for continuity under the weak* topology of $L^{2,\infty}(R^2)$. Since $L^{2,\infty}(R^2)$ is the dual space of the Lorentz space $L^{2,1}(R^2)$ (see [4]) the weak* topology is well defined in that space.

Lemma 4.1. *Suppose that* $u \in L^\infty(0, T; L^{2,\infty}(R^2))$ *with* $\nabla \cdot u = 0$ *and that* $v = \nabla \times u$ *is continuous from* $[0, T]$ *to* $\mathcal{M}$ *under the topology of weak convergence of* $\mathcal{M}$. *Then* u, *modified if necessary on a set of Lebesgue measure zero in* $[0, T]$, *is continuous from* $[0, T]$ *to* $L^{2,\infty}(R^2)$ *in the weak* topology.*

Proof. By Lemma 2.2 (iii), $K * v \in L^\infty(0\ T; L^{2,\infty}(R^2))$ and $u - K * v = 0$ a.e. in $[0\ T]$, as an element of $L^{2,\infty}(R^2)$. The assertion of the lemma will thus be verified if we show that $U = K * v$ is continuous. Take an arbitrary sequence t_l in $[0, T]$ with $t_l \to t$ as $l \to \infty$. By the Banach-Alaoglu theorem we can extract a subsequence, which is again denoted by t_l such that $U(t_l) \to U_\infty$ weakly* in $L^{2,\infty}(R^2)$. By assumption, $\nabla \times U(t_l) = (\nabla \times u)(t_l) \to (\nabla \times u)(t)$ in the weak topology of measures. On the other hand, weak* convergence in $L^{2,\infty}(R^2)$ implies the convergence in the topology of distributions; thus $\nabla \times U(t_l) \to \nabla \times U_\infty$ as $l \to \infty$. Hence $\nabla \times U_\infty = (\nabla \times u)(t) = v(t)$ and therefore $U_\infty = K * v(t) = U(t)$ does not depend on the choice of t_l. □

Theorem 4.2. (*Existence for the Navier-Stokes system*). *Suppose that* $a \in L^{2,\infty}(R^2)$, $\nabla \cdot a = 0$ *and that* $\nabla \times a$ *is a finite measure. Then problem* (1) *has a global solution* u *which is smooth for* $t > 0$ *such that*

(i) $u: [0, \infty) \to L^{2,\infty}(R^2)$ *is bounded and continuous under the weak* topology and* $u(\cdot, 0) = a$.

(ii) $v = \nabla \times u: [0, \infty) \to \mathcal{M}$ *is bounded and continuous under the weak topology and* $v(\cdot, 0) = \nabla \times a$.

(iii) *The estimates*

$$\|u\|_r(t) \leqq Ct^{1/r-1/2} \quad \text{for } t > 0, \quad 2 < r \leqq \infty; \tag{4.1}$$

$$\|\nabla u\|_r(t) \leqq Ct^{-1+1/r} \quad \text{for } t > 0, \quad 1 < r < \infty \tag{4.2}$$

hold with C *depending only on* r, ν *and* $\|\nabla \times a\|_{\mathcal{M}}$.

(iv) *For* $0 < \varepsilon < T$ *and nonnegative integers* k, h, *there is a constant* C *such that*

$$\sup_{[\varepsilon,T]} \|\nabla^k \partial_t^h u\|_\infty (t) \leqq C.$$

with C *depending only on* ε, k, h, T, ν *and on a bound for* $\|\nabla \times a\|_{\mathcal{M}}$.

(v) *The function* $u(t) = u(\cdot, t)$ *solves the integral equation* (1.1) *in* $L^{2,\infty}(R^2)$.

Proof. Define $a_\eta = e^{\nu\eta\Delta} a$ for $\eta > 0$. By the generalized Young's inequality and properties of the heat kernel, we obtain that $\nabla^k a_\eta \in L^p(R^2)$, $k = 0, 1, \ldots$ for all $p > 2$, and that $\nabla \times a_\eta \in L^q(R^2)$ for all $q \geqq 1$. Hence, by Theorem 2.4 a unique global smooth solution u_η of (1) with $u_\eta(\cdot, 0) = a_\eta$ exists. Since $\|\nabla \times a_\eta\|_1 \leqq \|\nabla \times a\|_{\mathcal{M}}$, the estimate (3.8) guarantees that there is a subsequence $u_{\eta'}$ converging to a function $u(x, t)$ uniformly on every compact subset in $(0, \infty) \times R^2$ together with all its derivatives, as $\eta' \to 0$. The asserted estimates for u in (iii), (iv) above now follow from (3.7c), (3.7b) and (3.8) by the lower semicontinuity of integrals. Since each u_η solves (1) for $t > 0$, it is clear that the limit $u(x, t)$ solves (1) for $t > 0$.

