BASIC HYPERGEOMETRIC SERIES

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My education was not much different from that of most mathematicians of my generation. It included courses on modern algebra, real and complex variables, both point set and algebraic topology, some number theory and projective geometry, and some specialized courses such as one on Riemann surfaces. In none of these courses was a hypergeometric function mentioned, and I am not even sure if the gamma function was mentioned after an advanced calculus course. The only time Bessel functions were mentioned was in an undergraduate course on differential equations, and the only thing done with them was to find a power series solution for the general Bessel equation. It is small wonder that with a similar education almost all mathematicians think of special functions as a dead subject which might have been interesting once. They have no idea why anyone would care about it now.

Fortunately there was one part of my education which was different. As a junior in college I read Widder's book The Laplace Transform and the manuscript of its very important sequel, Hirschman and Widder's The Convolution Transform. Then as a senior, I.I. Hirschman gave me a copy of a preprint of his on a multiplier theorem for Legendre series and suggested I extend it to ultraspherical series. This forced me to become acquainted with two other very important books, Gabor Szeg\'o's great book Orthogonal Polynomials, and the second volume of Higher Transcendental Functions, the monument to Harry Bateman which was written by Arthur Erd\'elyi and his co-workers W. Magnus, F. Oberhettinger and F.G. Tricomi.

From this I began to realize that the many formulas that had been found, usually in the 18th or 19th century, but once in a while in the early 20th century, were useful, and started to learn about their structure. However, I had written my Ph.D. thesis and worked for three more years before I learned that not every fact about special functions I would need had already been found, and it was a couple of more years before I learned that it was essential to understand hypergeometric functions. Like others, I had been put off by all the parameters. If there were so many parameters that it was necessary to put subscripts on them, then there has to be a better way to solve a problem than this. That was my initial reaction to generalized hypergeometric functions, and a very common reaction to judge from the many conversations I have had on these functions in the last twenty years. After learning a little more about hypergeometric functions, I was very surprised to realize that they had occurred regularly in first year calculus. The reason for the subscripts on the parameters is that not all interesting polynomials are of degree one or two.
a generalized hypergeometric function has a series representation

\[ \sum_{n=0}^{\infty} \frac{c_n}{n!} \]

with \( c_{n+1}/c_n \) a rational function of \( n \). These contain almost all the examples of infinite series introduced in calculus where the ratio test works easily. The ratio \( c_{n+1}/c_n \) can be factored, and it is usually written as

\[ \frac{c_{n+1}}{c_n} = \frac{(n + a_1) \cdots (n + a_p)x}{(n + b_1) \cdots (n + b_q)(n + 1)} \]

Introduce the shifted factorial

\[ (a)_0 = 1, \]

\[ (a)_n = a(a + 1) \cdots (a + n - 1), \quad n = 1, 2, \ldots \]

Then if \( c_0 = 1 \), equation (2) can be solved for \( c_n \) as

\[ c_n = \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!} \]

and

\[ \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!} \]

is the usual notation.

The first important result for a \( _pF_q \) with \( p > 2 \), \( q > 1 \) is probably Pfaff's sum

\[ _3F_2 \left[ \begin{array}{c} -n, a, b \\ c, a + b + 1 - c - n \end{array} \right] = \frac{(c - a)_n(c - b)_n}{(c)_n(c - a - b)_n}, \quad n = 0, 1, \ldots \]

This result from 1797, see Pfaff [1797], contains as a limit when \( n \to \infty \) another important result usually attributed to Gauss [1813]:

\[ _2F_1 \left[ \begin{array}{c} a, b \\ c \end{array} \right] = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \text{Re} (c - a - b) > 0. \]

The next instance is a very important result of Clausen [1828]:

\[ _2F_1 \left[ \begin{array}{c} a, b \\ a + b + \frac{1}{2} \end{array} \right]^2 = _3F_2 \left[ \begin{array}{c} 2a, 2b, a + b \\ a + b + \frac{1}{2}, 2a + 2b \end{array} \right] \]

Some of the interest in Clausen's formula is that it changes the square of a class of \( _2F_1 \)'s to a \( _3F_2 \). In this direction it is also interesting because it was probably the first instance of anyone finding a differential equation satisfied by \( y(x)^2, y(x)z(x) \) and \( [z(x)]^2 \) when \( y(x) \) and \( z(x) \) satisfy

\[ a(x)y'' + b(x)y' + c(x)y = 0. \]

This problem was considered for (9) by Appell, see Watson [1952], but the essence of his general argument occurs in Clausen's paper. This is a common phenomenon, which is usually not mentioned when the general method is introduced to students, so they do not learn how often general methods come from specific problems or examples. See D. and G. Chudnovsky [1988] for an instance of the use of Clausen's formula, where a result for \( _2F_1 \) is carried to a \( _3F_2 \) and from that to a very interesting set of expansions of \( \pi^{-1} \). Those identities were first discovered by Ramanujan. Here is Ramanujan's most impressive example:

\[ \frac{9801}{2\pi\sqrt{2}} = \sum_{n=0}^{\infty} \frac{[1103 + 26390n]((1/4)_n(1/2)_n(3/4)_n)}{n!} \frac{1}{(99)^{n+1}}. \]

There is another important reason why Clausen's formula is important. It leads to a large class of \( _3F_2 \)'s that are nonnegative for the power series variable between \(-1 \) and \( 1 \). The most famous use of this is in the final step of de Branges' solution of the Bieberbach conjecture, see de Branges [1985]. The integral of the \( _2F_1 \) or Jacobi polynomial he had is a \( _3F_2 \), and its positivity is an easy consequence of Clausen's formula, as Gasparr had observed ten years earlier. There are other important results which follow from the positivity in Clausen's identity.

Once Kummer [1836] wrote his long and important paper on \( _2F_1 \)'s and \( _1F_1 \)'s, this material became well known. It has been reworked by others. Riemann redid the \( _2F_1 \) using his idea that the singularities of a function go a long way toward determining the function. He showed that if the differential equation (9) has regular singularities at three points, and every other point in the extended complex plane is an ordinary point, then the equation is equivalent to the hypergeometric equation

\[ x(1-x)y'' + [c-(a+b+1)x]y' - aby = 0, \]

which has regular singular points at \( x = 0, 1, \infty \). Riemann's work was very influential, so much so that much of the mathematical community that considered hypergeometric functions studied them almost exclusively from the point of view of differential equations. This is clear in Klein's book [1933], and in the work on multiple hypergeometric functions that starts with Appell in 1980 and is summarized in Appell and Kampé de Fériet [1926].

The integral representations associated with the differential equation point of view are similar to Euler integral representations. This is

\[ _2F_1 \left[ \begin{array}{c} a, b \\ c \end{array} \right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 (1 - xt)^{-a+b-1}(1-t)^{-c+b-1}dt, \]

\[ |x| < 1, \text{ Re } c > Re b > 0, \text{ and includes related integrals with different contours.} \]

The differential equation point of view is very powerful where it works, but it does not work well for \( p \geq 3 \) or \( q \geq 2 \) as Kummer discovered. Thus there is a need to develop other methods to study hypergeometric functions.
In the late 19th and early 20th century a different type of integral representation was introduced. These two different types of integrals are best represented by Euler’s beta integral

\[
\int_0^1 t^{a-1}(1-t)^{b-1} dt = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a+b)}, \quad \text{Re} \,(a, b) > 0
\]

and Barnes’ beta integral

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(a+it) \Gamma(b+it) \Gamma(c-it) \Gamma(d-it)}{\Gamma(a+b+c+d)} dt = \frac{\Gamma(a+c) \Gamma(b+d) \Gamma(c+b) \Gamma(d+a)}{\Gamma(a+b+c+d)}, \quad \text{Re} \,(a, b, c, d) > 0.
\]

There is no direct connection with differential equations for integrals like (14), so it stands a better chance to work for larger values of \( p \) and \( q \).

While Euler, Gauss, and Riemann and many other great mathematicians wrote important and influential papers on hypergeometric functions, the development of basic hypergeometric functions was much slower. Euler and Gauss did important work on basic hypergeometric functions, but most of Gauss’ work was unpublished until after his death and Euler’s work was more influential on the development of number theory and elliptic functions.

Basic hypergeometric series are series \( \sum c_n \) with \( c_{n+1}/c_n \) a rational function of \( q^n \) for a fixed parameter \( q \), which is usually taken to satisfy \( |q| < 1 \), but at other times is a power of a prime. In this Foreword \( |q| < 1 \) will be assumed.

Euler summed three basic hypergeometric series. The one which had the largest impact was

\[
\sum_{n=0}^{\infty} (-1)^n q^{(3n^2-n)/2} = (q; q)_\infty,
\]

where

\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - a q^n),
\]

If

\[
(a; q)_n = (a; q)_\infty / (aq^n; q)_\infty
\]

then Euler also showed that

\[
\frac{1}{(x; q)_\infty} = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n}, \quad |x| < 1,
\]

and

\[
(x; q)_\infty = \sum_{n=0}^{\infty} (-1)^n q^{(n)} x^n.
\]

Eventually all of these were contained in the \( q \)-binomial theorem

\[
\frac{(a; q)_\infty}{(x; q)_\infty} = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} x^n, \quad |x| < 1.
\]

While (18) is clearly the special case \( a = 0 \), and (19) follows easily on replacing \( x \) by \( xq^{-1} \) and letting \( a \to \infty \), it is not so clear how to obtain (15) from (20). The easiest way was discovered by Cauchy and many others. Take \( a = q^{-2N} \), shift \( n \) by \( N \), rescale and let \( N \to \infty \). The result is called the triple product, and can be written as

\[
(x; q)_\infty (q x^{-1}; q)_\infty (q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(2)} x^n.
\]

Then \( q \to q^3 \) and \( x = q \) gives Euler’s formula (15).

Gauss used a basic hypergeometric series identity in his first proof of the determination of the sign of the Gauss sum, and Jacobi used some to determine the number of ways an integer can be written as the sum of two, four, six and eight squares. However, this particular aspect of Gauss’ work on Gauss sums was not very influential, as his hypergeometric series work had been, and Jacobi’s work appeared in his work on elliptic functions, so its hypergeometric character was lost in the great interest in the elliptic function work. Thus neither of these led to a serious treatment of basic hypergeometric series. The result that seems to have been the crucial one was a continued fraction of Eisenstein. This along with the one hundredth anniversary of Euler’s first work on continued fractions seems to have been the motivating forces behind Heine’s introduction of a basic hypergeometric extension of \( 2F_1(a, b; c; x) \). He considered

\[
2\Phi_1 \left[ \begin{array}{c} a, \ b \\ \ q^c \\ \ q^c \end{array} ; q, x \right] = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(q^c; q)_n (q; q)_n} x^n, \quad |x| < 1.
\]

Observe that

\[
\lim_{q \to 1} \frac{(q^a; q)_n}{(1-q)_n} = (a)_n,
\]

so

\[
\lim_{q \to 1} 2\Phi_1 \left[ \begin{array}{c} a, \ b \\ \ q^c \\ \ q^c \end{array} ; q, x \right] = 2F_1 \left[ \begin{array}{c} a, \ b \\ \ c \\ \ x \end{array} \right].
\]

Heine followed the pattern of Gauss’ published paper on hypergeometric series, and so obtained contiguous relations and from them continued fraction expansions. He also obtained some series transformations, and the sum

\[
2\Phi_1 \left[ \begin{array}{c} a, \ b \\ \ q^c \\ \ q^{c-a-b} \end{array} ; q^c ; q \right] = \frac{(q^{c-a}; q)_\infty (q^{c-b}; q)_\infty}{(q^c; q)_\infty (q^{c-a-b}; q)_\infty}, \quad |q^{c-a-b}| < 1.
\]

This sum becomes (7) when \( q \to 1 \).
As often happens to path breaking work, this work of Heine was to a large extent ignored. When writing the second edition of *Kugelfunctionen* (Heine [1878]) Heine decided to include some of his work on basic hypergeometric series. This material was printed in smaller type, and it is clear that Heine included it because he thought it was important, and he wanted to call attention to it, rather than because he thought it was directly related to spherical harmonics, the subject of his book. Surprisingly, his inclusion of this material led to some later work, which showed there was a very close connection between Heine's work on basic hypergeometric series and spherical harmonics. The person Heine influenced was L.J. Rogers, who is still best known as the first discoverer of the Rogers-Ramanujan identities. Rogers tried to understand this aspect of Heine's work, and one transformation in particular. Thomae [1879] had observed this transformation of Heine could be written as an extension of Euler’s integral representation (12), but Rogers was unaware of this explanation, and so discovered a second reason. He was able to modify the transformation so it became the permutation symmetry in a new series. While doing this he introduced a new set of polynomials which we now call the continuous $q$-Hermite polynomials. In a very important set of papers which were unjustly neglected for decades, Rogers discovered a more general set of polynomials and found some remarkable identities they satisfy, see Rogers [1893, 1894, 1895]. For example, he found the linearization coefficients of these polynomials which we now call the continuous $q$-ultraspherical polynomials. These polynomials contain many of the spherical harmonics Heine studied. Contained within this product identity is the special case of the square of one of these polynomials as a double series. As Gasper and Rahman have observed, one of these series can be summed, and the resulting identity is an extension of Clausen’s sum in the terminating case. Earlier, others had found a different extension of Clausen’s identity to basic hypergeometric series, but the resulting identity was not satisfactory. The identity had the product of two functions, the same functions but one evaluated at $x$ and the other at $qz$, and so was not a square. Thus the nonnegativity that is so useful in Clausen’s formula was not true for the corresponding basic hypergeometric series. Rogers’ result for his polynomials led directly to the better result which contains the appropriate nonnegativity. From this example and many others, one sees that orthogonal polynomials provide an alternative approach to the study of hypergeometric and basic hypergeometric functions. Both this approach and that of differential equations are most useful for small values of the degrees of the numerator and denominator polynomials in the ratio $c_{n+1}/c_n$, but orthogonal polynomials work for a larger class of series, and are much more useful for basic hypergeometric series. However, neither of these approaches is powerful enough to encompass all aspects of these functions. Direct series manipulations are surprisingly useful, when done by a master, or when a computer algebra system is used as an aid. Gasper and Rahman are both experts at symbolic calculations, and I regularly marvel at some of the formulas they have found. As quantum groups become better known, and as Baxter’s work spreads to other parts of mathematics as it has started to do, there will be many people trying to learn how to deal with basic hypergeometric series. This book is where I would start.

For many years people have asked me what is the best book on special functions. My response was George Gasper’s copy of Bailey’s book, which was heavily annotated with useful results and remarks. Now others can share the information contained in these margins, and many other very useful results.

Richard Askey
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basic hypergeometric series from a contour integral point of view, an idea first introduced by Barnes in 1907.

Chapter 5 is devoted to bilateral basic hypergeometric series, where the most fundamental formula is Ramanujan's \( \psi \) summation formula. Substantial contributions were also made by Bailey, M. Jackson, Slater and others, whose works form the basis of this chapter.

During the 1960's R. P. Agarwal and Slater each published a book partially devoted to the theory of basic hypergeometric series, and G. E. Andrews initiated his work in number theory, where he showed how useful the summation and transformation formulas for basic hypergeometric series are in the theory of partitions. Andrews gave simpler proofs of many old results, wrote review articles pointing out many important applications and, during the mid 1970's, started a period of very fruitful collaboration with R. Askey. Thanks to these two mathematicians, basic hypergeometric series is an active field of research today. Since Askey's primary area of interest is orthogonal polynomials, \( q \)-series suddenly provided him and his co-workers with a very rich environment for deriving \( q \)-extensions of beta integrals and of the classical orthogonal polynomials of Jacobi, Gegenbauer, Laguerre and Hermite. Askey and his students and collaborators who include W. A. Al-Salam, M. E. H. Ismail, T. H. Koornwinder, W. G. Morris, D. Stanton, and J. A. Wilson have produced a substantial amount of interesting work over the past fifteen years. This flurry of activity has been so infectious that many researchers found themselves hopelessly trapped by this alluring "\( q \)-disease", as it is affectionately called.

Our primary motivation for writing this book was to present in one modest volume the significant results of the past two hundred years so that they are readily available to students and researchers, to give a brief introduction to the applications to orthogonal polynomials that were discovered during the current renaissance period of basic hypergeometric series, and to point out important applications to other fields. Most of the material is elementary enough so that persons with a good background in analysis should be able to use this book as a textbook and a reference book. In order to assist the reader in developing a deeper understanding of the formulas and proof techniques and to include additional formulas, we have given a broad range of exercises at the end of each chapter. Additional information is provided in the Notes following the Exercises, particularly in relation to the results and relevant applications contained in the papers and books listed in the References. Although the References may have a bulky appearance, it is just an introduction to the vast literature available. Appendices I, II, and III are for quick reference, so that it is not necessary to page through the book in order to find the most frequently needed identities, summation formulas, and transformation formulas. It can be rather tedious to apply the summation and transformation formulas to the derivation of other formulas. But now that several symbolic computer algebraic systems are available, persons having access to such a system can let it do some of the symbolic manipulations, such as computing the form of Bailey's \( 10 \phi_9 \) transformation formula when its parameters are replaced by products of other

Preface
parameters. Due to space limitations, we were unable to be as comprehensive in our coverage of basic hypergeometric series and their applications as we would have liked. In particular, we could not include a systematic treatment of basic hypergeometric series in two or more variables, covering F. H. Jackson's work on basic Appell series and the works of R. A. Gustafson and S. C. Milne on \( U(n) \) multiple series generalizations of basic hypergeometric series referred to in the References. But we do highlight Askey and Wilson's fundamental work on their beautiful \( q \)-analogue of the classical beta integral in Chapter 6 and develop its connection with very-well-poised \( \phi_7 \) series. Chapter 7 is devoted to applications to orthogonal polynomials, mostly developed by Askey and his collaborators. We conclude the book with some further applications in Chapter 8, where we present part of our work on product and linearization formulas, Poisson kernels, and nonnegativity, and we also manage to point out some elementary facts about applications to the theory of partitions and the representations of integers as sums of squares of integers. The interested reader is referred to the books and papers of Andrews and N. J. Fine for additional applications to partition theory, and recent references are pointed out for applications to affine root systems (Macdonald identities), association schemes, combinatorics, difference equations, Lie algebras and groups, physics (such as representations of quantum groups and R. J. Baxter's work on the hard hexagon model of phase transitions in statistical mechanics), statistics, etc.

We use the common numbering system of letting (k.m.n) refer to the n-th numbered display in Section m of Chapter k, and letting (I.n), (II.n), and (III.n) refer to the n-th numbered display in Appendices I, II, and III, respectively. To refer to the papers and books in the References, we place the year of publication in square brackets immediately after the author's name. Thus Bailey [1935] refers to Bailey's 1935 book. Suffixes a, b, ... are used after the years to distinguish different papers by an author that appeared in the same year. Papers that have not yet been published are referred to with the year 1989, even though they might be published later due to the backlogs of journals. Since there are three Agarwals, two Chiharas and three Jacksons listed in the References, to minimize the use of initials we drop the initials of the author whose works are referred to most often. Hence Agarwal, Chihara, and Jackson refer to R. P. Agarwal, T. S. Chihara, and F. H. Jackson, respectively.

We would like to thank the publisher for their cooperation and patience during the preparation of this book. Thanks are also due to R. Askey, W. A. Al-Salam, R. P. Boas, T. S. Chihara, B. Gasper, R. Holt, M. E. H. Ismail, T. Koornwinder, and B. Nassrallah for pointing out typos and suggesting improvements in earlier versions of the book. We also wish to express our sincere thanks and appreciation to our T\( \text{eX} \)pyist, Diane Berezowski, who suffered through many revisions of the book but never lost her patience or sense of humor.

George Gasper
Mizan Rahman

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1 BASIC HYPERGEOMETRIC SERIES

1.1 Introduction

Our main objective in this chapter is to present the definitions and notations for hypergeometric and basic hypergeometric series, and to derive the elementary formulas that form the basis for most of the summation, transformation and expansion formulas, basic integrals, and applications to orthogonal polynomials and to other fields that follow in the subsequent chapters. We begin by defining Gauss' \( _2F_1 \) hypergeometric series, the \( _nF_m \) (generalized) hypergeometric series, and pointing out some of their most important special cases. Next we define Heine's \( _2\phi_1 \) basic hypergeometric series which contains an additional parameter \( q \), called the base, and then give the definition and notations for \( _r\phi_s \) basic hypergeometric series. Basic hypergeometric series are called \( q \)-analogues (basic analogues or \( q \)-extensions) of hypergeometric series because an \( _nF_m \) series can be obtained as the \( q \rightarrow 1 \) limit case of an \( _r\phi_s \) series.

Since the binomial theorem is at the foundation of most of the summation formulas for hypergeometric series, we then derive a \( q \)-analogue of it, called the \( q \)-binomial theorem, and use it to derive Heine's \( q \)-analogues of Euler's transformation formulas, Jacobi's triple product identity, and summation formulas that are \( q \)-analogues of those for hypergeometric series due to Chu and Vandemonde, Gauss, Kummer, Pfaff and Saalschütz, and to Carlsson and Minton. We also introduce \( q \)-analogues of the exponential, gamma and beta functions, as well as the concept of a \( q \)-integral that allows us to give a \( q \)-analogue of Euler's integral representation of a hypergeometric function. Many additional formulas and \( q \)-analogues are given in the exercises at the end of the chapter.

1.2 Hypergeometric and basic hypergeometric series

In 1812, Gauss presented to the Royal Society of Sciences at Göttingen his famous paper (Gauss [1813]) in which he considered the infinite series

\[
1 + \frac{ab}{1 \cdot c} z + \frac{a(a+1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} z^2 + \frac{a(a+1)(a+2)b(b+1)(b+2)}{1 \cdot 2 \cdot 3 \cdot c(c+1)(c+2)} z^3 + \cdots \quad (1.2.1)
\]

as a function of \( a, b, c, z \), where it is assumed that \( c \neq 0, -1, -2, \ldots \), so that no zero factors appear in the denominators of the terms of the series. He showed that the series converges absolutely for \( |z| < 1 \), and for \( |z| = 1 \) when \( \text{Re} (c - a - b) > 0 \), gave its (contiguous) recurrence relations, and derived his famous formula (eq. (1.2.11) below) for the sum of this series when \( z = 1 \) and \( \text{Re} (c - a - b) > 0 \).
Basic Hypergeometric Series

Although Gauss used the notation \( F(a, b, c, z) \) for his series, it is now customary to use \( F(a, b; c; z) \) or either of the notations

\[
\text{\( 2F_1(a, b; c; z) \ glimpse \ F \left[ \frac{a, b}{c}; z \right] \) for this series (and for its sum when it converges), because these notations separate the numerator parameters \( a, b \) from the denominator parameter \( c \) and the variable \( z \). In view of Gauss’ paper, his series is frequently called Gauss’ series. However, since the special case \( a = 1, b = c \) yields the geometric series

\[
1 + z + z^2 + z^3 + \cdots,
\]

Gauss’ series is also called the \( \text{(ordinary) hypergeometric series or the Gauss hypergeometric series.} \)

Some important functions which can be expressed by means of Gauss’ series are

\[
(1 + z)^b = F(-a, b; b; -z),
\]

\[
\log(1 + z) = zF(1, 1; 2; -z),
\]

\[
\sin^{-1} z = zF(1/2, 1/2; 3/2; z^2),
\]

\[
\tan^{-1} z = zF(1/2, 1/2; 3/2; -z^2),
\]

\[
e^z = \lim_{a \to \infty} F(a, b; b; z/a),
\]

where \(|z| < 1\) in the first four formulas. Also expressible by means of Gauss’ series are the classical orthogonal polynomials, such as the \( \text{Tchebichef polynomials of the first and second kinds} \)

\[
T_n(x) = F(-n, n; 1/2; (1 - x)/2),
\]

\[
U_n(x) = (n + 1)F(-n, n + 2; 3/2; (1 - x)/2),
\]

the \( \text{Legendre polynomials} \)

\[
P_n(x) = F(-n, n + 1; 1; (1 - x)/2),
\]

the \( \text{Gegenbauer (ultraspherical) polynomials} \)

\[
C_n^\lambda(x) = \left( \frac{2\lambda n!}{n!} \right) F(-n, n + 2\lambda; \lambda + 1/2; (1 - x)/2),
\]

and the more general \( \text{Jacobi polynomials} \)

\[
P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} F(-n, n + \alpha + \beta + 1; \alpha + 1; (1 - x)/2),
\]

where \( n = 0, 1, \ldots \), and \( (a)_n \) denotes the \( \text{shifted factorial defined by} \)

\[
(a)_0 = 1, \quad (a)_n = (a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}, \quad n = 1, 2, \ldots.
\]

Before Gauss, Chu [1303] (see Needham [1959, p.138], Takács [1973] and Askey [1975 p.59]) and Vandermonde [1772] had proved the summation formula

\[
F(-n, b; c; 1) = \frac{(c - b)_n}{(c)_n}, \quad n = 0, 1, \ldots,
\]

which is now called \( \text{Vandermonde’s formula or the Chu-Vandermonde formula} \).

Euler [1748] had derived several results for hypergeometric series, including his transformation formula

\[
F(a, b; c; z) = (1 - z)^{c-1-a-b} F(c - a, c - b; c; z), \quad |z| < 1.
\]

Formula (1.2.9) is the terminating case \( a = n \) of the summation formula

\[
F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \text{Re}(c - a - b) > 0,
\]

which Gauss proved in his paper.

Thirty-three years after Gauss’ paper, Heine [1846, 1847, 1878] introduces the series

\[
1 + \frac{(1 - q^a)(1 - q^b)}{(1 - q)(1 - q^{a+1})} z + \frac{(1 - q^a)(1 - q^{a+1})(1 - q^b)(1 - q^{b+1})}{(1 - q)(1 - q^2)(1 - q)(1 - q^{a+1})} z^2 + \cdots,
\]

where it is assumed that \( c \neq 0, -1, -2, \ldots \). This series converges absolutely for \(|z| < 1\) when \(|q| < 1\) and it tends (at least termwise) to Gauss’ series as \( q \to 1 \), because

\[
\lim_{q \to 1} \frac{1 - q^a}{1 - q} = a.
\]

The series in (1.2.12) is usually called \( \text{Heine’s series or, in view of the base } q, \text{ the basic hypergeometric series or q-hypergeometric series.} \)

Analogous to Gauss’ notation, Heine used the notation \( \phi(a, b, c; q, z) \) for his series. However, since one would like to also be able to consider the case when \( q \) to the power \( a, b, \) or \( c \) is replaced by zero, it is now customary to define the \( \text{basic hypergeometric series by} \)

\[
\phi(a, b; c; q, z) \equiv 2\phi_1(a, b; c; q, z) = 2\phi_1 \left[ \frac{a, b}{c}; q, z \right]
\]

\[
= \sum_{n=0}^{\infty} \frac{(a; q)_n(b; q)_n}{(c; q)_n q^n} z^n,
\]

where

\[
(a; q)_n = \begin{cases} 1, & n = 0, \\ (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), & n = 1, 2, \ldots. \end{cases}
\]

is the \( q \)-shifted factorial and it is assumed that \( c \neq q^{-m} \) for \( m = 0, 1, \ldots \). Some other notations that have been used in the literature for the produc \( (a; q)_n \) are \( (a)_n, [a]_n \), and even \( (a)_n \) when (1.2.8) is not used and the base is not displayed.

Another generalization of Gauss’ series is the \( \text{(generalized) hypergeometric series with } r \text{ numerator parameters } a_1, \ldots, a_r \) and \( s \) denominator parameter \( b_1, \ldots, b_s \) defined by

\[
\sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n \cdots (a_r)_n}{n!(b_1)_n \cdots (b_s)_n} z^n.
\]

(1.2.16)
Basic Hypergeometric Series

Some well-known special cases are the exponential function

\[ e^z = {}_0F_0(-; -; z), \]

(1.2.17)

the trigonometric functions

\[ \sin z = {}_0F_1(-; 3/2; -z^2/4), \]
\[ \cos z = {}_0F_1(-; 1/2; -z^2/4), \]

(1.2.18)

the Bessel function

\[ J_{\alpha}(z) = (z/2)^{\alpha} {}_0F_1(-; \alpha + 1; -z^2/4)/\Gamma(\alpha + 1), \]

(1.2.19)

where a dash is used to indicate the absence of either numerator (when \( r = 0 \)) or denominator (when \( s = 0 \)) parameters. Some other well-known special cases are the Hermite polynomials

\[ H_n(x) = (2x)^n {}_2F_0(n/2, -n/2; -x^2), \]

(1.2.20)

and the Laguerre polynomials

\[ L_n^\alpha(x) = \frac{(\alpha + 1)_n}{n!} F_1(-n; \alpha + 1; x). \]

(1.2.21)

Generalizing Heine’s series, we shall define an \( r \), \( \phi_s \) basic hypergeometric series by

\[ r, \phi_s(a_1, a_2, \ldots, a_r; b_1, \ldots, b_s; q, z) \equiv \sum_{n=0}^{\infty} \left( \frac{a_1, a_2, \ldots, a_r}{b_1, \ldots, b_s} \right)_n q^n z^n \]

(1.2.22)

with \( \left( \begin{array}{c} n \\ 0 \end{array} \right) = n(n-1)/2 \), where \( q \neq 0 \) when \( r > s + 1 \).

In (1.2.16) and (1.2.22) it is assumed that the parameters \( b_1, \ldots, b_s \) are such that the denominator factors in the terms of the series are never zero. Since

\[ (-m)_n = (q^{-m}; q)_n = 0, \quad n = m + 1, m + 2, \ldots, \]

(1.2.23)

an \( F_s \) series terminates if one of its numerator parameters is zero or a negative integer, and an \( r, \phi_s \) series terminates if one of its numerator parameters is of the form \( q^{-m} \) with \( m = 0, 1, 2, \ldots \), and \( q \neq 0 \). Basic analogues of the classical orthogonal polynomials will be considered in Chapter 7 as well as in the exercises at the ends of the chapters.

Unless stated otherwise, when dealing with nonterminating basic hypergeometric series we shall assume that \( |q| < 1 \) and that the parameters and variables are such that the series converges absolutely. Note that if \( |q| > 1 \), then we can perform an inversion with respect to the base by setting \( p = q^{-1} \) and using the identity

\[ (a; q)_n = (a^{-1}; p)_n (q^{-n}; q^{-1}) \]

(1.2.24)

to convert the series (1.2.22) to a similar series in base \( p \) with \( |p| < 1 \). The inverted series will have a finite radius of convergence if the original series does.

Observe that if we denote the terms of the series (1.2.16) and (1.2.22) which contain \( q^n \) by \( u_n \) and \( v_n \), respectively, then

\[ \frac{v_{n+1}}{v_n} = \frac{(a_1 + n)(a_2 + n) \cdots (a_r + n)}{(1 + n)(b_1 + n) \cdots (b_s + n)} q^n \]

(1.2.25)

is a rational function of \( q \), and

\[ \frac{v_{n+1}}{v_n} = \frac{(1 - a_1 q^n)(1 - a_2 q^n) \cdots (1 - a_r q^n)}{(1 - q^n)(1 - b_1 q^n) \cdots (1 - b_s q^n)} (-q^n)^{1+s-r} q^n \]

(1.2.26)

is a rational function of \( q^n \). Conversely, if \( \sum_{n=0}^{\infty} u_n \) and \( \sum_{n=0}^{\infty} v_n \) are power series with \( u_0 = v_0 = 1 \) such that \( u_{n+1}/u_n \) is a rational function of \( n \) and \( v_{n+1}/v_n \) is a rational function of \( q^n \), then these series are of the forms (1.2.16) and (1.2.22), respectively.

By the ratio test, the \( r, F_s \) series converges absolutely for all \( z \) if \( r \leq s \), and for \( |z| < 1 \) if \( r = s + 1 \). By an extension of the ratio test (Bromwich [1959, p.241]), it converges absolutely for \( |z| = 1 \) if \( r = s + 1 \) and \( \text{Re} [b_1 + \cdots + b_s - (a_1 + \cdots + a_r)] > 0 \). If \( r > s + 1 \) and \( z \neq 0 \) or \( r = s + 1 \) and \( |z| > 1 \), then this series diverges, unless it terminates.

If \( 0 < |q| < 1 \), the \( r, \phi_s \) series converges absolutely for all \( z \) if \( r \leq s \) and for \( |z| < 1 \) if \( r = s + 1 \). This series also converges absolutely if \( |q| > 1 \) and \( |z| < \sum_{n=0}^{\infty} b_1 b_2 \cdots b_s |a_1 a_2 \cdots a_r| \). It diverges for \( z \neq 0 \) if \( 0 < |q| < 1 \) and \( r > s + 1 \), and if \( |q| > 1 \) and \( |z| > \sum_{n=0}^{\infty} b_1 b_2 \cdots b_s |a_1 a_2 \cdots a_r| \), unless it terminates.

As is customary, the \( r, F_s \) and \( r, \phi_s \) notations are also used for the sums of these series inside the circle of convergence and for their analytic continuations (called hypergeometric functions and basic hypergeometric functions, respectively) outside the circle of convergence.

Observe that the series (1.2.22) has the property that if we replace \( z \) by \( z/a_r \) and let \( a_r \rightarrow \infty \), then the resulting series is again of the form (1.2.22) with \( r \) replaced by \( r - 1 \). Because this is not the case for the \( r, \phi_s \) series defined without the factors \( (1/1-q^n/2)^{1+s-r} \) in the books of Bailey [1935] and Slater [1966] and we wish to be able to handle such limit cases, we have chosen to use the series defined in (1.2.22). There is no loss in generality since the Bailey and Slater series can be obtained from the \( r = s + 1 \) case of (1.2.22) by choosing \( s \) sufficiently large and setting some of the parameters equal to zero.

An \( r+1, F_s \) series is called \( k \)-balanced if \( b_1 + b_2 + \cdots + b_s = k + a_1 + a_2 + \cdots + a_{r+1} \) and \( z = 1 \); \( a \)-balanced series is called balanced (or Saalschützian). Analogously, an \( r+1, \phi_s \) series is called \( k \)-balanced if \( b_1 b_2 \cdots b_s = q^k a_1 a_2 \cdots a_{r+1} \) and \( z = q \), and \( a \)-balanced series is called balanced (or Saalschützian). We will first encounter balanced series in §1.7, where we derive a summation formula for such a series.

For negative subscripts, the shifted factorial and the \( q \)-shifted factorials are defined by

\[ (a)_{-n} = \frac{1}{(a - 1)(a - 2) \cdots (a - n)} = \frac{1}{(a - n)_{n}} = \frac{(-1)^n q^n}{(1 - a)_n}, \]

(1.2.27)
Basic Hypergeometric Series

\[(a; q)_n = \frac{1}{(1 - a q^{-1})(1 - a q^{-2}) \cdots (1 - a q^{-n})} = (a q^{-n}; q)_n = \frac{(-q/a)^n q^n}{(q/a; q)_n}, \quad (1.2.28)\]

where \(n = 0, 1, \ldots\). We also define

\[(a; q)_\infty = \prod_{k=0}^{\infty} (1 - a q^k) \quad (1.2.29)\]

for \(|q| < 1\). Since the infinite product in (1.2.29) diverges when \(a \neq 0\) and \(|q| \geq 1\), whenever \((a; q)_\infty\) appears in a formula, we shall assume that \(|q| < 1\). The following easily verified identities will be frequently used in this book:

\[
(a; q)_n = \frac{(a; q)_\infty}{(a; q)_\infty}, \quad (1.2.30)
\]

\[
(a^{-1} q^{1-n}; q)_n = (a; q)_n (-a^{-1})^n q^{-n} (-q)_n^2, \quad (1.2.31)
\]

\[
(a; q)_n-k = \frac{(a; q)_n}{(a^{-1} q^{1-n}; q)_k} (-a^{-1})^k q^2-k = (a; q)_n-k, \quad (1.2.32)
\]

\[
(a; q)_{n+k} = (a; q)_n (aq^n; q)_k, \quad (1.2.33)
\]

\[
(a q^n; q)_k = \frac{(a; q)_k (aq^k; q)_n}{(a; q)_n}, \quad (1.2.34)
\]

\[
(a q^k; q)_{n-k} = \frac{(a; q)_n}{(a; q)_k}, \quad (1.2.35)
\]

\[
(a q^{2k}; q)_{n-k} = \frac{(a; q)_n (aq^{2n}; q)_k}{(a; q)_{2k}} \quad (1.2.36)
\]

\[
(q^{-n}; q)_k = \frac{(q; q)_n}{(q; q)_{n-k}} (-1)^k q^{k^2} = (-1)^k q^{k^2} = (q^{-n}; q)_{n-k}, \quad (1.2.37)
\]

\[
(a q^{-n}; q)_k = \frac{(a; q)_k (qa^{-1}; q)_n}{(a^{-1} q^{1-k}; q)_n} q^{-n} = (a q^{-n}; q)_k, \quad (1.2.38)
\]

\[
(a; q)_{2n} = (a; q^2)_n (aq; q^2)_n, \quad (1.2.39)
\]

\[
(a^2; q^2)_n = (a; q)_n (-a; q)_n, \quad (1.2.40)
\]

where \(n\) and \(k\) are integers. A more complete list of useful identities is given in Appendix I at the end of the book.

Since products of \(q\)-shifted factorials occur so often, to simplify them we shall frequently use the more compact notations

\[
(a_1, a_2, \ldots, a_m; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_m; q)_n, \quad (1.2.41)
\]

\[
(a_1, a_2, \ldots, a_m; q)_\infty = (a_1; q)_\infty (a_2; q)_\infty \cdots (a_m; q)_\infty, \quad (1.2.42)
\]

The \(q\)-binomial theorem

1.3 The \(q\)-binomial theorem

One of the most important summation formulas for hypergeometric series is given by the binomial theorem:

\[2 F_1(a, c; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} \frac{z^n}{n!} = (1 - z)^{-a}, \quad (1.3.1)\]

where \(|z| < 1\). We shall show that this formula has the following \(q\)-analogue

\[1 F_0(a; q; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(q)_n} \frac{z^n}{n!} = \frac{(aq; q)_\infty}{(z; q)_\infty}, \quad |z| < 1, \quad |q| < 1, \quad (1.3.2)\]

which was derived by Cauchy [1843], Heine [1847] and by other mathematicians.

Heine's proof of (1.3.2), which can also be found in the books Heine [1878] Bailey [1935, p.66] and Slater [1966, p.92], is better understood if one first follows Askey's [1980a] approach of evaluating the sum of the binomial series in (1.3.1), and then carries out the analogous steps for the series in (1.3.2).

Let us set

\[f_a(z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} z^n. \quad (1.3.3)\]

Since this series is uniformly convergent in \(|z| \leq \varepsilon\) when \(0 < \varepsilon < 1\), we may differentiate it termwise to get

\[f_a'(z) = \sum_{n=1}^{\infty} \frac{n(a)_n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{(a)_n+1}{n!} z^n = f_{a+1}(z). \quad (1.3.4)\]

Also

\[f_a(z) - f_{a+1}(z) = \sum_{n=1}^{\infty} \frac{(a)_n - (a+1)_n}{n!} z^n = \sum_{n=1}^{\infty} \frac{(a+1)_n}{n!} z^n = -\sum_{n=1}^{\infty} \frac{n(a+1)_n}{n!} z^n = -\sum_{n=0}^{\infty} \frac{(a+1)_n}{n!} z^{n+1} = -zf_{a+1}(z). \quad (1.3.5)\]

Eliminating \(f_{a+1}(z)\) from (1.3.4) and (1.3.5), we obtain the first order differential equation

\[f_a'(z) = \frac{a}{1 - z} f_a(z), \quad (1.3.6)\]

subject to the initial condition \(f_a(0) = 1\), which follows from the definition (1.3.3) of \(f_a(z)\). Solving (1.3.6) under this condition immediately gives the \(f_a(z) = (1 - z)^{-a}\) for \(|z| < 1\).
Analogously, let us now set

\[
  h_a(z) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n, \quad |z| < 1, \quad |q| < 1.
\]  

(1.3.7)

Clearly, \( h_{aq}(z) \to f_a(z) \) as \( q \to 1 \). Since \( h_{aq}(z) \) is a \( q \)-analogue of \( f_{a+1}(z) \), we first compute the difference

\[
  h_a(z) - h_{aq}(z) = \sum_{n=1}^{\infty} \frac{(a;q)_n - (aq;q)_n}{(q;q)_n} z^n
\]

\[
= \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} (1 - a - (1 - aq^n)) z^n
\]

\[
= -a \sum_{n=1}^{\infty} \frac{(1 - q^n)(aq;q)_{n-1}}{(q;q)_n} z^n
\]

\[
= -a \sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}}{(q;q)_n} z^n = -az h_{aq}(z), \quad (1.3.8)
\]

giving an analogue of (1.3.5). Observing that

\[
f'(z) = \lim_{q \to 1} \frac{f(z) - f(qz)}{(1 - q)z} \quad (1.3.9)
\]

for a differentiable function \( f(z) \), we next compute the difference

\[
h_a(z) - h_a(qz) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} (z^n - q^n z^n)
\]

\[
= \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \sum_{n=0}^{\infty} \frac{(a;q)_{n+1}}{(q;q)_n} z^{n+1}
\]

\[
= (1 - a) z h_a(qz), \quad (1.3.10)
\]

Eliminating \( h_{aq}(z) \) from (1.3.8) and (1.3.10) gives

\[
h_a(z) = \frac{1 - az}{1 - z} h_a(qz).
\]  

(1.3.11)

Iterating this relation \( n - 1 \) times and then letting \( n \to \infty \) we obtain

\[
h_a(z) = \frac{(az;q)_\infty}{(z;q)_\infty} h_a(q^nz)
\]

\[
= (az; q)_\infty h_a(0) = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad (1.3.12)
\]

since \( q^n \to 0 \) as \( n \to \infty \) and \( h_a(0) = 1 \) by (1.3.7), which completes the proof of (1.3.2).