We next prove (i) and (ii). By Proposition 3.5, a subsequence of $\nabla \times u_{\eta'}(\cdot, t)$ converges to $\nabla \times u(\cdot, t)$ uniformly on $[0, T]$, as $\eta' \to 0$, in the weak topology of $\mathcal{M}$. We conclude that $v = \nabla \times u$ is continuous from $[0, \infty)$ to $\mathcal{M}$ in the weak topology of $\mathcal{M}$ and $v(x, 0) = \nabla \times a(x)$. By (3.5) we see that $\|v\|_{\mathcal{M}}(t)$ is bounded on $[0, \infty)$. This completes the proof of (ii). Since $\{u_\eta\}$ is bounded in $L^\infty(0, \infty; L^{2,\infty}(R^2))$ by (3.5), a subsequence $\{u_{\eta'}\}$ converges to u weakly* in $L^\infty(0, \infty; L^{2,\infty}(R^n))$. Since $v = \nabla \times u$ satisfies (ii), applying Lemma 4.1 now yields (i).

It remains to prove (v), *i.e.*

$$u(t) = e^{\nu t\Delta} a - \int_0^t \nabla \cdot e^{\nu(t-s)\Delta} P(u \otimes u)(s)\, ds \quad \text{in } L^{2,\infty}(R^2).$$

For $\varepsilon > 0$ our solution $u(t)$ solves

$$u(t) = e^{\nu(t-\varepsilon)\Delta} u(\varepsilon) - \int_\varepsilon^t \nabla \cdot e^{\nu(t-s)\Delta} P(u \otimes u)(s)\, ds, \quad t \geqq \varepsilon$$

in all $L^p(R^2)$, $p > 2$. By (4.1) with $r = 4$ and (1.3) with $r = s = 2$, we have

$$\int_\varepsilon^t \nabla \cdot e^{\nu(t-s)\Delta} P(u \otimes u)(s)\, ds \to \int_0^t \nabla \cdot e^{\nu(t-s)\Delta} P(u \otimes u)(s)\, ds \quad \text{as } \varepsilon \to 0$$

in $L^2(R^2)$ and, therefore, in $L^{2,\infty}(R^2)$, because $L^2(R^2)$ is continuously embedded to $L^{2,\infty}(R^2)$. Hence we need only show that, for each fixed $t > 0$

$$e^{\nu(t-\varepsilon)\Delta} u(\varepsilon) \to e^{\nu t\Delta} a \quad \text{weakly* in } L^{2,\infty}(R^2) \text{ as } \varepsilon \to 0. \tag{*}$$

Assertion (i) and the boundedness of the operators $e^{t\Delta}$ in $L^{2,\infty}(R^2)$ together imply that $e^{\nu(t-\varepsilon)\Delta} u(\varepsilon)$ is bounded in $L^{2,\infty}(R^2)$ for each fixed $t > 0$. On the other hand, it is easily verified that

$$(e^{\nu(t-\varepsilon)\Delta} u(\varepsilon), \phi) \to (e^{\nu t\Delta} a, \phi) \quad \text{as } \varepsilon \to 0$$

for any smooth vector function ϕ with compact support in R^2. Since such functions ϕ are dense in the Lorentz space $L^{2,1}(R^2)$ (see *e.g.* [4]), and since $L^{2,\infty}(R^2)$ is the dual of the space $L^{2,1}(R^2)$, (*) follows from the Banach-Steinhaus theorem. □

The next proposition discusses properties of the vorticity $v = \nabla \times u$ of the solution u obtained in Theorem 4.2. The main assertion is that v has an integral representation in terms of a well behaved function $\Gamma(x, t; y, s)$, $t > s \geqq 0$, which is obtained as a limit of the fundamental solutions of parabolic operators L_{u_η} with smooth u_η. This representation plays an important role in discussing the uniqueness for solutions constructed in Theorem 4.2.

Theorem 4.3. (*Integral representation for* $\nabla \times u$). *Under the assumption of Theorem* 4.2, *the vorticity* $v = \nabla \times u$ *is expressed as*

$$v(x, t) = \int_{R^2} \Gamma(x, t; y, 0)\, (\nabla \times a)\, (dy), \qquad t > 0, \tag{4.3}$$

in terms of a continuous function $\Gamma(x, t; y, s)$, $x, y \in R^2$, $t > s \geqq 0$, *with the following properties* (4.4)–(4.6):

$$\int_{R^2} \Gamma(x, t; y, s)\, dy = \int_{R^2} \Gamma(x, t; y, s)\, dx = 1, \qquad t > s \geqq 0; \tag{4.4}$$

$$\Gamma(x\ t; y\ s) = \int_{R^2} \Gamma(x, t; z, t')\, \Gamma(z\ t'; y, s)\, dz, \qquad t > t' > s \geqq 0; \tag{4.5}$$

$$\begin{aligned} &C_1(t-s)^{-1} \exp\left[-C_2\, |x-y|^2/(t-s)\right] \\ &\leqq \Gamma(x, t; y, s) \leqq C_3(t-s)^{-1} \exp\left[-C_4\, |x-y|^2/(t-s)\right], \qquad t > s \geqq 0, \end{aligned} \tag{4.6}$$

with C_j, $j = 1, 2, 3, 4$, *depending only on* ν *and on a bound for* $\|\nabla \times a\|_{\mathcal{M}}$.

Moreover, the estimate

$$\|v\|_r\,(t) \leqq C t^{-1+1/r}\, \|\nabla \times a\|_{\mathcal{M}}, \qquad t > 0, \qquad 1 \leqq r \leqq \infty \tag{4.7}$$

holds with C *depending only on* r, ν *and on an upper bound of* $\|\nabla \times a\|_{\mathcal{M}}$.

Proof. As in the proof of Theorem 4.2, we consider the functions u_η and $v_\eta = \nabla \times u_\eta$. By (2.4)

$$v_\eta(x, t) = \int_{R^2} \Gamma_{u_\eta}(x, t; y, 0)\, (\nabla \times a_\eta)\, (y)\, dy, \qquad t > 0, \tag{4.8}$$

where Γ_{u_η} is the fundamental solution of $L_{u_\eta} = \partial_t - \nu\Delta + (u_\eta \cdot \nabla)$. Since $u_\eta = K * v_\eta$ and $\|v_\eta\|_1\,(t) \leqq \|\nabla \times a\|_{\mathcal{M}}$, Lemma 3.3 implies that the estimates (3.3) and (3.4) with $b = u_\eta$ are uniform in η. We can thus apply Ascoli's theorem to conclude that, by passing to a subsequence of $\{u_{\eta'}\}$,

$$\Gamma_{u_{\eta''}}(x, t; y, s) \to \Gamma(x, t; y, s) \qquad \text{as } \eta'' \to 0 \tag{4.9}$$

uniformly on compact subsets of points $(x, t; y, s)$ with $t > s \geqq 0$, and that the limit function Γ satisfies (4.6). Further, since $\nabla \times a_\eta \to \nabla \times a$ as $\eta \to 0$, weakly in $\mathcal{M}$, (4.8), (4.9) together with (3.3) for $b = u_\eta$, yield (4.3). Identities (4.4) and (4.5) are obtained in a similar fashion, since they hold for the fundamental solutions Γ_{u_η} (see (2.2), (2.3) and (3.9)). Finally, (4.7) follows from (4.3) and (4.6). □

We next consider the question of uniqueness for the solution obtained in this section. Let us recall that by the Lebesgue decomposition of a finite Radon measure μ (see [28, volume I, p. 22, Theorem 1.13]), μ is written uniquely as

$$\mu = \mu_{pp} + \mu_c$$

where μ_c is the continuous part, *i.e.*, $\mu_c(\{x\}) = 0$ for all $x \in R^2$ and μ_{pp} is the purely atomic part, *i.e.*, $\mu_{pp} = \sum_{j=1}^{\infty} \alpha_j\, \delta(x - z_j)$, $\alpha_j \in R$, $z_j \in R^2$. This is easily verified by defining $\mu_{pp} = E \lrcorner\, \mu$ with $E = \{x \in R^2; \mu(\{x\}) \neq 0\}$ and proving that E is a countable set. Here $E \lrcorner\, \mu$ denotes the Borel measure defined by $E \lrcorner\, \mu(A) = \mu(A \cap E)$.