One consequence of (1.3.2) is the product formula

\[
\phi_0(a; -; q, z) \phi_0(b; -; q, az) = \phi_0(ab; -; q, z),
\]  

(1.3.13)

which is a \( q \)-analogue of \((1 - z)^{-a}(1 - z)^{-b} = (1 - z)^{-a-b}\).
Hence, for $|z| < 1$ and $|b| < 1$,
\[
2\phi_1(a, b; c; q, z) = \frac{(b; q)\infty}{(c; q)\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n (cq^n; q)\infty}{(q; q)_n (bq^n; q)\infty} z^n
\]

\[
= \frac{(b; q)\infty}{(c; q)\infty} \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n \sum_{m=0}^{\infty} \frac{(c/b; q)_m (qz^m; q)\infty}{(q; q)_m} (bq^m)^n
\]

\[
= \frac{(b; q)\infty}{(c; q)\infty} \sum_{m=0}^{\infty} \frac{(c/b; q)_m}{(q; q)_m} \sum_{n=0}^{\infty} \frac{(a; q)_n (qz^n; q)\infty}{(q; q)_n} (azq^m; q)\infty
\]

\[
= \frac{(b, a; q; z; q)\infty}{(c, z; q)\infty} 2\phi_1(c/b, z; az, q; b)
\]

by (1.3.2), which gives (1.4.1).
Heine also showed that Euler's transformation formula
\[
2F_1(a, b; c; z) = (1 - z)^{-a-b} 2F_1(c - a, c - b; c; z)
(1.4.2)
\]
has a $q$-analogue of the form
\[
2\phi_1(a, b; c; q, z) = \frac{(abz/c; q)\infty}{(c; q)\infty} 2\phi_1(c/a, c/b; c; abz/c).
(1.4.3)
\]
A short way to prove this formula is just to iterate (1.4.1) as follows
\[
2\phi_1(a, b; c; q, z) = \frac{(b, a; z; q)\infty}{(c, z; q)\infty} 2\phi_1(c/b, z; az; q, b)
(1.4.4)
\]

\[
= \frac{(c/b, bz; q)\infty}{(c, z; q)\infty} 2\phi_1(abz/c, b; bz; q, c/b)
(1.4.5)
\]

\[
= \frac{(abz/c; q)\infty}{(c, z; q)\infty} 2\phi_1(c/a, c/b; c; abz/c).
(1.4.6)
\]

## 1.5 Heine's $q$-analogue of Gauss' summation formula

In order to derive Heine's [1847] $q$-analogue of Gauss' summation formula (1.2.11) it suffices to set $z = c/ab$ in (1.4.1), assume that $|b| < 1, |c/ab| < 1$, and observe that the series on the right side of
\[
2\phi_1(a, b; c; q, c/ab) = \frac{(b, c/b; q)\infty}{(c, c/ab; q)\infty} 1\phi_0(c/ab; c/q, b)
\]
can be summed by (1.3.2) to give
\[
2\phi_1(a, b; c; q, c/ab) = \frac{(c/a, c/b; q)\infty}{(c, c/ab; q)\infty}. \quad (1.5.1)
\]
By analytic continuation, we may drop the assumption that $|b| < 1$ and require only that $|c/ab| < 1$ for (1.5.1) to be valid.

Heine's summation formula

For the terminating case when $a = q^{-n}$, (1.5.1) reduces to
\[
2\phi_1(q^{-n}, b; c; q^{-n}/b) = \frac{(c/b; q)\infty}{(c; q)\infty}. \quad (1.5.2)
\]
By inversion or by changing the order of summation it follows from (1.5.2) that
\[
2\phi_1(q^{-n}, b; c; q) = \frac{(c/b; q)\infty}{(c; q)\infty} b^n. \quad (1.5.3)
\]
Both (1.5.2) and (1.5.3) are $q$-analogues of Vandermonde's formula (1.2.9).
These formulas can be used to derive other important formulas such as, for example, Jackson's [1910a] transformation formula
\[
2\phi_1(a, b; c; q, z) = \frac{(a, c/b; q)_k}{(z; q)_k} \sum_{k=0}^{\infty} \frac{(a, c, az; q)_{k}}{(q, c, az; q)_{k}} \frac{(-b)_k z^k}{k!} 2\phi_2(1, a, c - b; c, az; q, bz). \quad (1.5.4)
\]
This formula is a $q$-analogue of the Pfaff-Kummer transformation formula
\[
2F_1(a, b; c; z) = (1 - z)^{-a} 2F_1(a, c - b; c; z/(z - 1)). \quad (1.5.5)
\]
To prove (1.5.4), we use (1.5.2) to write
\[
\frac{(b; q)\infty}{(c; q)\infty} = \sum_{n=0}^{k} \frac{(q^{-n}, c/b; q)_n}{(q, c; q)_n} (bq^n)^n
\]
and hence
\[
2\phi_1(a, b; c; q, z)
\]

\[
= \sum_{k=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k \sum_{n=0}^{k} \frac{(q^{-n}, c/b; q)_n}{(q, c; q)_n} (bq^n)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(a; q)_k}{(q; q)_k} z^k \sum_{n=0}^{k} \frac{(q^{-n}, c/b; q)_n}{(q, c; q)_n} (bq^n)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{(a, c/b; q)_n}{(q, c; q)_n} (-b)_n z^n \sum_{n=0}^{k} \frac{(q^{-n}, q)_k}{(q; q)_k} z^k
\]

\[
= \sum_{n=0}^{\infty} \frac{(a, c/b; q)_n}{(q, c; q)_n} (-b)_n z^n \sum_{n=0}^{k} \frac{(aq^n; q)_k}{(q; q)_k} z^k
\]

by (1.3.2). Also see Andrews [1973]. If $a = q^{-n}$, then the series on the right side of (1.5.4) can be reversed (by replacing $k$ and $n - k$) to yield Sears' [1951c] transformation formula
\[
2\phi_1(q^{-n}, b; c; q, z)
\]

\[
= \frac{(c/b; q)_n}{(c; q)_n} \left( \frac{b}{q} \right)^n 3\phi_2(q^{-n}, q/z, c^{-1}q^{1-n}; bc^{-1}q^{1-n}, 0; q, q). \quad (1.5.6)
\]
1.6 Jacobi’s triple product identity
and the theta functions

Jacobi’s [1829] well-known triple product identity (see Andrews [1971])

\[ (q^2, q^2/z; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2} z^n \]  
(1.6.1)

can be easily derived by using Heine’s summation formula (1.5.1). First, set \( c = bzq^2 \) in (1.5.1) and then let \( b \to 0 \) and \( a \to \infty \) to obtain

\[ \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q; q)_n} z^n = (zq^2; q)_\infty. \]  
(1.6.2)

Similarly, setting \( c = zq \) in (1.5.1) and letting \( a \to \infty \) and \( b \to \infty \) we get

\[ \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q, zq; q)_n} = \frac{1}{(zq; q)_\infty}. \]  
(1.6.3)

Now use (1.6.2) to find that

\[ (zq^2, q^2/z; q)_\infty = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{m+n} q^{(m^2+n^2)/2}}{(q; q)_m(q; q)_n} z^{m-n} \]
\[ = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q; q)_n} z^n \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q, q^{n+1}; q)_k} q^{nk} \]
\[ + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n^2/2}}{(q; q)_n} z^{-n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q, q^{n+1}; q)_k} q^{nk}. \]  
(1.6.4)

Formula (1.6.1) then follows from (1.6.3) by observing that

\[ \frac{1}{(q; q)_n} \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q, q^{n+1}; q)_k} q^{nk} = \frac{1}{(q; q)_n(q^{n+1}; q)_\infty} = \frac{1}{(q; q)_\infty}. \]

An important application of (1.6.1) is that it can be used to express the theta functions (Whittaker and Watson [1965, Chapter 21])

\[ \vartheta_1(x) = 2 \sum_{n=0}^{\infty} (-1)^n q^{(n+1/2)^2} \sin(2n+1)x, \]  
(1.6.5)

\[ \vartheta_2(x) = 2 \sum_{n=0}^{\infty} q^{(n+1/2)^2} \cos(2n+1)x, \]  
(1.6.6)

\[ \vartheta_3(x) = 1 + 2 \sum_{n=1}^{\infty} q^n \cos 2nx, \]  
(1.6.7)

\[ \vartheta_4(x) = 1 + 2 \sum_{n=1}^{\infty} (-1)^n q^n \cos 2nx \]  
(1.6.8)

in terms of infinite products. Just replace \( q \) by \( q^2 \) in (1.6.1) and then set \( z \) equal to \( qe^{2ix}, -qe^{2ix}, -e^{2ix}, e^{2ix} \), respectively, to obtain

\[ \vartheta_1(x) = 2q^{1/4} \sin x \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n} \cos 2x + q^{4n}), \]  
(1.6.9)

\[ \vartheta_2(x) = 2q^{1/4} \cos x \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n} \cos 2x + q^{4n}), \]  
(1.6.10)

\[ \vartheta_3(x) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 + 2q^{2n-1} \cos 2x + q^{4n-2}), \]  
(1.6.11)

and

\[ \vartheta_4(x) = \prod_{n=1}^{\infty} (1 - q^{2n})(1 - 2q^{2n-1} \cos 2x + q^{4n-2}). \]  
(1.6.12)

1.7 A \( q \)-analogue of Saalschütz’s summation formula

Pfaff [1979] discovered the summation formula

\[ _3F_2(a, b, c; -a-c+1, 1; 1) = \frac{(c-a)_n(c-b)_n}{(c)_n(-a)_n}, \quad n = 0, 1, \ldots, \]  
(1.7.1)

which sums a terminating balanced \(_3F_2(1)\) series with argument 1. It was rediscovered by Saalschütz [1890] and is usually called \( \text{Saalschütz’s formula} \) or the \( \text{Pfaff–Saalschütz formula} \); see Askey [1975]. To derive a \( q \)-analogue of (1.7.1), observe that since, by (1.3.2),

\[ \frac{(ab/z; c)_\infty}{(z; q)_\infty} = \sum_{k=0}^{\infty} \frac{(ab/c; q)_k z^k}{(q; q)_k} \]

the right side of (1.4.3) equals

\[ \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(ab/c; q)_k(c/a, c/b; q)_m}{(q; q)_k(q; c, cq^{-1}/ab; q)_m} \left( \frac{ab}{c} \right)^m z^{k+m}, \]

and hence, equating the coefficients of \( z^n \) on both sides of (1.4.3) we get

\[ \sum_{j=0}^{n} \frac{(a, c^{-1}b; q)_j}{(q, c, cq^{-1}/ab; q)_j} q^j = \frac{(a, b; c)_n}{(c, ab/c; q)_n}. \]

Replacing \( a, b \) by \( c/a, c/b \), respectively, this gives the following sum of a terminating balanced \(_3\phi_2 \) series

\[ _3\phi_2(a, b, c; -a-c+1, 1; q, q) = \frac{(c/a, c/b; q)_n}{(c, ab/c; q)_n}, \quad n = 0, 1, \ldots, \]  
(1.7.2)

which was first derived by Jackson [1910a]. It is easy to see that (1.7.1) follows from (1.7.2) by replacing \( a, b, c \) in (1.7.2) by \( q^a, q^b, q^c \), respectively, and letting \( q \to 1 \). Note that letting \( a \to \infty \) in (1.7.2) gives (1.5.2), while letting \( a \to 0 \) gives (1.5.3).
1.8 The Bailey-Daum summation formula

Bailey [1941] and Daum [1942] independently discovered the summation formula

\[ 2\phi_1\left(a; b; \frac{aq/b}{q}; \frac{-q/b}{q}\right) = \frac{\left(-q; q\right)_\infty \left(aq, aq^2/b^2, q^2\right)_\infty}{\left(aq/b, -q/b; q\right)_\infty}, \tag{1.8.1} \]

which is a q-analogue of Kummer's formula

\[ 2F_1(a; b; 1 + a - b; -1) = \frac{\Gamma(1 + a - b)\Gamma(1 + \frac{1}{2}a)}{\Gamma(1 + a)\Gamma(1 + \frac{3}{2}a - b)} \tag{1.8.2} \]

Formula (1.8.1) can be easily obtained from (1.4.1) by using the identity (1.2.40) and a limiting form of (1.2.39), namely, \( (a; q)_\infty = (a, aq; q^2)_\infty \), to see that

\[ 2\phi_1\left(a; b; \frac{aq/b}{q}; \frac{-q/b}{q}\right) \]

\[ = \frac{\left(-q; q\right)_\infty \left(aq/b, -q/b; q\right)_\infty}{\left(aq/b, -q/b; q\right)_\infty} \left(a, aq^2/b^2, q^2\right)_\infty \]

\[ = \frac{\left(-q; q\right)_\infty \left(aq^2/b^2, q^2\right)_\infty}{\left(aq/b, -q/b; q\right)_\infty} \]

by (1.3.2).

1.9 q-Analogues of the Karlsson-Minton summation formulas

Minton [1970] showed that if \( a \) is a negative integer and \( m_1, m_2, \ldots, m_r \) are nonnegative integers such that \( -a \geq m_1 + \cdots + m_r \), then

\[ \frac{r+2}{r+1} F_{r+1} \left[ \begin{array}{c} a, b, b_1 + m_1, \ldots, b_r + m_r, 1 \\ b_1, \ldots, b_r \end{array} \right] = \frac{\Gamma(b + 1)\Gamma(1 - a)\Gamma(b_1 - b)m_1 \cdots (b_r - b)m_r}{\Gamma(1 + a - b)\Gamma(1 + b_1 + \cdots + b_r)m_1 \cdots (b_r + b_1)m_r} \tag{1.9.1} \]

where, as usual, it is assumed that none of the factors in the denominators of the terms of the series is zero. Karlsson [1971] showed that (1.9.1) also holds when \( a \) is not a negative integer provided that the series converges, i.e., if \( \Re(-a) > m_1 + \cdots + m_r - 1 \), and he deduced from (1.9.1) that

\[ (r+1) F_r \left[ \begin{array}{c} a, b + m_1, \ldots, b + m_r, 1 \\ b_1, \ldots, b_r \end{array} \right] = 0, \quad \Re(-a) > m_1 + \cdots + m_r, \tag{1.9.2} \]

\[ (r+1) F_r \left[ \begin{array}{c} -(m_1 + \cdots + m_r), b_1 + m_1, \ldots, b_r + m_r, 1 \\ b_1, \ldots, b_r \end{array} \right] = (-1)^{m_1 + \cdots + m_r} \frac{m_1 + \cdots + m_r}{(b_1)_m \cdots (b_r)_m}, \tag{1.9.3} \]

\[ \frac{r+1}{r} F_r \left[ \begin{array}{c} a, aq^{m_1}, \ldots, aq^{m_r}, q^{-1} \right] \right] = \frac{1}{(b_1)_m \cdots (b_r)_m}, \tag{1.9.4} \]

This expansion formula is a q-analogue of a formula used by Minton [1970, (4)].

1.9.1 q-Karlsson-Minton formulas

These formulas are particularly useful for evaluating sums that appear as solutions to some problems in theoretical physics such as the Racah coefficients. They were also used by Gaser [1981b] to prove the orthogonality on \( 0, 2\pi \) of certain functions that arose in Greiner's [1980] work on spherical harmonic on the Heisenberg group. Here we shall present Gaser's [1981a] derivation of q-analogues of the above formulas. Some of the formulas derived below will be used in Chapter 7 to prove the orthogonality relation for the continuous q-ultraspherical polynomials.

Observe that if \( m \) and \( n \) are nonnegative integers with \( m \geq n \), then

\[ 2\phi_1\left(q^{-m}, q^{-m}; b_r; q\right) = \frac{(b_r q^{m}; q)_n}{(b_r; q)_n} q^{-mn} \]

by (1.5.3), and hence

\[ (r+1) \phi_r \left[ \begin{array}{c} a_1, \ldots, a_r, b_r q^m \\ b_1, \ldots, b_r \end{array} \right] = \frac{1}{q} \frac{a_1, \ldots, a_r, b_r q^m}{b_1, \ldots, b_r} \cdot \frac{z^n}{(q - b_1, \ldots, b_r, q)} \cdot \frac{q^{mn} \cdot (z)^m}{(q; q)_n - q^{mn}} \cdot \frac{a_r, \ldots, a_1, b_1 q^m}{a_1, \ldots, a_r, b_1 q^m} \cdot \frac{q^{mn}}{q^{mn} - (z)^m}, \tag{1.9.4} \]

where \( z \in (0, 1) \).

This expansion formula is a q-analogue of a formula used by Minton [1970, (4)].

When \( r = 2 \), formulas (1.9.4), (1.5.1) and (1.5.3) yield

\[ 3\phi_2 \left[ \begin{array}{c} a, b; q^{m_1} \\ b_1, b_2 \end{array} \right] = \frac{(q, (b_1)_m, b_1 m_1; b_2)_m}{(q, (b_2)_m, b_2 m_2; b_1)_m} \cdot \frac{2\phi_1\left(\frac{b_1 m_1}{b_2}; q, b_1 q^{m_1}; b_2 q^{m_2}; q\right)}{(b_1, b_2; q)_m (b_1, b_2)_m}, \tag{1.9.5} \]

provided that \( |q^{-1} q^{1-m_n}| < 1 \). By induction it follows from (1.9.4) and (1.9.5) that if \( m_1, \ldots, m_r \) are nonnegative integers and \( |q^{-1} q^{1-(m_1 + \cdots + m_r)}| < 1 \), the

\[ (r+2) \phi_{r+1} \left[ \begin{array}{c} a, b, b_1 q^{m_1}, \ldots, b_1 q^{m_r}, q^{-1} \right] \right] = \frac{1}{(b_1)_m \cdots (b_r)_m} \cdot \frac{1}{(b_1, b_2, q^{m_1}, \ldots, q^{m_r})} \cdot \frac{1}{(b_1, b_2, q^{m_1}, \ldots, q^{m_r})}, \tag{1.9.6} \]

which is a q-analogue of (1.9.1). Formula (1.9.1) can be derived from (1.9.6) by replacing \( a, b, b_1, \ldots, b_r \) by \( q^{12}, q^{12}, \ldots, q^{12} \), respectively, and letting \( q - 1 \rightarrow 1 \).

Setting \( b_r = b, m_r = 1 \) and then replacing \( r + 1 \) in (1.9.6) gives

\[ (r+1) \phi_r \left[ \begin{array}{c} a, b, b_1 q^{m_1}, \ldots, b_r q^{m_r}, q^{-1} \right] \right] = \frac{1}{(b_1)_m \cdots (b_r)_m}, \tag{1.9.7} \]
while letting \( b \to \infty \) in the case \( a = q^{-(m_1 + \cdots + m_r)} \) of (1.9.6) gives

\[
\psi_{r+1} \left[ \frac{q^{-(m_1 + \cdots + m_r)}, b_1 q^{m_1}, \ldots, b_r q^{m_r}}{b_1, \ldots, b_r}; q, 1 \right] = \frac{(-1)^{m_1 + \cdots + m_r} (q; q)_{m_1 + \cdots + m_r} q^{-(m_1 + \cdots + m_r)(m_1 + \cdots + m_r + 1)/2}}{(b_1; q)_{m_1} \cdots (b_r; q)_{m_r}}, \tag{1.9.8}
\]

which are \( q \)-analogues of (1.9.2) and (1.9.3). Another \( q \)-analogue of (1.9.3) can be derived by letting \( b \to 0 \) in (1.9.6) to obtain

\[
\psi_{r+1} \left[ \frac{a, b_1 q^{m_1}, \ldots, b_r q^{m_r}}{b_1, \ldots, b_r}; q, a^{-1}q^{-(m_1 + \cdots + m_r)} \right] = \frac{(-1)^{m_1 + \cdots + m_r} (q; q)_{m_1} \cdots (b_1; q)_{m_1} \cdots (b_r; q)_{m_r}}{(a; q)_{\infty} (b_1; q)_{m_1} \cdots (b_r; q)_{m_r}}, \tag{1.9.9}
\]

when \( |a^{-1}q^{-(m_1 + \cdots + m_r)}| < 1 \).

In addition, if \( a = q^{-n} \) and \( n \) is a nonnegative integer then we can reverse the order of summation of the series in (1.9.6), (1.9.7) and (1.9.9) to obtain

\[
\psi_{r+2} \phi_{r+1} \left[ \frac{q^{-n}, b_1 b_1 q^{m_1}, \ldots, b_r q^{m_r}}{b_1 q, b_1, \ldots, b_r}; q, q \right] = \frac{b^n (q; q)_n (b_1; q)_{m_1} \cdots (b_r; q)_{m_r}}{(b_1 q; q)_n (b_1; q)_{m_1} \cdots (b_r; q)_{m_r}}, \quad n \geq m_1 + \cdots + m_r, \tag{1.10.10}
\]

and the following generalization of (1.9.8)

\[
\psi_{r+1} \phi_{r+1} \left[ \frac{q^{-n}, b_1 b_1 q^{m_1}, \ldots, b_r q^{m_r}}{b_1, \ldots, b_r}; q, q \right] = \frac{(-1)^{n} (q; q)_n q^{-n(n+1)/2}}{(b_1; q)_{m_1} \cdots (b_r; q)_{m_r}}, \tag{1.12.12}
\]

where \( n \geq m_1 + \cdots + m_r \), which also follows by letting \( b \to \infty \) in (1.9.10).

### 1.10 The \( q \)-gamma and \( q \)-beta functions

The \( q \)-gamma function

\[
\Gamma_q(x) = \frac{(q; q)_x}{(1 - q)^{1-x}}, \quad 0 < q < 1, \tag{1.10.11}
\]

was introduced by Thomae [1869] and later by Jackson [1904e]. Heine [1847] gave an equivalent definition, but without the factor \((1-q)^{1-x}\). When \( x = n+1 \) with \( n \) a nonnegative integer, this definition reduces to

\[
\Gamma_q(n+1) = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}), \tag{1.10.2}
\]

which clearly approaches \( n! \) as \( q \to 1^- \). Hence \( \Gamma_q(n+1) \) tends to \( \Gamma(n+1) = n! \) as \( q \to 1^- \). The definition of \( \Gamma_q(x) \) can be extended to \( |q| < 1 \) by using the principal values of \( q^x \) and \((1-q)^{1-x}\) in (1.10.1).

To show that

\[
\lim_{q \to 1^-} \Gamma_q(x) = \Gamma(x) \tag{1.10.3}
\]

we shall give a simple proof due to Wm. Gosper; see Andrews [1986]. From (1.10.1),

\[
\Gamma_q(x+1) = \frac{(q; q)_x}{(1 - q)^{1-x}} \frac{(1 - q)^{-x}}{(q^{1-x}; q)_x} = \left( \frac{(1 - q)^{1-x}}{(1 - q^{1-x})} \right)^x \prod_{n=1}^{\infty} \frac{(1 - q)^{1-x}}{(1 - q^{-n+1})^x} \frac{1}{(1 - q^n)^x}.
\]

Hence

\[
\lim_{q \to 1^-} \Gamma_q(x+1) = \left( \lim_{q \to 1^-} \frac{n}{n + x} \right)^x = \left[ x \left( \prod_{n=1}^{\infty} \left( 1 + \frac{x}{n} \right)^{-1} \right) \left( 1 + \frac{1}{n} \right) \right]^x = x \Gamma(x+1)
\]

by Euler's product formula (see Whittaker and Watson [1965, §12.11]) and the well-known functional equation for the gamma function

\[
\Gamma(x+1) = x \Gamma(x), \quad \Gamma(1) = 1. \tag{1.10.4}
\]

For a rigorous justification of the above steps see Koornwinder [1989a]. From (1.10.1) it is easily seen that, analogous to (1.10.4), \( \Gamma_q(x) \) satisfies the functional equation

\[
f(x+1) = \frac{1 - q^x}{1-q} f(x), \quad f(1) = 1. \tag{1.10.5}
\]

Askey [1978] derived analogues of many of the well-known properties of the gamma function, including its log-convexity (see the exercises at the end of this chapter), which show that (1.10.1) is a natural analogue of \( \Gamma(x) \).

It is obvious from (1.10.1) that \( \Gamma_q(x) \) has poles at \( x = 0, -1, -2, \ldots \). The residue of \( \Gamma_q(x) \) at \( x = -n \) is

\[
\lim_{x \to -n} (x+n) \Gamma_q(x) = \frac{(1-q)^{n+1}}{(1 - q^{-n})(1 - q^{-1})(1 - q^{-2}) \cdots (1 - q^{-n})} \lim_{x \to -n} \frac{x+n}{1 - q^{-x+n}} = \frac{(1-q)^{n+1}}{(q^{-n}; q)_n \log q^{-1}}. \tag{1.10.6}
\]

The \( q \)-gamma function has no zeros, so its reciprocal is an entire function with zeros at \( x = 0, -1, -2, \ldots \). Since

\[
\frac{1}{\Gamma_q(x)} = (1 - q)^{-x-1} \prod_{n=0}^{\infty} \frac{1 - q^{n+x}}{1 - q^{n+1}}, \tag{1.10.7}
\]

the function \( 1/\Gamma_q(x) \) has zeros at \( x = -n \pm 2\pi i/k \log q \), where \( k \) and \( n \) are nonnegative integers.

A \( q \)-analogue of Legendre's duplication formula

\[
\Gamma(2x) \Gamma_x^{1/2} = 2^{2x-1} \Gamma(x) \Gamma(x+1/2) \tag{1.10.8}
\]
Basic Hypergeometric Series

can be easily derived by observing that
\[
\frac{\Gamma_q(x)\Gamma_q(x + \frac{1}{2})}{\Gamma_q(\frac{1}{2})} = \frac{(q;q^2;q^2)_\infty}{(q^{2x};q^2)_\infty} (1 - q^2)^{1-2x} (1 - q^2)^{1-2x} = (1 + q)^{1-2x} \Gamma_q(2x)
\]
and hence
\[
\Gamma_q(2x)\Gamma_q(x + \frac{1}{2}) = (1 + q)^{2x-1} \Gamma_q(x)\Gamma_q(x + \frac{1}{2}).
\]
(1.10.9)

Similarly, it can be shown that the Gauss multiplication formula
\[
\Gamma(nz)(2\pi)^{(n-1)/2} = n^{nz-\frac{1}{2}} \Gamma(x)\Gamma(x + \frac{1}{n}) \ldots \Gamma(x + \frac{n-1}{n})
\]
(1.10.10)

has a q-analogue of the form
\[
\Gamma_q(nz)\Gamma_q\left(\frac{x}{n}\right) \Gamma_q\left(\frac{x}{n}\right) \ldots \Gamma_q\left(\frac{n-1}{n}\right) = (1 + q + \ldots + q^{n-1})^{nz-1} \Gamma_q(x)\Gamma_q(x + \frac{1}{n}) \ldots \Gamma_q(x + \frac{n-1}{n})
\]
(1.10.11)

with \(r = q^n\); see Jackson [1904e, 1905c]. The q-gamma function for \(q > 1\) is considered in Exercise 1.23. For other interesting properties of the q-gamma function see Askey [1978] and Moak [1980a,b] and Ismail, Lorch and Muldoon [1986].

Since the beta function is defined by
\[
B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},
\]
(1.10.12)
it is natural to define the q-beta function by
\[
B_q(x, y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}
\]
(1.10.13)

which tends to \(B(x, y)\) as \(q \to 1^-\). By (1.10.1) and (1.3.2),
\[
B_q(x, y) = (1 - q)\frac{(q, q^{1+y}; q)_\infty}{(q^x, q^y; q)_\infty}
\]
\[
= (1 - q)\frac{q^x q^y \sum_{n=0}^{\infty} (q^x; q)_n q^{nx}}{q^x q^y \sum_{n=0}^{\infty} (q^y; q)_n q^{ny}}
\]
\[
= (1 - q)\sum_{n=0}^{\infty} \frac{(q^{n+1}; q)_\infty}{(q^{n+y}; q)_\infty} q^{nx}, \quad \text{Re} \ x, \text{Re} \ y > 0.
\]
(1.10.14)

This series expansion will be used in the next section to derive a q-integral representation for \(B_q(x, y)\).

The q-integral

1.11 The q-integral

Thoma [1969, 1970] and Jackson [1910c] introduced the q-integral
\[
\int_0^1 f(t) \, dq(t) = (1 - q) \sum_{n=0}^{\infty} f(q^n)q^n
\]
(1.11.1)

and Jackson gave the more general definition
\[
\int_a^b f(t) \, dq(t) = \int_a^b f(t) \, dq(t) - \int_0^a f(t) \, dq(t),
\]
(1.11.2)

where
\[
\int_0^a f(t) \, dq(t) = \alpha(1 - q) \sum_{n=0}^{\infty} f(q^n)q^n
\]
(1.11.3)

Jackson also defined an integral on \((0, \infty)\) by
\[
\int_0^\infty f(t) \, dq(t) = (1 - q) \sum_{n=-\infty}^{\infty} f(q^n)q^n
\]
(1.11.4)

The bilateral q-integral is defined by
\[
\int_{-\infty}^{\infty} f(t) \, dq(t) = (1 - q) \sum_{n=-\infty}^{\infty} [f(q^n) + f(-q^n)]q^n
\]
(1.11.5)

If \(f\) is continuous on \([0, a]\), then it is easily seen that
\[
\lim_{q \to 1^-} \int_0^a f(t) \, dq(t) = \int_0^a f(t) \, dt
\]
(1.11.6)

and that a similar limit holds for (1.11.4) and (1.11.5) when \(f\) is suitably restricted. By (1.11.1), it follows from (1.10.14) that
\[
B_q(x, y) = \int_0^1 t^{x-1} (tq^y; q)_\infty \, dq(t), \quad \text{Re} \ x > 0, \quad y \neq 0, -1, -2, \ldots,
\]
(1.11.7)

which clearly approaches the beta function integral
\[
B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt, \quad \text{Re} \ x, \text{Re} \ y > 0,
\]
(1.11.8)
as \(q \to 1^-\). Thoma [1969] rewrote Heine’s formula (1.4.1) in the q-integral form
\[
{}_2\phi_1(q^a, b; q^c; q, q) = \frac{\Gamma_q(c)}{\Gamma_q(c-b) \Gamma_q(c)} \int_0^1 t^{b-1} (tz^a, t^c-b; q)_\infty \, dq(t),
\]
(1.11.9)

which is a q-analogue of Euler’s integral representation
\[
{}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b) \Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a} \, dt,
\]
(1.11.10)

where \(|\arg(1-z)| < \pi\) and \(\text{Re} \ c > \text{Re} \ b > 0\).

The q-integral notation is, as we shall see later, quite useful in simplifying and manipulating various formulas involving sums of series.
Exercises 1

1.1 Verify the identities (1.2.30)–(1.2.40), and show that

(i) \( (aq^{-n}; q)_n = (q/a; q)_n \left( -\frac{a}{q} \right)^n q^{-n(\frac{1}{2})} \),
(ii) \( (aq^{-k-n}; q)_n = \left( \frac{q/a; q}{q/a; q} + k \right)^n q^{-n(\frac{1}{2})} \),
(iii) \( \frac{(a; q)_n}{(a; q)_n} = 1 - a^{-2n} \),
(iv) \( (a; q)_\infty = (a^\frac{1}{2}, -a^\frac{1}{2}, (aq)^\frac{1}{2}, -aq)q \).

1.2 The \( q \)-binomial coefficient is defined by

\[ \frac{n}{k}_q = \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} \]

for \( k = 0, 1, \ldots, n \), and by

\[ \frac{\alpha}{\beta}_q = \frac{(q^\beta + 1, q^{\alpha - \beta + 1}; q)_\infty}{(q, q^{\alpha + 1}; q)_\infty} = \frac{\Gamma_q(\alpha + 1)}{\Gamma_q(\beta + 1)\Gamma_q(\alpha - \beta + 1)} \]

for complex \( \alpha \) and \( \beta \) when \( |q| < 1 \). Verify that

(i) \( \frac{n}{k}_q = \left[ \frac{n}{n-k}_q \right] \),
(ii) \( \frac{\alpha}{k}_q = \left( \frac{q^{-\alpha}; q}{q^{-\alpha}; q} k \right)^{-\alpha} q^{-\alpha}(-q^k)q^{-\frac{k}{2}} \),
(iii) \( \frac{k + \alpha}{k}_q = \frac{(q^{(k+1)}; q)_k}{(q; q)_k} \),
(iv) \( \frac{-\alpha}{k}_q = \left( \frac{\alpha + k - 1}{k}_q \right)^{-\alpha} q^{-\alpha}(-q^k)q^{-\frac{k}{2}} \),
(v) \( \frac{\alpha + 1}{k}_q = \left[ \alpha \right]_q k^{\alpha} + \alpha \left[ k - 1 \right]_q q^\alpha k^{\alpha - k} \),
(vi) \( (z; q)_n = \sum_{k=0}^{n} \left[ \frac{n}{k}_q \right] (q)_k (z^k)q^k \),

when \( k \) and \( n \) are nonnegative integers.

1.3 (i) Show that the binomial theorem

\( (a + b)^n = \sum_{k=0}^{n} \left( \frac{n}{k}_q \right) a^k b^{n-k} \)

where \( n = 0, 1, \ldots \), has a \( q \)-analogue of the form

\( (ab; q)_n = \sum_{k=0}^{n} \left[ \frac{n}{k}_q \right] (a^k b^{n-k})q(q)_n k_n \).

Exercises 1

(ii) Extend the above formula to the \( q \)-multinomial theorem

\[ (a_1 a_2 \cdots a_{m+1}; q)_n \]

\[ = \sum_{k_1 + \cdots + k_{m+1} = n} \frac{n}{k_1 \cdots k_{m+1}} \left( \frac{q}{a_1; q} \frac{q}{a_2; q} \cdots \frac{a_{m+1}; q}{q} \right) (a_1; q)_k \cdots (a_{m+1}; q)_k \]

where \( m = 1, 2, \ldots \), \( n = 0, 1, \ldots \), and

\[ \left( \frac{a_1; q}{k_1}, \ldots, \frac{a_{m+1}; q}{k_{m+1}} \right)_n \]

is the \( q \)-multinomial coefficient.

1.4 (i) Prove the inversion formula

\[ \phi_1 \left[ \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; q, z \right] \]

\[ = \sum_{n=0}^{\infty} \frac{(a, q)_n}{(q, b_1, \ldots, b_s; q)_n} \left( \frac{a_1 \cdots a_r z}{b_1 \cdots b_s q} \right)^n \]

(ii) By reversing the order of summation, show that

\[ \phi_1 \left[ \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; q, z \right] \]

\[ = \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left( \frac{z}{q} \right)^n \left( -1 \right) q \]

\[ \cdot \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, q^1/n, a_1, \ldots, q^1/n, a_r; q)_n} \left( \frac{b_1 \cdots b_s q^{n+1}}{a_1 \cdots a_r z} \right)^n \]

when \( n = 0, 1, \ldots \).

1.5 Show that

\[ \frac{(c, bq^n; q)_m}{(b; q)_m} = \frac{(b/c; q)_n}{(b; q)_m} \sum_{k=0}^{n} \frac{(q^n, c; q)_k}{(q, cq^{1-n}; q)_k} (cq^k; q)_m. \]

1.6 Prove the summation formulas

(i) \( \phi_1(q^{-n}, q^{1-n}; q^2; q^2, q^2) = \frac{(b^2, q^2)_n}{(b^2; q^2)_n} q^{-\frac{n}{2}} \)

(ii) \( \phi_1(a; c; q, a/c)_q = \frac{(c/a; q)_\infty}{(c; q)_\infty} \)

1.7 Show that, for \( |z| < 1 \),

\[ \phi_1(a^2, aq; a; q, z) = (1 + az) \frac{(a^2 q^2; q)_\infty}{(z; q)_\infty}. \]
1.8 Show that, when $|a| < 1$ and $|bq/a^2| < 1$,

$$2\phi_1(a^2, a^2/b; b, q^2, bq/a^2) = \frac{\Gamma(a^2; q^2)\Gamma(2bq/a^2; q^2)}{\Gamma(2bq/a^2; q^2)} \left[ \frac{\Gamma(ba; q)\Gamma(-ba; q)}{\Gamma(a; q)} + \frac{\Gamma(-ba; q)}{\Gamma(a; q)} \right].$$

(Andrews and Askey [1977])

1.9 Let $\phi(a, b, c)$ denote the series $2\phi_1(a, b; c; q, z)$. Verify Heine’s [1847] $q$-contiguous relations:

(i) $\phi(a, b, cq^{-1}) = \phi(a, b, c) = cz(1-a)(1-b)/(q-c)(1-c) \phi(aq, bq, cq)$,

(ii) $\phi(aq, b, c) - \phi(a, b, c) = az(1-b)/(1-c) \phi(aq, bq, cq)$,

(iii) $\phi(aq, b, cq) - \phi(a, b, c) = az(1-b)(1-c)/\Gamma(aq, bq, cq)$,

(iv) $\phi(aq, bq^{-1}, c) - \phi(a, b, c) = az(1-b)/\Gamma(aq, bq, cq)$.

1.10 Denoting $2\phi_1(a, b; c; q, z)$, $2\phi_1(aq^{1/2}, b, c; q, z)$, $2\phi_1(a, bq^{1/2}; c; q, z)$ and $2\phi_1(a, b; cq^{1/2}; q, z)$ by $\phi_1, \phi(q^{1/2})$, $\phi(bq^{1/2})$ and $\phi(cq^{1/2})$, respectively, show that

(i) $b(a - b)\phi(aq) - (a - 1)\phi(bq) = (b - a)\phi$,

(ii) $a(1-b)c\phi(bq^{-1}) - (1-a)\phi(aq^{-1}) = (a-b)(1-abz/c)\phi$,

(iii) $q(1-a)\phi(aq^{-1}) + (1-a)(1-abz/c)\phi(aq)
= [1 + q - a - az(1-b)/c] \phi(aq$),

(iv) $(1-c)\phi(cq^{-1}) + (c-a)(c-b)z\phi(cq)
= (c-1)\phi(cq - z) + (c-a)\phi(cq)$.

(Heine [1847])

1.11 Let $g(\theta; \lambda, \mu, \nu) = (\lambda e^{i\theta}, \mu e^{i\theta}, \nu e^{i\theta}, q, x, e^{i\theta})$. Prove that $g(\theta; \lambda, \mu, \nu)$ is symmetric in $\lambda, \mu, \nu$ and is even in $\theta$.

1.12 Let $D_q$ be the $q$-derivative operator defined for fixed $q$ by

$$D_qf(z) = \frac{f(z) - f(qz)}{1-qz},$$

and let $D^n_qu = D_q(D_{q^{-1}}u)$ for $n = 1, 2, \ldots$. Show that

(i) $D^2_qf(z) = \frac{d^2}{dz^2}f(z)$ if $f$ is differentiable at $z$,

(ii) $D^n_q2\phi_1(a, b; c; q, dz) = \frac{\Gamma(a, b; q)}{\Gamma(c; q)} \frac{\Gamma(aq^n, bq^n; c; q)}{\Gamma(q^n, cq^n; q)} \frac{\Gamma(a, b; c; q, z^q^n)}{\Gamma(c; q)\Gamma(cq^n; q)} \frac{\Gamma(aq^n, bq^n; c; q, z^q^n)}{\Gamma(c; q)\Gamma(cq^n; q)}$.

1.13 Show that $u(z) = 2\phi_1(a, b; c; q, z)$ satisfies (for $|z| < 1$ and in the formal power series sense) the second order $q$-differential equation

$$z(c - abqz)2\phi_1^{(2)}u + \frac{1-c}{1-q} \left[ \frac{1}{1-q} \right] D_qu
- \left[ \frac{1}{1-q} \right] D_{q^{-1}}u = 0,$$

where $D_q$ is defined as in Ex. 1.12. By replacing $a, b, c$, respectively, by $a^q, b^q, c^q$ and then letting $q \to 1^-$ show that the above equation tends to the second order differential equation

$$z(1 - z)v'' + [c - (a + b + 1)z]v' - abu = 0$$

for the hypergeometric function $v(z) = 2F_1(a, b; c; z)$, where $|z| < 1$. (Heine [1847])

1.14 Let $|z| < 1$ and let $e_q(x)$ and $E_q(x)$ be as defined in §1.3. Define

$$\sin_q(x) = e_q(ix) - e_q(-ix) = \sum_{n=0}^{\infty} \frac{(-1)^n x}{q^n} e^{-n},$$

$$\cos_q(x) = e_q(ix) + e_q(-ix) = \sum_{n=0}^{\infty} \frac{(-1)^n x}{q^n} e^{2n}.$$

Also define

$$\sin_q(x) = \frac{E_q(ix) - E_q(-ix)}{2i}, \quad \cos_q(x) = \frac{E_q(ix) + E_q(-ix)}{2}.$$

Show that

(i) $e_q(ix) = \cos_q(x) + i\sin_q(x)$,

(ii) $E_q(ix) = \cos_q(x) + i\sin_q(x)$,

(iii) $\sin_q(x)\sin_q(x) + \cos_q(x)\cos_q(x) = 1$,

(iv) $\sin_q(x)\cos_q(x) - \sin_q(x)\cos_q(x) = 0$.

For these identities and other identities involving $q$-analogues of sin $x$ and cos $x$, see Jackson [1904a] and Hahn [1949c].