Lemma 4.4. *For any finite Radon measure μ on R^2 we have*

$$\limsup_{t \downarrow 0} t^{1-1/r} \| e^{t\Delta} \mu \|_r \leqq C_r \| \mu_{pp} \|_{\mathcal{M}} \quad \textit{for all } r > 1,$$

where C_r depends only on r.

Proof. We first assert the estimate

$$\| e^{t\Delta} \mu \|_r \leqq C_r t^{-1+1/r} \| \mu \|_{\mathcal{M}}.$$

Indeed, since the linear operator $Af = f * \mu$ is bounded in both L^1 and L^∞ with operator-norm not exceeding $\|\mu\|_{\mathcal{M}}$, applying the Riesz-Thorin theorem ([4], [28, Volume II]) to A yields the above estimate if we take f as the heat kernel.

This estimate shows that to complete the proof of the lemma we need only prove that

$$\lim_{t \downarrow 0} t^{1-1/r} \| e^{t\Delta} \mu \|_r = 0 \quad \text{for all } r > 1, \tag{4.10}$$

under the assumption that μ is continuous, *i.e.*, $\mu(\{x\}) = 0$ for any $x \in R^2$. Without loss of generality we may assume that $\mu \geqq 0$. For any fixed $\varepsilon > 0$ we take $N > 0$ so that, writing $B(0, N) = \{x; |x| \leqq N\}$, we conclude that $\mu[R^2 \setminus B(0, N)] < \varepsilon$ and hence $\mu_2 = (R^2 \setminus B(0, N)) \lrcorner\, \mu$ satisfies

$$t^{1-1/r} \| e^{t\Delta} \mu_2 \|_r \leqq C_r \varepsilon \quad \text{for all } r > 1. \tag{4.11}$$

The support of the measure $\mu_1 = \mu - \mu_2$ is contained in $B(0, N)$ and a direct calculation gives

$$\begin{aligned}(t^{1-1/r} \| e^{t\Delta} \mu_1 \|_r)^r &= C_r' t^{-1} \int_{R^2} \left(\int_{|y| \leqq N} \exp\,[-\,|x - y|^2/4t]\, \mu_1(dy) \right)^r dx \\ &= C_r' t^{-1} \left(\int_{|x| > 2N} + \int_{|x| \leqq 2N} \right) \left(\int_{|y| \leqq N} \exp\,[-\,|x - y|^2/4t]\, \mu_1(dy) \right)^r dx \\ &= I_1(t) + I_2(t).\end{aligned} \tag{4.12}$$

Since $|x - y| > |x|/2$ if $|x| > 2N$ and $|y| \leqq N$,

$$I_1(t) \leqq C_r' \| \mu_1 \|_{\mathcal{M}}^r\, t^{-1} \int_{|x| > 2N} \exp\,[-r\,|x|^2/16t]\, dx \to 0 \quad \text{as } t \to 0. \tag{4.13}$$

For $I_2(t)$, applying Minkowski's inequality yields

$$I_2(t) \leqq C'_r t^{-1} \int_{|x| \leqq 2N} \left(\int_{|x-y|>\delta} \exp\left[-|x-y|^2/4t\right] \mu_1(dy) \right)^r dx$$

$$+ C'_r t^{-1} \int_{|x| \leqq 2N} \left(\int_{|x-y| \leqq \delta} \exp\left[-|x-y|^2/4t\right] \mu_1(dy) \right)^r dx$$

$$\equiv I_{21}(t) + I_{22}(t), \tag{4.14}$$

where $\delta > 0$ is to be chosen later. Obviously, for any fixed $\delta > 0$,

$$I_{21}(t) \leqq C'_r \operatorname{mea}[B(0, 2N)] \|\mu_1\|^r_{\mathscr{M}} t^{-1} \exp(-r\delta^2/4t) \to 0, \text{ as } t \to 0 \tag{4.15}$$

where mea is the Lebesgue measure on R^2. On the other hand, Hölder's inequality yields

$$I_{22}(t) \leqq C'_r \int_{|x| \leqq 2N} [\mu_1 B(x,\delta)]^{r-1} \left(\int_{|x-y| \leqq \delta} \exp\left[-r|x-y|^2/4t\right] \mu_1(dy) \right) dx/t$$

$$\leqq C''_r \sup_{|x| \leqq 2N} [\mu_1 B(x,\delta)]^{r-1} \|e^{r^{-1}t\Delta} \mu_1\|_1$$

$$\leqq C''_r \|\mu\|_{\mathscr{M}} \times \sup_{|x| \leqq 2N} [\mu_1 B(x,\delta)]^{r-1}, \tag{4.16}$$

where $B(x, \delta) = \{y; |y-x| \leqq \delta\}$. We shall now show that

$$\mu_1[B(x,\delta)] \to 0, \quad \text{as } \delta \to 0, \quad \text{uniformly in } |x| \leqq 2N. \tag{4.17}$$

The desired result (4.10) will then follow from (4.11)–(4.15) by taking δ such that $I_{22}(t) < \varepsilon^r$ and recalling that ε is arbitrary.