1.15 Prove the transformation formulas

(i) $2\phi_1 \left[ a^{-n}, b; c, q, z \right] = \frac{b^q z^{-n}/c^{-n} \Gamma(q^n, cq^n; q)}{\Gamma(q^n, cq^n; q)} \frac{\Gamma(a^q, b^q, c; q, z^q)}{\Gamma(a^q, b^q, c; q, z^q)} \frac{\Gamma(a^q, b^q, c; q, z^q)}{\Gamma(a^q, b^q, c; q, z^q)},$

(ii) $2\phi_1 \left[ a^{-n}, b; c, q, z \right] = \frac{c^n b^{-n} \Gamma(q^n, cq^n; q, z^q)}{\Gamma(q^n, cq^n; q, z^q)} \frac{\Gamma(a^q, b^q, c; q, z^q)}{\Gamma(a^q, b^q, c; q, z^q)},$

(iii) $2\phi_1 \left[ a^{-n}, b; c, q, z \right] = \frac{c^n b^{-n} \Gamma(q^n, cq^n; q, z^q)}{\Gamma(q^n, cq^n; q, z^q)} \frac{\Gamma(a^q, b^q, c; q, z^q)}{\Gamma(a^q, b^q, c; q, z^q)}.$

(Jackson [1905a, 1927])
1.16 Show that
\[ \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} q^{n(n+1)/2} = (q; q)_\infty (aq; q^2)_\infty. \]

1.17 Show that
\[ \sum_{k=0}^{n} \frac{(a, b; q)_k}{(q; q)_k} (-ab)^{n-k} q^{(n-k)(n+k-1)/2} \]
\[ = (a; q)_{n+1} \sum_{k=0}^{n} \frac{(-b)_k (q^{1/2})}{(q; q)_k (q; q)_{n-k} (1 - aq^{n-k})}. \] (Carlitz [1974])

1.18 Show that
\[ \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q, a^2; q)_n} q^{n^2} (a/2z)^n \phi_1(q^{-n}, a; q^{-n}/a; q, qz^2/a) \]
\[ = (z; q)_\infty \phi_1(a, a/z^2; a^2; q, -zt), \quad |zt| < 1. \]

1.19 Using (1.5.4) show that
\((i)\) \[ \phi_2 \left[ \begin{array}{c} a, q/a \\ -q, b \\ -q, -b \end{array} \right] = \frac{(ab, bq/a; q^2)_\infty}{(b; q)_\infty}, \]
\((ii)\) \[ \phi_2 \left[ \begin{array}{c} a^2, b^2 \\ a^2 b; q^3 \\ -a^2 b \end{array} \right] = \frac{(a^2, b^2; q^3)_\infty}{(a^2 b^2; q^3)_\infty} \]
(Andrews [1973])

1.20 Prove that if \(\text{Re } x > 0\) and \(0 < q < 1\), then
\((i)\) \[ \Gamma_q(x) = (q; q)_\infty (1 - q)^{1-x} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n}, \]
\((ii)\) \[ \Gamma_q(x) = \frac{(1 - q)^{x-1}}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{nx}}{(q; q)_n} q^{n^2/2}. \]

1.21 For \(0 < q < 1\) and \(x > 0\), show that
\[ \frac{d^2}{dx^2} \log \Gamma_q(x) = (\log q)^2 \sum_{n=0}^{\infty} \frac{q^{n^2} x}{(1 - q^{n+1})^2}, \]
which proves that \(\log \Gamma_q(x)\) is convex for \(x > 0\) when \(0 < q < 1\).

1.22 Conversely, prove that if \(f(x)\) is a function which satisfies
\[ f(x+1) = \frac{1-q^x}{1-q} f(x) \text{ for some } q, 0 < q < 1, \]
\[ f(1) = 1, \]
and \(\log f(x)\) is convex for \(x > 0\), then \(f(x) = \Gamma_q(x)\). This is Askey's [1978] \(q\)-analogue of the Bohr-Mollerup [1922] theorem for \(\Gamma(x)\). For two extensions to the \(q > 1\) case (with \(\Gamma_q(x)\) defined as in the next exercise), see Moak [1980b].

1.23 For \(q > 1\) the \(q\)-gamma function is defined by
\[ \Gamma_q(x) = \frac{(q^{-1}; q)_\infty}{(q^{-x}; q)_\infty} (q-1)^{1-x} q^{(x-1)/2}. \]
Show that this function also satisfies the functional equation (1.10.5) and that \(\Gamma_q(x) \rightarrow \Gamma(x)\) as \(q \rightarrow 1^+\). Show that for \(q > 1\) the residue of \(\Gamma_q(x)\) at \(x = -n\) is
\[ \frac{(q - 1)^{n+1} q^{n+1}}{(q; q)_n} log q. \]

1.24 Jackson [1905a,b,e] gave the following \(q\)-analogues of Bessel functions:
\[ J_q^{(1)}(x; q) = \frac{(q^{x+1}; q)_\infty}{(q; q)_\infty} (x/2)^x \phi_1(0, 0; q^{x+1}, q, -x^2/4), \]
\[ J_q^{(2)}(x; q) = \frac{(q^{x+1}; q)_\infty}{(q; q)_\infty} (x/2)^x \phi_0 \left( -; q^{x+1}, q, -x^2 q^{x+1}/4 \right), \]
where \(0 < q < 1\). The above notations for the \(q\)-Bessel functions are due to Ismail [1981, 1982].
Show that
\[ J_q^{(2)}(x; q) = (-x^2/4; q)_\infty J_q^{(1)}(1; q), \]
and
\[ \lim_{q \rightarrow 1} J_q^{(k)}(x(1-q); q) = J_\nu(x), \quad k = 1, 2. \]

1.25 For the \(q\)-Bessel functions defined as in Exercise 1.24 prove that
\((i)\) \[ q^{1-q} J_q^{(1)}(x; q) = \frac{2(1-q^x)}{x} J_q^{(k)}(x; q) - J_q^{(k)}(1; q), \]
\((ii)\) \[ J_q^{(1)}(x q^{1/2}; q) = q^{-x/2} \left( J_q^{(1)}(x; q) + \frac{x}{2} J_q^{(1)}(1; q) \right); \]
\((iii)\) \[ J_q^{(1)}(x q^{1/2}; q) = q^{-x/2} \left( J_q^{(1)}(x; q) - \frac{x}{2} J_q^{(1)}(1; q) \right). \]

1.26 Following Ismail [1982], let
\[ f_\nu(x) = J_\nu^{(1)}(x; q) J_q^{(1)}(x q^{1/2}; q) - J_\nu^{(1)}(x; q) J_q^{(1)}(x q^{1/2}; q). \]
Show that
\[ f_\nu(x q^{1/2}) = \left( 1 + \frac{x^2}{4} \right) f_\nu(x), \]
and deduce that, for non-integral \(\nu\),
\[ f_\nu(x) = q^{-\nu/2} (q^{1-\nu}; q)_\infty (q, q, -x^2/4; q)_\infty. \]
1.27 Show that
\[
\sum_{n=-\infty}^{\infty} t^n J_{n}^{(2)}(x; q) = \left(-x^2/4; q\right)_\infty e_q(x/t)e_q(x/2t).
\]
This is a \(q\)-analogue of the generating function
\[
\sum_{n=-\infty}^{\infty} t^n J_n(x) = e^{x(t-t^{-1})/2}.
\]

1.28 The continuous \(q\)-Hermite polynomials are defined in Askey and Ismail [1983] by
\[
H_n(x|q) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k} e_{n-k}(x-2k)\theta,
\]
where \(x = \cos \theta\); see Szegö [1926], Carlitz [1955, 1957a, 1958, 1960] and Rogers [1894, 1917]. Derive the generating function
\[
\sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q; q)_n} t^n = \frac{1}{(te^{\theta}, te^{-\theta}; q)_{\infty}}, \quad |t| < 1. \quad (\text{Rogers [1984]})
\]

1.29 The continuous \(q\)-ultraspherical polynomials are defined in Askey and Ismail [1983] by
\[
C_n(x; \beta|q) = \sum_{k=0}^{n} \frac{\beta_k(q; \beta; q)_{n-k}}{(q; q)_k} e_{n-k}(x-2k)\theta,
\]
where \(x = \cos \theta\). Show that
\[
C_n(x; \beta|q) = \frac{(\beta; q)_n e_{m\theta}}{(q; q)_n} \begin{cases} \frac{q^{-n} \beta^{-1} q^{-1-n}}{q, \beta^{-1} e^{-2i\theta}} & \text{if } \beta = 0, \\
\frac{q^n \beta^n e^{i\theta}}{(q; q)_n} & \text{if } \beta = 0
\end{cases}
\]
\[
= \frac{(\beta; q)_n e_{m\theta}}{(q; q)_n} \begin{cases} q^{-n} \beta^{-1} q^{-1-n} & \text{if } \beta = 0, \\
\beta^n e^{i\theta} & \text{if } \beta = 0
\end{cases}
\]
\[
\lim_{q \to 1} C_n(x; \beta|q) = C_n(\beta|x),
\]
and
\[
\sum_{n=0}^{\infty} C_n(x; \beta|q) t^n = \frac{(\beta t e^{i\theta}, \beta t e^{-i\theta}; q)_{\infty}}{(te^{i\theta}, te^{-i\theta}; q)_{\infty}}, \quad |t| < 1. \quad (\text{Rogers [1985]})
\]

1.30 Show that if \(m_1, \ldots, m_r\) are nonnegative integers, then
\[
(i) \quad r+1 \phi_{r+1} \left[ \begin{array}{c}
\left( q \right)_\infty \left( b_1, b_2, \ldots, b_r \right) \vspace{1mm}
\end{array} \right] \\
= \left( \frac{(q; q)_\infty (b_1/b_2; q)_{m_2} \ldots (b_r/b_1; q)_{m_r}}{(b_1; q)_\infty (b_1, b_2, \ldots, b_r; q)_{m_r}} \right) \left( q, q^{1-(m_1+\cdots+m_r)} \right),
\]

(ii) \[ r \phi \left[ \begin{array}{c}
\left( q \right)_\infty \left( b_1, b_2, \ldots, b_r \right) \vspace{1mm}
\end{array} \right] = 0,
\]

(iii) \[ r \phi \left[ \begin{array}{c}
\left( q \right)_\infty \left( b_1, b_2, \ldots, b_r \right) \vspace{1mm}
\end{array} \right] = \left( -1 \right)^{m_1+\cdots+m_r} \left( q \right)_\infty \left( b_1, b_2, \ldots, b_r \right) \left( q, q^{1-(m_1+\cdots+m_r)} \right),
\]

1.31 Let \(\Delta_b\) denote the \(q\)-difference operator defined for a fixed \(q\) by
\[
\Delta_b f(x) = b f(qx) - f(x)
\]
and let \(\Delta = \Delta_1\). Show that if
\[
v_n(z) = \frac{(a_1, \ldots, a_r; q)_n (-1)^{1+s-r} q^{r(1+s-r)n(n-1)/2} z^n}{(q, b_1, \ldots, b_s; q)_n},
\]
then
\[
\left( \Delta \Delta_{b_1} \Delta_{b_2} \cdots \Delta_{b_s} \right) v_n(z) = z \left( \Delta_{a_1} \Delta_{a_2} \cdots \Delta_{a_r} v_{n-1} \right) (x q^{1+s-r}), \quad n = 1, 2, \ldots
\]
Use this to show that the basic hypergeometric series
\[
v(z) = r \phi_a (a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)
\]
satisfies (in the sense of formal power series) the \(q\)-difference equation
\[
\left( \Delta \Delta_{b_1} \Delta_{b_2} \cdots \Delta_{b_s} \right) v(z) = z \left( \Delta_{a_1} \Delta_{a_2} \cdots \Delta_{a_r} \right) v(x q^{1+s-r}).
\]

This is a \(q\)-analogue of the formal differential equation for generalized hypergeometric series given, e.g., in Henrici [1974, Theorem (1.5)] and Slater [1966, (2.1.2.1)].

1.32 The little \(q\)-Jacobi polynomials are defined by
\[
p_n(x; a, b; q) = 2 \phi_1 (q^{-n}, abq^{n+1}; aq; q, qx)
\]
Show that these polynomials satisfy the orthogonality relation
\[
\sum_{n=0}^{\infty} \frac{(bq; q)_n}{(q; q)_n} p_n(q^2; a, b; q)p_m(q^2; a, b; q) = \begin{cases} 0, & \text{if } m \neq n, \\
\frac{(q, bq; q)_n (1 - abq^n)(aq^n; q)_\infty}{(aq, abq; q)_n (1 - abq^{n+1})(aq; q)_\infty}, & \text{if } m = n.
\end{cases}
\]
(Andrews and Askey [1977])

1.33 Show for the above little \(q\)-Jacobi polynomials that the formula
\[
p_n(x; c, d; q) = \sum_{k=0}^{n} a_{k,n} p_k(x; a, b; q)
\]
holds with
\[
a_{k,n} = \frac{(q^{-n}, aq, cdq^{n+1}; q)_k}{(q, cq, abq^{k+1}; q)_k} \left( \frac{q^{k-n}, cdq^{n+k+1}, aq^{k+1}}{cq^{k+1}, abq^{2k+2}; q, q} \right)_{k+1}.
\]
(Andrews and Askey [1977])

1.34 (i) If \(m, m_1, m_2, \ldots, m_r\) are arbitrary nonnegative integers and
Basic Hypergeometric Series

$$|a^{-1}q^{m+1-(m_{1}+\cdots+m_{r})}| < 1$$, show that

$$r+2\phi_{r+1} \begin{bmatrix} a, b_{1}q^{m_{1}}, \ldots, b_{r}q^{m_{r}} \\ b_{q_{1}}^{l+1}, b_{1}, \ldots, b_{r} \\ q, q^{-1}q^{m+1-(m_{1}+\cdots+m_{r})} \end{bmatrix} = \frac{(q, bq/a; q)_{\infty}(bq; q)_{m_{1}}(b_{1}; q)_{m_{1}} \cdots (b_{r}; q)_{m_{r}}}{(bq, q/a; q)_{\infty}(q; q)_{m_{1}}(b_{1}; q)_{m_{1}} \cdots (b_{r}; q)_{m_{r}}}$$

$$r+2\phi_{r+1} \begin{bmatrix} b_{q}^{m_{1}}, b_{1}/q, \ldots, b_{r}/b_{r} \\ b_{q}/a, b_{q}^{l+1}/b_{1}, \ldots, b_{q}/b_{r} \\ q, q^{-1}q^{m+1-(m_{1}+\cdots+m_{r})} \end{bmatrix} ;$$

(ii) if $m_{1}, m_{2}, \ldots, m_{r}$ are nonnegative integers and $|a^{-1}q^{1-(m_{1}+\cdots+m_{r})}| < 1$, $|cq| < 1$, show that

$$r+2\phi_{r+1} \begin{bmatrix} a, b_{1}q^{m_{1}}, \ldots, b_{r}q^{m_{r}} \\ bq, b_{1}, \ldots, b_{r} \\ q, q^{-1}q^{1-(m_{1}+\cdots+m_{r})} \end{bmatrix} = \frac{(bq/a, cq; q)_{\infty}(b_{1}/b_{1}; q)_{m_{1}}(b_{1}; q)_{m_{1}} \cdots (b_{r}; q)_{m_{r}}}{(bcq, q/a; q)_{\infty}(b_{1}; q)_{m_{1}}(b_{1}; q)_{m_{1}} \cdots (b_{r}; q)_{m_{r}}}$$

$$r+2\phi_{r+1} \begin{bmatrix} c^{-1}, b, bq/b_{1}, \ldots, bq/b_{r} \\ bq/a, bq^{l+1}/b_{1}, \ldots, bq^{1-m_{r}}/b_{r} \\ q, cq \end{bmatrix} .$$

(Gasper [1981a])

1.35 Use Ex 1.2 (v) to prove that if $x$ and $y$ are indeterminates such that $xy = qyx$, $q$ commutes with $x$ and $y$, and the associativity law holds, then

$$(x + y)^{n} = \sum_{k=0}^{n} \binom{n}{k}_{q} y^{k} x^{n-k} = \sum_{k=0}^{n} \binom{n}{k}_{q} q^{-1} x^{k} y^{n-k}.$$ 

(See Cigler [1979], Feinsilver [1982], Koornwinder [1988c], Schützenberger [1953], and Yang [1989])

Notes 1

§1.1 and 1.2 For additional material on hypergeometric series and orthogonal polynomials see, e.g., the books Erdélyi [1953], Rainville [1960], Szegő [1975], Whittaker and Watson [1965], Agarwal [1963], Carlson [1977], T.S. Chihara [1978], Henrici [1974], Luke [1969], Miller [1968], Nikiforov and Uvarov [1988], Vilenkin [1968], and Watson [1952]. Some techniques for using symbolic computer algebraic systems such as Mathmatica and Macsyma to derive formulas containing hypergeometric and basic hypergeometric series are discussed in Gasp[1989].

§1.3–1.5 The $q$-binomial theorem was also derived in Jacobi [1846], along with the $q$-Vandermonde formula. Bijective proofs of the $q$-binomial theorem, Heine’s $\phi_{1}$ transformation and $q$-analogue of Gauss’ summation formula, the $q$-Saalschütz formula, and of other formulas are presented in Joichi and Stanton [1987]. Bender [1971] used partitions to derive an extension of the $q$-Vandermonde sum in the form of a generalized $q$-binomial Vandermonde convolution.

§1.6 Other proofs of Jacobi’s triple product identity and/or applications of it are presented in Adiga et al. [1985], Andrews [1965], Cheema [1964], Ewell [1981], Gustafson [1989b], Joichi and Stanton [1989], Kac [1978, 1985], Lepowsky and Milne [1978], Lewis [1984], Macdonald [1972], Menon [1965], Milne [1985a], Sudler [1966], Sylvester [1892], and Wright [1965]. Concerning theta functions, see Adiga et al. [1985], Askey [1989c], Bellman [1961], and Jensen’s use of theta functions in Pólya [1927] to derive necessary and sufficient conditions for the Riemann hypothesis to hold.

§1.7 Some applications of the $q$-Saalschütz formula are contained in Carlitz [1969b] and Wright [1968].

§1.9 Formulas (1.9.3) and (1.9.8) were rediscovered by Gustafson [1987a, Theorems 3.15 and 3.18] while working on multivariable orthogonal polynomials.

§1.11 Also see Jackson [1917, 1951] and, for fractional $q$-integrals and $q$-derivatives, Al-Salam [1966] and Agarwal [1969b]. Toepfitz [1963], pp.53-55 pointed out that around 1650 Fermat used a $q$-integral type Riemann sum to evaluate the integral of $x^{k}$ on the interval $[0, b]$.

Ex. 1.2 The $q$-binomial coefficient $\binom{n}{k}_{q}$, which is also called the Gaussian binomial coefficient, counts the number of $k$ dimensional subspaces of an $n$ dimensional vector space over a field with $q$ elements (Goldman and Rota [1970]), and it is the generating function, in powers of $q$, for partitions into at most $k$ parts not exceeding $n - k$ (Sylvester [1882]). It arises in such diverse fields as analysis, computer programming, geometry, number theory, physics, and statistics. See, e.g., Aigner [1979], Andrews [1971a, 1976], Baker and Coon [1970], Baxter and Pearce [1983], Berman and Fytre [1972], Dowling [1973], Dunkl [1981], Garvan and Stanton [1989], Handa and Mohanty [1980], Ihrig and Ismail [1981], Jimbo [1985, 1986], van Kampen [1961], Kendall and Stuart [1979, §3.1–3], Knuth [1971, 1973], Pólya [1970], Pólya and Alexanderson [1970], Szegö [1975, §2.7], and Zaslavsky [1987]. Sylvester [1987] used the invariant theory that he and Cayley developed to prove that the coefficients of the Gaussian polynomial $\binom{n}{k}_{q} = \sum a_{j} q^{j}$ are unimodal. A constructive proof was recently given by O’Hara [1989]. Also see Bressoud [1989], Pólya [1970], and Zeilberger [1989b,c,d]. The unimodality of the sequence $\left( \binom{n}{k}_{q} : k = 0, 1, \ldots, n \right)$ is explicitly displayed in Aigner [1979, Proposition 3.13], and Macdonald [1979, p. 67].

Ex. 1.3 Cigler [1979] derived an operator form of the $q$-binomial theorem. MacMahon [1916, Arts. 105–107] showed that if a multiset is permuted, then the generating function for inversions is the $q$-multinomial coefficient. Also see Carlitz [1963a], Kadell [1985a], and Knuth [1973, p. 33, Ex. 16]. Gasper derived the $q$-multinomial theorem in part (ii) several years ago by using the $q$-binomial theorem and mathematical induction. G.E. Andrews observed in a 1988 letter that it can also be derived by using the expansion formula for the $q$-Lauricella function $\Phi_{P}$ stated in Andrews [1972, (4.1)] and the $q$-Vandermonde sum. Some sums of $q$-multinomial coefficients are considered in Bressoud [1978].
SUMMATION, TRANSFORMATION, AND EXPANSION FORMULAS

2.1 Well-poised, nearly-poised, and very-well-poised hypergeometric and basic hypergeometric series

The hypergeometric series

\[ \sum_{r=0}^{\infty} \frac{a_1, a_2, \ldots, a_{r+1}, z}{b_1, \ldots, b_r} \]

is called well-poised if its parameters satisfy the relations

\[ 1 + a_1 = a_2 + b_1 = a_3 + b_2 = \ldots = a_{r+1} + b_r, \]

and it is called nearly-poised if all but one of the above pairs of parameters (regarding 1 as the first denominator parameter) have the same sum. The series (2.1.1) is called a nearly-poised series of the first kind if

\[ 1 + a_1 \neq a_2 + b_1 = a_3 + b_2 = \ldots = a_{r+1} + b_r, \]

and it is called a nearly-poised series of the second kind if

\[ 1 + a_1 = a_2 + b_1 = a_3 + b_2 = \ldots = a_r + b_{r-1} \neq a_{r+1} + b_r. \]

The order of summation of a terminating nearly-poised series can be changed, without altering the series, so that the resulting series is either of the first kind or of the second kind.

Kummer's summation formula (1.8.2) gives the sum of a well-poised \( _2F_1 \) series with argument \(-1\). Another example of a summable well-poised series is provided by Dixon's [1903] formula

\[ \frac{\binom{a}{c}}{\Gamma(1 + \frac{1}{2}a, \Gamma(1 + a - b, (1 + a - c)\Gamma(1 + \frac{1}{2}a - c)\Gamma(1 + a - b - c)} \]

Re \((1 + \frac{1}{2}a - b - c) > 0\), which reduces to Kummer's formula (1.8.2) by letting \(c \to -\infty\).

If the series (2.1.1) is well-poised and \(a_2 = 1 + \frac{1}{2}a_1\), then it is called a very-well-poised series. Dougall's [1907] summation formulas

\[ \sum_{n=0}^{\infty} \frac{a, 1 + \frac{1}{2}a, b, c, d, e, -n}{\frac{1}{2}a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e, 1 + a + n + 1, 1, 1} \]

\[ = \frac{(1 + a)_n}{(1 + a - b)_n(1 + a - c)_n(1 + a - d)_n(1 + a - c - d)_n} \]

(2.1.6)
when the series is 2-balanced (i.e., $1 + 2a + n = b + c + d + e$), and
\[
\begin{align*}
\, _3F_4 \left[
\begin{array}{c}
a_1, a_1 + \frac{1}{2} a_2, b, c, d \\
\frac{1}{2} a_1, 1 + a - b, 1 + a - c, 1 + a - d
\end{array}
\mid q, z
\right] &= \\
\frac{\Gamma(1 + a - b)\Gamma(1 + a - c)\Gamma(1 + a - d)}{\Gamma(1 + a + b)\Gamma(1 + a + c)\Gamma(1 + a + d)}
\end{align*}
\tag{2.1.7}
\]
when $Re(1 + a - b - c - d) > 0$, illustrate the importance of very-well-poised hypergeometric series. Note that Dixon's formula (2.1.5) follows from (2.1.7) by setting $d = \frac{1}{2} a$.

Analogous to the hypergeometric case, we shall call the basic hypergeometric series
\[
\sum_{r=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q, z)}{(b_1, \ldots, b_r; q, z)} (q, a_0, a_1, \ldots, a_r; q, z)
\tag{2.2.8}
\]
well-poised if the parameters satisfy the relations
\[
qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r;
\tag{2.2.9}
\]
very-well-poised if, in addition, $a_2 = qa_1^\frac{1}{2}$, $a_3 = -qa_1^\frac{1}{2}$; a nearly-poised series of the first kind if
\[
qa_1 \neq a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r,
\tag{2.2.10}
\]

and a nearly-poised series of the second kind if
\[
qa_1 = a_2b_1 = a_3b_2 = \cdots = a_{r+1}b_r.
\tag{2.2.11}
\]

In this chapter we shall be primarily concerned with the summation and transformation formulas for very-well-poised basic hypergeometric series. To help simplify some of the displays involving very-well-poised $r+1\phi_r$ series which arise in the proofs of this and the subsequent chapters we shall frequently replace
\[
\begin{align*}
\sum_{r=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q, z)}{(b_1, \ldots, b_r; q, z)} (q, a_0, a_1, \ldots, a_r; q, z)
\end{align*}
\tag{2.2.11}
\]
by the more compact notation
\[
\sum_{r=0}^{\infty} \frac{(a_1, a_2, \ldots, a_{r+1}; q, z)}{(b_1, \ldots, b_r; q, z)} (q, a_0, a_1, \ldots, a_r; q, z)
\tag{2.2.11}
\]

In the displays of the main formulas, however, we shall continue to use the $r+1\phi_r$ notation, since in most applications one needs to know the denominator parameters.

### 2.2 A general expansion formula

Let $a, b, c$ be arbitrary parameters and $k$ be a nonnegative integer. Then, by the $q$-Saalschütz formula (1.7.2)
\[
\begin{align*}
\sum_{k=0}^{\infty} \frac{(q, q^{-1-k}; q, q^{-k}; q, q)}{(q, q^{-1}; q, q^{-k}; q, q)} (a, b, c, q^{-k}; q, q)
\end{align*}
\tag{2.2.12}
\]

so that
\[
\begin{align*}
\sum_{k=0}^{n} \frac{(b, c, q^{-n}; q, q^{-k}; q, q)}{(q, aq, qa; q, q)} (a, b, c, q^{-k}; q, q)
\end{align*}
\tag{2.2.13}
\]

where $\{A_k\}$ is an arbitrary sequence. This is equivalent to Bailey's [19] lemma. Choosing
\[
\begin{align*}
A_k = \frac{(a, a_1, \ldots, a_r; q, q)}{(b_1, b_2, \ldots, b_{r+1}; q, q)} (a, b, c, q^{-k}; q, q)
\end{align*}
\tag{2.2.14}
\]

we obtain the expansion formula
\[
\begin{align*}
\sum_{k=0}^{n} \frac{(q, aq, qa; q, q)}{(q, aq, qa; q, q)} (a, b, c, q^{-k}; q, q)
\end{align*}
\tag{2.2.15}
\]

This is a $q$-analogue of Bailey's formula [1935, 4.3(1)]. The most important property of (2.2.4) is that it enables one to reduce the $r+4\phi_{r+3}$ series to a $r+2\phi_{r+1}$ series. Consequently, if the above $r+2\phi_{r+1}$ series is summable some values of the parameters then (2.2.4) gives a transformation formula the corresponding $r+4\phi_{r+3}$ series in terms of a single series.

### 2.3 A summation formula for a terminating very-well-poised $4\phi_3$ series

Setting $b = qa_1^\frac{1}{2}, c = -qa_1^\frac{1}{2}$ and $a_k = b_k, k = 1, 2, \ldots, r, b_{r+1} = aq^{n+1},$ obtain from (2.2.4) that
\[
\begin{align*}
\sum_{k=0}^{n} \frac{(a, qa, qa, q^{-n}; q, q)}{(q, aq, qa, q^{-n+1}; q, q)} (a, b, c, q^{-k}; q, q)
\end{align*}
\tag{2.2.16}
\]

where
\[
\begin{align*}
\phi_1 (a, b, c, q^{-k}; q, q^{-n+1}, q, q^{-1}, q^{-j})
\end{align*}
\tag{2.2.17}
\]
If $z = q^n$, then the above $2\phi_1$ series can be summed by means of the Bailey-Douma summation formula (1.8.1), which gives

$$2\phi_1 \left( a q^{ji}, q^{-i-n}; a q^{j+n+1}; q, -q^{-i+n-j} \right) = (q^{-i}; q)_\infty \frac{(a q^{j+1}; q)_\infty (a q^{2n+2}; q)_\infty}{(a q^{n+j}; q)_\infty (q^{-i-n}; q)_\infty}.$$  

(2.3.2)

Hence, using the identities (1.2.32), (1.2.39) and (1.2.40), and simplifying, we obtain the transformation formula

$$4\phi_3 \left[ a, q a^{3/2}, q^{-3/2}; a q^{-1}, q, q^{-n} \right]_{a^{1/2}, -a^{1/2}, a q^{n+1}; q, q^{-n}} = \frac{(q a^{-1}; q)_n}{(q a^{1/2}; q)_n} 2\phi_1 \left( q^{-n}, -q^{-1}; -q^{-n}; q, q \right).$$  

(2.3.3)

Clearly, both sides of (2.3.3) are equal to 1 when $n = 0$. By (1.5.3) the $2\phi_1$ series on the right of (2.3.3) has the sum $(q^{-n}; q)_n / (-q^{-i}; q)_n$ when $n = 0, 1, \ldots, 1$. Since $(q^{-n}; q)_n = 0$ unless $n = 0$, it follows that

$$4\phi_3 \left[ a, q a^{3/2}, q^{-3/2}; a q^{-1}, q, q^{-n} \right]_{a^{1/2}, -a^{1/2}, a q^{n+1}; q, q^{-n}} = \delta_{m,n},$$  

(2.3.4)

where

$$\delta_{m,n} = \begin{cases} 1, & m = n, \\ 0, & m \neq n. \end{cases}$$  

(2.3.5)

is the Kronecker delta function. This summation formula will be used in the next section to obtain the sum of a $4\phi_3$ series.

### 2.4 A summation formula for a terminating very-well-poised $4\phi_3$ series

Let us now set $a_1 = q a^{3/2}$, $a_2 = -q a^{3/2}$, $b_1 = a^{1/2}$, $b_2 = -a^{1/2}$, $b_{r+1} = a q^{n+1}$ and $b_r = b_r$, for $r = 3, 4, \ldots, r$. Then (2.2.4) gives

$$\phi_3 \left[ \begin{matrix} a, q a^{3/2}, q^{-3/2}, b, c, q^{-n} \\ a^{1/2}, -a^{1/2}, a q/b, a q/c, a q^{n+1}; q, z \end{matrix} \right]\ = \ \sum_{j=0}^{n} \frac{(a q/b, q a^{3/2}, -q a^{3/2}, q^{-n}; q)_j (a q; 2q)_{2j}}{(q, a^{1/2}, -a^{1/2}, a q/b, a q/c, a q^{n+1}; q)_j} \cdot \frac{(-b c x)^j}{a q} q^{-j}.$$  

(2.2.4)

If $z = a q^{n+1}/bc$, then we can sum the above $4\phi_3$ series by means of (2.3.4) and obtain the summation formula

$$\phi_3 \left[ \begin{matrix} a, q a^{3/2}, q^{-3/2}, b, c, q^{-n} \\ a^{1/2}, -a^{1/2}, a q/b, a q/c, a q^{n+1}; q, \frac{a q^{n+1}}{bc} \end{matrix} \right]\ = \ \frac{(a q/b, q a^{3/2}, -q a^{3/2}, q^{-n}; q)_n (a q; 2a)_n}{(q, a^{1/2}, -a^{1/2}, a q/b, a q/c, a q^{n+1}; q)_n} \left( -1 \right)^n q^{n(n+1)/2}.$$  

(2.4.2)

Note the special relationship that the argument of the above $6\phi_3$ series must have with the parameters of the series in order that the series be summable, namely, it is the positive square root of $q$ times the product of the denominator parameters divided by the product of the numerator parameters, when one temporarily assumes that $q$ and the parameters are positive. This relationship between the argument and the parameters also holds for the $4\phi_3$ series in (2.3.4).

### 2.5 Watson’s transformation formula for a terminating very-well-poised $s\phi_7$ series

We shall now use (2.4.2) to prove Watson’s [1929a] transformation formula for a terminating very-well-poised $s\phi_7$ series as a multiple of a terminating balanced $4\phi_3$ series:

$$s\phi_7 \left[ \begin{matrix} a, q a^{3/2}, q^{-3/2}, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, a q/b, a q/c, a q/d, a q/e, a q^{n+1}; q \end{matrix} \right]\ = \ \frac{(a q, a q/de; q)_n}{(a q/d, a q/e; q)_n} 4\phi_3 \left[ \begin{matrix} q^{-n}, d, e, a q/bc \\ a q/b, a q/c, de q^{-n}/a; q \end{matrix} \right].$$  

(2.5.1)

It suffices to observe that from (2.2.4) we have

$$s\phi_7 \left[ \begin{matrix} a, q a^{3/2}, q^{-3/2}, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, a q/b, a q/c, a q/d, a q/e, a q^{n+1}; q \end{matrix} \right]\ = \ \sum_{j=0}^{n} \left( a q/b, q a^{3/2}, -q a^{3/2}, q^{-n}; q \right)_j (a q; 2q)_{2j} \left( -a q^{n+1} \right)^j q^{-j}.$$  

(2.5.2)

which gives formula (2.5.1) by using (2.4.2) to sum the above $s\phi_5$ series.

Note that the argument of the $s\phi_7$ series in (2.5.1) is related to the parameters in exactly the same way as stated for (2.3.4) and (2.4.2) at the end of §2.4.

### 2.6 Jackson’s sum of a terminating very-well-poised balanced $s\phi_7$ series

The $s\phi_7$ series in (2.5.1) is balanced when the six parameters $a, b, c, d, e$ and $n$ satisfy the relation

$$a^2 q^{n+1} = bcde.$$  

(2.6.1)

For such a series Jackson [1921] showed that

$$s\phi_7 \left[ \begin{matrix} a, q a^{3/2}, q^{-3/2}, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, a q/b, a q/c, a q/d, a q/e, a q^{n+1}; q \end{matrix} \right]\ = \ \frac{(a q, a q/bc, a q/bd, a q/cd; q)_n}{(a q/b, a q/c, a q/d, a q/bc; q)_n},$$  

(2.6.2)
when \( n = 0, 1, 2, \ldots \). This formula follows directly from (2.5.1), since the \( 4\phi_3 \) series on the right of (2.5.1) becomes a balanced \( 3\phi_2 \) series when (2.6.1) holds, and therefore can be summed by the \( q \)-Saalschütz formula. Note that (2.6.2) is a \( q \)-analogue of (2.1.9), as can be seen by replacing \( a, b, c, d, e \) by \( q^a, q^b, q^c, q^d, q^e \), respectively, and then letting \( q \to 1 \). It should be observed that the series \( \phi \psi \) in (2.6.2) is balanced, while the limiting series \( \tau \phi \) in (2.1.9) is 2-balanced. The reason for this apparent discrepancy is that the appropriate \( q \)-analogue of the term \((1 + 1/2a_k)/(1 + 1/2a_k)\) in the \( \tau \phi \) series is not \((qa^2/a; q)_k/(qa^2/a; q)_k = (1 - a^2q^k)/(1 - a^2q^k)\) but \((qa^2/a; q)_k/(qa^2/a; q)_k = (1 - a^2q^k)/(1 - a^2q^k)\) but \((qa^2/a; q)_k/(qa^2/a; q)_k = (1 - a^2q^k)/(1 - a^2q^k)\), which introduces an additional \( q \)-factor in the ratio of the products of the numerator and denominator parameters.

### 2.7 Some special and limiting cases of Jackson’s and Watson’s formulas: the Rogers-Ramanujan identities

Many of the known summation formulas for basic hypergeometric series are special or limiting cases of Jackson’s formula (2.6.2). For example, if we take \( d = -\infty \) in (2.6.2) we get (2.4.2). On the other hand, taking the limit \( a \to 0 \) after replacing \( d \) by \( aq/d \) gives the \( q \)-Saalschütz formula (1.7.2). Let us now take the limit \( n \to \infty \) in (2.6.2). This gives

\[
\phi_3 \left[ \begin{array}{c} a, qa^2, -qa^2, b, c, d \\ a^2, -a^2, aq/b, aq/c, aq/d \end{array} \right; q, aq/\gcd(a, b, c, d)] = (aq, aq/bc, aq/bd, aq/cd; q)_\infty, \tag{2.7.1}
\]

provided \(|aq/\gcd(a, b, c, d)| < 1\). This is clearly a \( q \)-analogue of Dougall’s formula (2.1.9). Setting \( d = a^2 \) in (2.7.1), we get a \( q \)-analogue of Dixon’s formula (2.1.9)

\[
\phi_3 \left[ \begin{array}{c} a, qa^2, b, c, d \\ a^2, -a^2, aq/b, aq/c \end{array} \right; q, qa^2/\gcd(a, b, c)] = (aq, aq/bc, qa^2/b, qa^2/c; q)_\infty, \tag{2.7.2}
\]

provided \(|qa^2/\gcd(a, b, c)| < 1\).

Watson [1929a] used his transformation formula (2.5.1) to give a simple proof of the famous Rogers-Ramanujan identities (Hardy [1937]):

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2;q)_n} = \frac{(q^2, q^3, q^5; q)^{\infty}}{(q; q)^{\infty}}, \tag{2.7.3}
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q^2;q)_n} = \frac{(q, q^4, q^6; q)^{\infty}}{(q; q)^{\infty}}, \tag{2.7.4}
\]

where \(|q| < 1\). First let \( b, c, d, e \to \infty \) in (2.5.1) to obtain

\[
\sum_{k=0}^{n} \frac{(a; q)_k}{(q; q)_k} (1 - aq^{2k}) (q^{-n}; q)_k q^{2k} (a^2q^{n})^k
\]

### 2.8 Bailey’s transformation formulas for terminating \( 5\phi_4 \) and \( \tau \phi_0 \) series

Using Jackson’s formula (2.6.2), it can be easily shown that

\[
\frac{(a, b, c; q)_k}{(q, aq/b, aq/c; q)_k} = \frac{(\lambda bc/a; q)_k}{(qa^2/\lambda bc; q)_k} \tag{2.8.1}
\]

where \( \lambda \) is an arbitrary parameter. If \( \{A_k\} \) is an arbitrary sequence, it follows that

\[
\sum_{k=0}^{\infty} \frac{(a, b, c; q)_k}{(q, aq/b, aq/c; q)_k} A_k
\]
Bailey's transformation formula

Next, let us choose \( A_k = \frac{q^k(1-aq^{2k})(d,q^{-n};q)_k/(1-a)(aq/d, a^2q^{2n}/\lambda^2;q)_k}{\lambda = qa^2/bcd} \) in (2.8.2) so that the inner series on the right side takes the form

\[
q^{-n} \binom{1-aq^2}{d, q^{-n}; q} \binom{1-a}{aq/d, a^2d^2n/\lambda^2; q}.
\]

This \( s\phi_4 \) series is a special case of the \( s\phi_4 \) series on the left side of (2.8.2); in fact, the \( 12s\phi_{11} \) series on the right side of (2.8.3) in this case reduces to a terminating balanced very-well-poised \( \phi_7 \) series which we can sum by Jackson's formula (2.6.2). Carrying out the straightforward calculations, we get Bailey's [1947] second transformation formula

\[
s\phi_4 \left[ \frac{a, q^2a, -qa^2}{aq/b, aq/c, aq/d, a^2q^{-n}/\lambda^2; q} \right] = \frac{(\lambda q^2/a^2; q)_n}{(\lambda q^2/a^2; q)_n} \binom{1-a}{aq/d, a^2d^2n/\lambda^2; q},
\]

where \( \lambda = qa^2/bcd \).

Note that the \( s\phi_4 \) series on the left is balanced and nearly-poised of the second kind, while the \( 12s\phi_{11} \) series on the right is balanced and very-well-poised. Note also that a terminating nearly-poised series of the second kind can be expressed as a multiple of a nearly-poised series of the first kind by simply reversing the series.

By proceeding as in the proof of (2.8.3), one can obtain the following variation of (2.8.3)

\[
s\phi_4 \left[ \frac{q^{-n}, b, c, d, e}{q^{-n}/b, q^{-n}/c, q^{-n}/d, eq^{-2n}/\lambda^2; q} \right] = \frac{(\lambda q^2q^{-n+1}/e, q; q)_n}{(\lambda q^2q^{-n+1}/e, q; q)_n} \binom{1-a}{aq/d, a^2d^2n/\lambda^2; q},
\]

where \( \lambda = q^{1-2n}/bcd \).

One of the most important transformation formulas for basic hypergeometric series is Bailey's [1929] formula transforming a terminating \( 10s\phi_9 \) series, which is both balanced and very-well-poised, into a series of the same type: \( a, qa^4, -qa^4, b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n} \)

\[
10\phi_9 \left[ \frac{a, q^4a^4, -a^4q^4, b, c, d, e, f, \lambda aq^{n+1}/ef, q^{-n}}{aq/b, aq/c, aq/d, aq/e, aq/f, efq^{-n}/\lambda, aq^{n+1}/q, q} \right] = \frac{(aq/e, aq/f, \lambda q^4/ef; q)_n}{(aq/e, aq/f, \lambda^2q^4/ef; q)_n} \binom{1-a}{aq/d, a^2d^2n/\lambda^2; q},
\]

where \( \lambda = qa^2/bcd \).

To derive this formula, first observe that by (2.6.2)

\[
s\phi_7 \left[ \frac{\lambda q^2a^2, -\lambda q^2b, \lambda b/a, \lambda c/a, \lambda d/a, \lambda aq^{m-n}}{\lambda^2, -\lambda^2, aq/b, aq/c, aq/d, \lambda q^m/a, \lambda q^{m+1}/q, q} \right] = \frac{(aq/b, aq/c, aq/d, a^2q^m/a, \lambda q^{-m})}{(aq/b, aq/c, aq/d, a\lambda, q^{-m})},
\]

where \( \lambda = q^{1-2n}/bcd \).
and hence the left side of (2.9.1) equals
\[
\sum_{m=0}^{\infty} (a; q)_{m} (1 - a q^{2m}) (e, f, \lambda q^{m+1} / e f, q^{-n}; q)_{m} (a / \lambda; q)_{m} q^{m} \\
= \sum_{j=0}^{\infty} (\lambda ; q)_{j} (1 - \lambda q^{2j}) (\lambda b / a, \lambda c / a, \lambda d / a, \lambda q^{m}, q^{-n}; q)_{j} (a / \lambda ; q)_{m+j} (\lambda; q)_{m} (1 - \lambda) (a q^{j} / e f, q^{-n} ; \lambda, \lambda q^{m+1} / e f, q^{-n}; q)_{j} \\
= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a ; q)_{m+j} (1 - a q^{2m}) (e, f, \lambda q^{m+1} / e f, q^{-n}; q)_{m} (a / \lambda; q)_{m+j} (\lambda ; q)_{m} (1 - \lambda) (a q^{j} / e f, q^{-n} ; \lambda, \lambda q^{m+1} / e f, q^{-n}; q)_{j}}{(q; q)_{j} (a ; q)_{j} (\lambda ; q)_{j} (1 - \lambda) (a q^{j} / e f, q^{-n} ; \lambda, \lambda q^{m+1} / e f, q^{-n}; q)_{j}} \\
= \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(a ; q)_{j} (1 - a q^{2j}) (\lambda b / a, \lambda c / a, \lambda d / a, e, f, \lambda q^{m+1} / e f, q^{-n}; q)_{j}}{(q; q)_{j} (\lambda ; q)_{j} (1 - \lambda) (a q^{j} / e f, q^{-n} ; \lambda, \lambda q^{m+1} / e f, q^{-n}; q)_{j}} \\
= \sum_{j=0}^{\infty} \frac{(a q^{j} / \lambda ; q)_{j} (a q^{j} / \lambda ; q)_{j}}{(\lambda ; q)_{j}^{2}} a W_{7} (a q^{j} ; q^{-n}, e f^{j}, q^{j}, a / \lambda, \lambda q^{m+1} / e f, q^{-n}; q) \\
= \frac{a W_{7} (a q^{j} ; q^{-n}, e f^{j}, q^{j}, a / \lambda, \lambda q^{m+1} / e f, q^{-n}; q)}{(\lambda ; q)_{j}^{2}} .}
\]

where the \( a W_{7} \) series is defined as in \( \S 2.1 \). Summing the above \( a W_{7} \) series by means of (2.6.2) and simplifying the coefficients, we obtain (2.9.1). It is sometimes helpful to rewrite (2.9.1) in a somewhat more symmetrical form:
\[
10 W_{9} (a ; b, c, d, e, f, g, h, q, q) \\
= \sum_{m=0}^{\infty} (a q^{m} / b ; c, a, q^{m} / e f, g, h, q, q)_{m} (a q^{m} / b ; c, a, q^{m} / e f, g, h, q, q)_{m} \\
= \sum_{m=0}^{\infty} \frac{(a q^{m} / b ; c, a, q^{m} / e f, g, h, q, q)_{m} (a q^{m} / b ; c, a, q^{m} / e f, g, h, q, q)_{m}}{\cdot \cdot \cdot}
\]

where at least one of the parameters \( e, f, g, h \) is of form \( q^{-n}, n = 0, 1, 2, ..., \) and \( q^{2} a^{3} = b c d e f g h \).

2.10 Limiting cases of Bailey's \( 10 \phi_{9} \) transformation formula

A number of the known transformation formulas for basic hypergeometric series follow as limiting cases of the transformation formula (2.9.1). If we let \( b, c, \) or \( d \to \infty \) in (2.9.1), we obtain Watson's formula (2.5.1). On the other hand, if we take the limit \( n \to \infty \), we get the transformation formula for a nonterminating \( \phi_{7} \) series
\[
\phi_{7} \left[ a, q a, a q, a q^{2}, b, c, d, e, f, \frac{a^{2} q^{2}}{q} \right] \\
= \frac{(a q/e, f; \lambda, f; q)_{\infty}}{(q/e, f; q)_{\infty}} \\
= \frac{a q / e, f; \lambda, f; q_{\infty}}{(q/e, f; q_{\infty})} \\
= \frac{a q / e, f; \lambda, f; q_{\infty}}{\cdot \cdot \cdot}
\]

which is a \( q \)-analogue of Whipple's [1926b] formula
\[
\phi_{7} \left[ a^{2} q^{2} \right] \\
= \frac{(a ; q)_{n}(f - a)}{(a ; q)_{n}} \\
= \frac{(a ; q)_{n}(f - a)}{(a ; q)_{n}} f_{n} \\
= \frac{d, 1 + a - e - n, 1 + a - e - n}{d, 1 + a - e - n, 1 + a - e - n} .}
\]

Limiting cases of \( 10 \phi_{9} \) transformation formula

where \( \lambda = qa^{2}/bcde \) and
\[
\text{max} \left( |aq/ef|, |\lambda q/ef| \right) \leq 1 .
\]

The convergence of the two series in (2.10.1) is ensured by the inequalities (2.10.2) which, of course, are not required if both series terminate. For example, if \( f = q^{-n}, n = 0, 1, 2, ..., \) then (2.10.1) becomes
\[
\phi_{7} \left[ \frac{a, q a, a q, a q^{2} \cdot b, c, d, e, f}{a^{2} q^{2}} \right] \\
= \frac{(a q/ef; \lambda, f; q_{\infty})}{(a q/ef; q_{\infty})} \\
= \frac{(a q/ef; \lambda, f; q_{\infty})}{(a q/ef; q_{\infty})} \\
= \frac{(a q/ef; \lambda, f; q_{\infty})}{(a q/ef; q_{\infty})} .}
\]

This identity expresses one terminating very-well-poised \( \phi_{7} \) series in terms of another. However, it should be noticed that in both terminating and nonterminating cases of the \( \phi_{7} \) transformation derived above the arguments of the series are related to the parameters in the same way as described in \( \S 2.4 \).

Using (2.5.1) we can now express (2.10.3) as a transformation formula between two terminating balanced \( 4 \phi_{3} \) series:
\[
\phi_{3} \left[ \frac{a q^{-n}, a, b, c}{d, e, f} ; q, q \right] \\
= \frac{(e/a, f/a; q)_{n}}{(e/a, f/a; q)_{n}} a^{n} \phi_{3} \left[ \frac{a^{2} q^{2} \cdot d, a^{2} q^{2} \cdot d, a^{2} q^{2} \cdot d}{a^{2} q^{2} \cdot d} ; q, q \right] ,}
\]

where \( abc = defq^{a-n} \). This is a very useful formula which was first derived by Sears [1951], and hence is called the Sears’ \( 4 \phi_{3} \) transformation formula. It is a \( q \)-analogue of Whipple’s [1926b] formula
\[
\phi_{3} \left[ \frac{1}{q, q} ; d, e, f \right] \\
= \frac{(a - e - a - d - b - c)}{(e/a, f/a; q)_{n}} \\
= \frac{(a - e - a - d - b - c)}{(e/a, f/a; q)_{n}} \\
= \frac{(d, 1 + a - e - n, 1 + a - e - n)}{d, 1 + a - e - n, 1 + a - e - n} .}
\]
provided \(|aq/de| < 1\) to ensure that the nonterminating series on the right side converges. Use of (2.5.1) then leads to the formula

\[
4φ_3 \left[ q^{-n}, a, b, c, d, e, f ; q, q \right] = \frac{(deq^n/a, deq^n/b, deq^n/c, deq^n/abc ; q)_{\infty}}{(deq^n/a, deq^n/b, deq^n/ac, deq^n/bc ; q)_{\infty}}
\]

\[
φ_7 \left[ \begin{array}{c}
deq^{-n}, q (deq^{-n})^{\frac{1}{2}}, -q (deq^{-n})^{\frac{1}{2}}, \\
(deq^{-n})^{\frac{1}{2}}, -q (deq^{-n})^{\frac{1}{2}}, \\
(a, b, c, d, e, f, q)_{\infty} \\
deq/a, deq/b, deq/c, e, d \end{array} \right] \frac{d}{abc}, \tag{2.10.7}
\]

provided \(def = q^{-n}abc\) and \(|de/abc| < 1\).

As another limiting case of (2.9.1) Bailey [1935, 8.5 (3)] found a nonterminating extension of (2.5.1) that expresses a very-well-poised \(sφ_7\) series in terms of two balanced \(qφ_3\) series. First use (2.9.1) to get

\[
10W_9 \left( a, b, c, d, e, f, a^2q^{-n} ; bdef, q^{-n} ; q, q \right) = \frac{(aq, aq/deq, aq/def, aq/eq, q)_{\infty}}{(aq, aq/deq, aq/def, aq/eq, q)_{\infty}}
\]

\[
\cdot 10W_9 \left( deq^{-n} ; a, eq/def, eq/q, q \right) = \frac{(aq, aq/deq, aq/def, aq/eqf, q)_{\infty}}{(aq, aq/deq, aq/def, aq/eqf, q)_{\infty}}
\]

\[
\cdot 10W_9 \left( deq^{-n} ; a, eq/def, eq/q, q \right). \tag{2.10.8}
\]

Clearly, the \(10W_9\) on the left side of (2.10.8) tends to the \(sφ_7\) series on the left side of (2.10.1) as \(n \to \infty\). However, the terms near both ends of the series on the right side of (2.10.8) are large compared to those in the middle for large \(n\), which prevents us from taking the term-by-term limit directly. To circumvent this difficulty, Bailey chose \(n\) to be an odd integer, say \(n = 2m + 1\) (this is not necessary, but it makes the analysis simpler), and divided the series on the right into two halves, each containing \(m + 1\) terms, and then reversed the order of the second series. The procedure is schematized as follows:

\[
\sum_{k=0}^{2m+1} \lambda_k = \sum_{k=0}^{m} \lambda_k + \sum_{k=m+1}^{2m+1} \lambda_k
\]

\[
= \sum_{k=0}^{m} \lambda_k + \sum_{k=m}^{2m+1} \lambda_{2m+1-k}, \tag{2.10.9}
\]

where \(\{\lambda_k\}\) is an arbitrary sequence. Letting \(m \to \infty\) (and hence \(n \to \infty\)), it follows from (2.10.8) that

\[
φ_7 \left[ a, qa^k, -qa^k, b, c, d, e, f, a^2q^2 ; \frac{a^2q^2}{bdef} \right] = \frac{(aq, aq/deq, aq/def, aq/eq, q)_{\infty}}{(aq, aq/deq, aq/def, aq/eq, q)_{\infty}} 4φ_3 \left[ \begin{array}{c}
(aq, aq/deq, aq/def, aq/eq, q)_{\infty} 4φ_3 \left[ \begin{array}{c}
(aq, aq/deq, aq/def, aq/eq, q)_{\infty}
\end{array} \right]
\end{array} \right]
\]

\[
\cdot 2φ_3 \left[ a^2q^2 / bdef, a^2q^2 / cdef, aq^2 / def ; q, q \right]. \tag{2.10.10}
\]

where \(|a^2q^2/bdef| < 1\), if the \(sφ_7\) series does not terminate. Note that if either \(b\) or \(c\) is of the form \(q^{-n}, n = 0, 1, 2, \ldots\), then the \(sφ_7\) series on the left side terminates but the series on the right side do not necessarily terminate. On the other hand if one of the numerator parameters (except \(a^2q^2/bdef\)) in either of the above \(sφ_7\) series in (2.10.10) is of the form \(q^{-n}\), then the coefficient of the other \(4φ_3\) series vanishes and we get either (2.5.1) or (2.10.7).

If \(aq/de, aq/df\) or \(aq/ef\) equals 1, then (2.10.10) reduces to the \(sφ_7\) summation formula (2.7.1). If, on the other hand, \(aq/df = 1\) then the \(sφ_7\) series in (2.10.10) reduces to a \(sφ_3\) which, via (2.7.1), leads to the summation formula

\[
\frac{(aq, aq/de, aq/df, aq/ef, q)_{\infty}}{(aq, aq/de, aq/df, aq/ef, q)_{\infty}}
\]

\[
= \frac{(aq, c/e, c/f, aq/ef, q)_{\infty}}{(aq, c/e, c/f, aq/ef, q)_{\infty}} 3φ_2 \left[ \begin{array}{c}
aq/bc, e, f \end{array} ; q, q \right]
\]

\[
= \frac{(aq, aq/ef, e, f, aq/bc, aq/cq, ef, q)_{\infty}}{(aq, aq/ef, e, f, aq/bc, aq/cq, ef, q)_{\infty}} 3φ_2 \left[ \begin{array}{c}
aq/bc, ef, aq/cq, ef ; q, q \right]
\]

\[
= \frac{(aq, aq/ef, ef, aq/cq, ef, q)_{\infty}}{(aq, aq/ef, ef, aq/cq, ef, q)_{\infty}} 3φ_2 \left[ \begin{array}{c}
q^2, ef, q ; q, q \right]. \tag{2.10.11}
\]

Solving for the first \(3φ_2\) series on the right and relabelling the parameters we get the following nonterminating extension of the \(q\)-Saalschütz formula

\[
3φ_2 \left[ a, b, c, e, f ; q, q \right] = \frac{(q/e, a, b, c, ef, q)_{\infty}}{(q/e, a, b, c, ef, q)_{\infty}} 3φ_2 \left[ \begin{array}{c}
q^2, ef, q ; q, q \right], \tag{2.10.12}
\]

where \(ef = abcq\). Sears [1951a, (5.2)] derived this formula by a different method. If \(a, b, c\) or \(c\) is of the form \(q^{-n}, n = 0, 1, 2, \ldots\), then (2.10.12) reduces to (1.7.2).

A special case of (2.10.12) which is worth mentioning is obtained by setting \(c = 0, f = 0\), and then replacing \(e\) by \(c\) to get

\[
2φ_1 \left( a, b, c, q \right) = \frac{(q/c, aq/bc, q)_{\infty}}{(q/c, aq/bc, q)_{\infty}} 2φ_1 \left( aq/c, bc/q, q^2 / q, q \right) \tag{2.10.13}
\]

If \(a\) or \(b\) is of the form \(q^{-n}, n = 0, 1, 2, \ldots\), then (2.10.13) reduces to (1.5.3). In general, a \(2φ_1\) \((a, b; c, q)\) series does not have a sum which can be written as a ratio of infinite products. However, we can still express (2.10.13) as the summation formula for a single bilateral infinite series in the following way.

First, use Heine's transformation formula (1.4.1) to transform both \(2φ_1\) series in (2.10.13):
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\[
\begin{align*}
\phi_2(aq/c; bq/c, q^2/c; q, q) &= \frac{(aq/c; bq^2/c; q)_\infty}{(q^2/c; q, q)_\infty} \sum_{n=0}^{\infty} \frac{(q/a; q)_n (aq)}{(bq^2/c; q)_n} Q^n. \\
\end{align*}
\]  

Next, note that

\[
\sum_{n=0}^{\infty} \frac{(q/a; q)_n (aq)}{(bq^2/c; q)_n} Q^n = \frac{(aq)}{c(1 - bq/c)} \sum_{n=1}^{\infty} \frac{(1/a; q)_n (aq)}{(bq/c; q)_n} Q^n.
\]

Using (2.10.14) and the identity (2.10.16) in (2.10.13) we obtain

\[
\sum_{n=-\infty}^{\infty} \frac{(c/bq; q)_n (aq)}{(aq)_n} Q^n = \frac{(c, q/c, abq/c, q; q)_\infty}{(bq/c, bq/c, q; q)_\infty},
\]

which is Ramanujan's sum (see Chapter 5 and Andrews and Askey [1978]). However, the conditions under which (2.10.17) is valid, namely, \(|q| < 1, |b| < 1, |aq/c| < 1, |bq/c| < 1, a > 0\), and \(b > 0\), are more restrictive than those for (2.10.13). Note that (2.10.17) tends to Jacobi's triple product identity (1.6.1) when \(a = 0\) and \(b = 0\). We shall give an alternative derivation of this important sum in Chapter 5 where bilinear sums are considered.

As was pointed out by Al-Salam and Verma [1982a], both (2.10.10) and (2.10.12) can be conveniently expressed as \(q\)-integrals. Thus (2.10.11) is equivalent to

\[
\int_a^b \frac{(q/t, q/t, q/t, q; q)_\infty}{(dt, rt, qt, rt, q; q)_\infty} dt = \frac{(aq)}{c} (q, bq/a, a/b, c/d, c/e, c/f; q)_\infty,
\]

where \(c = abdef\), while (2.10.10) is equivalent to

\[
\int_a^b \frac{(q/t, q/t, q/t, q/t, q; q)_\infty}{(dt, rt, qt, rt, q; q)_\infty} dt = \frac{(aq)}{c} (q, bq/a, ab/a, b/c/d, c/e, c/f; q)_\infty.
\]

By substituting \(c = abdef\) into (2.10.18), letting \(f \to 0\) and then replacing \(a, d, e\) by \(-a, -c, a, d, b\), respectively, we obtain Andrews and Askey's [1981] formula

\[
\int_a^b \frac{(-qt/a, qt/b, qt/c; q)_\infty}{(-ct/a, ct/b, ct/c; q)_\infty} dt = \frac{(aq)}{c} (q, -a/b, -q/a, c/d, c/e; q)_\infty.
\]
where

\[ f(t) = \frac{(t/a, qt, def, aqt, bdef, aqt, cdef; q)_{\infty}}{(t/d, te, tf, def, aqt, bdef; q)_{\infty}}, \quad (2.11.4) \]

and

\[ \int_{aq}^{bdef/a} f(t) \, dt = \frac{bdef(1 - q)_{aq, bdef, a^2, q^2 / bdef, bq / d, bq / e, bq / f, bq / a, bq / c; q)_{\infty}}{a(q/a, aq, ef, ba, de, be, af / a, bq / ef, bq / c, q^2 / a; q)_{\infty}}, \quad (2.11.5) \]

the advantage of our use of the q-integral notation can be seen by comparing the above proof with that given in Bailey [1936].

The special case \( qa^2 = bdef \) is particularly important since the series on the left side of (2.11.1) and the second series on the right become balanced, while the first series on the right becomes a \( \phi_4 \) series with sum

\[ \frac{(aq / ce, af / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

provided \( |aq / ce| < 1 \). This gives Bailey's summation formula:

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]

where \( qa^2 = bdef \), which is a nonterminating extension of Jackson's formula (2.6.2). This can also be written in the following equivalent form

\[ \sum_{n=0}^{\infty} \frac{(bq / a, q, b, ce, c / ef, f / c, q, c; q)_{\infty}}{(aq / e, c / ef, aq / ce, q, c; q)_{\infty}}, \]
and the expression in (2.12.3) simplifies to
\[
\frac{(b^{2}q/a, \lambda q^{m+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)}{(b^{2}q/a, \lambda q^{m+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)} \quad \Rightarrow \quad -\frac{x}{\lambda \cdot W_{\lambda}(b^{2}q/a, b^{m}/h, b^{m}/a, b^{m}/h, a^{m}/h, q)}
\]

(2.12.5)

since, by (2.12.4), \(aq/fg = bh/h, \quad aq/fh = bg/h, \quad \text{and} \quad aq/gh = bf/h\).

We now turn to the double sum that corresponds to the lower limit, \(\lambda\), in the \(q\)-integral (2.12.1). This leads to the series
\[
-\lambda(1 - q)(q, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h; a, \lambda q^{m+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

\[
-\lambda(1 - q)(q, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h; a, \lambda q^{m+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

\[
-\lambda(1 - q)(q, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h; a, \lambda q^{m+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

\[
-\lambda(1 - q)(q, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h; a, \lambda q^{m+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

(2.12.6)

The last \(q\)-integrals in (2.12.6) are balanced and nonterminating, so we may use (2.11.7) to get
\[
-\lambda(1 - q)(q, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h; a, \lambda q^{n+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

\[
-\lambda(1 - q)(q, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h; a, \lambda q^{n+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

\[
-\lambda(1 - q)(q, qa/c, qa/d, qa/e, qa/f, qa/g, qa/h; a, \lambda q^{n+1}/b, bq/m, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

(2.12.7)

Use of this breaks up the double series in (2.12.6) into two parts:
\[
-\lambda(1 - q)(q, b^{2}q/a, \lambda q^{m+1}/b, b^{m}/a, b^{m}/h, a^{m}/h, b^{2}q/a, \lambda q^{m+1}/b, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

\[
-\lambda(1 - q)(q, b^{2}q/a, \lambda q^{m+1}/b, b^{m}/a, b^{m}/h, a^{m}/h, b^{2}q/a, \lambda q^{m+1}/b, b^{m}/a, b^{m}/h, a^{m}/h)_{\infty}
\]

(2.12.8)

Summing the last \(q\)-integrals by (2.6.2) we find that the sum over \(n\) in (2.12.8) equals \(10W_{\lambda}(a, b, c, d, e, f, g, h, q)\), which is, of course, balanced by virtue of (2.12.4). Equating the expression in (2.12.2) with the sum of those in (2.12.5) and (2.12.8), and simplifying the coefficients, we finally obtain Bailey's [1947b] four-term transformation formula

\[
10q \sum_{n=0}^{\infty} \frac{(a; q)_{n}(1 - aq^{n})(b, f, g, h, a/\lambda; q)_{n}}{(q; q)_{n}(1 - a)(aq/b, aq/f, aq/g, aq/h, a; q)_{n}} q^{n}
\]

\[
= \frac{\lambda}{10q} \sum_{n=0}^{\infty} \frac{(a; q)_{n}(1 - aq^{n})(b, f, g, h, a/\lambda; q)_{n}}{(q; q)_{n}(1 - a)(aq/b, aq/f, aq/g, aq/h, a; q)_{n}} q^{n}
\]

(2.12.9)

In terms of the \(q\)-integrals this can be written in a more compact form:
\[
\int_{a}^{b} \frac{(qt/a, b, ta^{-1} - ta^{-1}, a, t, q/t, q, e, f, g, h; q)_{\infty}}{(b/a, ta^{-1} - ta^{-1}, a, t, q/t, q, e, f, g, h; q)_{\infty}} dt
\]

\[
= \frac{\lambda}{10q} \sum_{n=0}^{\infty} \frac{(a; q)_{n}(1 - aq^{n})(b, f, g, h, a/\lambda; q)_{n}}{(q; q)_{n}(1 - a)(aq/b, aq/f, aq/g, aq/h, a; q)_{n}} q^{n}
\]

(2.12.10)

where \(\lambda = qa^{2}/cd\) and \(a^{2}q^{2} = bcd\cdot ef\cdot gh\).
Exercises 2

2.1 Show that
\[ 3_2 \phi_2 \left[ \frac{a, qa^3, -qa^3}{a^3, -a^3} ; q, t \right] = \frac{(atq^2; q)_\infty}{(t; q)_\infty (t; q)_\infty} \quad |t| < 1. \]

2.2 Show that, for \( \max(|t|, |aq|) < 1 \),
\[ 4_3 \phi_3 \left[ \frac{a, qa^3, -qa^3, b}{a^3, -a^3, ab/b^2; q, t} \right] = \frac{(aq, bq; q)_\infty}{(t, aq/b; q)_\infty} \quad 2_1 \phi_1 \left( b^{-1}; t; bq; q, aq \right). \]

2.3 Give an alternate proof of the \( 6_5 \phi_5 \) summation formula (2.4.2) by first using (2.2.4) to derive a terminating form of the \( q \)-Dixon formula (2.7.2) and then using it along with the \( q \)-Saalschütz formula (1.7.2).

2.4 Prove Sears' identity (2.10.4) by using (1.4.3) and the coefficients in the power series expansion of the product
\[ 2_1 \phi_1 (a, b, c; q, z) 2_1 \phi_1 (d, e; abde/c; q, abz/c). \]

2.5 Prove that the sum of the first \( n + 1 \) terms of the series
\[ \sum_{k=0}^\infty \frac{(a; q)_k (1 - aq^{2k})(b, c, a/bc; q)_k}{(q; q)_k (1 - a)(aq/b, aq/c, bcq; q)_k} q^k \]
is
\[ \frac{(aq, bq, cq, aq/bc; q)_n}{(aq/b, aq/c, bcq; q)_n}. \]

2.6 Show that
\[ 4_3 \phi_3 \left[ \frac{q^{-n}, b, c, -q^{-n}/bc}{q^{-1}}, a/q^{-1}/bc; q, q \right] = \begin{cases} \begin{align*} (q, b^2, c^m, 2m)_{bcq; q}_{2m} & \quad n = 2m + 1, \\ 0 & \quad n = 2m, \end{align*} \end{cases} \]
where \( m = 0, 1, 2, \ldots \). (Bailey [1941], Carlitz [1969a])

2.7 Derive Jackson's terminating \( q \)-analogue of Dixon's sum:
\[ 3_2 \phi_2 \left[ \frac{q^{-2n}, b, c}{q^{-1} - q^{-1}/bc}, q^{2n}/bc; q, q \right] = \frac{(bc; q)_n (bc; q)_{2n}}{(q, bc; q)_{2m}} \quad \text{where} \quad n = 0, 1, 2, \ldots \quad \text{(See Jackson [1921, 1941], Bailey [1941], and Carlitz [1969a])}. \]

2.8 If \( b = q^{-n}, n = 0, 1, 2, \ldots \), show that
\[ 4_3 \phi_3 \left[ \frac{a, b, c, -c^2}{(abq)^{1/2}, -aq^{1/2}, c; q, q} \right] = \frac{(aq, bq, cq/a, cq/b; q^2)_\infty}{(q, abq, cq/aq; q^2)_\infty} a^{n/2}. \quad \text{(Andrews [1976a])} \]

2.9 Prove that
\[ 4_3 \phi_3 \left[ \frac{a, b, -b, aq/c^2}{aq/c, -aq/c, b^2; q, q} \right] = \frac{(aq/c, -aq/c, b^2; q, q)}{(aq/c, -aq/c, b^2; q, q)} \quad \infty \]
\[ + \frac{(b^2/q, q/b, -aq/c, -aq/c, b^2; q, q)}{(aq/c, -aq/c, b^2; q, q)} \quad \infty \]
\[ \cdot 4_3 \phi_3 \left[ \frac{q/b, -q/b, aq/b^2, aq/b^2, q^2 b^2}{q/b, -q/b, aq/b^2, aq/b^2, q^2 b^2} \right] = \frac{(aq/b^2, -aq/b^2, aq/b^2, aq/b^2, q^2 b^2; q, q)}{(aq/b^2, -aq/b^2, aq/b^2, aq/b^2, q^2 b^2; q, q)} \quad \infty \]
\[ \cdot 4_3 \phi_3 \left[ \frac{q/b, -q/b, aq/b^2, aq/b^2, q^2 b^2}{q/b, -q/b, aq/b^2, aq/b^2, q^2 b^2} \right] = \frac{(aq/b^2, -aq/b^2, aq/b^2, aq/b^2, q^2 b^2; q, q)}{(aq/b^2, -aq/b^2, aq/b^2, aq/b^2, q^2 b^2; q, q)} \quad \infty \]

2.10 The \( q \)-Racah polynomials, which were introduced by Askey and Wilson [1979], are defined by
\[ W_n(x; a, b, c, N; q) = W_n \left[ \frac{q^n, abq^{n+1}, q^{-n}, cq^{n+1}}{aq, bcq, q^{-n}} \right], \]
where \( n = 0, 1, 2, \ldots, N \). Show that
\[ W_n(x; a, b, c, N; q) = \frac{(aq/c, bq; q)_n}{(aq/c, bq; q)_n} c^n W_n(N-x; b, a, c^{-1}, N; q). \]

2.11 The Askey-Wilson polynomials are defined in Askey and Wilson [1985] by
\[ p_n(x; a, b, c, d | q) = \frac{a^{-n}(ab, ac, ad; q)_n}{ab, ac, ad} 4_3 \phi_3 \left[ \frac{q^{-n}, abdq^{n-1}, ae^{-i\theta}, ae^{-i\theta}}{ab, ac, ad; q} \right], \]
where \( x = \cos \theta \). Show that
(i) \( p_n(x; a, b, c, d | q) = p_n(x; b, a, c, d | q) \),
(ii) \( p_n(-x; a, b, c, d | q) = (-1)^n p_n(x; -a, -b, -c, -d | q) \).

2.12 Show that
\[ 10 W_9(a; b^4, -b^4, (aq)^{1/2}, -aq^{1/2}, a/b, a^2 q^{n+1}/b, q^{-n-1}; q, q) = \frac{(aq, a^2 q/b^2; q)_n}{(aq/b, a^2 q/b^2; q)_n}, \quad n = 0, 1, 2, \ldots \]

2.13 If \( \lambda = qa^2/bcd \) and \( |\lambda/a| < 1 \), prove that
(i) \[ 4_3 \phi_3 \left[ \frac{a, b, c, d}{aq/b, aq/c, aq/d; q^2, q^2} \right] = \frac{(\lambda q/a, \lambda q/a, \lambda q/a; q)_\infty}{(\lambda q/a, \lambda q/a, \lambda q/a; q)_\infty} \quad \infty \]
\[ \cdot 10 W_9 \left( \lambda; a^4, -a^4, (aq)^{1/2}, -aq^{1/2}, a/b, \lambda c/a, \lambda d/a; q, q \right), \]
(ii) \[ 4_3 \phi_3 \left[ \frac{a, b, c, d}{aq/b, aq/c, aq/d; q^2, q^2} \right] = \frac{(aq, -q, aq^{1/2}, -aq^{1/2}; q)_\infty}{(\lambda q, \lambda q/a, \lambda q/a, \lambda q/a; q)_\infty} \quad \infty \]
\[ \cdot 9 W_7 \left( \lambda; a^4, -a^4, a/b, \lambda c/a, \lambda d/a; q, q \right). \]
2.14 (i) Show that
\[
4\phi_3 \left[ \frac{a, qa^{1/2}, b, q^{-n}}{a^{1/2}, b q/b, b^2 q^{-1} a^{1/2}; q, q} \right] = (ab^{-2}, b^{-1}, - q b^{-1} a^{3/2}; q)_n
\]
which is a q-analogue of Bailey [1935, 4.5(1.3)].

(ii) Using (i) in the formula (2.8.2) prove the following q-analogue of Bailey [1935, 4.5(4)]:
\[
6\phi_5 \left[ \frac{a q a^{1/2}, b, c, d, q^{-n}}{a^{1/2}, a q b, a q c, a q d, a^2 q^{-1} - \lambda^2; q, q} \right] = \frac{(\lambda/a, \lambda^2/a^2, -\lambda q a^{1/2}; q)_n}{(\lambda, \lambda^2/a^2, -\lambda a^{1/2}; q)_n} \omega_{\lambda \eta \epsilon \nu \bar{\lambda}}
\]
\[
\lambda = a q \sqrt{b c d}
\]
This formula is equivalent to Jain's [1982, (4.6)] transformation formula.

2.15 By taking suitable q-integrals of the function
\[
f(t) = \frac{(qt/b, qt/c, qt/d, qt/e, q^2/t^2 bcdef; q)_\infty}{(at/bcdeq, qt/bce, qt/bcf; q)_\infty}
\]
prove Bailey's [1936, (4.6)] identity
\[
a^{-1} (a q g/d, a q e/d, a q f/d, q/adeq, q/aeq, q/aqf; q)_\infty
\]
\[
\times \left( q a^2, ab, ac, ad, ae, af; q, aq^2/abcdef \right)
\]
\[
+ b^{-1} (b q/d, b q/e, b q/f, q/bedq, q/bdeq, q/bcf; q)_\infty
\]
\[
\times \left( q b^2, ba, bc, bd, be, bf; q, q^2/abcdef \right)
\]
\[
+ c^{-1} (c q/d, c q/e, c q/f, q/cdeq, q/cqe; q, q^2/cdef; q)_\infty
\]
\[
\times \left( q c^2, ca, cb, cd, ce, cf; q, q^2/abcdef \right)
\]
\[
= 0,
\]
provided \( |q^2/abcdef| < 1 \).

2.16 Let \( S(\lambda, \mu, \nu, \rho) = (\lambda, q/\lambda, \mu, q/\mu, \nu, q/\nu, \rho, q/\rho; q)_\infty \). Using Ex. 2.15 prove that
\[
S(x, \lambda, x/\lambda, \mu, \mu/\lambda) - S(x, x/\lambda, \mu, \mu/\lambda)
\]
\[
= \frac{4}{\lambda} S(x \mu, x/\mu, \nu, \mu/\lambda),
\]
where \( x, \lambda, \mu, \nu \) are non-zero complex numbers. (Sears [1951c,d], Bailey [1936])

2.17 Show that
(i) \( s8\phi_7 \left[ \frac{\lambda, \lambda^2/\mu, -\lambda^2/\mu, a, b, c, -c, -\lambda q/\mu^2}{\lambda^2, -\lambda^2, \lambda q/a, \lambda q/b, \lambda q/c, -\lambda q/c, -\lambda q/a, c^2, c^2 q/b, c^2 q/c; q, \lambda q/\mu^2} \right] = (\lambda q, c^2/\mu; q)_\infty (\lambda q, b q, c^2 q/a, c^2 q/b; q^2)_\infty \)
\( (\lambda q/a, \lambda q/b; q)_\infty (q, ab, c^2 q/a, c^2 q/b; q^2)_\infty \)
where \( \lambda = -c(ab/q)^{1/2} \) and \( |q\lambda/ab| < 1 \);
(ii) \( s8\phi_7 \left[ \frac{-c, q(-q)^{3/2}, q(-c)^{3/2}, a, q/a, c, -d, -q/d}{(c)^{1/2}, -(c)^{1/2}, -c q/a, -c q/c, -q, c^2 q/c, c^2 q/a, c^2 q/c; q^2}_\infty \right] = (c^2 q/c, c^2 q/a, c^2 q/c, c^2 q/a; q^2)_\infty \)
\( (c^2 q/c, c^2 q/a, c^2 q/c, c^2 q/a; q^2)_\infty \)
Verify that (i) is a q-analogue of Watson's summation formula (Bailey [1935, 3.3(1)]) while (ii) is a q-analogue of Whipple's formula (Bailey [1935, 3.4(1)]). (See Jain and Verma [1985] and Gasper and Rahman [1986]).

2.18 In the \( 8\phi_8 \) summation formula (2.7.1) let \( b, c, d \rightarrow \infty \). Then set \( a = 1 \) to prove Euler's [1748] identity
\[
1 + \sum_{n=1}^{\infty} (-1)^n \left( q^n (n^2-1)/2 + q^{n(3n+1)/2} \right) = (q, q)_\infty
\]

2.19 Show that
\[
_{10}W_9 \left[ a; b, c, d, e, f, g, q^{-n}; q, q \right]
\]
\[
= (aq, aq/e, aq/d, aq/ef, aq/eg/b; q)_n e^n
\]
\( (aq/c, aq/d, aq/e, aq/f, aq/g, b/e; q)_n \)
\( _{10}W_9 \left( q^{-n}/b, e, aq/bc, aq/bd, aq/bf, aq/bg, eq^{-n}/a, q^{-n}; q, q \right) \),
where \( a^3 q^{n+2} = bcdefg \) and \( n = 0, 1, 2, \ldots \).

2.20 Prove that
\[
_{10}W_9 \left[ a; b, c, d, e, f, g, q^{-n}; q, a^3 q^{n+3}/bcdefg \right]
\]
\[
= (aq, aq/fg; q)_n \sum_{j=0}^{n} \left( f^{-n} q, g, aq/def; q \right) (f q^{-n}/a; q)_j
\]
\( (aq/f, aq/g; q)_n \sum_{j=0}^{n} (q, aq/def; q) (f q^{-n}/a; q)_j \)
\( 4\phi_3 \left[ q^{-j}, d, e, aq/bc \right]
\]
\( aq/b, aq/c, e q^{-j}/a; q, q \),
for \( n = 0, 1, 2, \ldots \).
2.21 Show that
\[
10W_0 (a; b, c, d, e, f, g, h; q; a^3 q^3 / bcdefgh)
\]
\[
= \sum_{n=0}^{\infty} \frac{(\lambda; q)_n (1 - \lambda q^{2n}) (\lambda b/a, \lambda c/a, \lambda d/a, e, f, g, h; q)_n (aq; q)_n}{(q; q)_n (1 - \lambda) (aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)_n (\lambda b; q)_n (\lambda c; q)_n (\lambda d; q)_n}
\]
\[
\times \frac{aq^n}{2^n} sW_7 (aq^{2n}; a/\lambda, eq^n, f q^n, gq^n, hq^n; q, a^3 q^3 / bcdefgh),
\]
where \( \lambda = qa^2 / bc \) and \( a^3 q^3 / bcdefgh < 1 \).

2.22 Prove that
\[
\sum_{n=0}^{\infty} \frac{(a; q)_n (1 - a q^{2n}) (b, c, d, e; q)_n}{(q; q)_n (1 - a) (aq/b, aq/c, aq/d, aq/e; q)_n} (-\frac{a^2 q^2}{bcde})^n q^n (f; q)_n
\]
\[
= \frac{(aq, aq/de; q)_\infty}{(aq/d, aq/e; q)_\infty} 3 \phi_2 \begin{bmatrix} \frac{aq/bc, d, e}{aq/b, aq/c; q} \\ \frac{aq/de}{aq/e} \end{bmatrix} [a/q; a/q; q]_\infty,
\]
where \( \frac{|aq/de|}{|aq/d, aq/e|} < 1 \).

Deduce that
\[
\sum_{n=0}^{\infty} \frac{(a; q)_n (1 - a q^{2n}) (d; e; q)_n}{(q; q)_n (1 - a) (dq/de; q)_n} (-\frac{a^2 q^2}{bcde})^n q^n (f; q)_n
\]
\[
= \frac{(aq, aq/de; q)_\infty}{(aq/d, aq/e; q)_\infty}.
\]

2.23 Prove that
\[
\sum_{j=0}^{\infty} \frac{(ab, ac, ad; q)_j}{(abcd, aqz, aqz/; q)_j} q^j f \]
\[
= \frac{(1 - z/a)(1 - abcz)}{(1 - bz)(1 - cz)} sW_7 (abcz; a, bc, cz/d, d; q; dz)
\]
\[
- \frac{(ab, ac, ad; q)_{n+1}}{(abcd, aqz, aqz/z; q)_{n+1}} (1 - a q^{n+1} / z)(1 - abcz q^{n+1})
\]
\[
- sW_7 (abcz q^{n+1}; a q^{n+1}, aqz, bc, qz, dz).
\]

2.24 Show that
\[
\begin{array}{c}
\phi_4 [a, b, c, d, e, f, g, h; q, a^3 q^3 / bcdefgh] \\
= \frac{(\lambda q/a, \lambda q/e, \lambda q^2/a, aq/f; q)_\infty}{(\lambda q/a, \lambda q/e, \lambda q^2/a, aq/f, \lambda q^2/f; q)_\infty}
\end{array}
\]
\[
\times \frac{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)_\infty}
\]
\[
\times \frac{(aq^2/b, aq^2/c, aq^2/d, aq^2/e, aq^2/f, aq^2/g, aq^2/h; q)_\infty}{(aq^2/b, aq^2/c, aq^2/d, aq^2/e, aq^2/f, aq^2/g, aq^2/h; q)_\infty}
\]
\[
\times \frac{(q/z, q/aqz, q/aqz/z; q)_\infty}{(q/z, q/aqz, q/aqz/z; q)_\infty}
\]
\[
\times \frac{(aq, aq/de; q)_\infty}{(aq/d, aq/e; q)_\infty} 3 \phi_2 \begin{bmatrix} \frac{aq/bc, d, e}{aq/b, aq/c; q} \\ \frac{aq/de}{aq/e} \end{bmatrix} [a/q; a/q; q]_\infty.
\]

2.25 By interchanging the order of summation in the double sum in Ex. 2.24 and using Bailey's summation formula (2.11.7), prove Jain and Verma's [1982, (7.1)] transformation formula
\[
\phi_4 [a, b, c, d, e, f, g, h; q, a^3 q^3 / bcdefgh]
\]
\[
= \frac{(a, b, c, d, e, f, g, h; q)\infty}{(a, b, c, d, e, f, g, h; q)\infty} \frac{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)\infty}
\]
\[
\times 3 \phi_2 \begin{bmatrix} \frac{aq/bc, d, e}{aq/b, aq/c; q} \\ \frac{aq/de}{aq/e} \end{bmatrix} [a/q; a/q; q]_\infty.
\]

where the parameters are related in the same way as in Ex. 2.24. Note that this is a nonterminating extension of (2.8.3) and that the first \( s \phi_4 \) series on the left is a nearly-poised series of the second kind while the second \( s \phi_4 \) series is a nearly-poised series of the first kind.