Suppose that (4.17) were false. Then there would exist $\eta > 0$, $\delta_l \downarrow 0$ and x_l with $|x_l| \leqq 2N$ such that

$$\mu_1[B(x_l, \delta_l)] \geqq \eta \quad \text{for all } l. \tag{4.18}$$

By passing to a subsequence we may assume that $x_l \to x$ as $l \to \infty$. For any $\delta > 0$, $B(x_l, \delta_l) \subset B(x, \delta)$ provided l is sufficiently large. Since $\mu_1(\{x\}) = 0$, we have $\lim_{\delta \downarrow 0} \mu_1[B(x,\delta)] = 0$, so $\lim_{l \to \infty} \mu_1[B(x_l, \delta_l)] = 0$, which contradicts (4.18). We thus obtain (4.17). □

Theorem 4.5. (*Uniqueness*). *Suppose that* $a \in L^{2,\infty}(R^2)$, $\nabla \times a \in \mathscr{M}$, *and* $\nabla \cdot a = 0$. *Take* $m > 0$ *so that* $\|\nabla \times a\|_{\mathscr{M}} \leqq m$ *and let u be the solution of* (1) *given in Theorem* 4.2. *Then*

(i) *For all* $p > 2$,

$$\limsup_{t \downarrow 0} t^{1/2-1/p} \|u\|_p(t) \leqq C \|(\nabla \times a)_{pp}\|_{\mathscr{M}} \tag{4.19}$$

with C depending only on p, m and ν.

(ii) *For each* $p > 2$ *there is a positive constant* $\varepsilon = \varepsilon(p, \nu, m)$ *such that if* $\|(\nabla \times a)_{pp}\|_{\mathscr{M}} < \varepsilon$, *then the solution u is unique in the class of functions w with the following properties*:

(a) $w: [0, \infty) \to L^{2,\infty}(R^2)$ *is weakly* continuous and* $w(\cdot, 0) = a$;

(b) $w:(0,\infty)\to L^p(R^2)$ *is continuous and satisfies* (4.19) *for* $p>2$.

(c) w *solves* (1.1) *in* $L^{2,\infty}(R^2)$.

In particular the solution u is unique provided that $\nabla\times a$ *is a continuous measure.*

Proof. (i) Since $p>2$ and $u=K*v$ with $v=\nabla\times u$, we get by (2.6a)

$$\|u\|_p(t)\leqq C\|K\|_{2,\infty}\|v\|_q(t),\qquad 1/q=1/p+1/2$$

with C depending only on p. By (4.3), (4.6) and Lemma 4.4 we see that

$$\limsup_{t\downarrow 0} t^{1-1/q}\|v\|_q(t)\leqq C'\|(\nabla\times a)_{pp}\|_{\mathcal{M}}$$

where C' depends only on q, m and ν. Combining these two estimates gives (4.19).

(ii) Let $\tilde u$ be another solution of (1) with the same initial data a satisfying properties (a) and (b) above. By (c) the difference $w=u-\tilde u$ satisfies

$$w(t)=-\int_0^t \nabla\cdot e^{\nu(t-s)\Delta}P[w\otimes u(s)+\tilde u\otimes w(s)]\,ds,$$

so that, as in the proof of Lemma 1.1 (i),

$$\|w\|_p(t)\leqq M\int_0^t (t-s)^{-1/p-1/2}[\|u\|_p+\|\tilde u\|_p](s)\,\|w\|_p(s)\,ds.$$