2.26 If \( a = q^{-n}, n = 0, 1, 2, ..., \) prove that
\[
\phi_4 [a, b, c, d, e, f, g, h; q, a^3 q^3 / bcdefgh]
\]
\[
= \frac{(a, b, c, d, e, f, g, h; q)\infty}{(a, b, c, d, e, f, g, h; q)\infty} \frac{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)\infty}
\]
\[
\times 3 \phi_2 \begin{bmatrix} \frac{aq/bc, d, e}{aq/b, aq/c; q} \\ \frac{aq/de}{aq/e} \end{bmatrix} [a/q; a/q; q]_\infty.
\]

(Sears [1951a, (4.1)], Carlitz [1969a, (2.4)])

2.27 Show that
\[
\phi_4 [q^{-n}, a, b, c, d, e, f, g, h; q, a^3 q^3 / bcdefgh]
\]
\[
= \frac{(a, b, c, d, e, f, g, h; q)\infty}{(a, b, c, d, e, f, g, h; q)\infty} \frac{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q)\infty}
\]
\[
\times 3 \phi_2 \begin{bmatrix} \frac{aq/bc, d, e}{aq/b, aq/c; q} \\ \frac{aq/de}{aq/e} \end{bmatrix} [a/q; a/q; q]_\infty.
\]

where \( \lambda = qa^2 / bc \) and \( f = ea^2 / \lambda^2 \). Note that this reduces to (2.8.3) when \( e = q^{-n}, n = 0, 1, 2, ... \).
2.32 Use (2.10.18), the \(q\)-binomial theorem, and the generating function in E. 1.29 to derive the formula

\[
C_n(\cos \theta; \beta|q) = \frac{2i \sin \theta}{1 - q} \left( \frac{\beta q^2}{(q, \beta^2, \beta e^{-2i\theta}, e^{2i\theta}; q)_{\infty}} \right) \\
\frac{1}{(q, q)_{\infty}} \sum_{a \in \mathbb{Z}} \left( gte^{i\theta}, gte^{-i\theta}; q \right)_{\infty} e^{a\theta} dt, \quad 0 < \theta < \pi.
\]

(Rahman and Verma [1986a])

**Notes 2**

\[\] 

§2.5 Some applications of Watson’s transformation formula (2.5.1) mock theta functions are presented in Watson [1936, 1937].


§2.9 Agarwal [1953a] showed that Bailey’s transformation (2.9.1) gives transformation formula for truncated 3φ7 series, where the sum of the first of the infinite series is called a truncated series.

§2.10 Many additional transformation formulas for hypergeometric series are derived in Whipple [1926a, b].

§2.11 Additional transformation formulas for \(3\phi_7\) series are derived Agarwal [1953c].

Ex. 2.6 Also see the summation formulas for very-well-poised series C.M. Joshi and Verma [1979].

Ex. 2.31 This exercise is the \(n = 3\) case of the Zeilberger and Brenewald [1985] theorem that if \(x_1, \ldots, x_n, q\) are commuting indeterminates or \(a_1, \ldots, a_n\) are nonnegative integers, then the constant term in the Laurent expansion of

\[
\prod_{1 \leq i < j \leq n} \frac{(x_j/x_j; q)_{a_j}(q x_j/x_i; q)_{a_j}}{(q; q)_{a_1} \cdots (q; q)_{a_n}}
\]

is equal to the \(q\)-multinomial coefficient

\[
\frac{(q; q)_{a_1 + \cdots + a_n}}{(q; q)_{a_1} \cdots (q; q)_{a_n}}.
\]

This was called the Andrews’ \(q\)-Dyson conjecture because Andrews [1975] had conjectured it as a \(q\)-analogue of a previously proved conjecture of Dyson.
that the constant term in the Laurent expansion of
\[
\prod_{1 \leq i \neq j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i}
\]
is equal to the multinomial coefficient
\[
\frac{(a_1 + \cdots + a_n)!}{a_1! \cdots a_n!}
\]
The \( n = 4 \) case of the Andrews' \( q \)-Dyson conjecture was proved independently by Kaddell [1985b]. Additional constant term results are derived in Bressoud and Goulden [1985], Evans, Ismail and Stanton [1982], Kaddell [1989a-c], Macdonald [1972–1989], Morris [1982], Stanton [1986b, 1989], and Zeilberger [1987, 1988, 1989a].

\[
3F2 \left[ \begin{array}{c}
-n, a, b, \frac{1}{c, d} \\
\end{array} ; \frac{1}{x} \right] = \frac{(d-b)_n}{(d)_n} 3F2 \left[ \begin{array}{c}
-n, c-a, b \\
\end{array} ; c, 1+b-d-n, 1 \right],
\]
\( n = 0, 1, 2, \ldots, \)
\[
3F2 \left[ \begin{array}{c}
a, b, c, 1 \\
d, e \\
\end{array} ; 1 \right] = \frac{\Gamma(d) \Gamma(e) \Gamma(s)}{\Gamma(a) \Gamma(s+b) \Gamma(s+c)} 3F2 \left[ \begin{array}{c}
d-a, e-a, 1 \\
s+b, s+c \\
\end{array} ; 1 \right],
\]
s = d + e - a - b - c,
\[
3F2 \left[ \begin{array}{c}
a, b, c \\
d, e \\
\end{array} ; 1 \right] = \frac{\Gamma(1-a) \Gamma(d) \Gamma(c-b)}{\Gamma(d-b) \Gamma(e-b) \Gamma(1+b-a) \Gamma(c)} 3F2 \left[ \begin{array}{c}
b, b-d+1, b-e+1 \\
1+b-c, 1+b-a \\
\end{array} ; 1 \right] + \text{idem } (b; c),
\]
where the symbol "idem \((b; c)\)" after an expression means that the preceding expression is repeated with \( b \) and \( c \) interchanged.

The main topic of this chapter, however, will be the \( q \)-analogues of a large class of transformations known as quadratic transformations. Two functions \( f(z) \) and \( g(w) \) are said to satisfy a quadratic transformation if \( z \) and \( w \) identically satisfy a quadratic equation and \( f(z) = g(w) \). Among the important examples of quadratic transformation formulas are
\[
(1+z)^a \, 2F1(a, b; 1+a-b; -z) = 2F1 \left( \frac{a}{2}, \frac{a+1}{2}; -b; 1+a-b; \frac{4z}{1+z} \right),
\]
\( (1-z)^a \, 2F1(a, b; 2b; 2z) = 2F1 \left( \frac{a}{2}, \frac{a+1}{2}; b; \frac{1}{2}; \frac{z^2}{(1-z)^2} \right), \)
\[
(1-z)^a \, 2F1(2a, a+b; 2a+2b; z) = 2F1 \left( a, b; a+b+\frac{1}{2}; \frac{z^2}{4(z-1)} \right),
\]
Two-term transformation formulas for $\phi_2$ series

with $n = 0, 1, 2, \ldots$

Keeping $n$ fixed and choosing special or limiting values of one of the other parameters leads to transformation formulas for terminating $\phi_2$ series. Let $\nu$ consider this class of formulas first.

Case (i) Letting $c \to 0$ in (3.2.1) we get

$$3\phi_2 \left[ \begin{array}{c} q^{-n}, a, b \\ d, e \end{array} ; q, q \right] = \frac{(e/a; q)_n}{(e; q)_n} a^n 3\phi_2 \left[ \begin{array}{c} q^{-n}, a/b, bq \\ d, aq^{1-n}/e \end{array} ; q, q \right].$$

Note that the series on the left is of type I and that on the right is of type I Formula (3.2.2) is a $q$-analogue of (3.1.1).

Case (ii) Letting $a \to 0$ in (3.2.1) gives

$$3\phi_2 \left[ \begin{array}{c} q^{-n}, b, c \\ d, e \end{array} ; q, q \right] = \frac{(bc/d; q)_n}{(e; q)_n} \frac{(bc)}{(d)} 3\phi_2 \left[ \begin{array}{c} q^{-n}, d/b, dc \\ d, de/bc \end{array} ; q, q \right].$$

If we let $c \to 0$ in (3.2.3) we obtain

$$2\phi_1 (q^{-n}, d/b; d; q, bq/e) = (-1)^n q^{n(2n)} (e; q)_n e^n 3\phi_2 \left[ \begin{array}{c} q^{-n}, b, 0 \\ d, e \end{array} ; q, q \right],$$

which may be written in the form (Ex. 1.15(i))

$$2\phi_1 (a, b; c; q, z) = \frac{(abz/c; q)_\infty}{(bz/c; q)_\infty} 3\phi_2 \left[ \begin{array}{c} a, c/b, 0 \\ c, cq/bz \end{array} ; q, q \right],$$

where $a = q^{-n}$, $n = 0, 1, 2, \ldots$

Case (iii) Let $c \to \infty$ in (3.2.1). This gives Sears’ [1951b, (4.5)] formula

$$3\phi_2 \left[ \begin{array}{c} q^{-n}, a, b \\ d, e \end{array} ; q, q \right] = \frac{(e/a; q)_n}{(e; q)_n} 3\phi_2 \left[ \begin{array}{c} q^{-n}, a/b, d/c \\ d, d^{1-n}/e \end{array} ; q, q \right].$$

Note that there is no essential difference between (3.2.2) and (3.2.5) since one can be obtained from the other by a change of parameters.

Case (iv) Replacing $a$ by $aq^n$ in (3.2.1) and simplifying, we get

$$4\phi_3 \left[ \begin{array}{c} q^{-n}, aq^n, b, c \\ d, e, abq \end{array} ; q, q \right] = \frac{(aq/e, de/bc; q)_n}{(e, abc/q; q)_n} \frac{(bc)}{(d)} 4\phi_3 \left[ \begin{array}{c} q^{-n}, aq^n, d/b, dc \\ d, de/bc, aq/e \end{array} ; q, q \right].$$

Set $d = \lambda c$ and then let $c \to \infty$. In the resulting formula we replace $\lambda, e, aq^n/aq^n$ by $c, d$ and $e,$ respectively, to get

$$3\phi_2 \left[ \begin{array}{c} q^{-n}, aq^n, b \\ d, e \end{array} ; q, q \right] = \frac{(aq/d, aq/e; q)_n}{(d; e; q)_n} \frac{(de)}{(aq)} 3\phi_2 \left[ \begin{array}{c} q^{-n}, aq^n, abq \end{array} ; q, q \right].$$
which is a transformation formula between two terminating \( 3\phi_2 \) series of type II.

Let us now consider the class of transformation formulas that connect two nonterminating \( 3\phi_2 \) series.

**Case (v)** In (3.2.1) let us take \( n \to \infty \). A straightforward term-by-term limiting process gives the formula

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} ; q, \frac{de}{abc} \right] = \left( \frac{e(a, de/abc; q)_\infty}{(e, de/abc; q)_\infty} \right) 3\phi_2 \left[ \begin{array}{c} a, d-b, d-c \\ d, e \end{array} ; q, \frac{e}{a} \right]. \tag{3.2.7}
\]

Apart from the general requirement that no zero shall appear in the denominators of the two \( 3\phi_2 \) series, the parameters must be restricted by the convergence conditions: \( |de/abc| < 1 \) and \( |e/a| < 1 \). This formula is a \( q \)-analogue of the Kummer-Thomae-Whipple formula

\[
3F_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} ; 1 \right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} 3F_2 \left[ \begin{array}{c} a, d-b, d-c \\ d, d+e-b-c \end{array} ; 1 \right], \tag{3.2.8}
\]

where \( e-a > 0 \) and \( d+e-a-b-c > 0 \).

**Case (vi)** Iterating (3.2.1) once gives

\[
3\phi_2 \left[ \begin{array}{c} q^{-n}, a, b, c \\ d, e \end{array} ; q, \frac{de}{abc} \right] = \frac{(b, de/abc; q)_n}{(d, e/abc; q)_n} 3\phi_2 \left[ \begin{array}{c} q^{-n}, d/b, e/b, de/abc \\ d, e \end{array} ; q, \frac{q^n}{b} \right]. \tag{3.2.9}
\]

Let us assume that \( \max_{\mathbb{N}} ([b], |de/abc|) < 1 \). Then, taking the limit \( n \to \infty \), we obtain Hall’s [1936] formula

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} ; q, \frac{de}{abc} \right] = \frac{(b, de/abc; q)_\infty}{(d, e/abc; q)_\infty} 3\phi_2 \left[ \begin{array}{c} d/b, e/b, de/abc \\ d, e \end{array} ; q, 1 \right]. \tag{3.2.10}
\]

Note that this is a \( q \)-analogue of formula (3.1.2).

Before leaving this section it is worth mentioning that by taking the limit \( n \to \infty \) in Watson’s formula (2.5.1), we get another transformation formula:

\[
3\phi_2 \left[ \begin{array}{c} aq/de, d, e \\ aq/b, aq/c \end{array} ; q, \frac{aq}{de} \right] = \left( \frac{aq/d, aq/de; q}_\infty \right) \left( \frac{aq/b, aq/c; q}_\infty \right) \sum_{k=0}^{\infty} \frac{(a; q)_k (1-aq^{2k})(b, c, d; e, q)_k}{(q; q)_k (aq/b, aq/c, aq/d, aq/e; q)_k} q(k) \left( -\frac{a^2q^2}{bcde} \right), \tag{3.2.11}
\]

provided \( |aq/de| < 1 \). This is a \( q \)-analogue of the formula

\[
F_2 \left[ \begin{array}{c} 1 + a - b - c, d, e \\ 1 + a - b, 1 + a - c \end{array} ; 1 \right] = \frac{\Gamma(1+a)\Gamma(1+a-d-e)}{\Gamma(1+a-d)\Gamma(1+a-e)}
\cdot a, 1 + \frac{1}{2}a, b, c, d, e
\cdot \frac{1}{3}a, 1 + a - b, 1 + a - c, 1 + a - d, 1 + a - e; -1 \right]. \tag{3.2.12}
\]

where \( Re \left( 1 + a - d - e \right) > 0 \); see Bailey [1935, 4.4(2)].

### 3.3 Three-term transformation formulas for \( 3\phi_2 \) series

In (2.10.10) let us replace \( a, b, c, d, e, f \) by \( Aq^N, Bq^N, C, D, E, Fq^N \), respectively, and then let \( N \to \infty \). In the resulting formula replace \( C, D, E, Aq/B \) and \( Aq/F \) by \( a, b, c, d \) and \( e, f \), respectively, to obtain

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} ; q, \frac{de}{abc} \right] = \frac{(e/b, e/c, q; q)_\infty}{(e, e/bc, q; q)_\infty} 3\phi_2 \left[ \begin{array}{c} d/a, b, c \\ d, bce/e, q, q \end{array} \right]
+ \frac{(d/a, b, c, de/abc; q)_\infty}{(d/e, b, e/c, de/abc; q)_\infty} 3\phi_2 \left[ \begin{array}{c} e/b, e/c, q; q, q \end{array} \right], \tag{3.3.1}
\]

where \( |de/abc| < 1 \) and \( bc/e \) is not an integer power of \( q \). This expresses a \( 3\phi_2 \) series of type II in terms of a \( 3\phi_2 \) series of type I. As a special case of (3.3.1), let \( a = q^{-n} \) with \( n = 0, 1, 2, \ldots \). Then

\[
3\phi_2 \left[ \begin{array}{c} q^{-n}, b, c \\ d, e \end{array} ; q, \frac{deq^n}{bc} \right]
= \frac{(e/b, e/c, q; q)_\infty}{(e, e/bc, q; q)_\infty} 3\phi_2 \left[ \begin{array}{c} b, c, dq^n \\ d/e, d, q \end{array} \right]
+ \frac{(b, c; q)_\infty}{(e/bc, q; q)_\infty} \frac{(de/abc; q)_\infty}{(d/e, q; q)_\infty} 3\phi_2 \left[ \begin{array}{c} e/b, e/c, dq^n/bc \\ q, q \end{array} \right]. \tag{3.3.2}
\]

Setting \( n = 0 \) in (3.3.2) gives the summation formula (2.10.13).

We shall now obtain a transformation formula involving three \( 3\phi_2 \) series of type II. We start by replacing \( a, b, c, d, e, f \) in (2.11.1) by \( Aq^N, Bq^N, C, D, E, Fq^N \), respectively, and then taking the limit \( N \to \infty \). In the resulting formula we replace \( C, D, E, Aq/B, Aq/F \) by \( a, b, c, d \), respectively, and obtain

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} ; \frac{de}{abc} \right]
= \frac{(e/b, e/c, q; q; d; q)_\infty}{(e, eq/d, q, a; eq/bq, bcq^2/eq; q)_\infty} 3\phi_2 \left[ \begin{array}{c} c, d/a, cq/e \\ bq \end{array} \right]
+ \frac{(q/d, eq/d, d, c, d, a, de/boq, bcp^2/eq; q)_\infty}{(d/e, eq/d, q, d, a; eq/bc, bcp^2/eq; q)_\infty} 3\phi_2 \left[ \begin{array}{c} aq/d, bq/d, cq/d \\ d/e, eq/d \end{array} ; \frac{de}{abc} \right], \tag{3.3.3}
\]

provided \( |boq/d| < 1 \), \( |de/abc| < 1 \) and none of the denominator parameters on either side produces a zero factor. If \( |q| < |de/abc| < 1 \), then

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ d, e \end{array} ; \frac{de}{abc} \right]
= \frac{(e/b, e/c, q; d; bq/a, cq/a, abcq/de; q)_\infty}{(e, eq/boq, a, bq/d, cq/d, de/boq, bcp^2/eq; q)_\infty} 3\phi_2 \left[ \begin{array}{c} q/a, d/a, e/a \\ bq/a, cq/a, \frac{abcq}{de} \right].
\]
Additional Summation, Transformation, and Expansion Formulas

\[
\frac{(b, c, q/d, d/a, eq/d, de/bcq, bcaq^2/d; q)_\infty}{(e, bcq/a, bq/d, eq/d, bcaq^2/e, d/q; q)_\infty} 3\phi_2 \left[ \begin{array}{c} aq/d, bq/d, cq/d \\ q^2/d, eq/d \end{array} ; q, abc \right] \frac{de}{q, abcd}.
\]

(3.3.4)

by observing that from (3.2.7)

\[
3\phi_2 \left[ \begin{array}{c} c, d/a, eq/c/a; q, bq \\ cq/a, bcq/e; q, d \end{array} \right] = \frac{(abcq/de, bq/a; q)_\infty}{(bcq/e, bq/d; q)_\infty} 3\phi_2 \left[ \begin{array}{c} q/a, d/a, e/a \\ bq/a, cq/a \end{array} ; q, abc \right] \frac{de}{q, abcd}.
\]

If we set \( e = xc \) in (3.3.3), let \( c \to 0 \) and then replace \( d \) and \( \lambda \) by \( c \) and \( abc/c, \) respectively, where \( |z| < 1, \) then \( |bq/c| < 1, \) we obtain

\[
2\phi_1(a, b; c; q, z) = \frac{(abz/c, q/c; q)_\infty}{(az/c, q/a; q)_\infty} 2\phi_2(c/a, eq/abz; cq/az; q, bq/c) - \frac{(bcz/c, a/az; q^2/az; q)_\infty}{(c/q, bq/c, q/a, az/c; q)_\infty} 2\phi_2(a/c, bq/c, q^2/c; q, z).
\]

(3.3.5)

Sears' [1951c, p.173] four-term transformation formulas involving \( 3\phi_2 \) series of types I and II can also be derived by a combination of the formulas obtained in the previous section. Some of these transformation formulas also arise as special cases of the more general formulas that we shall obtain in the next chapter by using contour integrals.

### 3.4 Transformation formulas for well-poised \( 3\phi_2 \) and very-well-poised \( 5\phi_4 \) series with arbitrary arguments.

Gasper and Rahman [1986] found the following formula connecting a well-poised \( 3\phi_2 \) series with two balanced \( 5\phi_4 \) series:

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ \frac{aqz}{bc} \end{array} ; q, \frac{aqz}{bc} \right] = (azq; q)_\infty 5\phi_4 \left[ \begin{array}{c} aq^2; azq, q^2/azq, q; q \\ a/b, aq/c, azq, q/x \end{array} ; q, q \right] + (aq/b, aq/c, azq/b, azq/c; q)_\infty 5\phi_4 \left[ \begin{array}{c} xaq^2, -xq^2, xaq; xaq^2, q; q \\ axq, bxq, xaq^2 \end{array} ; q, q \right] + (aq/b, aq/c, azq/b, azq/c; q)_\infty 5\phi_4 \left[ \begin{array}{c} xaq^2, -xq^2, xaq; xaq^2, q; q \\ axq, bxq, xaq^2 \end{array} ; q, q \right].
\]

(3.4.1)

Convergence of the \( 3\phi_2 \) series on the left requires that \( |aqz/bc| < 1 \). It is also essential to assume that \( x \) does not equal \( q^{2j}, j = 0, 1, 2, \ldots \), because one of the factors \((xq; q)_\infty\) and \((x^{-1}; q)_\infty\) appearing in the denominators on the right side of (3.4.1). Note that if either \( a \) or \( aq/bc \) is 1 or a negative integer power of \( q \), then the coefficient of the second \( 5\phi_4 \) series on the right vanishes, so that (3.4.1)

reduces to the Sears-Carlitz formula (Ex. 2.26). An important application of (3.4.1) is given in §8.8.

To prove (3.4.1) we replace \( d \) by \( dq^n \) in (2.8.3) and then let \( n \to \infty \). This gives

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ \frac{aq/b, aq/c; q, \frac{d}{q} \to \infty} \right] = \frac{(bcd/\frac{aq/bc}{q}; q)_\infty}{(bcd/\frac{aq/bc}{q}; q)_\infty} \lim_{n \to \infty} 12W_{11} (2aq^{1-n}/bcd; a^{1}, -a^{1}, (aq)^{1}, -(aq)^{1}, aq^{1-n}/bc, aq^{-n}/bd, aq^{-n}/cd, aq^{-n}/cd^2, q^n, q) \quad (3.4.2)
\]

To take the limit on the right side of (3.4.2) it suffices to proceed as in (2.10.9) to obtain

\[
\lim_{n \to \infty} 12W_{11} \left[ \begin{array}{c} \frac{a^1, -a^1, (aq)^{1}, -(aq)^{1}, aq/bc}{aq/b, aq/c, bcd/\frac{aq/bc}{q}; \frac{d}{q} \to \infty} \\ \frac{aq/b, aq/c, bcd/\frac{aq/bc}{q}; \frac{d}{q} \to \infty} \end{array} \right] = \frac{b\cd\left(b\cd/a^2, bd/a, cd/a, aq/bc, aq/c; q\right)_\infty}{a^2\left(d/a, aq/b, aq/c, bcd/a, a^2q^2/\frac{bcd}{q}; q\right)_\infty} \cdot 5\phi_4 \left[ \begin{array}{c} a/b, bcd/\frac{aq/bc}{q}, -bcd/\frac{aq/bc}{q}, a^{1/2}b, -bcd/\frac{aq/bc}{q}, a^{1/2}b \\ bd/a, cd/a, bcd/a^2, \frac{bcd}{q^2}; \frac{d}{q} \to \infty \end{array} ; q, q \right].
\]

(3.3.3)

Using this in (3.4.2) and replacing \( d \) by \( qxz^2/\frac{bc}{q} \), we get (3.4.1).

If we now replace \( d \) by \( dq^n \) in (2.8.5) and then let \( n \to \infty \), we obtain the transformation formula

\[
5\phi_4 \left[ \begin{array}{c} a, qaq^{1}, -qaq^{1}, b, c \\ a^{1}, -a^{1}, \frac{aq/b, aq/c; q, \frac{d}{q} \to \infty}{q, q} \end{array} \right] = (1 - x^2)(\frac{aq}{x}; q)_\infty (\frac{x(aq)^{1/2}}{x}; q)_\infty 5\phi_4 \left[ \begin{array}{c} (aq)^{1/2} - aq^{1/2}, \frac{aq}{x}; q, q \\ aq/a, qa^{1/2}, xaq^{1/2}, aq/bc \end{array} ; q, q \right] + \frac{(aq/b, aq/c, xaq^{1/2}/bc, aq^{1/2}/x; q)_\infty}{aq/b, aq/c, xaq^{1/2}/bc, aq^{1/2}/x; q)_\infty} 5\phi_4 \left[ \begin{array}{c} \frac{aq}{x}, x, xaq^{1/2}, -xaq^{1/2}, xaq^{1/2}/bc \\ xaq^{1/2}/bc, x(aq)^{1/2}/bc, x(aq)^{1/2}/bc^2 \end{array} ; q, q \right].
\]

(3.4.4)

In terms of \( q \)-integrals formulas (3.4.1) and (3.4.4) are equivalent to

\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ \frac{aq/b, aq/c; q, \frac{d}{q} \to \infty} \right] = \frac{(a, aq/bc; q)_\infty}{s(1 - q)(q, aq/b, aq/c, q/x; q)_\infty}
\]

\[
\frac{(a, aq/bc; q)_\infty}{s(1 - q)(q, aq/b, aq/c, q/x; q)_\infty}
\]
3.5 Transformations of series with base $q^2$ to series with base $q$

If in Sears' summation formula (2.10.12) we set $b = -c$, $c = -q$, replace $a$ by $aq^r$, $r = 0, 1, 2, \ldots$, multiply both sides by

$$\frac{(x^2, y^2, q^2)_r}{(-q; q)_r (x^2 y^2 b^2; q^2)_r} b^{2r} q^r$$

and then sum over $r$ from 0 to $\infty$, we get

$$\frac{(1 - x^2)(a, aq/2; q)_\infty}{s(1 - q)(a, aq/b, aq/c, z(qa)^{-1}; q)_\infty}$$

$$\int_{a^2} a^2 (a, aq/2; q)_\infty \int_{aq} \frac{(qaq)^{1/2} / s, -u(aq)^{1/2} / s, uaq^{1/2} / s, -uaq^{1/2} / s, aq/bc; q)_\infty}{d_q u,}$$

respectively, where $s \neq 0$ is an arbitrary parameter.

If we now set $c = (aq)^{1/2}$ in (3.4.5), replace $x$ by $x/b(aq)^{1/2}$, and use (2.10.19), then we get

$$\frac{(xq/b, aq/2; q^2)_\infty}{a, aq/2; q^2} = \frac{(a, aq/2; q^2)_\infty}{a, aq/2; q^2}$$

$$(xq/b, aq/2; q^2)_\infty \int_{a^2} a^2 (a, aq/2; q)_\infty \int_{aq} \frac{(qaq)^{1/2} / s, -u(aq)^{1/2} / s, uaq^{1/2} / s, -uaq^{1/2} / s, aq/bc; q)_\infty}{d_q u,}$$

provided $|xz/b^2| < 1$ when the series do not terminate.

Similarly, setting $c = (aq)^{1/2}$ and replacing $x$ by $x/q$ in (3.4.6) we obtain

$$\frac{(a, aq/2; q^2)_\infty}{a, aq/2; q^2} = \frac{(a, aq/2; q^2)_\infty}{a, aq/2; q^2}$$

$$(xz/b^2, x/q; q)_\infty \int_{a^2} a^2 (a, aq/2; q)_\infty \int_{aq} \frac{(qaq)^{1/2} / s, -u(aq)^{1/2} / s, uaq^{1/2} / s, -uaq^{1/2} / s, aq/bc; q)_\infty}{d_q u,}$$

provided $|x/qb^2| < 1$ when the series do not terminate.

assuming that $|qb^2| < 1$ when the series on the left is nonterminating.

Since the two $\phi_2$ series on the right side can be summed by the $q$-Saalschütz formula (1.7.2) with the base $q$ replaced by $q^2$, it follows from (3.5.1) that

$$\frac{(a^2; q^2)_\infty}{(b^2; q^2)_\infty} \phi_2 \left[ a^2, a^2; q^2, q^2 b^2 \right]$$

$$= \left( -a, ab; q^2 \right)_\infty \phi_2 \left[ a^2, a^2; q^2, q^2 b^2 \right]$$

and then sum over $r$ from 0 to $\infty$, we get

$$\int_{a^2} a^2 (a, aq/2; q)_\infty \int_{aq} \frac{(qaq)^{1/2} / s, -u(aq)^{1/2} / s, uaq^{1/2} / s, -uaq^{1/2} / s, aq/bc; q)_\infty}{d_q u,}$$

provided $|xz/b^2| < 1$ when the series do not terminate.

Note that one of the terms on the right side of (3.5.2) drops out when $a = \pm q^n$, $n = 0, 1, 2, \ldots$. Setting $y = ab$ and using (2.10.10) gives

$$\phi_2 \left[ a^2, a^2; q^2, q^2 b^2 \right] = \left( b^2, a^2 b^2 z^2; q^2 \right)_\infty \phi_2 \left[ a^2, a^2 z^2 & q^2 b^2 \right]$$

$$= \left( -a, ab; q^2 \right)_\infty \phi_2 \left[ a^2, a^2; q^2, q^2 b^2 \right]$$

where $|qb^2| < 1$ and $|ab^2| < 1$ when the series do not terminate. By applying Heine's transformation formula (1.4.1) twice to the $\phi_2$ series above and replacing $b$ by $q^2/b$ we find that

$$\frac{(a^2, a^2; q^2, q^2 b^2; q^2)_\infty}{(a^2 b^2, b^2 z^2; q^2)_\infty} \phi_2 \left[ a^2, a^2; q^2, q^2 b^2 \right]$$

$$= \frac{(x^2, y^2, q^2)_r}{(x^2 z^2; q^2)_r} \phi_2 \left[ a^2, a^2; q^2, q^2 b^2 \right]$$

$$= \frac{(x^2, y^2, q^2)_r}{(x^2 z^2; q^2)_r} \phi_2 \left[ a^2, a^2; q^2, q^2 b^2 \right]$$

and then sum over $r$ from 0 to $\infty$, we get

$$\int_{a^2} a^2 (a, aq/2; q)_\infty \int_{aq} \frac{(qaq)^{1/2} / s, -u(aq)^{1/2} / s, uaq^{1/2} / s, -uaq^{1/2} / s, aq/bc; q)_\infty}{d_q u,}$$

provided $|xz/b^2| < 1$ when the series do not terminate.
with the usual understanding that if the $10\phi_9$ series on the left does not terminate then the convergence condition $|a^2q^3/bc^2d^2e^2| < 1$ must be assumed to hold.

First we rewrite (2.10.12) in the form

$$
\frac{\left(\frac{aq^4n+1}{cd}\right)}{\left(\frac{aq^2n, daq^2n, eq^2n}{aq/\ell q^2n, cdeq^2n, adq^2n}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{cq^2n, daq^2n, eq^2n}{aq/\ell q^2n, cdeq^2n, adq^2n}\right)}{g \left(\frac{aq^4n+1}{cd}\right)} \frac{q^n}{g^r}
$$

$$
+ \frac{\left(\frac{qa^2q^3}{cdeq^2n}\right)}{\left(\frac{qa^2q^3}{cdeq^2n}\right)} \sum_{n=0}^{\infty} \frac{\left(\frac{aq^2q^3}{cdeq^2n}\right)}{g \left(\frac{qa^2q^3}{cdeq^2n}\right)} \frac{q^n}{g^r}
$$

$$
= \frac{\left(\frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{q}\right)}{\left(\frac{aq}{c}, \frac{aq}{d}, \frac{aq}{e}, \frac{aq}{q}\right)} \frac{\left(\frac{c, d, e, q_2n}{a, b, c, d, e, q_2n}\right)}{g \left(\frac{c, d, e, q_2n}{a, b, c, d, e, q_2n}\right)} \frac{q^n}{g^r}
$$

where $n$ is a nonnegative integer. Using (1.2.39) and (1.2.40), multiplying both sides of (3.5.8) by

$$
\frac{\left(\frac{a, b, q^2}{c}, \frac{a, q^2}{b^2}, \frac{a, q^2}{b^2}\right)}{\left(\frac{a, b, q^2}{c}, \frac{a, q^2}{b^2}, \frac{a, q^2}{b^2}\right)} \frac{\left(\frac{1-aq^4n}{1-aq^4n}\right)}{g \left(\frac{1-aq^4n}{1-aq^4n}\right)} \frac{q^n}{g^r}
$$

and summing over $n$ from 0 to $\infty$, we get

$$
\frac{\left(\frac{a, q^2}{b^2}, \frac{a, q^2}{b^2}\right)}{\left(\frac{a, q^2}{b^2}, \frac{a, q^2}{b^2}\right)} \frac{\left(\frac{1-aq^4n}{1-aq^4n}\right)}{g \left(\frac{1-aq^4n}{1-aq^4n}\right)} \frac{q^n}{g^r}
$$

The first double series on the right side of (3.5.9) easily transforms to

$$
\sum_{m=0}^{\infty} \frac{\left(\frac{c, d, e, q}{g^m}\right)}{\left(\frac{c, d, e, q}{g^m}\right)} \frac{\left(\frac{a, b, q^4n+1}{c, d, e, q^4n+1}\right)}{g \left(\frac{a, b, q^4n+1}{c, d, e, q^4n+1}\right)} \frac{q^n}{g^r}
$$

which, by (2.4.2), equals

$$
\frac{\left(\frac{a, b, q^4n+1}{c, d, e, q^4n+1}\right)}{\left(\frac{a, b, q^4n+1}{c, d, e, q^4n+1}\right)} \frac{\left(\frac{a, b, q^4n+1}{c, d, e, q^4n+1}\right)}{g \left(\frac{a, b, q^4n+1}{c, d, e, q^4n+1}\right)} \frac{q^n}{g^r}
$$

Similarly we can express the second double series on the right side of (3.5.9) as a single balanced $\phi_9$ series. Combining the two we get (3.5.7).

The special case of (3.5.7) that results from setting $e = (aq)^{\frac{1}{2}}$ is particularly interesting because both $\phi_9$ series on the right side become balanced $\phi_9$ series. 
series which, via (2.10.10), combine into a single $s\phi_7$ series with base $q$. Thus we have the formula

$$s\phi_7 \left[ a, a_q^{-1}, a_q^{-2}, b, c, c_q, d, d_q \right] = \begin{cases} \frac{(aq, aq/b, aq/cd, -aq/cd, aq/db \setminus -aq/db \setminus ; q)_\infty}{(aq/b, aq/c, aq/d, -aq/d, aq/cdb \setminus -aq/cdb \setminus ; q)_\infty} & \text{if } |a^2q^2/bc^2d^2| < 1 \text{ and } |aq/bc| < 1 \text{ when the series do not terminate.} \\ \end{cases}$$

(3.5.10)

### 3.6 Bibasic summation formulas

Our main objective in this section is to derive summation formulas containing two independent bases. Let us start by observing that when $d = a/bc$, Jackson's $s\phi_7$ summation formula (2.6.2) reduces to the following sum of a truncated series

$$\sum_{k=0}^{n} \frac{1 - aq^{2k}}{1 - a} \frac{(a, b, c, a/bc; q)_k}{(aq/b, aq/c, aq/bc; q)_k} q^k = \frac{(aq, bq, cq, aqc, q/a; q)_n}{(aq/b, aq/c, bcq; q)_n}$$

(3.6.1)

where $n = 0, 1, \ldots$. Notice that this series telescopes, for if we set $\sigma_{-1} = 0$ and

$$\sigma_k = \frac{(aq, bq, cq, aqc, q/a; q)_k}{(aq/b, aq/c, bcq; q)_k}$$

(3.6.2)

for $k = 0, 1, \ldots$, and apply the difference operator $\Delta$ defined by $\Delta u_k = u_k - u_{k-1}$ to $\sigma_k$, then we get

$$\Delta \sigma_k = \frac{(1 - aq^{2k})(a, b, c, a/bc; q)_k}{(1 - a) (aq/b, aq/c, aq/bc; q)_k} q^k,$$

(3.6.3)

which gives (3.6.1), since

$$\sum_{k=0}^{n} \Delta u_k = u_n - u_{-1}$$

(3.6.4)

for any sequence $\{u_k\}$. These observations and the bibasic extension

$$\tau_k = \frac{(ap, bp; p)_k (q, aq/bc; q)_k}{(aq/b; q)_k (ap/c, bc/bp; p)_k}$$

(3.6.5)

of $\sigma_k$ were used in Gasper [1989a] to show that

$$\Delta \tau_k = \frac{(1 - aq^{2k})(1 - bp^{2k}q^{-k})}{(1 - a)(1 - b)} \frac{(a, b; p)_k (c, a/bc; q)_k}{(aq/b; q)_k (ap/c, bc/bp; p)_k} q^k,$$

(3.6.6)

which, by (3.6.4), gave the indefinite bibasic summation formula

$$\sum_{k=0}^{n} \frac{(1 - aq^{k})(1 - bp^{k}q^{-k})}{(1 - a)(1 - b)} \frac{(a, b; p)_k (c, a/bc; q)_k}{(aq/b; q)_k (ap/c, bc/bp; p)_k} q^k$$

(3.6.7)

for $n = 0, 1, \ldots$. Notice that the part of the series on the left side of (3.6.7) containing the $q$-shifted factorials is split-poised in the sense that $aq = b(abc)$ and $c(abc) = (a/bc)(bcp) = ap$, while the expression on the right side is balanced and well-poised since

$$\frac{(ap, bp; p)_n (cq, aqc, q/a; q)_n}{(aq/b; q)_n (ap/c, bc/bp; p)_n}$$

and

$$\frac{(ap, bp; p)_n (cq, aqc, q/a; q)_n}{(aq/b; q)_n (ap/c, bc/bp; p)_n}$$

The $b \to 0$ case of (3.6.7)

$$\sum_{k=0}^{n} \frac{1 - aq^{2k}}{1 - a} \frac{(a, b; p)_k (c; q)_k}{(aq/b; q)_k (ap/c, bc/bp; p)_k} q^{-k} = \frac{(ap, bp; p)_n (cq, aqc, q/a; q)_n}{(aq/b; q)_n (ap/c, bc/bp; p)_n}$$

(3.6.9)

for $k = 0, \pm 1, \pm 2, \ldots$, and observed that

$$\Delta s_k = s_k - s_{k-1}$$

(3.6.10)

Since (3.6.4) extends to

$$\sum_{k=m}^{n} \Delta u_k = u_n - u_{m-1},$$

(3.6.11)

where we employed the standard convention of defining

$$\sum_{k=m}^{n} a_k = \begin{cases} \sum_{k=m}^{n} a_k & m \leq n, \\
0 & m = n + 1, \\
-(a_{n+1} + a_{n+2} + \cdots + a_{m-1}) & m \geq n + 2, 
\end{cases}$$

(3.6.12)
for \( n, m = 0, \pm 1, \pm 2, \ldots \), it follows from (3.6.10) that (3.6.7) extends to the indefinite bibasic summation formula

\[
\sum_{k=-m}^{n} \frac{(1 - adk^b q^k)(1 - bp^k/dq^k)}{(1 - ad)(1 - b/d)} \frac{(a, b; p)_k (c, ad^2/bc; q)_k}{(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k
\]

\[
= (1 - a)(1 - b)(1 - c)(1 - ad^2/bc)
\]

\[
\begin{aligned}
&= \frac{1}{d(1 - ad)(1 - bd)(1 - c/d)(1 - ad^2/bc)} \\
&\cdot \left\{ \frac{(ap, bp; p)_n (cq, ad^2q/bc; q)_n}{(dq, adq/b; q)_n (adp/c, bcp/d; p)_n} - \frac{(c, ad, d/bc; p)_{m+1}(1, d/b; ad; q)_{m+1}}{(c, ad/bc^2; q)_{m+1}(1/a, 1/b; p)_{m+1}} \right\}
\end{aligned}
\]

(3.6.13)

for \( n, m = 0, \pm 1, \pm 2, \ldots \), by applying the identity (1.2.28). Observe that (3.6.7) is the case \( d = 1 \) of (3.6.13) and that the right side of (3.6.9) is balanced and well-poised since

\[
(ap/bp)(cq/adq/bc) = (dq/adq/b)(adp/c)(bcp/d)
\]

and

\[
(ap/dq) = (bp/adq/b) = (cq/acp/c) = (adq/bc)(bcp/d).
\]

It is these observations and the factorization that occurred in (3.6.10) which motivated the choice of \( s_k \) in (3.6.9).

If \( |p| < 1 \) and \( |q| < 1 \), then by letting \( n \) or \( m \) tend to infinity in (3.6.13) we find that (3.6.13) also holds with \( n \) or \( m \) replaced by \( \infty \). In particular, this yields the following evaluation of a bilateral bibasic series

\[
\sum_{k=-\infty}^{\infty} \frac{(1 - adk^b q^k)(1 - bp^k/dq^k)}{(1 - ad)(1 - b/d)} \frac{(a, b; p)_k (c, ad^2/bc; q)_k}{(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k
\]

\[
= (1 - a)(1 - b)(1 - c)(1 - ad^2/bc)
\]

\[
\begin{aligned}
&= \frac{1}{d(1 - ad)(1 - bd)(1 - c/d)(1 - ad^2/bc)} \\
&\cdot \left\{ \frac{(ap, bp; p)_\infty (cq, ad^2q/bc; q)_\infty}{(dq, adq/b; q)_\infty (adp/c, bcp/d; p)_\infty} - \frac{(c, ad, d/bc; p)_\infty(1, d/b; ad; q)_\infty}{(c, ad/bc^2; q)_\infty(1/a, 1/b; p)_\infty} \right\}
\end{aligned}
\]

(3.6.14)

where \( |p| < 1 \) and \( |q| < 1 \).

In §3.8 we shall use the \( m = 0 \) case of (3.6.13) in the form

\[
\sum_{k=0}^{n} \frac{(1 - adk^b q^k)(1 - bp^k/dq^k)}{(1 - ad)(1 - b/d)} \frac{(a, b; p)_k (c, ad^2/bc; q)_k}{(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k
\]

\[
= (1 - a)(1 - b)(1 - c)(1 - ad^2/bc)
\]

\[
\begin{aligned}
&= \frac{1}{d(1 - ad)(1 - bd)(1 - c/d)(1 - ad^2/bc)} \\
&\cdot \left\{ \frac{(ap, bp; p)_n (cq, ad^2q/bc; q)_n}{(dq, adq/b; q)_n (adp/c, bcp/d; p)_n} - \frac{(c, ad, d/bc; p)_{m+1}(1, d/b; ad; q)_{m+1}}{(c, ad/bc^2; q)_{m+1}(1/a, 1/b; p)_{m+1}} \right\}
\end{aligned}
\]

(3.6.15)

There is no loss in generality since, by setting \( k = j - m \) in (3.6.13), it is seen that (3.6.13) is equivalent to (3.6.15) with \( n, a, b, c, d \) replaced by \( n + m, ap^{-m}, bp^{-m}, cq^{-m}, dq^{-m} \), respectively. We shall also use the special case \( c = q^{-n} \) of (3.6.15) in the form

\[
\sum_{k=0}^{n} \frac{(1 - adk^b q^k)(1 - bp^k/dq^k)}{(1 - ad)(1 - b/d)} \frac{(a, b; p)_k (c, ad^2q^n/bq^n; q)_k}{(dq, adq/b; q)_k (adp/c, bcp/d; p)_k} q^k
\]

\[
= (1 - d)(1 - ad/b)(1 - adq^n)(1 - dq^n/b)
\]

\[
= (1 - ad)(1 - d/b)(1 - dq^n)(1 - adq^n/b)
\]

(3.6.16)

where \( n = 0, 1, \ldots \). The \( d \to 1 \) limit case of (3.6.16) gives

\[
\sum_{k=0}^{n} \frac{(1 - ap^k q^k)(1 - bp^k q^{-k})}{(1 - a)(1 - b)} \frac{(a, b; p)_k (c, ad^2 q^n/bq^n; q)_k}{(q, aq/b; q)_k (apq^n, bqp^{-n}; p)_k} q^k
\]

\[
= \delta_{n,0},
\]

(3.6.17)

where \( \delta_{n,0} \) is the Kronecker delta function and \( n = 0, 1, \ldots \), was derived independently by Bressoud [1988], Gasper [1989], and Krattenthaler [1989b].

If we replace \( n, a, b \) and \( d \) in (3.6.17) by \( n = m, ap^{-m}, bp^{-m}, q^{-m} \) and \( j - m \), respectively, we obtain the orthogonality relation

\[
\sum_{j=m}^{\infty} a_{nj} b_{jm} = \delta_{n,m}
\]

(3.6.18)

with

\[
a_{nj} = \frac{(-1)^{n+j} (1 - ap^j q^{-j})(1 - bp^j q^{-j})(apq^n, bqp^{-n}; p}_{n-1}(q; q)_{n-1}(apq^n, bqp^{-n}; p)_n(bq^{-2n} / a; q)_{n-j}
\]

(3.6.19)

\[
b_{jm} = \frac{(ap^m q^{-m}, bpq^{-m}; p)_j \cdot \cdot \cdot (- a^{-1} b q^{-2m})^{j-m} q^{-j(m-2)}}{(q, aq^{j+2m}/b; q)_{j-m}}
\]

(3.6.20)

This shows that the triangular matrix \( A = (a_{nj}) \) is inverse to the triangular matrix \( B = (b_{jm}) \). Since inverse matrices commute, by computing the \( j^{th} \) term of \( B \), we obtain the orthogonality relation

\[
\sum_{n=0}^{j} \frac{(1 - ap^k q^{-k})(1 - bp^k q^{-k})(apq^k, bqp^{-k}; p)_{j-k-1}}{(q; q)_{n-1}(apq^k, bqp^{-k}; p)_{j-k-1}(aq^{1+k}/bq^{-k}; q)_{j-k-1}}
\]

\[
\cdot \left(1 - a^{-1} b^{-1} q^{-2k} \right)^{(-1)^n q^{-2(m-2)-(j-k-1)+i(j-k-1)}} = \delta_{j,k},
\]

(3.6.21)

which, by replacing \( j, n, a, b \) by \( n + k, k, ap^{-k} q^{-k}, bp^{-k} q^{-k} \), respectively, yields the bibasic summation formula

\[
\left(1 - \frac{a}{p}\right) \left(1 - \frac{b}{p}\right) \sum_{k=0}^{n} (aq^k, bq^{-k}; p)_{n-1}(1 - aq^k/b)(1 - aq^k/b)^{-1} q^{-k} = \delta_{n,0}
\]

(3.6.22)

for \( n = 0, 1, \ldots \). The \( b \to 0 \) limit case of (3.6.22) was derived in Al-Salam and Verma [1984] by using the fact that the \( n^{th} \) difference of a polynomial in \( q \) of degree less than \( n \) is equal to zero. For applications to \( q \)-analogues of Lagrange inversion, see Gessel and Stanton [1983, 1986] and Gasper [1989a]. Formulas (3.6.17) and (3.6.22) will be used in §3.7 to derive some useful general expansion formulas.
3.7 Bibasic expansion formulas

One of the most important general expansion formulas for hypergeometric series is the Fields and Wimp [1961] expansion

\[
\begin{align*}
\Phi_{r+2}^{(u)}(a_R, c_T; xw) &= \sum_{n=0}^{\infty} \frac{(a_R)_n (\alpha)_n (\beta)_n (-x)^n}{(b_S)_n (\gamma + n)_n n!} \\
&= \frac{n + \alpha, n + \beta, n + a_R; x}{1 + 2n + \gamma, n + b_S} \\
&= \frac{-n, n + \gamma, c_T; w}{\alpha, \beta, d_U},
\end{align*}
\]

where we employed the contracted notation of representing \(a_1, \ldots, a_r\) by \(a_R\), \(a_1, \ldots, a_r\) by \((a_R)_n\), and \(n + a_1, \ldots, n + a_r\) by \(n + a_R\). In (3.7.1), as elsewhere, either the parameters and variables are assumed to be such that the (multiple) series converge absolutely or the series are considered to be formal power series in the variables \(x\) and \(w\). Special cases of (3.7.1) were employed, e.g., in Gasper [1975a] to prove the nonnegativity of certain sums (kernels) of Jacobi polynomials and to give additional proofs of the Askey and Gasper [1976] inequalities that de Branges [1985] used at the last step in his proof of the Bieberbach conjecture.

Verma [1972] showed that (3.7.1) is a special case of the expansion

\[
\sum_{n=0}^{\infty} A_n B_n \frac{(n)!}{n!} = \sum_{n=0}^{\infty} \frac{(-x)^n}{(\gamma + n)_n} \sum_{k=0}^{n} \frac{(\alpha + k)_n (\beta)_n k!}{n!} A_n B_{n+k} x^k
\]

and derived the \(q\)-analogue

\[
\sum_{n=0}^{\infty} A_n B_n \frac{(q^n)_n}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-x)^n}{(\gamma + n)_n} \sum_{k=0}^{n} \frac{(\alpha + k)_n (\beta)_n k!}{n!} A_n B_{n+k} x^k
\]

To derive a bibasic extension of (3.7.3) we first observe that, by (3.6.17),

\[
\sum_{j=0}^{m} \frac{(1 - \gamma p^{r+j} q^{r+j}) (1 - \sigma p^{r+j} q^{r+j})}{(1 - \gamma p^{r+j} q^{r+j}) (1 - \sigma p^{r+j} q^{r+j})} \frac{q_j}{(\gamma p^{r+j} q^{r+j}, \sigma p^{r+j} q^{r+j}; p)_j} q^j = \delta_{m,0}
\]

where we employed the contracted notation of representing \(a_1, \ldots, a_r\) by \(a_R\), \(a_1, \ldots, a_r\) by \((a_R)_n\), and \(n + a_1, \ldots, n + a_r\) by \(n + a_R\). In (3.7.1), as elsewhere, either the parameters and variables are assumed to be such that the (multiple) series converge absolutely or the series are considered to be formal power series in the variables \(x\) and \(w\). Special cases of (3.7.1) were employed, e.g., in Gasper [1975a] to prove the nonnegativity of certain sums (kernels) of Jacobi polynomials and to give additional proofs of the Askey and Gasper [1976] inequalities that de Branges [1985] used at the last step in his proof of the Bieberbach conjecture.

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\]

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\]

where we employed the contracted notation of representing \(a_1, \ldots, a_r\) by \(a_R\), \(a_1, \ldots, a_r\) by \((a_R)_n\), and \(n + a_1, \ldots, n + a_r\) by \(n + a_R\). In (3.7.1), as elsewhere, either the parameters and variables are assumed to be such that the (multiple) series converge absolutely or the series are considered to be formal power series in the variables \(x\) and \(w\). Special cases of (3.7.1) were employed, e.g., in Gasper [1975a] to prove the nonnegativity of certain sums (kernels) of Jacobi polynomials and to give additional proofs of the Askey and Gasper [1976] inequalities that de Branges [1985] used at the last step in his proof of the Bieberbach conjecture.

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\]

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\]

To derive a bibasic extension of (3.7.3) we first observe that, by (3.6.17),

\[
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\]

where we employed the contracted notation of representing \(a_1, \ldots, a_r\) by \(a_R\), \(a_1, \ldots, a_r\) by \((a_R)_n\), and \(n + a_1, \ldots, n + a_r\) by \(n + a_R\). In (3.7.1), as elsewhere, either the parameters and variables are assumed to be such that the (multiple) series converge absolutely or the series are considered to be formal power series in the variables \(x\) and \(w\). Special cases of (3.7.1) were employed, e.g., in Gasper [1975a] to prove the nonnegativity of certain sums (kernels) of Jacobi polynomials and to give additional proofs of the Askey and Gasper [1976] inequalities that de Branges [1985] used at the last step in his proof of the Bieberbach conjecture.

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\[
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\]

and derived the \(q\)-analogue

\[
\sum_{n=0}^{\infty} A_n B_n \frac{(q^n)_n}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-x)^n}{(\gamma + n)_n} \sum_{k=0}^{n} \frac{(\alpha + k)_n (\beta)_n k!}{n!} A_n B_{n+k} x^k
\]

To derive a bibasic extension of (3.7.3) we first observe that, by (3.6.17),

\[
\sum_{j=0}^{m} \frac{(1 - \gamma p^{r+j} q^{r+j}) (1 - \sigma p^{r+j} q^{r+j})}{(1 - \gamma p^{r+j} q^{r+j}) (1 - \sigma p^{r+j} q^{r+j})} \frac{q_j}{(\gamma p^{r+j} q^{r+j}, \sigma p^{r+j} q^{r+j}; p)_j} q^j = \delta_{m,0}
\]
Bibasic expansion formulas

\[ \sum_{j=0}^{\infty} \frac{(ap^k q^k; p)_j}{(q; q)_j} B_{j+k}(-x)^j q^{j}. \]  

(3.7.11)

The \( p = q \) case of (3.7.11) is due to Jackson [1910a].

In order to employ (3.6.22) to extend (3.7.10), replace \( n \) in (3.6.22) by \( j \), multiply both sides by \( B_{n+j} x^{n+j} (a/b) q^j \), sum from \( j = 0 \) to \( \infty \), change the order of summation and then replace \( k \) by \( n-k \) and \( j \) by \( j+k-n \) to obtain

\[ B_n x^n = \left( 1 - \frac{a}{p} \right) \left( 1 - \frac{b}{p} \right) \sum_{k=0}^{\infty} \frac{1 - a q^{2k-2n}/b^n}{(q; q)_{k-n}} \sum_{j=0}^{\infty} \frac{aq^{k-n} b^{q-n-k} p_{j+k-n-1} - a^{j+k-n}}{b} \cdot (-x)^j B_{j+k} q^{k(n-j-k-n+1)+j(2+j-k-n+1)}. \]  

(3.7.12)

Next we replace \( a \) by \( ap^{n+1} q^n \), \( b \) by \( b^{n+1} q^{-n} \), multiply both sides by \( A_n w^n \) and then sum from \( n = 0 \) to \( \infty \) to get

\[ \sum_{n=0}^{\infty} A_n B_n (xw)^n = \sum_{k=0}^{\infty} \frac{(ap^k q^k; p)_{k-1} x^k}{(aq^k/b; q)_{k-1}} \sum_{n=0}^{\infty} \frac{1 - ap^n q^n}{(aq^k/b; q)_{k-n}} \sum_{j=0}^{\infty} \frac{(ap^k q^k; p)_j}{(aq^k/b; q)_{j}} B_{j+k}(-x)^j q^{j}. \]  

(3.7.13)

This formula tends directly to (3.7.11) as \( b \to 0 \). By replacing \( A_n, B_n, x, w \) by suitable multiples, we may change (3.7.13) to an equivalent form which tends to (3.7.11) as \( b \to \infty \). In addition, by replacing \( A_n, B_n, x, w \) by \( A_n q^{2n} / b \), \( B_n q^{-2n} / b, a \cdot w / b, \) respectively, we can write (3.7.13) in the simpler looking equivalent form

\[ \sum_{n=0}^{\infty} A_n B_n x^n w^n = \sum_{k=0}^{\infty} \frac{(ap^k q^k; p)_{k-1} (-x)^k q^{k+1}}{(aq^k/b; q)_{k-1}} \sum_{n=0}^{\infty} \frac{1 - ap^n q^n}{(aq^k/b; q)_{k-n}} \sum_{j=0}^{\infty} \frac{(ap^k q^k; p)_j}{(aq^k/b; q)_{j}} B_{j+k} x^j q^j. \]  

(3.7.14)

As in the derivation of (3.7.6), one may extend (3.7.14) by replacing \( B_{j+k} \) by \( B_{j+k} C_{n+j+k-n} \) with \( C_{n,0} = 1 \) for \( n = 0, 1, \ldots \). Multivariable expansions, which are really special cases of (3.7.6) and (3.7.14), may be obtained by replacing \( A_n \) and \( B_n \) in (3.7.6) and (3.7.14) by multiple power series, see, e.g.
3.8 Quadratic, cubic, and quartic summation and transformation formulas

By setting \( p = q^j \) or \( q = p^j, j = 2, 3, \ldots \), in the bibasic summation formulas of §3.7 and using summation and transformation formulas for basic hypergeometric series, one can derive families of quadratic, cubic, etc. summation, transformation and expansion formulas. To illustrate this we shall derive a quadratic transformation formula containing five arbitrary parameters by starting with the case of \( q^2 \):

\[
\sum_{k=0}^{n} \frac{(1 - adp^{2k})(1 - b/dp^{2k})}{(1 - ad)(1 - b/d)} \frac{(a,b;p)_k (p^{-2n}a^2 b^{2n}/b p^2)_k}{(1 - d)(1 - ad/b)(1 - adp^{2n})(1 - dp^{2n}/d)\frac{p^{2k}}{p^k}} \frac{q^k}{(a^2 q^2/b;q)_k}
\]

where \( n = 0, 1, \ldots \).

Change \( p \) to \( q \) and \( d \) to \( c \) in (3.8.1), multiply both sides by

\[
\frac{(ac^2/b; q^2)_m (c/b; q)_2n}{(q^2; q^2)_m (acq; q)_2n} C_n
\]

and sum over \( n \) to get

\[
\sum_{n=0}^{\infty} \frac{(ac^2/b; q^2)_n (cq/b; q)_{2n}(1 - c)(1 - ac/b)}{(q^2; q^2)_n (acq; q)_{2n}(1 - ac^2 q/b)} C_n
\]

\[
= \sum_{n=0}^{\infty} \frac{(1 - acq^{2k})(1 - b/cq^k)}{(1 - ac)(1 - b/c)} \frac{(a,b,q)_{k} (ac^2/b; q^2)_m (c/q)_n (acq;q)_{2n+k}}{(q^2; q^2)_m (acq; q)_{2n+k}} C_{n+k+m}
\]

(3.8.2)

Setting

\[
C_n = \frac{(1 - ac^2 q^{2n}/b)(d,e,f;q^2)_n (a^2 q^4/def)^n}{(1 - ac^2/b)(ac^2 q^2/bd, ac^2 q^2/be, ac^2 q^2/bf; q^2)_n},
\]

it follows from (3.8.2) that

\[
t_0 W_q(ac^2/b; ac/b, c, cq/b, cq^2/b, d, e, f; q^2, a^2 c^2 q^4/def)
\]

Quadratic, cubic, and quartic formulas

\[
= \sum_{k=0}^{\infty} \frac{(1 - acq^{2k})(1 - b/cq^k)(a,b,c,b;q)_k (ac^2 q^4/b; q^2)_k}{(1 - ac)(1 - b/c)} \frac{(cq^2, ac^2 q^2/b; q^2)_k}{(acq; q)_k}
\]

\[
\frac{(d,e,f;q^2)_k (a^2 c^2 q^4/def)^k}{(ac^2 q^2/bd, ac^2 q^2/be, ac^2 q^2/bf; q^2)_k}
\]

\[
\cdot s W_q(ac^2 q^4k/b; c q^k/b, cq^k; b, dq^k, eq^k, f q^k; q^2, a^2 c^2 q^4/def).
\]

(3.8.3)

If we now assume that

\[
a^2 c^2 q = def,
\]

then we can apply (2.11.7) to get

\[
s W_q(ac^2 q^4k/b; c q^k/b, cq^k; b, dq^k, eq^k, f q^k; q^2, q^k)
\]

\[
= \frac{(ac^2 q^{4k+2}/b, bf/ac^2 q^{2k}, abq^{2k+1}, acq^{k+2}/d; q)_\infty}{(acq^{k+2}/ac^2 q^{k+2}/bd, ac^2 q^{2k+2}/be; q)_\infty}
\]

\[
\cdot \frac{(aq^{k+2}/e, acq^{k+1}/ac, acq^{k+2}/de; q^2)_\infty}{(b e f/ac^2, b d f/ac^2, f/ac^2, f/ac^2, f/ac^2, f/ac^2, f/ac^2)}
\]

\[
+ \frac{bfq^{-2k}(ac^2 q^{4k+2}/b, c q^k/b, cq^k; b, dq^{k+1}, eq^{k+1}; q)_\infty}{(ac^2 q^{4k+2}/bd, ac^2 q^{4k+2}/be, ac^2 q^{4k+2}/cf; q^2)_\infty}
\]

\[
\frac{(fg^2/e, fg^2/d, bfq^{-2k}/ac^2, b f q^{k+2}/ac^2, f q^{k+2}/cf, q^2)_\infty}{(ac^2 q^{4k+2}/bf, b e f/ac^2, b d f/ac^2, f/ac^2, b f q^2/ac^2, f/ac^2, f/ac^2)}
\]

\[
\cdot s W_q(b f^2/ac^2, f q^2, b f/ac^2, b f/ac^2, f/ac^2, f/ac^2, f/ac^2, f/ac^2; q^2, q^2).
\]

(3.8.5)

which, combined with (3.8.3), gives

\[
t_{0} W_q(ac^2/b; c, d, e, f, ac/b, cq/b, cq^2/b; q^2, q^2)
\]

\[
= \frac{(ac^2 q^{2k}/b, ac^2 q^{2k}/bd, abq/b, bf/ac^2, q^2)_\infty (acq/d, acq/e, q^2)_\infty}{(ac^2 q^{2k}/bd, ac^2 q^{2k}/be, bd f/ac^2, ac^2 q^{2k}/cf; q^2)_\infty (acq, f/ac^2, q^2)_\infty}
\]

\[
\sum_{k=0}^{\infty} \frac{(1 - acq^{2k})(1 - b/cq^k)(a,b,c,b;q)_k (ac^2 q^4/b; q^2)_k}{(1 - ac)(1 - b/c)} \frac{(cq^2, ac^2 q^2/b; q^2)_k}{(acq; q)_k}
\]

\[
\cdot \frac{bfq^{-2k}(ac^2 q^{4k+2}/b, c q^k/b, cq^k; b, dq^{k+1}, eq^{k+1}; q)_\infty}{(ac^2 q^{4k+2}/bd, ac^2 q^{4k+2}/be, ac^2 q^{4k+2}/cf; q^2)_\infty}
\]

\[
\frac{bfq^{-2k}/ac^2, b f q^{k+2}/ac^2, f q^{k+2}/cf, q^2)_\infty}{(ac^2 q^{4k+2}/bf, b e f/ac^2, b d f/ac^2, f/ac^2, b f q^2/ac^2, f/ac^2)}
\]

\[
\cdot (b f q/c, f/c, q^2)_\infty
\]

\[
\cdot (acq, f/ac^2, q^2)_\infty
\]

\[
\cdot s W_q(b f^2/ac^2, f q^2, b f/ac^2, b f/ac^2, f/ac^2, f/ac^2; q^2, q^2).
\]

(3.8.6)
The last sum over \( k \) in (3.8.6) is

\[
\sum_{j=0}^{\infty} \frac{(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
\sum_{k=0}^{\infty} \frac{(1 - acq^{3k})(1 - b/cq^k)}{(1 - ac)(1 - b/c)} \left( \frac{ac^2/bf\ q^2_j}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} \right)_{k} (1 - b^2q^2/ac^2)_{-2}^k
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/fq^2)(1 - b/fq^2)_{c/(1 - ac)}(1 - b/c)(1 - fq^2/j/(1 - ac/bfq^2))
\]

\[
\left\{ \frac{(a, b, q)_{\infty}}{(c, ac/b, q^2)_{\infty}}(f\ q^2, ac^2/bf\ q^2; q^2)_{\infty} \right\}
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/f)(1 - b/fac)
\]

\[
\cdot 10W_9(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, f^2/ac; q^2, q^4)
\]

\[
+ \frac{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - b/fac)}{(a, b, q)_{\infty}(f, ac^2/bf; q^2)_{\infty}}
\]

\[
\left( \frac{ac^2, ac^2/bf, q^2}{ac^2/fq, q^2} \right)_{\infty}
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/f)(1 - b/fac)
\]

\[
\cdot 10W_9(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, f^2/ac; q^2, q^4)
\]

\[
\cdot f + \frac{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - b/fac)}{(a, b, q)_{\infty}(f, ac^2/bf; q^2)_{\infty}}
\]

\[
\left( \frac{ac^2, ac^2/bf, q^2}{ac^2/fq, q^2} \right)_{\infty}
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/f)(1 - b/fac)
\]

\[
\cdot 10W_9(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, f^2/ac; q^2, q^4)
\]

\[
+ \frac{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - b/fac)}{(a, b, q)_{\infty}(f, ac^2/bf; q^2)_{\infty}}
\]

\[
\left( \frac{ac^2, ac^2/bf, q^2}{ac^2/fq, q^2} \right)_{\infty}
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/f)(1 - b/fac)
\]

\[
\cdot 10W_9(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, f^2/ac; q^2, q^4)
\]

\[
+ \frac{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - b/fac)}{(a, b, q)_{\infty}(f, ac^2/bf; q^2)_{\infty}}
\]

\[
\left( \frac{ac^2, ac^2/bf, q^2}{ac^2/fq, q^2} \right)_{\infty}
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/f)(1 - b/fac)
\]

\[
\cdot 10W_9(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, f^2/ac; q^2, q^4)
\]

\[
+ \frac{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - b/fac)}{(a, b, q)_{\infty}(f, ac^2/bf; q^2)_{\infty}}
\]

\[
\left( \frac{ac^2, ac^2/bf, q^2}{ac^2/fq, q^2} \right)_{\infty}
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/f)(1 - b/fac)
\]

\[
\cdot 10W_9(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, f^2/ac; q^2, q^4)
\]

\[
+ \frac{ac(1 - ac)(1 - c/b)(1 - f/c)(1 - b/fac)}{(a, b, q)_{\infty}(f, ac^2/bf; q^2)_{\infty}}
\]

\[
\left( \frac{ac^2, ac^2/bf, q^2}{ac^2/fq, q^2} \right)_{\infty}
\]

\[
= \sum_{j=0}^{\infty} \frac{(b^2/ac^2, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, q^2_j)_{j}}{(q^2, f^2/d, q^2/e, bfq/ac, q^2_j)_{j}} (1 - b^2q^2/ac^2)_{-2}^j
\]

\[
(1 - c)(1 - ac/b)(1 - ac/f)(1 - b/fac)
\]

\[
\cdot 10W_9(b^2/ac^2, f, bdf/ac^2, be/f/ac^2, f/ac, fq/ac, f^2/ac; q^2, q^4)
\]
Additional Summation, Transformation, and Expansion Formulas

Note that the first two terms on the left side of (3.6.11) containing the $10W_9$ series can be transformed to another pair of $10W_9$ series by applying the four-term transformation formula (2.12.9). Since the $q^2r$ series in (3.6.11) is balanced it can be summed by (1.7.2) whenever it terminates. When $c = 1$ formula (3.6.11) reduces to the quadruple summation formula

$$
\sum_{k=0}^{\infty} \frac{1 - aq^{3k}}{1 - a} \left( q^2, aq^2q/b, aq/b; q^2 \right)_k \left( aq, f/a, f/b, f/aq, af/bq^2 \right) = \frac{(aq, f/a, f/b, f/aq, af/bq^2)}{(aq, f/aq, af/bq^2, f/bq^2)}.
$$

(3.6.12)

By multiplying both sides of (3.6.11) by $(f/a; q)_\infty$ and then setting $f = ac$ we obtain Rahman's [1989] quadruple transformation formula

$$
\sum_{k=0}^{\infty} \frac{(a; q^2)_k (1 - aq^{3k}) (d, aq/d; q^2)_k (b, c, c/b; q)_k}{(q; q)_k (1 - a) (aq/d, d, q^2; q^2)_k} = \frac{(aq^2, b, c, c/b, q^2)}{(aq^2/b, c, q^2; q^2)}.
$$

(3.6.13)

Also, the case $d = q^n$ of (3.6.11) gives

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k}}{1 - ac} \left( aq^2, acq^2/b, acq^2/bq^2 \right)_k \left( acq^2/bq^2, f/aq^2/bq^2, f/aq^2/b \right) = \frac{(acq^2/bq^2, f/aq^2/bq^2, f/aq^2/b)}{(aq^2/bq^2, f/aq^2/b, f/aq^2)}.
$$

(3.6.14)

and the case $b = cq^{n+1}$ gives

$$
\sum_{k=0}^{n} \frac{1 - aq^{3k}}{1 - ac} \left( d, aq^2q/dq^2; q^2 \right)_k (a, cq^n, n; q, q^2) = \frac{(aq^2/dq^2, aq, dq^2; q^2)}{(aq^2/q^2, aq^2/dq^2, dq^2; q^2)}.
$$

(3.6.15)

for $n = 0, 1, \ldots$.

Similarly, the special case

$$
\sum_{k=0}^{n} \frac{(1 - aq^{4k})(1 - b/cq^2k)}{1 - ac} \left( a, b; q \right)_k (q^{-3n}, ac, aq^2/b, q^2) = \frac{(1 - b/cq^2k)}{(1 - ac)(1/c/bq^{3n})/n = 0, 1, 2, \ldots}
$$

(3.6.16)

of (3.6.16) is used in Gasper and Rahman [1989] to show that all cases in (3.6.12) have a $q$-analogue of the form

$$
\sum_{k=0}^{n} \frac{1 - acq^{3k}}{1 - ac} \left( aq^2, acq^2/b, dq^2, dq^2/bq^2 \right)_k (aq^2/bq^2, dq^2/bq^2, dq^2/b) = \frac{(aq^2/bq^2, dq^2/bq^2, dq^2/b)}{(aq^2/bq^2, dq^2/b, dq^2)}.
$$

(3.6.17)

and to derive the extension $10W_9(ac^2/cd; d, c, a^2b^2d, ac/cd, c^2/bd, cd/b^2, q^2)$
\[
(1-c)(1-ac/b)(1-acc^4/b)(1-cq^n/b) \\
= (1-acc^2/b)(1-acc^4/b) \\
= (1-ac)(1-c/b)(1-cq^2/b)(1-acq^4/b), \quad n = 0, 1, 2, \ldots, (3.8.20)
\]

of (3.6.16) was used in Gasper and Rahman [1989a] to derive the quartic transformation formula
\[
10W_9(a^2c^2/b, a^2d/b^2; q^2, ac/b, c, cq/b, c^2q/b, cq^2/b, cq^4/b; q, q^4)
\]
\[
= (1-c)(1-ac/b)(a^2b^2/cq^4, cq/b, q)/(ac^2q^4/b, ab^2c^2q^2q^4; q^4) \sum_{k=0}^{\infty} \frac{1-acc^4 q^{3k}}{1-ac q^k} (a/bc^2q^4/c^2q^2; q^2q^4; q^4)\]
\[
= \Phi \left\{ a_1, \ldots, a_r; c_1, \ldots, c_{r+1}; q, q_1, \ldots, q_m; z \right\} (3.9.1)
\]

series \(\sum_{n=0}^{\infty} v_n\) will be called a bibasic hypergeometric series in bases \(p\) and \(q\) if \(v_{n+1}/v_n\) is a rational function of \(p^n\) and \(q^n\), and \(p\) and \(q\) are independent. More generally, we shall call a series \(\sum_{n=0}^{\infty} v_n\) a multibasic (or \(m\)-basic) hypergeometric series in bases \(q_1, \ldots, q_m\) if \(v_{n+1}/v_n\) is a rational function of \(q_1^n, \ldots, q_m^n\), and \(q_1, \ldots, q_m\) are independent. Similarly a bilateral series \(\sum_{n=-\infty}^{\infty} v_n\) will be called a bilateral multibasic (or \(m\)-basic) hypergeometric series in bases \(q_1, \ldots, q_m\) if \(v_{n+1}/v_n\) is a rational function of \(q_1^n, \ldots, q_m^n\), and \(q_1, \ldots, q_m\) are independent. Multibasic series are sometimes called polybasic series.

Since a multibasic series in bases \(q_1, \ldots, q_m\) may contain products and quotients of \(g\)-shifted factorials \((a; q_1)_n\) with \(q\) replaced by \(q_1^k \cdots q_m^k\) where \(k_1, \ldots, k_m\) are arbitrary integers, the form of such a series could be quite complicated. Therefore, in working with multibasic series either the series are displayed explicitly or notations are employed which apply only to the series under consideration. For example, to shorten the displays of many of the formulas derived in §3.8 we employ the notation
\[
\Phi \left\{ a_1, \ldots, a_r; c_1, \ldots, c_{r+1}; q, q_1, \ldots, q_m; z \right\}
\]

\[
\sum_{n=0}^{\infty} \frac{1-acc^4 q^{3k}}{1-ac q^k} (a/bc^2q^4/c^2q^2; q^2q^4; q^4)\]
\[
= \Phi \left\{ a, b, c; c_1, \ldots, c_{r+1}; q_1, \ldots, q_m; z \right\} (3.8.21)
\]

When \(b = q^2/a\) the sum of the two \(10W_9\) series in (3.8.21) reduces to a sum of two \(9W_7\) series, which can be summed by (2.11.7) to obtain the quartic summation formula
\[
= \Phi \left\{ a, b, c; c_1, \ldots, c_{r+1}; q_1, \ldots, q_m; z \right\} (3.9.2)
\]

\[
\Phi \left\{ a : c_1; \ldots, c_m; q, q_1, \ldots, q_m; z \right\} (3.9.3)
\]

denote the series in (3.9.2).

If in (2.2.2) we set
\[
A_k = \frac{(a; q)_k}{(q^{-n}; q)_k} \sum_{j=1}^{m} (c_{j+1}, \ldots, c_{j+r}; q_j)_k (-1)^j q_j^k s_j \cdot r_j
\]

then we obtain the expansion
\[
\Phi \left\{ a, b, c; c_1, \ldots, c_{r+1}; q, q_1, \ldots, q_m; z \right\}
\]
\[
= \sum_{n=0}^{\infty} \frac{aq/b, aq/c; d_1, \ldots, d_{s_1}; \ldots, d_m, \ldots, d_{s_m}; q, q_1, \ldots, q_m; z}{(q, aq/b, aq/c; q)_n (-bcz)^n q^{-n}}
\]
\[
= \sum_{n=0}^{\infty} \frac{(aq/b, aq/c; d_1, \ldots, d_{s_1}; \ldots, d_m, \ldots, d_{s_m}; q, q_1, \ldots, q_m; z}{(q, aq/b, aq/c; q)_n (-bcz)^n q^{-n}}
\]
\[
= \sum_{n=0}^{\infty} \frac{(aq/b, aq/c; d_1, \ldots, d_{s_1}; \ldots, d_m, \ldots, d_{s_m}; q, q_1, \ldots, q_m; z}{(q, aq/b, aq/c; q)_n (-bcz)^n q^{-n}}
\]
\[
= \sum_{n=0}^{\infty} \frac{(aq/b, aq/c; d_1, \ldots, d_{s_1}; \ldots, d_m, \ldots, d_{s_m}; q, q_1, \ldots, q_m; z}{(q, aq/b, aq/c; q)_n (-bcz)^n q^{-n}}
\]
\[
= \sum_{n=0}^{\infty} \frac{(aq/b, aq/c; d_1, \ldots, d_{s_1}; \ldots, d_m, \ldots, d_{s_m}; q, q_1, \ldots, q_m; z}{(q, aq/b, aq/c; q)_n (-bcz)^n q^{-n}}
\]
Transformations of series

\[
\Phi \left[ a^2q^{\frac{a}{2}}, -aq^{2i+2} : a_1q^{i}, \ldots, a_rq^{i}, q^{-n} \right] = \Phi \left[ a^2q^{\frac{b}{2}}, -aq^{2j+2} : b_1q^{j}, \ldots, b_rq^{j}, b_{r+1}q^{j} \right] = q^2, q^2, \frac{b^2c^2}{a} 
\]

(3.10)

and

\[
\Phi \left[ a^2, -aq^2, b^2, c^2, d^2 : a_1, \ldots, a_r, q^{-n} \right] = \Phi \left[ a^2, -aq^2 / b^2, a^2q^2 / c^2, a^2q^2 / d^2 : b_1, \ldots, b_{r+1}q^2 \right] = q^2, q^2, q^2 
\]

(3.10)

respectively, where \( \lambda = qa^2 / bcd \).

Let us first consider the \( r = 1 \) case of (3.10.1). If we set \( a_1 = -aq \), \( b_1 = w, b_2 = -aq^{n+1} \), \( z = awq^{n+1} / b^2c^2 \) and apply (1.240), the series on the right side of (3.10.1) reduces to a very well-poised \( \psi_4 \) series in base \( q \), hence can be summed by (2.4.2). This gives the transformation formula

\[
\Phi \left[ a^2, -aq^2, b^2, c^2 : -aq^{n}/w, q^{-n} \right] = \Phi \left[ a^2, -aq^2 / b^2, a^2q^2 / c^2 : w, -aq^{n+1}q^2 \right] = q^2, q^2, \frac{awq^{n+2}}{b^2c^2} 
\]

(3.11)

Note that the above \( \psi_4 \) series is balanced and that the \( \Phi \) series on left side of (3.10.3) can be written as

\[
\psi_4 \left[ -aq^{n}/w, a, b, c, -c, aq / w, q^{-n}, q, awq^{n+2} / b^2c^2 \right] 
\]

(3.11)

Formula (3.10.2) is a \( q \)-analogue of Bailey [1935, 4.5(1)]. By reversing the sequence on both sides of (3.10.3) and relabelling the parameters, this formula can be written, as in Jain and Verma [1980], in the form

\[
\psi_4 \left[ a, qa^{\frac{-1}{2}}, -aq^{\frac{-1}{2}}, b, x, -x, y, -y, q^{-n}, q^{-n} \right] = \psi_4 \left[ a, qa^{\frac{-1}{2}}, -aq^{\frac{-1}{2}}, -aq^{2n+1} / b^2c^2, -aq^{2n+1} / b^2c^2, -aq^{2n+1} / b^2c^2, -aq^{2n+1} / b^2c^2 \right] 
\]

(3.11)

For a nonterminating extension of (3.10.4) see Jain and Verma [1982].
Since the \( s \phi_4 \) series on the right side of (3.10.3) is balanced, it can be summed by (1.7.2) whenever it reduces to a \( s \phi_2 \) series. Thus, we obtain the summation formulas:

\[
\Phi \left[ a^2, aq^2, -aq^2 : -aq/w, q^{-n} \right] = \frac{(w/a, -aq/^q)_{n}(wq^{-n-1}/a, aq^{-n}/w; q^2)_{n}}{(w, -q^2)_{n}(aq^{-n}/w, wq^{-n}/a; q^2)_{n}},
\]

(3.10.5)

\[
\Phi \left[ a^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(a^2 q^2/b^2, -aq^2/b^2 : b^2 q^{-n-1}, -aq^2/b^2 : b^2 q^{-n}/w; q^2)_{n}}{(a^2 q^2/b^2, -aq^2/b^2 : b^2 q^{-n-1}, -aq^2/b^2 : b^2 q^{-n}/w; q^2)_{n}},
\]

(3.10.6)

\[
\Phi \left[ a^2, aq^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(-aq, a^2 q^2/b^2 : b^2 q^{-n-1}, -aq^2/b^2 : b^2 q^{-n}/w; q^2)_{n}}{(-q, 1/b^2 : b^2 q^{-n}/w; q^2)_{n}},
\]

(3.10.7)


\[
\Phi \left[ a^2, aq^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(-aq, a^2 q^2/b^2 : b^2 q^{-n-1}, -aq^2/b^2 : b^2 q^{-n}/w; q^2)_{n}}{(-q, 1/b^2 : b^2 q^{-n}/w; q^2)_{n}},
\]

(3.10.8)

These are \( q \)-analogues of formulas 4.5(1.1) - 4.5(1.4) in Bailey [1935]. Since the series on the left sides of (3.10.5) and (3.10.6) can also be written as very-well-poised \( \phi_7 \) series in base \( q \), which are transformable to balanced \( 4 \phi_3 \) series by Watson's formula (2.5.1), formulas (3.10.5) and (3.10.6) are equivalent to the summation formulas

\[
\Phi \left[ a^2, aq^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(w/a, -aq/^q)_{n}(wq^{-n-1}/a, aq^{-n}/w; q^2)_{n}}{(w, -a^{-1} : q^2)_{n}(aq^{-n}/w, wq^{-n}/a; q^2)_{n}},
\]

(3.10.9)

and

\[
\Phi \left[ a^2, aq^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(1 + 1/b)(1 + a^2 q^n/b)(a/b^2, q^n/b, 1/b ; q)_{n}}{(1 + a^2 q^n/b)(1 + a^2 q^n/b)(a/b^2, q^n/b, 1/b ; q)_{n}},
\]

(3.10.10)

respectively, which are closer to what one would expect \( q \)-analogues of formulas 4.5(1.1) and 4.5(1.2) in Bailey [1935] to look like.

It is also of interest to note that if we set \( c^2 = aq \) in (3.10.3), rewrite the \( \Phi \) series on the left side as an \( s \phi_7 \) series in base \( q \) and transform it to a balanced \( 4 \phi_3 \) series, we obtain

\[
\Phi \left[ a^2, aq^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(w/a, -aq^2/b^2 ; q^2)_{n}(aq^2/b^2, aq^2/b^2 : aq^2/b^2 : aq^2/b^2 ; q^2)_{n}}{(w, -aq^2/b^2 ; q^2)_{n}(aq^2/b^2, aq^2/b^2 : aq^2/b^2 : aq^2/b^2 ; q^2)_{n}},
\]

(3.10.11)

which is a \( q \)-analogue of the \( c = (1 + a)/2 \) case of Bailey [1935, 4.5(1)]. Using (2.10.4), the left side of (3.10.11) can be transformed to give

\[
\Phi \left[ a^2, aq^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(w/a, -aq^2/b^2 ; q^2)_{n}(aq^2/b^2, aq^2/b^2 : aq^2/b^2 : aq^2/b^2 ; q^2)_{n}}{(w, -aq^2/b^2 ; q^2)_{n}(aq^2/b^2, aq^2/b^2 : aq^2/b^2 : aq^2/b^2 ; q^2)_{n}},
\]

(3.10.12)

This formula was first proved by Singh [1959] and more recently by Askey and Wilson [1985]. The latter authors also wrote it in the form

\[
\Phi \left[ a^2, aq^2, -aq^2, b^2 : -aq^2/b^2, q^{-n} \right] = \frac{(-aq, a^2 q^2/b^2 : b^2 q^{-n-1}, -aq^2/b^2 : b^2 q^{-n}/w; q^2)_{n}}{(-q, 1/b^2 : b^2 q^{-n}/w; q^2)_{n}},
\]

(3.10.13)

provided that both series terminate.

Now that we have the summation formulas (3.10.5)-(3.10.8), we can use them to produce additional transformation formulas. Set \( r = 3 \) and \( a_1 = -aq^2, a_3 = -aq^2, a_1 = -b_2 = a^3, b_4 = -aq^{n+1} \) in (3.10.1). The \( \Phi \) series on the right side can now be summed by (3.10.5) and this leads to the following \( q \)-analogue of Bailey [1935, 4.5(2)]

\[
\Phi \left[ a^2, aq^2, -aq^2, b^2, c^2 : -aq^2/w, q^{-n} \right] = \frac{(w/a, -aq^2/b, a^2 q^2/c^2 ; q^2)_{n}(aq^2/b, a^2 q^2/c^2 ; q^2)_{n}}{(w, -aq^2/b, a^2 q^2/c^2 ; q^2)_{n}(a^2 q^2/b, a^2 q^2/c^2 ; q^2)_{n}},
\]

(3.10.14)

Let us now turn to applications of (3.10.2). If we set \( r = 1, a_1 = -\lambda q^{n+1}, a_1 = -\lambda q^{n+1}, a_1 = -\lambda q^{n+1}, z = q \\ in (3.10.2), where \( \lambda = qa^2/bcd^2, \) then the \( \Phi \) series on the right side reduces to a balanced very-well-poised \( s \phi_7 \) series in base \( q \) which can be summed by Jackson's formula (2.6.2). Thus, we obtain the transformation formula

\[
\Phi \left[ a^2, -aq^2, b^2, c^2, d^2 : -a^2 q^{n+1} \right] = \frac{(a^2, -aq^2/b, a^2 q^2/c^2, a^2 q^2/d^2 ; q^2)_{n}}{(a^2 q^2/b, a^2 q^2/c^2, a^2 q^2/d^2 ; q^2)_{n}}.
\]

(3.10.15)
Exercises 3

3.1 Deduce from (3.10.13) that

\[ 3\phi_2 \begin{bmatrix} a^2, b^2, z \\ abq, -abq \end{bmatrix} : q, q = 3\phi_2 \begin{bmatrix} a^2, b^2, z \\ a^2b^2q, 0 \end{bmatrix} : q^2, q^2, \]

provided that both series terminate. Show that this formula is a \(q\)-analogue of Gauss' quadratic transformation formula (3.1.7) when the series terminate.