Thus, $\|w\|_{p,T}\equiv\sup_{0<t\leqq T} t^{1/2-1/p}\|w\|_p(t)$ satisfies

$$\|w\|_{p,T}\leqq MB(1/2-1/p,2/p)\,[\|u\|_{p,t}+\|\tilde u\|_{p,T}]\,\|w\|_{p,T} \tag{4.20}$$

where B is the beta function. We here assume that $(\nabla\times a)_{pp}$ satisfies

$$2CMB(1/2-1/p,2/p)\,\|(\nabla\times a)_{pp}\|_{\mathcal{M}}<1 \tag{4.21}$$

where C is the constant in (4.19). Estimates (4.19)–(4.21) together imply that if we take $T>0$ sufficiently small, then $\|w\|_{p,T}\leqq c\|w\|_{p,T}$ for some $c<1$, and this yields $w=0$ on $[0,T]$ since $\|w\|_{p,T}$ is finite. On the interval $[T,\infty)$, both u and $\tilde u$ are classical solutions belonging to L^p, so we get $w=0$ on $[T,\infty)$ by the result on uniqueness established in Proposition 1.2. □

Theorem 4.5 shows, in particular, that the solution is unique whenever $\nabla\times a$ is a continuous measure. When the measure $\nabla\times a$ has a density, *i.e.*, when $\nabla\times a$ is in $L^1(R^2)$, we can also prove additional regularity at $t=0$, as shown in the following theorem.

Theorem 4.6. *If* $a\in L^{2,\infty}(R^2)$, $\nabla\cdot a=0$ *and if* $\nabla\times a\in L^1(R^2)$, *then the* (*unique*) *solution u of* (1) *belongs to* $\mathrm{BC}([0,\infty);L^{2,\infty}(R^2))$.

Proof. By assumption, $e^{\nu t\Delta}(\nabla\times a)$ is in $\mathrm{BC}([0,\infty);L^1(R^2))$. Thus, by (2.6b), the function $e^{\nu t\Delta}a=e^{\nu t\Delta}K*(\nabla\times a)=K*[e^{\nu t\Delta}(\nabla\times a)]$ belongs to

$BC([0,\infty); L^{2,\infty}(R^2))$. By (v) of Theorem 4.2 it suffices, therefore, to show that the function

$$S[u](t) = -\int_0^t \nabla \cdot e^{\nu(t-s)\Delta} P(u \otimes u)(s)\, ds$$

is in $BC([0,\infty); L^{2,\infty}(R^2))$. By (1.3) with $r = s = 2$, and (4.1) with $r = 4$,

$$\begin{aligned} \|S[u]\|_{2,\infty}(t) &\leqq \|S[u]\|_2(t) \\ &\leqq C\int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}}\, ds = CB(1/2, 1/2) \quad \text{for } t > 0, \end{aligned}$$

which implies that u is bounded and continuous for $t > 0$. On the other hand, since $\nabla \times a$ contains no pure point part, Theorem 4.5 (i) yields

$$\lim_{t\downarrow 0} t^{1/2-1/p} \|u\|_p(t) = 0 \qquad \text{for all } p > 2. \tag{4.22}$$

Hence, using again (1.3) with $r = s = 2$ and (4.1) with $r = 4$ we get, as $t \to 0$,

$$\begin{aligned} \|S[u]\|_{2,\infty}(t) &\leqq \|S[u]\|_2(t) \\ &\leqq CB(1/2, 1/2)\, \|u\|_{4,t} \to 0 \end{aligned}$$

because $\|u\|_{4,t} \equiv \sup_{0<s\leqq t} s^{\frac{1}{4}} \|u\|_4(s) \to 0$ as $t \to 0$, by (4.22). This shows that u is also continuous at $t = 0$. □

Remark. BENFATTO, ESPOSITO & PULVIRENTI [3] prove existence and uniqueness of solutions to (1) under initial data a such that

$$\nabla \times a = \sum_{j=1}^{m} \alpha_j\, \delta(x - z_j), \qquad \alpha_j \in R^1, \qquad z_j \in R^2$$

where $\Sigma_j[\alpha_j|$ is sufficiently small. Here $\delta(x - z_j)$ is the Dirac measure supported by z_j. Our theorem of uniqueness covers that of [3]; moreover, our theorem of existence improves that of [3] since no restriction is imposed here on either the size or the form of the measure $\nabla \times a$.

Note. After this paper was submitted, the authors learned that COTTET [37] proved a theorem similar to Theorem 4.2. He constructs a weak solution of the vorticity equation when initial vorticity is a finite measure. However, his method is different from ours and he does not take up regularity of weak solutions. Our theorem of uniqueness (Theorem 4.5) is stronger than his because he needs to assume that the total variation of the initial vorticity itself is small.

The research reported here was partially supported by the Japan Ministry of Education, Science and Culture.

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(Received March 23, 1988)