3.2 Using the sum

\[ 2\phi_1 \left( q^{-n}, q^{1-n}; qb^2, q^2, q^2 \right) = \frac{(b^2; q^2)_n}{(b; q)_n} \left( q^{-2} \right), \]

prove that

(i) \( 3\phi_2 \begin{bmatrix} a, b, -b \\ abq, ax \end{bmatrix} : q, q = (z; q)_\infty \frac{(z; q)_\infty}{(az; q)_\infty} 2\phi_1 \left( a, bq^2; q^2, q^2 \right), \) \( |z| < 1. \)

Use this formula to prove that

(ii) \( 3\phi_2 \begin{bmatrix} a^2, ab, -ab \\ a^2b^2, ax^2 \end{bmatrix} : q, q = \frac{(a^2x^2; q^2)_\infty}{(z, -a^2z; q)_\infty} 2\phi_2 \begin{bmatrix} a^2, b^2 \\ a^2b^2q, a^2x^2q^2 \end{bmatrix}, \) \( |z| < 1. \)

3.3 Show that

\[ 3\phi_2 \begin{bmatrix} a, q/a, z \\ c, q \end{bmatrix} = \frac{(-1, -qz/c; q)\infty}{(-q/c, -z; q)\infty} 3\phi_2 \begin{bmatrix} c/a, ac/q, z^2 \\ c^2, 0 \end{bmatrix} : q^2, q^2, \]

when the series terminate. This is a \(q\)-analogue of

\[ \frac{\psi}{a, 1 - a; c; z) = \left( 1 - z \right)^{c-1} \psi \left( (c - a)/2, (a + 1)/2; c; 4z(1 - z) \right) \]

when the series terminate.

3.4 Show that

\[ \sum_{k=0}^{n} \frac{(q^{-n}, b, -b; q^2)k}{(q, b^2; q)^k} q^{nk-\binom{k}{2}} \]

vanishes if \( n \) is an odd integer. Evaluate the sum when \( n \) is even. Hence, or otherwise, show that

\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a, c, q)_m(q^2; q^2)_m}{(q, q)_{m+n}(b^2; q_{m+n})} (-z)^m z^n \]

\[ = 4\phi_3 \left[ \begin{array}{c} a, ac, c \\ d, dq, q^2 \end{array} : q^2, q^2 \right], \] \( |z| < 1. \)

Deduce that

\[ 4\phi_3 \left[ \begin{array}{c} q^{-n}, q^{1-n}, a, q^2 \\ q^{-2}, d, dq \end{array} : q^2, q^2 \right] \]

\[ = \frac{(d/a; q)_n a^n 4\phi_3 \left[ \begin{array}{c} q^{-n}, a, -b \\ b^2, q^{-1-n}/d \end{array} : q, -q/d \right]}{(d, q)_n} \]

(Jain [1981])

3.5 By using Sears' summation formula (2.10.12) show that

\[ \sum_{r=0}^{\infty} \frac{(a, aq/e; q)_r}{(q, abq/e, acq/e; q)_r} A_r \]

\[ = \frac{(aq/e, bq/e, cq/e, abce/q; q)_\infty}{(q/e, abce/q, bcq/e; q)_\infty} \sum_{k=0}^{\infty} \frac{(aq^k; q)_r}{(q, aq; q^2; q)_k} \sum_{r=0}^{\infty} \frac{(aq^{k+1}; q)_r}{(q, abcq^{k+1}/e; q)_r} A_r \]

\[ + \frac{(a, b, c, abcq^2/e^2; q)_\infty}{(e/q, abq/e, acq/e, bcq/e; q)_\infty} \sum_{k=0}^{\infty} \frac{(aq^k; q)_r}{(q, aq^2; e^2; q)_k} q^k \sum_{r=0}^{\infty} \frac{(aq^{k+1}; q)_r}{(q, abcq^{k+2}/e^2; q)_r} A_r, \]
where \(a, b, c, e\) are arbitrary parameters such that \(e \neq q\), and \(\{A_n\}\) is an arbitrary sequence such that the infinite series on both sides converge absolutely.

3.6 Prove that

\[
\sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, e} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) = \frac{(q/e, aq/c, e, q/e, e; q, q)\infty}{(aq/c, e, q/e, e, q/e, e; q, q)\infty} \sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) \frac{q^k}{q^k, q^2}.
\]

provided \(|bcq/e| < 1\).

3.7 Prove that

\[
\sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) = \frac{(q/e, aq/c, e, q/e, e; q, q)\infty}{(aq/c, e, q/e, e, q/e, e; q, q)\infty} \sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) \frac{q^k}{q^k, q^2}.
\]

3.8 Assuming that \(|x| < 1\) and \(a/b \neq q^j\), \(j = 0, \pm 1, \pm 2, \ldots\), prove that

\[
\sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) = \frac{(b, c/a, ax/q; q)\infty}{(b/a, c, x, q; q)\infty} \sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) \frac{q^k}{q^k, q^2}.
\]

Show that this is a \(q\)-analogue of the formula

\[
\sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) = \frac{(b, c/a, ax/q; q)\infty}{(b/a, c, x, q; q)\infty} \sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) \frac{q^k}{q^k, q^2}.
\]

\[3.10\text{ Show that } \sum_{n=0}^{\infty} \frac{(\lambda^2/ab^2, q^2/ab^2; q, q)\infty}{(q^2 / q^2) / q^{2n+2} / q^{2n+2}} < 1.\]

\[3.11\text{ Derive Jackson's [1941] product formula}
\]

\[
\Phi \left[ \frac{q^{2n} - q^{2n+1} / w}{q^{2n} - q^{2n+1} / w} \right] = \frac{(-1, w/d; q)\infty}{(-1, w/d; q)\infty} \sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) \frac{q^k}{q^k, q^2}.
\]

\[3.12\text{ If } \lambda = a^2q^2 / b^2 c^2 d^2, \text{ show that}
\]

\[\Phi \left[ \frac{a^2 - a^2 q / b^2 c^2 d^2 ; q, q}{a^2 - a^2 q / b^2 c^2 d^2 ; q, q} \right] = \frac{(-a, a^2 q^2 / b^2, a^2 q^2 / c^2 ; q, q)\infty}{(-a, a^2 q^2 / b^2, a^2 q^2 / c^2 ; q, q)\infty} \sum_{k=0}^{\infty} \left( \frac{a, b, c, q}{e, aq/c, e, q/e, e; q, q} \frac{q^{2k}}{q^{2k} / q^{2k+2}} \right) \frac{q^k}{q^k, q^2}.
\]
3.14 Using (3.4.7) show that the $q$-Bessel function defined in Ex. 1.24 can be expressed as

\[ J^{(3)}_\nu(x; q) = \frac{(q^{\nu+1}; q)_\infty}{(q^{-\nu-1/4}; q)_\infty} \left( \frac{2}{x} \right)^\nu (q^{\nu+1/4}; q)_\infty \sum_{n=0}^{\infty} \left( -x^2 q^n/4 \right)_n (1 + x^2 q^{2n+\nu}/4)(-x^2/4;q)_n \left( -x^2 q^n/4 \right)_n q^{2n}. \]

3.15 Using (3.5.4) show that

\[ J^{(2)}_\nu(x; q) = \frac{1}{\Gamma(q+1)} \left( \frac{ix}{2}; q^6 \right)_\infty \left( \frac{x}{2(1-q)} \right)^\nu \right) \cdot \phi_1 \left[ \frac{q^{\nu+1/4}; q^6}{q^{\nu+1/2}; q^6}; \frac{ix}{2} \right], \quad |x| < 2. \]

3.16 Show that

\[ \phi_1 \left[ \frac{a, -qa^4, b, c, axzq}{aq/b, aq/c, q, axzq} \right] = \frac{(1 - xa^4)(aqzq; q)_\infty}{(aqzq; q)_\infty} \phi_4 \left[ \frac{a^4, -qa^4, (aqzq; q)^4, aq/bc; q, q, q}{aq/b, aq/c, q, axzq} \right] + \frac{(aqzbq; q, axzq; q, axzq; q)_\infty}{(aqzbq; q, axzq; q, axzq; q)} \phi_5 \left[ \frac{xaqzq, axzq, xqzq, xzq; q, axzq}{aq/b, aq/c, q, q, q, q} \right]. \]

3.17 If \( (cz/ab; q)_\infty (z; q^2)_\infty \) show that

\[ \phi_1 \left[ \frac{c, a/c, b/c; czq^2, x}{czq^2, x, czq^2, czq^2} \right] = \sum_{n=0}^{\infty} (c; q^2)_n a_n x^n. \]

(Singh [1959], Nassrallah [1982])

3.18 If \( (cqz/ab; q)_\infty (z; q^2)_\infty \) show that

\[ \phi_1 \left[ \frac{cq/a, c, qz/b; cq^2, z}{cq^2, c, qz^2, cz} \right] = \sum_{n=0}^{\infty} (c; q^2)_n a_n x^n. \]

(Singh [1959], Nassrallah [1982])

3.19 If \( (cz/ab; q)_\infty (z; q^2)_\infty \) show that

\[ \phi_1 \left[ \frac{cq/a, c, qz/b; cq^2, z}{cq^2, c, qz^2, cz} \right] \equiv \phi_1 \left[ \frac{a, b, c, q^2}{cz, cz} \right]. \]

(Singh [1959], Nassrallah [1982])

3.20 Prove that

\[ \sum_{k=0}^{\infty} \frac{1 - apkq^k}{1 - a} \left( \frac{ap}{q} \right)_k (bq^k; q)_k = 0 \]

when \( |ap|, |aq|, |bq| < 1 \), and extend this to the bibasic transformation formulas

\[ \sum_{k=0}^{\infty} \frac{1 - apkq^k}{1 - a} \left( \frac{ap}{q} \right)_k (b^q; b^p)_k = 0. \]

(Gasper [1989a])

3.21 Derive the quadbasic transformation formula

\[ \sum_{k=0}^{\infty} \frac{1 - apkq^k}{1 - a} \left( \frac{ap}{q} \right)_k (b^q; b^p)_k = \sum_{k=0}^{\infty} \frac{1 - apkq^k}{1 - a} \left( \frac{ap}{q} \right)_k (b^q; b^p)_k. \]

for \( n = 0, 1, \ldots. \) Use it to derive the mixed bibasic and hypergeometric transformation formula

\[ \sum_{k=0}^{\infty} \frac{1 - apkq^k}{1 - a} \left( \frac{ap}{q} \right)_k (a, b; c, d; q)_k. \]
which is equivalent to (3.7.14) and extends Srivastava [1984, (10)].
(Gasper [1989a])

3.23 Prove the following q-Lagrange inversion theorem:
If
\[ G_n(x) = \sum_{j=0}^{\infty} b_{jn} x^j, \]
where \( b_{jn} \) is as defined in (3.6.20), and if
\[ f(x) = \sum_{j=0}^{\infty} f_j x^j = \sum_{n=0}^{\infty} c_n G_n(x), \]
then
\[ f_j = \sum_{n=0}^{\infty} c_n b_{jn}, \]
and, vice versa,
\[ c_n = \sum_{j=0}^{n} a_{nj} f_j, \]
where \( a_{nj} \) is as defined in (3.6.19).
(Gasper [1989a])


3.25 Prove Gasper’s formula
\[
\sum_{n=0}^{\infty} \frac{(a^2 b^2 c^2 q^{-1}; q^2)_n (1 - a^2 b^2 c^2 q^{n-1}) (a b c d; a b c d^{-1}; q^2)_n (a^2 b^2 c^2 q_n)}{(q; q)_n (1 - a^2 b^2 c^2 q^{-1}) (abcd; abcd; q^2)_n (a^2 b^2 c^2 q^2; q^2)_n}
\]
\[
= \frac{(a^2 b^2 c^2 q, a^2 b^2 c^2 q^2, dq/abc, bcda/q, cdaq/b, dbaq/c; q^2)_\infty}{(q, b^2 c^2 q, a^2 b^2 q, abcd, adqc/ab, bdaj/abc; q^2)_\infty}
\]
\[
- \frac{(a^2 b^2 c^2 q, q^2 a^2 b^2 q, abeq, abdc, abcd^{-1}; q^2)_\infty}{(aq/abc, bcda/q, cdaq/b, dbaq/c; q^2)_n}
\]
\[
\cdot \sum_{n=0}^{\infty} (1 - d^2 q^{4n+2}) \frac{(bcda/q, a/bcq, dba/q, cqd; q^2)_n}{(adqc/ab, dbaq/c, adqc/ab; q^2)_n+1} \left( -\frac{d}{abc} \right)^n q^{(n+1)^2},
\]
(Rahman [1989d])

3.26 Show that
\[
\sum_{k=0}^{\infty} \frac{(abcq)_k (1 - abcq^{3k+1}) (d, q, d; q)_k (abq, bcq, caq; q^2)_k}{(q^2, q^2)_k (1 - abcq)/d, (abcq^2; q^2)_k (cq, ab, bq; q)_k}
\]
\[
= s \phi_7 \frac{abcq, q^2 \sqrt{abcq}, -q^2 \sqrt{abcq}, d, q, d, abq, bcq, caq}{\sqrt{abcq}, -\sqrt{abcq}, abcq^2/d, abcq^2 q^2, aq^2, bq^2, q^2, q^2}
\]
\[
+ \frac{(abq, abq, bcq, caq, d, q, d; q^2)_\infty}{(q^2, aq^2, bq^2, cq^2, abcq^2/d, abdq^2; q^2)_\infty (1 - aq)(1 - bq)(1 - cq)}
\]
\[
\cdot 4 \phi_3 \left[ q^2, abcq^2, dq^2/d, aq^2, bq^2, cq^2; aq^2, bq^2, cq^2 \right].
\]
3.27 Prove
\[
\sum_{n=0}^{\infty} \frac{(bcdq^{-2}; q^3)_n(1 - bcdq^{4n-2})(b, c, d; q)_n}{(q; q)_n(1 - bcdq^{-2})(cdq, bdq, bcd; q^3)_n} q^{n^2} 
= \frac{(bcdq, bdq, bcd; q^3)_\infty}{(q, cdq, bdq, bcd; q^3)_\infty}.
\]
(Rahman [1989d])

3.28 Show that
\[
\sum_{n=0}^{\infty} \frac{(bcdq^{-1}; q^3)_n(1 - bcdq^{4n-1})(b, c, d; q)_n}{(q; q)_n(1 - bcdq^{-1})(cdq^2, bdq^2, bcdq^2, q^3)_n} q^{n^2+n} 
= \frac{(bcdq^2, bdq^2, bcdq^2; q^3)_\infty}{(q^2, cdq^2, bdq^2, bcdq^2; q^3)_\infty}.
\]
(Gasper [1989a])

3.29 Derive the summation formulas:
(i)
\[
\sum_{k=0}^{\infty} \frac{(1 - aq^{5k})(a, b, q^2; q)_k(ab^2/q^2; q^2)_k(q^2/b; q)_k}{(1 - a)(q^2, ab^2/q^2; q^2)_k(ab, ab^2q; q^2)_k} \left( - \frac{q}{b} \right)^k q^{(k+1)/2} 
= \frac{(aq^2, aq^2/b; q^2)_\infty(ab, ab^2q; q^2)_\infty}{(aq^2/b^2; q^2)_\infty(q^3, aq^2/b; q^2)_\infty}.
\]
(ii)
\[
\sum_{k=0}^{\infty} \frac{(1 - acq^{4k})(a, q^4; q^2)_k(q^2/ac; q^2)_k(ac/q; q)_k(a\sqrt{a}/a; q^2)_k}{(1 - a)(cq^3, a^2cq/q^2; q^2)_k(a^2c^2/q^2; q^2)_k} \left( - \frac{ac}{q^2} \right)^k q^{(k+1)/2} 
= \frac{(acq^2, acq^2/q^2)_\infty(q^2, acq^2/q^2, ac^2q, a^2c^2/q^2, a^4q^2/a; q^2)_\infty}{(q^2, a^2c^2/q^2; q^2)_\infty(cq^3, acq^2, a^2cq^2, a^2c^2q, ac, acq^2; q^2)_\infty}.
\]
(iii)
\[
\sum_{k=0}^{\infty} \frac{(1 - acq^{5k})(a, q^4/a; q^2)_k(q/ac; q^2)_k(ac/q; q)_k(a^2c^2/q^2; q^2)_k}{(1 - a)(cq^3, acq^2; q^2)_k(a^2c^2q; q^2)_k(a^2c^2q/q^2; q^2)_k} \left( -ac \right)^k q^{(k+1)/2} 
= \frac{(ac^2, acq^2; q^2)_\infty(q^2, a^2c^2q, acq^2, a^2c^2q, a^2c^2q/a; q^2)_\infty}{(q^2, a^2c^2q; q^2)_\infty(cq^3, acq^2, a^2cq^2, a^2c^2q, ac, acq^2, a^2c^2q; q^2)_\infty}.
\]
(Rahman [1989e])

3.30 Derive the quartic summation formula
\[
\sum_{k=0}^{\infty} \frac{1 - aq^{5k}(a, b; q)_k(q/b, q^2/b, q^3/b, q^2q^3)_k(a^2b^2/q^2; q^2)_k}{(1 - a)(q^4, aq^4/b; q^2)_k(abq, ab/b; q)_k(q^3/b; q^2)_k} \left( - \frac{a^2b^2}{q^2} \right)^k q^{k/2} 
+ \frac{ab^3}{q^2} \frac{(aq, bq, 1/b; q)_\infty(a^2b^2; q^2)_\infty}{(ab, q^3/a^2b^2; q_\infty(ab, q^2; q_\infty)^2(ab, q^4/b^2, q^3; q_\infty)} 
\cdot \phi_1 \left[ a^2b^2/q^2, a^2b^2q^2; q^4, a^2b^2q^2 \right] 
= \frac{(aq, ab^2q^2; q\infty)}{(ab, q^2\infty(ab/q; q)_\infty)^2(ab, q^4/b^2, q^3; q\infty)}.
\]
(Gasper [1989a])

3.31 Derive the cubic transformation formulas
\[
(i) \sum_{k=0}^{\infty} \frac{1 - acq^{4k}}{1 - ac} \frac{(a, b; q)_k(cq^3, acq^2b; q^2)_k(ab, q^2)_k(q^3/b; q^2)_k}{(aq^3, acq^2b; q^2)_k(abq, ab^2q; q^2)_k(q^3/b; q^2)_k} q^{k/3} 
= \frac{1 - acq^2}{1 - ac} \left( 1 - ab/q^2 \right) \left( 1 - abq/n \right) \left( 1 - acq^3/n^2 \right),
\]
\[
(ii) \sum_{k=0}^{\infty} \frac{1 - acq^{4k}}{1 - ac} \frac{(a, b; q)_k(acq^3, acq^2b; q^2)_k(ab, q^2)_k}{(aq, bgq; q)_k(cq^3, acq^2b; q^2)_k(abq, bcdq; q^2)_k} q^{k/3} 
= \frac{1 - acq^2}{1 - ac} (a, b; q)_k \left( acq^3, acq^2b; q^2 \right)_n q^{k/3} 
\cdot 10W_9(a/b, q^2a^2, c, a, bc, q^{n+1}, q^{n+2}, q, a^{n+3}, q^3, q^3),
\]
where \( n = 0, 1, \ldots \) (Gasper and Rahman [1989a])

3.32 Derive the cubic summation formula
\[
\sum_{k=0}^{\infty} \frac{1 - a^2q^{4k}}{1 - a^2} \frac{(b, q^2/b; q)_k(a^2; q^2; q^2)_k}{(a^2q^2/b, b^2q^2; q^2)_k(a^2q^2/b^2; q^2)_k} q^{k/3} 
= \frac{1 - a^2q^2/b, q^2/b^2; c, a^2; q^2/b^2; q^2, a^2q^2/b^2; q^2}_\infty \left( b^2q^2, c, q^2, b^2q^2, c, a^2q^2/b^2, q^2 \right)_\infty 
\cdot \frac{(q^2, c, q^2, b^2q^2, c, a^2q^2/b^2, q^2)_\infty}{(q^2, c, q^2, b^2q^2, c, a^2q^2/b^2, q^2)_\infty} \left[ b^2q^2, c, q^2, a^2q^2/b^2, q^2 \right]^2 q^{k/2} 
\cdot \left( a^2bq, bc/b^2q^2 \right)_\infty 2\phi_1 \left[ b^2q^2, c, q^2, a^2q^2/b^2, q^2 \right] q^{k/2}.
and show that it has the $q \to 1^-$ limit case

$$
\tau F_6 \left[ \begin{array}{c}
\frac{1}{2}, a - 1/2, b, c, (2a + 2 - 3c)/3, a/2 + 1 \\
3/2, (2a - b + 3)/3, (2a + b + 1)/3, 3c - 1, 2a + 1 - 3c, a/2 - 1
\end{array} \right] \\
= \frac{\Gamma \left( \frac{1}{2} \right) \Gamma \left( \frac{3}{3} \right) \Gamma \left( c - \frac{1}{2} \right) \Gamma \left( c + \frac{1}{2} \right) \Gamma \left( \frac{2a - b - 1}{3} \right)}{
\Gamma \left( \frac{1 + \frac{1}{2}}{3} \right) \Gamma \left( \frac{4a - 6c + 2}{3} \right) \Gamma \left( \frac{3a - 2c - 1}{3} \right) \Gamma \left( \frac{2a - 1}{3} \right) \Gamma \left( \frac{2a - b - 1}{3} \right) \Gamma \left( \frac{2a - 1}{3} \right)
}
.
$$

(Gasper and Rahman [1989a])

### 3.33

Derive the quartic transformation formula

$$
\sum_{k=0}^{\infty} \frac{1 - a^2 b^2 q^{2k-2}}{1 - a^2 b^2 q^2} \frac{(a, b; q)_k (ab/q, a, b; q^2)_k (a^2 b^2 q^2; q^4)_k}{(q, q)_k (abq, ab; q^2)_k (a^2 b^2 q^2; q^4)_k} = (aq, b; q)_\infty (abq; q^2)_\infty (a^2 b^2 q^2; q^4)_\infty 1^{101} \left[ \begin{array}{c}
aq^4 q^4, b q^4 \\
\end{array} \right]
$$

(Gasper and Rahman [1989a])

### Notes 3

§3.4 Bressoud [1987] contains some transformation formulas for terminating $r+1 \varphi_r(a_1, a_2, \ldots, a_{r+1}; b_1, \ldots, b_r; q; z)$ series that are almost poised in the sense that $b_k a_{k+1} = a_k q^k$ with $\delta_k = 0, 1$ or 2 for $1 \leq k \leq r$. Transformations for level basic series, that is, $r+1 \varphi_r$ series in which $a_1 b_k = q a_{k+1}$ for $1 \leq k \leq r$, are considered in Gasper [1985].

§3.6 Agarwal and Verma [1967a, b] derived transformation formulas for certain sums of bibasic series by applying the theorem of residues to contour integrals of the form (4.9.2) considered in Chapter 4. Inversion formulas are also considered in Carlitz [1973] and, in connection with the Bailey lattice, in A.K. Agarwal, Andrews and Bressoud [1987].

§3.7 Jackson [1928] applied his $q$-analogue of the Euler's transformation formula (the $p = q$ case of (3.7.11)) to the derivation of transformation formulas and theta function series. Jackson [1942, 1944] and Jain [1980a] also derived $q$-analogues of some of the double hypergeometric function expansions in Burchall and Chaundy [1940, 1941].

§3.8 Gosper [1988a] stated a strange $q$-series transformation formula containing bases $q^2, q^2$, and $q^6$. Krattenthaler [1989c] independently derived the terminating case of (3.8.18) and terminating special cases of some of the other summation formulas in this section.

§3.9 For multibasic series containing bibasic shifted factorials of the form $(a; p, q)_n = \prod_{j=0}^{n-1} (1 - ap^j q^k)$ and connections with Schur functions and permutation statistics, see Désarménien and Foata [1985–1988].


Ex. 3.161 $q$-Differential equations for certain products of basic hypergeometric series are considered in Jackson [1911].
4

**BASIC CONTOUR INTEGRALS**

### 4.1 Introduction

Our first objective in this chapter is to give q-analogues of Barnes' [1908] contour integral representation for the hypergeometric function

\[ _2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(a) \Gamma(b)} \int_{-\infty}^{\infty} \frac{\Gamma(a + s) \Gamma(b + s) \Gamma(-s)}{\Gamma(c + s)} (-z)^s \, ds, \quad (4.1.1) \]

where \(|\text{arg}(-z)| < \pi\), Barnes' [1908] first lemma

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(a + s) \Gamma(b + s) \Gamma(c - s) \Gamma(d - s)}{\Gamma(a + c) \Gamma(a + d) \Gamma(b + c) \Gamma(b + d)} \, ds \\
= \frac{\Gamma(a + c) \Gamma(a + d) \Gamma(b + c) \Gamma(b + d)}{\Gamma(a + b + c + d)}, \quad (4.1.2)
\]

and Barnes' [1910] second lemma

\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\Gamma(a + s) \Gamma(b + s) \Gamma(c + s) \Gamma(1 - d - s) \Gamma(-s)}{\Gamma(e + s) \Gamma(a) \Gamma(b) \Gamma(c) \Gamma(1 - d - a) \Gamma(1 - d - b) \Gamma(1 - d - c)} \, ds \\
= \frac{\Gamma(a) \Gamma(b) \Gamma(c) \Gamma(1 - d - a) \Gamma(1 - d - b) \Gamma(1 - d - c)}{\Gamma(e - a) \Gamma(e - b) \Gamma(e - c)}, \quad (4.1.3)
\]

where \(e = a + b + c + d + 1\).

In (4.1.1) the contour of integration is the imaginary axis directed upward with indentations, if necessary, to ensure that the poles of \(\Gamma(-s)\), i.e. \(s = 0, 1, 2, \ldots\), lie to the right of the contour and the poles of \(\Gamma(a + s) \Gamma(b + s)\), i.e. \(s = -a - n, -b - n\) with \(n = 0, 1, 2, \ldots\), lie to the left of the contour (as shown in Fig. 4.1 at the end of this section). The assumption that there exists such a contour excludes the possibility that \(a\) or \(b\) is zero or a negative integer. Similarly, in (4.1.2), (4.1.3) and the other contour integrals in this book it is assumed that the parameters are such that the contour of integration can be drawn separating the increasing and decreasing sequences of poles.

Barnes' first and second lemmas are integral analogues of Gauss' \(_2F_1\) summation formula (1.2.11) and Saalschütz's formula (1.7.1), respectively. In Askey and Roy [1986] it was pointed out that Barnes' first lemma is an extension of the beta integral (1.11.8). To see this, replace \(b\) by \(b + i\omega\), \(d\) by \(d + i\omega\) and then set \(s = \omega x\) in (4.1.2). Then let \(\omega \to \infty\) and use Stirling's formula to obtain the beta integral in the form

\[
\int_{-\infty}^{\infty} x^{a+c-1} (1-x)^{b+d-1} \, dx = B(a+c, b+d), \quad (4.1.4)
\]

where \(\text{Re}(a + c) > 0, \text{Re}(b + d) > 0\) and

\[
x_+ = \begin{cases} x & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \quad (4.1.5)
\]

It is for this reason that Askey and Roy call (4.1.2) Barnes' beta integral. Following Watson [1910], we shall give a \(q\)-analogue of (4.1.1) in §4.2, that is, a Barnes-type integral representation for \(_2\phi_1(a, b; c; q, z)\). It will be used in §4.3 to derive an analytic continuation formula for the \(_2\phi_1\) series. We shall give \(q\)-analogues of (4.1.2) and (4.1.3) in §4.4. The rest of the chapter will be devoted to generalizations of these integral representations, other types of basic contour integrals, and to the use of these integrals to derive general transformation formulas for basic hypergeometric series.
4.2 Watson’s contour integral representation for \( \mathbf{2\phi_1(a, b; c; q, z)} \) series

For the sake of simplicity we shall assume in this and the following five sections that \( 0 < q < 1 \) and write

\[ q = e^{-\omega}, \quad \omega > 0. \]  

(4.2.1)

This is not a severe restriction for most applications because the results derived for \( 0 < q < 1 \) can usually be extended to complex \( q \) in the unit disc by using analytic continuation. The restriction \( 0 < q < 1 \) has the advantage of simplifying the proofs by enabling one to use contours parallel to the imaginary axis. Extensions to complex \( q \) in the unit disc will be considered in §4.8.

For \( 0 < q < 1 \) Watson [1910] showed that Barnes’ contour integral in (4.1.1) has a \( q \)-analogue of the form

\[
\begin{align*}
2\phi_1(a, b; c; q, z) &= \left(\frac{a, b; q}{c, q}\right) \frac{(-1)}{2\pi i} \int_{-\infty}^{\infty} \frac{\left(q^{1+s}, cq^s; q\right)_{\infty}}{(aq^s, bq^s; q)^{\infty}} \frac{\pi(-z)^s}{\sin \pi s} \, ds,
\end{align*}
\]

(4.2.2)

where \( |z| < 1, |\arg(-z)| < \pi \). The contour of integration (denote it by \( C \)) runs from \(-\infty \) to \( \infty \) (in Watson’s paper the contour is taken in the opposite direction) so that the poles of \( \left(q^{1+s}, cq^s; q\right)_{\infty} / \sin \pi s \) lie to the right of the contour and the poles of \( 1/(aq^s, bq^s; q)^{\infty} \), i.e. \( s = \omega^{-1} \log a - n + 2\pi im\omega^{-1}, s = \omega^{-1} \log b - n + 2\pi im\omega^{-1} \) with \( n = 0, 1, 2, \ldots \), and \( m = 0, \pm 1, \pm 2, \ldots \), when \( a \) and \( b \) are not zero, lie to the left of the contour and are at least some \( e > 0 \) distance away from the contour.

To prove (4.2.2) first observe that by the triangle inequality,

\[ |1 - |a||e^{-\omega \Re(s)}| \leq |1 - aq^s| \leq 1 + |a||e^{-\omega \Re(s)}| \]

and so

\[
\frac{\left(q^{1+s}, cq^s; q\right)_{\infty}}{(aq^s, bq^s; q)_{\infty}} \leq \prod_{n=0}^{\infty} \frac{1 + e^{-(n+1+\Re(s))\omega}}{1 - |a||e^{-(n+\Re(s))\omega}|} \frac{1 + |c||e^{-(n+\Re(s))\omega}}{1 - |b||e^{-(n+\Re(s))\omega}|},
\]

(4.2.3)

which is bounded on \( C \). Hence the integral in (4.2.2) converges if \( \Re[s \log(-z) - \log(\sin \pi s)] < 0 \) on \( C \) for large \( |s| \), i.e. if \( |\arg(-z)| < \pi \).

Now consider the integral in (4.2.2) with \( C \) replaced by a contour \( C_R \) consisting of a large clockwise-oriented semicircle of radius \( R \) with center at the origin that lies to the right of \( C \), is terminated by \( C \) and is bounded away from the poles (as shown in Fig. 4.2).

![Fig. 4.2.](image)

Setting \( s = Re^{i\theta} \), we have for \( |z| < 1 \) that

\[
\Re \left[ \log \frac{(-z)^s}{\sin \pi s} \right] = R \cos \theta \log |z| - \sin \theta \arg(-z) - \pi |\sin \theta| + O(1)
\]

\[
\leq -R |\sin \theta \arg(-z) + \pi |\sin \theta| + O(1).
\]

Hence, when \( |z| < 1 \) and \( |\arg(-z)| < \pi - \delta, 0 < \delta < \pi \), we have

\[
\frac{(-z)^s}{\sin \pi s} = O[\exp(-\delta R |\sin \theta|)]
\]

(4.2.4)

on \( C_R \) as \( R \to \infty \), and it follows that the integral in (4.2.2) with \( C \) replaced by \( C_R \) tends to zero as \( R \to \infty \), under the above restrictions. Therefore, by applying Cauchy’s theorem to the closed contour consisting of \( C_R \) and that part of \( C \) terminated above and below by \( C_R \) and letting \( R \to \infty \), we obtain that \( -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \cdots ds \) equals the sum of the residues of the integrand at \( n = 0, 1, 2, \ldots \). Since

\[
\lim_{s \to n} \frac{(q^{1+s}, cq^s; q)_{\infty}}{(aq^s, bq^s; q)_{\infty}} \frac{\pi(-z)^s}{\sin \pi s} = \frac{(q^{1+n}, cq^n; q)_{\infty}}{(aq^n, bq^n; q)_{\infty}} z^n,
\]

this completes the proof of Watson’s formula (4.2.2).
4.3 Analytic continuation of $2\phi_1(a,b;c;q,z)$

Since the integral in (4.2.2) defines an analytic function of $z$ which is single-valued when $\arg(-z) < \pi$, the right side of (4.2.2) gives the analytic continuation of the function represented by the series $2\phi_1(a,b;c;q,z)$ when $|z| < 1$. As in the ordinary hypergeometric case, we shall denote this analytic continuation of $2\phi_1(a,b;c;q,z)$ to the domain $|\arg(-z)| < \pi$ again by $2\phi_1(a,b;c;q,z)$.

Barnes [1908] used (4.1.1) to show that if $|\arg(-z)| < \pi$ and $a,b,c,a-b$ are not integers, then the analytic continuation for $|z| > 1$ of the series which defines $2F_1(a,b;c;z)$ for $|z| < 1$ is given by the equation

$$2F_1(a,b;c;z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(z-a)^{-a}2F_1(a,1+a-c;1+a-b;z^{-1}) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(z-b)^{-b}2F_1(b,1+b-c;1+b-a;z^{-1}),$$

(4.3.1)

where, as elsewhere in this section, the symbol "-" is used in the sense "is the analytic continuation of". To illustrate the extension of Barnes' method to $2\phi_1$ series we shall now give Watson's [1910] derivation of the following $q$-analogue of (4.3.1):

$$2\phi_1(a,b;c;q,z) = \frac{(a,c/a;q)_{\infty}(az,q/az;q)_{\infty}}{(c,b/a;q)_{\infty}(z,q/z;q)_{\infty}}2\phi_1(a,aq/c;aq/b;q,cq/abz) + \frac{(a,c/b;q)_{\infty}(bz,q/bz;q)_{\infty}}{(c,a/b;q)_{\infty}(z,q/z;q)_{\infty}}2\phi_1(b,bq/c;bq/a;q,cq/abz),$$

(4.3.2)

provided that $|\arg(-z)| < \pi$, $c$ and $a/b$ are not integer powers of $q$, and $a,b,z \neq 0$.

First consider the integral

$$I_1 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+s},cq^{s};q)_{\infty}}{(aq^s,bq^{s};q)_{\infty}} \Gamma(1+s)(-z)^s ds$$

(4.3.3)

along three line-segments $A_1, A_2, B$, whose equations are:

$$A_1 : \text{Im}(s) = m_1, \quad A_2 : \text{Im}(s) = -m_2, \quad B : \text{Re}(s) = -M,$$

(4.3.4)

where $m_1, m_2, M$ are large positive constants chosen so that $A_1, A_2, B$ are at least a distance $\epsilon > 0$ away from each pole and zero of

$$g(s) = \frac{(q^{1+s},cq^{s};q)_{\infty}}{(aq^s,bq^{s};q)_{\infty}}$$

(4.3.5)

and it is assumed that $A_1, A_2, B$ are terminated by each other and by the contour of the integral in (4.2.2), i.e. $\text{Re}(s) = 0$ with suitable indentations.

From an asymptotic formula for $(a;q)_{\infty}$ with $q = e^{-\omega}$, $\omega > 0$, due to Littlewood [1907, §12], it follows that if $\text{Re}(s) \to -\infty$ with $|s - s_0| > \epsilon$ for some fixed $\epsilon > 0$ and any zero $s_0$ of $(q^s;q)_{\infty}$, then

$$\text{Re}\log(q^s;q)_{\infty} = \frac{\omega}{2} (\text{Re}(s))^2 + \frac{\omega}{2} \text{Re}(s) + O(1).$$

This implies that

$$g(s) = \frac{(ab,c/q)_{\infty}}{(a/b,cq)_{\infty}}^{\text{Re}(s)},$$

(4.3.7)

when $\text{Re}(s) \to -\infty$ with $s$ bounded away from the zeros and poles of $g(s)$. By using this asymptotic expansion and the method of §4.2 it can be shown that if $|z| > |cq/ab|$, then the value of the integral $I_1$ in (4.3.3) taken along the contours $A_1, A_2, B$ tends to zero as $m_1, m_2, M \to \infty$.

Hence, by Cauchy's theorem, the value of $I_1$, taken along the contour $C$ of §4.2, equals the sum of the residues of the integrand at its poles to the left of $C$ when $|z| > |cq/ab|$. Set $\alpha = -\omega^{-1} \log a, \beta = -\omega^{-1} \log b$ so that $a = q^\alpha, b = q^\beta$.

Since the residue of the integrand at $-\alpha - n + 2\pi inw^{-1}$ is

$$\frac{(a^{-1}q^{-n},ca^{-1}q^{-n},q^{n+1};q)_{\infty}}{(q,q,ba^{-1}q^{-n};q)_{\infty}} \pi \omega^{-1}(-z)^{-\alpha-n}q^{(n+1)/2}$$

$$\times \exp \{2m\pi i\omega^{-1}\log(-z)\} \csc (2m\pi i\omega^{-1} - \alpha),$$

we have

$$I_1 = \sum_{m=-\infty}^{\infty} \csc (2m\pi i\omega^{-1} - \alpha) \exp \{2m\pi i\omega^{-1}\log(-z)\}$$

$$\times \frac{\pi \omega^{-1}(c/a,q/a;q)_{\infty}}{(b/a,c,q)_{\infty}} (-z)^{-\alpha} 2\phi_1(a,aq/c;aq/b;q,cq/abz)$$

+ idem $(a,b)$.

(4.3.8)

Thus it remains to evaluate the above sums over $m$ when $|\arg(-z)| < \pi$. Letting $c = b$ in (4.3.8) and using (4.2.2), we find that the analytic continuation of $2\phi_1(a,b;b;q,z)$ is

$$\sum_{m=-\infty}^{\infty} \csc (\alpha - 2m\pi i\omega^{-1}) \exp \{2m\pi i\omega^{-1}\log(-z)\}$$

$$\times \frac{\pi \omega^{-1}(a,q/a;q)_{\infty}}{(q,q)_{\infty}} (-z)^{-\alpha} 2\phi_1(a,aq/b;aq/b;q,aq/az).$$
Since, by the \( q \)-binomial theorem,
\[
\phi_1(a, b; q, z) = \frac{(az; q)_\infty}{(z; q)_\infty}
\]
and the products converge for all values of \( z \), it follows that
\[
\sum_{m=\infty}^{\infty} \frac{\csc \left( \alpha \pi - 2m \pi^2 iw^{-1} \right) \exp \left\{ 2m \pi i w^{-1} \log(-z) \right\}}{\pi(a, q/a, z, q/z; q)_\infty} \cdot (-z)^{-\alpha}
= \frac{\omega(q, q, az, q/az; q)_\infty}{\pi(a, q/a, z, q/z; q)_\infty} (z; q)_\infty \quad (4.3.9)
\]
Using (4.3.9) in (4.3.8) we finally obtain (4.3.2).

### 4.4 \( q \)-Analogues of Barnes' first and second lemmas

Assume, as before, that \( 0 < q < 1 \), and consider the integral
\[
I_2 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1-c+i}, q^{1-d-i}; q)_\infty}{(q^{a+i}, q^{b+i}; q)_\infty} \frac{\pi q^c ds}{\sin \pi(c-s) \sin \pi(d-s)}, \quad (4.4.1)
\]
where, as usual, the contour of integration runs from \(-\infty\) to \( \infty \) so that the increasing sequences of poles of the integrand (i.e. \( c+n, d+n \) with \( n = 0, 1, 2, \ldots \)) lie to the right and the decreasing sequences of poles (i.e. the zeros of \( (q^{a+i}, q^{b+i}; q)_\infty \)) lie to the left of the contour. By using Cauchy's theorem as in §4.2 to evaluate this integral as the sum of the residues at the poles \( c+n, d+n \) with \( n = 0, 1, 2, \ldots \), we find that
\[
I_2 = \frac{\pi q^c}{\sin \pi(c-d)} \frac{(q, q^{1-c+d}; q)_\infty}{(q^{c+d}, q^{b+c}; q)_\infty} \phi_1(q^{a+c}, q^{b+c}; q^{1+c-d}; q, q;
+ \text{idem} (c; d)). \quad (4.4.2)
\]
Applying the formula (2.10.13) to (4.4.2), we get
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1-c+i}, q^{1-d-i}; q)_\infty}{(q^{a+i}, q^{b+i}; q)_\infty} \frac{\pi q^c ds}{\sin \pi(c-s) \sin \pi(d-s)} = \frac{q^c}{\sin \pi(c-d)} \frac{(q, q^{1+c-d}, q^{d-c}, q^{a+b+c+d}; q)_\infty}{(q^{a+c}, q^{b+d}, q^{c+a}, q^{d+b+c+d}; q)_\infty}, \quad (4.4.3)
\]
which is Watson's [1910] \( q \)-analogue of Barnes' first lemma (4.1.2), as can be seen by rewriting it in terms of \( q \)-gamma functions.

A \( q \)-analogue of Barnes' second lemma (4.1.3) can be derived by proceeding as in Agarwal [1933]. Set \( c = n \) and \( d = c - a - b \) in (4.4.3) to obtain
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1-c+i}, q^{1-d-i}; q)_\infty}{(q^{a+i}, q^{b+i}; q)_\infty} \frac{\pi q^c ds}{\sin \pi s \sin \pi(c-a-b-s)} = \csc \pi(c-a-b) \frac{(q^{1+a+b-c}, q^{c-a-b}, q^{d-c}, q^{a+b}; q)_\infty}{(q^a, q^b, q^{c+a-b}, q^{d-c}; q)_\infty} \cdot (-1)^n \frac{(q^a, q^b; q)_n}{(q^c; q)_n} n^{n(c-a-b)-1} (2), \quad (4.4.4)
\]

for \( n = 0, 1, 2, \ldots \) Next, replace \( c \) by \( d \) in (4.4.4), multiply both sides by \((-1)^n q^{n(c-a-b)+\binom{n}{2}} (q; q)_n (q, q; q)_n \), sum over \( n \) and change the order of integration and summation (which is easily justified if \(|q^{c-a-b}| < 1\) to obtain
\[
\csc \pi(d-a-b) \left\{ \left( \frac{q^{1+a+b-c}, q^{d-a-b}, q^{d-c}, q^{a+b}; q)_\infty}{(q^a, q^b, q^{c+a-b}, q^{d-c}; q)_\infty} \right) \cdot 2 \phi_2(q^a, q^b, q^c, q^d, q^{a+b}, q^{a+b+c}; q)_\infty \right\}
= 2 \pi i \int_{-\infty}^{\infty} \frac{(q^{1+s}, q^{1-d-s}; q)_\infty}{(q^{a+s}, q^{b+s}; q)_\infty} \frac{\pi q^d ds}{\sin \pi s \sin \pi(d-a-b-s)}
\]
\[
= \frac{(q^a, q^b, q^{c+a-b}, q^{d-c}; q)_\infty}{(q^a, q^b, q^{1+a+b-c}, q^{d-c}; q)_\infty} \pi q^d ds
= \frac{2 \pi i}{\sin \pi s \sin \pi(d-a-b-s)} \left\{ \left( \frac{q^{1+a+b-c}, q^{d-a-b}, q^{d-c}, q^{a+b}; q)_\infty}{(q^a, q^b, q^{c+a-b}, q^{d-c}; q)_\infty} \right) \cdot 2 \phi_2(q^a, q^b, q^c, q^d, q^{a+b}, q^{a+b+c}; q)_\infty \right\}
= \frac{(q^{a+c}, q^{a+c+d}; q)_\infty}{(q^a, q^b, q^{c+d}, q^{1+c+d}; q)_\infty} \pi q^d ds
= \frac{2 \pi i}{\sin \pi s \sin \pi(d-a-b-s)} \left\{ \left( \frac{q^{1+s}, q^{1-d-s}; q)_\infty}{(q^{a+s}, q^{b+s}; q)_\infty} \frac{(q^a, q^b, q^{c+a-b}, q^{d-c}; q)_\infty}{(q^a, q^b, q^{c+a-b}, q^{d-c}; q)_\infty} \pi q^d ds \right\}, \quad (4.4.6)
\]
where \( d + e = 1 + a + b + c \), which is Agarwal's \( q \)-analogue of Barnes' second lemma. This integral converges if \( q \) is so small that
\[
\Re \left\{ q \log q - \log(\sin \pi s \sin \pi(d+s)) \right\} < 0 \quad (4.4.7)
\]
on the contour for large \( |s| \).

### 4.5 Analytic continuation of \( r+1 \phi_r \) series

By employing Cauchy's theorem as in §4.2, we find that if \( |x| < 1 \) and \( |\arg(-x)| < \pi \), then
\[
r+1 \phi_r \left[ a_1, a_2, \ldots, a_{r+1} ; b_1, b_2, \ldots, b_r ; q, x \right] = \left( \frac{a_1, a_2, \ldots, a_{r+1} ; q}{b_1, b_2, \ldots, b_r ; q} \right)_\infty
\]
\[
= \left( \frac{a_1, a_2, \ldots, a_{r+1} ; q}{b_1, b_2, \ldots, b_r ; q} \right)_\infty
\]
\[
= (1) \int_{-\infty}^{\infty} \frac{(q^{1+s}, b_1 q^a, b_2 q^a, \ldots, b_r q^a ; q)_\infty}{(a_1 q^x, a_2 q^x, \ldots, a_{r+1} q^x ; q)_\infty} \frac{\pi(-x)^s ds}{\sin \pi s}, \quad (4.5.1)
\]
where, as before, only the poles of the integrand at \( 0, 1, 2, \ldots \), lie to the right of the contour. As in the \( r = 1 \) case, the right side of (4.5.1) gives the analytic continuation of the \( r+1 \phi_r \) series on the left side to the domain \( |\arg(-x)| < \pi \).
4.6 Contour integrals representing well-poised series

Let us replace $a, b, c, d$ and $e$ in (4.6.4) by $a + n, b + n, c + n, d + n$ and $e + 2n$, respectively, where

$$e = 1 + a + b + c - d,$$

and transform the integration variable $s$ to $s - n$, where $n$ is a non-negative integer. Then we get

$$\frac{1}{2\pi i} \int_{n-i\infty}^{n+i\infty} \frac{(q^{1+s-n}, q^{d+s}, q^{c+s+n}; q)_\infty}{(q^{1+s}, q^{d+s}, q^{c+s+n}; q)_\infty} \pi q^{-n} ds$$

$$= \csc \pi d \frac{(q^{a}, q^{b}, q^{c}, q^{d+e-a}, q^{c-e-b}, q^{a-b-c}; q)_\infty}{(q, q^{b}, q^{c}, q^{d-e-b}, q^{c-a-b}; q)_\infty} \pi q^{-n} ds.$$  

The limits of integration $n \pm i\infty$ can be replaced by $\pm i\infty$ because we always indent the contour of integration to separate the increasing and decreasing sequences of poles. Thus, it follows from (4.6.2) that

$$\begin{aligned}
\csc(\pi d) & \frac{(q^{a}, q^{b}, q^{c}, q^{d+e-a}, q^{c-e-b}, q^{a-b-c}; q)_\infty}{(q, q^{b}, q^{c}, q^{d-e-b}, q^{c-a-b}; q)_\infty} \pi q^{-n} ds
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+s}, q^{d+s}, q^{e+s}; q)_\infty}{(q^{1+s}, q^{d+s}, q^{e+s}; q)_\infty} \pi q^{(n+1)} ds
= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\pi q^{(n+1)} ds}{\sin \pi s \sin \pi (d + s)}.
\end{aligned}$$

where $B = 2 + 2A - a - b - c - d - e$, provided $Re B > 0$ and

$$Re \left[ s \log q - \log(\sin \pi s \sin \pi (a + b + c - A + s)) \right] < 0.$$
4.7 A contour integral analogue of Bailey's summation formula

By replacing $A, a, b, c, d, e$ in (4.6.5) by $a, d, e, f, b, c$, respectively, we obtain the formula

$$s W_7(q^{a} ; q^{b}, q^{c}, q^{d}, q^{e}, q^{f}; q, q) = \sin \pi (d + e + f - a)$$

$$\frac{(q; q^{a}, q^{b}, q^{c}, q^{d}, q^{e}, q^{f}; q, q)_{\infty}}{(q^{1+a-a}, q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+a-a}, q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}{\sin \pi s \sin \pi (d + e + f - a + s)} ds,$$

(4.7.1)

provided the series is balanced, i.e.,

$$1 + 2a = b + c + d + e + f.$$

(4.7.2)

Since $1 + 2(2b - a) = b + (b + c - a) + (b + d - a) + (b + e - a) + (b + f - a)$ and (4.7.2), it follows that (4.7.1) remains unchanged if we replace $a, c, d, e, f, b$ by $2b - a, b + c - a, b + d - a, b + e - a, b + f - a$, respectively, and keep $b$ unaltered. Then (4.7.1) gives

$$s W_7(q^{2b-a}; q^{b}, q^{c+b-c}, q^{d+b-d}, q^{e+b-e}, q^{f+b-f}, a, q) = \sin \pi c$$

$$\frac{(q^{1+2b-a}; q^{b+b-a}, q^{c+c-a}, q^{d+d-a}, q^{e+e-a}, q^{f+f-a}; q, q)_{\infty}}{(q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}{\sin \pi s \sin \pi (c - s)} ds,$$

(4.7.3)

where the second integral in (4.7.3) follows from the first by a change of the integration variable $s \rightarrow a - b + s$. Combining (4.7.1) and (4.7.3) and simplifying, we obtain

$$s W_7(q^{a} ; q^{b}, q^{c}, q^{d}, q^{e}, q^{f}; q, q)_{\infty}$$

$$= q^{b-a} \frac{(q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}{(q^{1+2b-a}, q^{b+c-a}, q^{d+b-d}, q^{e+b-e}, q^{f+b-f}; q, q)_{\infty}}$$

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}{\sin \pi s \sin \pi (c + b - a - s)} ds.$$

(4.7.4)

 Extensions to complex $q$ inside the unit disc

$$\phi_1(a, b, c, q, z) = \frac{(q^{1+a-a}, q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}{(q^{a}, q^{b}, q^{c}, q^{d}, q^{e}, q^{f}; q, q)_{\infty}}$$

$$\pi q^{a} \int_{0}^{\infty} \frac{(q^{1+a-a}, q^{1+a-b}, q^{1+a-c}, q^{1+a-d}, q^{1+a-e}, q^{1+a-f}; q, q)_{\infty}}{\sin \pi s \sin \pi (a - b + s)} ds.$$

(4.7.5)

where (4.7.2) holds.

Evaluating the above integral via (4.4.6), we obtain Bailey's summation formula (2.11.7).

Agarwal's [1953b] formula

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+a-a}, q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}{\sin \pi s \sin \pi (a - b + s)} ds.$$

(4.7.6)

on the contour for large $|s|$.

4.8 Extensions to complex $q$ inside the unit disc

The previous basic contour integrals can be extended to complex $q$ inside the unit disc by using suitable contours. For $0 < |q| < 1$, let

$$\log q = -\omega_1 + i\omega_2,$$

(4.8.1)

where $\omega_1 = -\log|q| > 0$ and $\omega_2 = -\text{Arg } q$.

Thus $q = e^{-\omega_1 + i\omega_2}$. Then a modification of the proof in §4.2 (see Watson [1910]) shows that if $0 < |q| < 1$ and $|z| < 1$, then formula (4.2.2) extends to

$$\phi_1(a, b, c; q, z) = \frac{(q^{1+a-a}, q^{1+b-b}, q^{1+c-c}, q^{1+d-d}, q^{1+e-e}, q^{1+f-f}; q, q)_{\infty}}{(q^{a}, q^{b}, q^{c}, q^{d}, q^{e}, q^{f}; q, q)_{\infty}}$$

$$\frac{1}{2\pi i} \int_{C} \frac{(q^{1+a-a}, q^{1+a-b}, q^{1+a-c}, q^{1+a-d}, q^{1+a-e}, q^{1+a-f}; q, q)_{\infty}}{\sin \pi s \sin \pi (a - b + s)} ds.$$

(4.8.2)
Basic Contour Integrals

where $C$ is an upward directed contour parallel to the line $\text{Re}(s(\omega_1 + i\omega_2)) = 0$ with indentations, to ensure that the increasing sequence of poles $0, 1, 2, \ldots$, of the integrand lie to the right, and the decreasing sequences of poles lie to the left of $C$.

Since the above integral converges if $\text{Re}[\log(-z) - \log(\sin \pi s)] < 0$ on $C$ for large $|z|$, i.e., if

$$|\arg(-z) - \omega_2 \omega_1^{-1} \log |z|| < \pi,$$

(4.8.3)

it is required that $z$ satisfies (4.8.3) in order for (4.8.2) to hold. This restriction means that the $z$-plane has a cut in the form of the spiral whose equation in polar coordinates is $r = e^{\omega_1 \theta/\omega_2}$.

Analogously, when $0 < |q| < 1$, the contours in the $q$-analogues of Barnes’ first and second lemmas given in §4.4 and the contours in the other integrals in §§4.4-4.7 must be replaced by upward directed contours parallel to the line $\text{Re}(s(\omega_1 + i\omega_2)) = 0$ with indentations to separate the increasing and decreasing sequences of poles.

4.9 Other types of basic contour integrals

Let $q = e^{-\omega}$ with $\omega > 0$ and suppose that

$$P(z) = \frac{(a_1 z, \ldots, a_A z, b_1/z, \ldots, b_B/z; q)_\infty}{(c_1 z, \ldots, c_C z, d_1/z, \ldots, d_D/z; q)_\infty}$$

(4.9.1)

has only simple poles. During the 1950’s Slater [1952c,d, 1955] considered contour integrals of the form

$$I_m \equiv I_m(A, B; C, D) = \frac{\omega}{2\pi i} \int_{-\infty}^{\infty} P(q^s)q^{ms} ds$$

(4.9.2)

with $m = 0$ or 1. However, here we shall let $m$ be an arbitrary integer. It is assumed that none of the poles of $P(q^s)$ lie on the lines $\text{Im} s = \pm \pi/\omega$ and that the contour of integration runs from $-i\pi/\omega$ to $i\pi/\omega$ and separates the increasing sequences of poles in $|\text{Im} s| < \pi/\omega$ from those that are decreasing.

By setting $i\theta = -\omega$ the integral $I_m$ can also be written in the “exponential” form

$$I_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(e^{i\theta}) e^{im\theta} d\theta$$

(4.9.3)

with suitable indentations, if necessary, in the contour of integration. Similarly, setting $z = q^s$ we obtain that

$$I_m = \frac{1}{2\pi i} \int_{K} P(z)z^{m-1} dz,$$

(4.9.4)

where the contour $K$ is a deformation of the (positively oriented) unit circle so that the poles of $1/(c_1 z, \ldots, c_C z; q)_\infty$ lie outside the contour and the origin and poles of $1/(d_1/z, \ldots, d_D/z; q)_\infty$ lie inside the contour. Special cases of (4.9.3) and (4.9.4) have been considered by Askey and Roy [1986].

Although each of the above three types of integrals can be used to derive transformation formulas for basic hypergeometric series, we shall prefer to mainly use the contour integrals of the type in (4.9.4) since they are easier to work with, especially when the assumption in Slater [1952c,d, 1966] that $0 < q < 1$ is replaced by only assuming that $|q| < 1$, which is the case we wish to consider in the remainder of this chapter.

4.10 General basic contour integral formulas

Our main objective in this section is to see what formulas can be derived by applying Cauchy’s theorem to the integrals $I_m$ in (4.9.4).

Let $|q| < 1$ and let $\delta$ be a positive number such that $\delta \neq |d_j q^n|$ for $j = 1, 2, \ldots, D,$ and $\delta \neq |c_j q^{-n}|$ for $j = 1, 2, \ldots, C$. Also let $C_N$ be the circle $|z| = \delta|q|^N$, where $N$ is a positive integer. Then $C_N$ does not pass through any of the poles of $P(z)$ and we have that

$$|P(q^N) (q^{-N})^{-1}| = \frac{|(a_1, \ldots, a_A b_1, \ldots, b_B; q)_\infty|}{|(c_1, \ldots, c_C b_1, \ldots, b_B; q)_\infty|}$$

$$= \frac{|(c_1, \ldots, c_C b_1, \ldots, b_B; q)_N (b_1 \ldots b_B q^{-m-1})^N}{|(a_1, \ldots, a_A b_1, \ldots, b_B; q)_N (b_1 \ldots b_B q^{m-1})^N|}$$

$$\cdot \delta^{m-1} |\delta^{N} q^{(N-1)/2} D-B = O \left( \frac{|b_1 \ldots b_B q^{m-1}|^N}{D_1 \ldots D_D} \right)^{D-B} \right) .$$

(4.10.1)

Since $C_N$ is of length $O(|q|^N)$ it follows from (4.10.1) that if $D > B$ or if $D = B$ and

$$\frac{|b_1 \ldots b_B q^{m-1}|^N}{D_1 \ldots D_D} < 1,$$

(4.10.2)

then

$$\lim_{N \to \infty} \int_{C_N} P(z)z^{m-1} dz = 0.$$  

(4.10.3)

Hence, by applying Cauchy’s residue theorem to the region between $K$ and $C_N$ for sufficiently large $N$ and letting $N \to \infty$, we find that if $D > B$ or if $D = B$ and (4.10.2) holds, then $I_m$ equals the sum of the residues of $P(z)z^{m-1}$ at the poles of $1/(d_1/z, \ldots, d_D/z; q)_\infty$. Therefore, since

$$\text{Residue at } z = \frac{1}{(d_z; q)_\infty} = (-1)^n q^n \binom{n}{2} (q; q)_n (q; q)_\infty, \quad n = 0, 1, 2, \ldots, $$

(4.10.4)

it follows that

$$I_m = \frac{1}{2\pi i} \int_{K} P(z)z^{m-1} dz,$$

(4.10.5)
if $D > B$, or if $D = B$ and (4.10.2) holds.

In addition, by considering the residues of $P(z)z^{m-1}$ outside of $K$ or by
just using the inversion $z \to z^{-1}$ and renaming the parameters, we obtain

\[
I_m = \frac{(b_{1}c_{1},, b_{B}c_{1}, a_{1}/c_{1},, a_{A}/c_{1}; q)_{\infty}}{(q, d_{1}c_{1},, d_{D}c_{1}, c_{2}/c_{1},, c_{C}/c_{1}; q)_{\infty}} c_{1}^{-m}
\sum_{n=0}^{\infty} \frac{(c_{1}q^{-1}a_{1},, c_{A}q^{-1}a_{A}/c_{A}; q)_{n}}{(q, b_{1}c_{1},, b_{B}c_{1}, c_{2}/c_{1},, c_{C}/c_{1}; q)_{n}}
\cdot \left(-c_{1}q^{(n+1)/2}\right)^{(C-A)} \left(\frac{a_{1}a_{A}q^{-m}/c_{1}^{m}}{c_{1}/c_{C}}\right)^{n}
+ \text{idem} (c_{1}, c_{2},, c_{C})
\]  

(4.10.6)

if $C > A$, or if $C = A$ and

\[
\frac{a_{1}a_{A}q^{-m}}{c_{1}/c_{C}} < 1.
\]  

(4.10.7)

In the special case when $C = A$ we can use the $\phi_{n}$ notation to write (4.10.5) in the form

\[
I_{m}(A, B; A, D) = \frac{(a_{1}d_{1},, a_{A}d_{1}, b_{1}/d_{1},, b_{B}/d_{1}; q)_{\infty}}{(q, c_{1}d_{1},, c_{A}d_{1}, d_{2}/d_{1},, d_{D}/d_{1}; q)_{\infty}} d_{1}^{-m}
\cdot \phi_{A+D-1}(c_{1}d_{1},, c_{A}d_{1}, qd_{1}/d_{1},, qd_{D}/d_{1}; q, (qd_{1})^{D-B})
+ \text{idem} (d_{1}, d_{2},, d_{D})
\]  

(4.10.8)

where $t = b_{1}b_{2}q^{n}/d_{1},, d_{D}$, if $D > B$, or if $D = B$ and (4.10.2) holds.

Similarly, from the $D = B$ case of (4.10.6) we have

\[
I_{m}(A, B; C, B) = \frac{(b_{1}c_{1},, b_{B}c_{1}, a_{1}/c_{1},, a_{A}/c_{1}; q)_{\infty}}{(q, d_{1}c_{1},, d_{B}c_{1}, c_{2}/c_{1},, c_{C}/c_{1}; q)_{\infty}} c_{1}^{-m}
\cdot \phi_{B+C-1}(d_{1}c_{1},, d_{B}c_{1}, qc_{1}/a_{1},, qc_{1}/c_{A}; q, u(qc_{1})^{C-A})
+ \text{idem} (c_{1}, c_{2},, c_{C})
\]  

(4.10.9)

if $C > A$, or if $C = A$ and (4.10.7) holds, where $u = a_{1}a_{A}q^{-m}/c_{1}/c_{C}$.

Evaluations of $I_{m}$ which follow from these formulas will be considered in
§4.11.

From (4.10.8) and (4.10.9) it follows that if $C = A$ and $D = B$, then we have the transformation formula

\[
\frac{(a_{1}d_{1},, a_{A}d_{1}, b_{1}/d_{1},, b_{B}/d_{1}; q)_{\infty}}{(c_{1}d_{1},, c_{A}d_{1}, d_{2}/d_{1},, d_{D}/d_{1}; q)_{\infty}} d_{1}^{-m}
\cdot \phi_{A+B}(d_{1}c_{1},, d_{D}c_{1}, qd_{1}/d_{1},, qd_{D}/d_{1}; q, b_{1}b_{B}q^{m})
+ \text{idem} (d_{1}, d_{2},, d_{B})
\]  

\[
= \frac{(b_{1}c_{1},, b_{B}c_{1}, a_{1}/c_{1},, a_{A}/c_{1}; q)_{\infty}}{(d_{1}c_{1},, d_{B}c_{1}, c_{2}/c_{1},, c_{C}/c_{1}; q)_{\infty}} c_{1}^{-m}
\cdot \left(-c_{1}q^{(n+1)/2}\right)^{(C-A)} \left(\frac{a_{1}a_{A}q^{-m}}{c_{1}/c_{C}}\right)^{n}
+ \text{idem} (c_{1}, c_{2},, c_{C})
\]  

(4.10.10)

provided that $|b_{1}b_{B}q^{m}| < |d_{1}d_{D}q|, |a_{1}a_{A}q^{-m}| < |c_{1}/c_{C}|$ and $m = 0, 1, 2, 3, . . . . . . . .

In some applications it is useful to have a variable $z$ in the argument of the series which is independent of the parameters in the series. This can be accomplished by replacing $A$ by $A + 1, B$ by $B + 1$ and setting $b_{B+1} = z$ and $a_{A+1} = q^{2}/z$ in (4.10.10). More generally, doing this to the $m = 0$ case of (4.10.5) and of (4.10.6) gives the rather general transformation formula

\[
\frac{(a_{1}d_{1},, a_{A}d_{1}, b_{1}/d_{1},, b_{B}/d_{1}, z/d_{1}, qd_{1}/z; q)_{\infty}}{(c_{1}d_{1},, c_{A}d_{1}, d_{2}/d_{1},, d_{D}/d_{1}; q)_{\infty}}
\cdot \sum_{n=0}^{\infty} \frac{(c_{1}d_{1},, c_{A}d_{1}, qd_{1}/b_{1},, qd_{1}/b_{B}; q)_{n}}{(q, a_{1}d_{1},, a_{A}d_{1}, d_{2}/d_{1},, d_{D}/d_{1}; q)_{n}}
\cdot \left(-d_{1}q^{(n+1)/2}\right)^{(D-B-1)} \left(\frac{b_{1}b_{B}z}{d_{1}}\right)^{n}
+ \text{idem} (d_{1}, d_{2},, d_{D})
\]  

(4.10.11)

where, for convergence,

(i) $D > B + 1$, or $D = B + 1$ and $|b_{1}b_{B}z| < 1$

and

(ii) $C > A + 1$, or $C = A + 1$ and $|a_{1}a_{A}q/z| < 1$.

This is formula (5.2.20) in Slater [1966]. Observe that by replacing $z$ in
(4.10.11) by $zq^{m}$ and using the identity

\[
(qd/q^{m}, zq^{m}/d; q)_{\infty} = (-1)^{m}(d/z)^{m}q^{-(7)}(qd/z, zd; q)_{\infty}
\]  

(4.10.12)

we obtain from (4.10.11) the formula that would have been derived by using
(4.10.5) and (4.10.6) with $m$ an arbitrary integer.
4.11 Some additional extensions of the beta integral

Askey and Roy [1986] used Ramanujan's summation formula (2.10.17) to show that

\[
\frac{1}{2\pi i} \int_{C} \frac{(ce^{i\theta}/\beta, qe^{i\theta}/\alpha, ce^{-i\theta}/q, qe^{-i\theta}/\beta; q)}{(ae^{i\theta}, be^{i\theta}, a^{-e^{-i\theta}}, be^{-i\theta}; q)} \, d\theta = \left(\frac{a_{0}b_{0}c_{0}d_{0} / q}{a_{0}b_{0}c_{0}d_{0} / \beta q}\right)_{\infty}, \tag{4.11.1}
\]

where \(\max\{\{|q|, |a|, |b|, |\alpha|, |\beta|\} < 1\) and \(\alpha \beta \neq 0\); and they extended it to the contour integral form

\[
\frac{1}{2\pi i} \int_{C} \frac{(cz/\beta, qz/\alpha, cza/\beta, z, qz/\alpha, cza/\beta; q)}{(az, bz, \alpha/\beta, z, \beta/\alpha, cza/\beta; q)} \, dz = \left(\frac{a_{0}b_{0}c_{0}d_{0} / q}{a_{0}b_{0}c_{0}d_{0} / \beta q}\right)_{\infty}, \tag{4.11.2}
\]

where \(a_{0}, a_{\beta}, b_{0}, b_{\beta} \neq q^{-n}, n = 0, 1, 2, \ldots, \alpha \beta \neq 0\), and \(K\) is a deformation of the unit circle as described in \(\S 4.9\). These formulas can also be derived from the \(A = B = D = 2, m = 0\) case of (4.10.8) by setting \(a_{0} = c/\beta, a_{2} = q/\alpha, b_{1} = b_{0}, b_{2} = b_{0}/\beta, c_{1} = a_{1}, c_{2} = c_{0}, d_{1} = d_{0} = \beta, d_{2} = \beta\) and then using the summation formula (2.10.13) for the sum of the two \(\varphi_{1}\) series resulting on the right side. In Askey and Roy [1986] it is also shown how Barnes' beta integral (4.2) can be obtained as a limit case of (4.11.1).

Analogously, application of the summation formula (2.10.11) to the \(A = 3, B = D = 2, m = 0\) case of (4.10.8) gives

\[
\frac{1}{2\pi i} \int_{K} \frac{(cz/\beta, qz/\alpha, cza/\beta, \gamma/z, q\alpha \beta /\gamma z; q)}{(az, bz, cz, \alpha/\beta, \gamma/\beta, z, \beta/\alpha, cza/\beta, \gamma/z; q)} \, dz = \left(\frac{a_{0}b_{0}c_{0}d_{0} / q}{a_{0}b_{0}c_{0}d_{0} / \beta q}\right)_{\infty}, \tag{4.11.3}
\]

where \(\delta = ab\alpha \beta, ab\alpha \beta \gamma \neq 0\), and

\[a_{0}, a_{\beta}, b_{0}, b_{\beta}, c_{0}, c_{\beta} \neq q^{-n}, n = 0, 1, 2, \ldots\]

Note that (4.11.2) follows from the \(c = 0\) case of (4.11.3).

In addition, application of Bailey's summation formula (2.11.7) gives the more general formula

\[
\frac{1}{2\pi i} \int_{K} \frac{(za_{1}^{+}, z^{-a_{1}^{+}}, qaz/\beta, qaz/\alpha, qaz/\beta, qaz/\alpha, qaz/\beta, qaz/\alpha; q)}{(aaz, a^{2}z, a/\beta, z, a/\beta, z, a/\beta, z; q)} \, dz = \left(\frac{a_{0}b_{0}c_{0}d_{0} / q}{a_{0}b_{0}c_{0}d_{0} / \beta q}\right)_{\infty}, \tag{4.11.4}
\]

where \(a_{q} = bcdf \alpha \beta, bcdf \alpha \beta \gamma \neq 0\),

\[a_{0}, a_{\beta}, b_{0}, b_{\beta}, c_{0}, c_{\beta}, d_{0}, d_{\beta}, e_{1}, d_{1}, e_{2}, d_{2} \neq q^{-n}, n = 0, 1, 2, \ldots, \]

and \(K\) is as described in \(\S 4.9\); see Gasper [1989c].

4.12 Sears' transformations of well-poised series

Sears [1951d, (7.2)] used series manipulations of well-poised series to derive the transformation formula

\[
\begin{align*}
\left(\frac{qa_{1}/a_{2}m+2, \ldots, qa_{1}/a_{2}m, q/a_{2}m+2, \ldots, q/a_{2}m, q^{2}/a_{2}, q^{2}/a_{2}, q^{2}/a_{2}; q, -x}{a_{1}, 1, a_{2}, 1, a_{2}m+1, a_{2}m+1, a_{2}m+1, a_{2}m+1; q}\right)_{\infty} \\

= \left(\frac{q^{2}/a_{2}m+2, \ldots, q^{2}/a_{2}m, q/a_{2}m+2, \ldots, q/a_{2}m+1, q^{2}/a_{2}m+1, q^{2}/a_{2}m+1}{a_{1}/a_{2}, q^{2}/a_{2}, q^{2}/a_{2}, a_{2}m+1, a_{2}m+1, a_{2}m+1, a_{2}m+1; q, 1-x}\right)_{\infty}
\end{align*}
\]

(4.12.1)

where \(q = (qa_{1})^{M}/a_{2}m+2 \ldots a_{2}m+2\). Slater [1952c] observed that this formula could also be derived from (4.10.10) by taking \(A = B = M + 1, m = 0\) and choosing the parameters such that \(P(z)\) in (4.9.1) becomes

\[
\begin{align*}
\left(\frac{qa_{1}/a_{2}m+2, \ldots, qa_{1}/a_{2}m, qz^{2}/a_{2}, qz^{2}/a_{2}, qz^{2}/a_{2}; q, -zq^{2}/a_{1}; q}\right)_{\infty} \\

= \left(\frac{qz^{2}/a_{2}m+2, \ldots, qz^{2}/a_{2}m, qz^{2}/a_{2}m+2, \ldots, qz^{2}/a_{2}m+1, qz^{2}/a_{2}m+1, \beta z^{2}/a_{1}; q, 1-z^{2}/a_{1}; q}\right)_{\infty}
\end{align*}
\]

(4.12.2)

and then using the fact that

\[
(a_{1}, -a_{1}/a_{1}, -a_{1}/a_{1}; q)_{\infty} \neq a^{-2}(qa_{1}, -qa_{1}, 1/a_{1}, -1/a_{1}; q)_{\infty} = 2(a_{1}, -a_{1}/a_{1}, -a_{1}/a_{1}; q)_{\infty}
\]

(4.12.3)

to combine the terms with the same \(2M_{\beta}2M_{-1}\) series.

Similarly, taking \(A = B = M + 2\) and \(m = 1\) in (4.10.10) and choosing the parameters such that \(P(z)\) in (4.9.1) becomes

\[
\begin{align*}
\left(\frac{qa_{1}/a_{2}m+3, \ldots, qa_{1}/a_{2}m, qz^{2}/a_{2}, qz^{2}/a_{2}, qz^{2}/a_{2}, qz^{2}/a_{2}, qz^{2}/a_{2}, qz^{2}/a_{2}; q, -z(a_{1})^{2}, q_{1}}{a_{1}z^{2}/a_{0}m+2; q}_{\infty} \\

= \left(\frac{qz^{2}/a_{2}m+3, \ldots, qz^{2}/a_{2}m, qz^{2}/a_{2}m+3, \ldots, qz^{2}/a_{2}m+2, qz^{2}/a_{2}m+2, qz^{2}/a_{2}m+2, qz^{2}/a_{2}m+2}{a_{1}z^{2}/a_{0}m+3; q}_{\infty}
\end{align*}
\]

(4.12.4)
where we obtain
\[
(qa_1/a_{M+3}, \ldots, qa_1/a_{2M}, q/a_{M+3}, \ldots, q/a_{2M}, q/a_1; q)_{\infty}
\]
\[
(1+z, a_1, a_2, \ldots, a_{M+2}; q)_{\infty}
\]
\[
(a_1 z, a_1, a_2, \ldots, a_{M+2}; q)_{\infty}
\]
\[
(q/a_{M+3}, \ldots, q/a_{2M+1}, 1/za_1^{1/2}, 1/za_1^{1/2}; q)_{\infty}
\]
\[
(1/za_1^{1/2}, za_1^{1/2}; q)_{\infty}
\]
(4.12.5)

where \(x = (qa_1)^M/a_1 \cdots a_{2M}\), which is formula (7.3) in Sears [1951d].

Finally, if we take \(A = B = M + 2\) and \(m = 1\) in (4.10.10) and choose the parameters such that \(P(z)\) in (4.9.1) becomes
\[
(qa_1z/a_{M+3}, \ldots, qa_1z/a_{2M+1}, q_2a_1^{1/2}, -q_2a_1^{1/2}, \pm q^2 za_1^{1/2}; q)_{\infty}
\]
\[
(a_1 z, a_1, a_2, \ldots, a_{M+2}; q)_{\infty}
\]
\[
(q/a_{M+3}, \ldots, q/a_{2M+1}, 1/za_1^{1/2}, 1/za_1^{1/2}; q)_{\infty}
\]
\[
(1/za_1^{1/2}, za_1^{1/2}; q)_{\infty}
\]
(4.12.6)

we obtain
\[
(qa_1/a_{M+3}, \ldots, qa_1/a_{2M+1}, q/a_{M+3}, \ldots, q/a_{2M+1}; q)_{\infty}
\]
\[
(1+z, a_1, a_2, \ldots, a_{M+2}; q)_{\infty}
\]
\[
(a_1, a_1, a_2, \ldots, a_{M+1}; q)_{\infty}
\]
\[
(qa_2/a_{M+3}, \ldots, qa_2/a_{2M+1}, qa_1/a_{2M+2}, q/a_{M+3}, \ldots, q/a_{2M+1}; q)_{\infty}
\]
\[
(a_1, a_1, a_2, \ldots, a_{M+2}; q)_{\infty}
\]
\[
(a_1^{1/2}, -a_1^{1/2}, za_1^{1/2}, -q^2 za_1^{1/2}; q)_{\infty}
\]
\[
(1/za_1^{1/2}, za_1^{1/2}; q)_{\infty}
\]
\[
(1/za_1^{1/2}, za_1^{1/2}; q)_{\infty}
\]
(4.12.7)

where \(y = (qa_1)^{M+1/2}/a_1 \cdots a_{2M+1}\), which are formulas (7.4) and (7.5) in Sears [1951d].

Exercises 4

4.1 Let \(\text{Re}\, c > 0\), \(\text{Re}\, d > 0\), and \(\text{Re}(x + y) > 1\). Show that Cauchy’s [1825] beta integral
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{ds}{(1 + cs)^x(1 - ds)^y} = \frac{\Gamma(x + 1)(1 + d/c)^{-x}(1 + c/d)^{-y}}{(c + d)\Gamma(x)\Gamma(y)}
\]
has a \(q\)-analogue of the form
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{ds}{(-csq^x, dsq^y; q)_{\infty}} = \frac{\Gamma_q(x + y - 1)}{\Gamma_q(x)\Gamma_q(y)} \frac{(-cq^x/d, -dq^y/c; q)_{\infty}}{(c + d)(-cq/d, -dq/c; q)_{\infty}}
\]
where \(0 < q < 1\).

(Wilson [1985])

4.2 Prove that
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+s}, -q^{a+s}, q^{a-b+1+s}, -q^{a-b+1+s}, q^{d+e-a+s}; q)_{\infty}}{(q^{-s}, q^{d+s}, q^{c+s}, -q^{c+1+s}, -q^{a-2b+s}; q)_{\infty}} \sin \pi s \sin \pi(c + d + e - a + s) \, ds
\]
\[
= \csc \pi(c + d + e - a)(q, q^{c+d+e-a}, q^{a-1-c-d-e}, q^{1+-a-b}; q)_{\infty}
\]
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+a-b}, -q^a, q^{1+a-b}, -q^{1+a-b}; q)_{\infty}}{(q^{1+a}, q^{a-1-b}, q^{1+a-b}, q^{1+a-b}; q)_{\infty}} \sin \pi s \sin \pi(c + d + e - a + s) \, ds
\]
\[
\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{(q^{1+a+b}, -q^a, q^{1+a+b}, -q^{1+a+b}; q)_{\infty}}{(q^{1+a}, q^{a-1-b}, q^{1+a-b}, q^{1+a-b}; q)_{\infty}} \sin \pi s \sin \pi(c + d + e - a + s) \, ds
\]
\[
\frac{1}{10} W_9 (q^a, q^{1+a/2}, -q^{1+a/2}, q^a, -q^a, q^a, q^a, q^a, q^{1+2a-2b-c-d-e})
\]
where \(1 + 2a - 2b > c + d + e\).

4.3 Show that
\[
\begin{aligned}
\phi_2 & \left[ \frac{ac, bc, ad}{ab, c, d}; q, g \right] \\
& = \frac{(f, q/\ell, cf/g, \eta/cf, ab, c, d; q)_{\infty}}{(f, q/\ell, cf/g, \eta/cf, ab, c, d; q)_{\infty}}
\end{aligned}
\]
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f e^{i\theta} / g, g e^{i\theta} / cf, d h e^{i\theta}, g e^{i\theta} / f, c e^{-i\theta} / g_{\infty}}{(a e^{i\theta}, b e^{i\theta}, c e^{i\theta}, g e^{i\theta}, a e^{-i\theta}, g^{-1} e^{i\theta} / g_{\infty})_{\infty}} \, d\theta.
\]
4.4 Prove that
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( f e^{i\theta}, ke^{i\theta}/d, qde^{-i\theta}/k, cke^{-i\theta}, qe^{i\theta}/ck, abdgh e^{i\theta}/f; q \right) \infty d\theta \]
\[ = (k, q/k, ck/d, qd/c, cf, df, acdg, bdgh, cdgh, abcd/f; q) \infty \]
\[ = q W_7(cdf/g, cg, df, af, hf, fh; q, abcdh/f). \]

4.5 Prove that
\[ 4\sqrt{q} \left[ \frac{q^{-n}, abcdq^{-n}, e^{i\theta}}{ab, ac, ad} : q; q \right] \]
\[ = \frac{(q; q) \infty}{(q^{1/2}, q^{1/2}; q) \infty} \left( \frac{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}) \infty}{(q^{1/2} e^{i\theta}; q) \infty} \right)^2 \]
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{d\phi}{(abce^{i\theta}/\sigma, ad; q) \infty} \left( \frac{2 e^{i\theta}}{\sigma} \right)^n d\phi \]
for \( n = 0, 1, 2, \ldots \).

4.6 Prove that
\[ a^{-1} \frac{(aq/e, aq/f, aq/g, aq/h, q/a, q/a, q/ah; q) \infty}{(aq^2, ab, ac, ad, b/a, c/a, d/a; q) \infty} \cdot 10 W_9(a^2; ab, ac, ad, ae, af, ag, ah; q, q^3 / abcdgfh) \]
\[ + \text{ident} (a; b, c, d) = 0, \]
where \(|q^3| < |abcdgfh|\).

4.7 Prove that
\[ a^{-1} \frac{(a_i q/b_1, a_1 q/b_2, \ldots, a_i q/b_r, q/a_i b_1, \ldots, q/a_i b_r; q) \infty}{(q^2 a_i, a_1 a_2, \ldots, a_i a_r, a_2/a_1, \ldots, a_r/a_i; q) \infty} \]
\[ \cdot 2r+2 W_{2r+1} \left( a_1^2; a_1 a_2, \ldots, a_i a_r, a_1 b_1, \ldots, a_i b_r, q^{-1}/a_1 \cdots a_r b_1 \cdots b_r \right) \]
\[ + \text{ident} (a_i; a_2, \ldots, a_r) = 0, \]
where \( r = 1, 2, \ldots \), and \(|q^{-1}| < |a_1 \cdots a_r b_1 \cdots b_r|\).

4.8 Show that
\[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\alpha}{-cs^m+1, bds, baxq^{-n}; q} \infty d\alpha = \frac{-c\alpha/c, -bd/c, q^{-1}/(c+d)}{(c+d)(-dq/c, -ba/cq; q) \infty (q, -cq/d; q) \infty}, \]
where \( \text{Re}(c, d, ba) > 0 \) and \( n = 0, 1, \ldots. \) Show that the \( q \)-Cauchy beta integral in Ex. 4.1 follows from this formula by letting \( n \to \infty \) and then setting \( b = q^\alpha, \alpha = -q^\beta \).

4.9 Extend the integral in Ex. 4.8 to
\[ \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{(-acs, a^2 s/\alpha, a^2 s/\beta, \alpha^2 s/\gamma, a^2 s/\delta, \alpha^2 s/\lambda; q) \infty}{(-cs, \alpha s, \beta s, \gamma s, \delta s, \lambda s; q) \infty} \]
\[ \cdot \frac{1 - a^2 c^2 s^2}{q} \frac{ds}{s} \]
\[ = \frac{(a/bq, -ac\alpha/\beta, ac\alpha/\gamma, -ac/\delta, -ac/\lambda; q) \infty}{c(q, \alpha/c, \beta/c, -\gamma/c, -\delta/c, -\lambda/c; q) \infty} \]
\[ \cdot \frac{(q^2/a, a^2/c\beta, \alpha^2/c\gamma, \alpha^2/c\delta; \gamma q, \gamma q; q) \infty}{(-cq/\alpha, -cq/\beta, -cq/\gamma, -a^2 c^2/\alpha \beta \gamma q; q) \infty}, \]
where \( \text{Re}(c, \alpha, \beta, \gamma, \delta, \lambda) > 0 \), \( a^2 c^2 = a\beta \gamma \delta \lambda q^2 \), \( ac = -\lambda q^{n+1} \), and \( n = 1, 2, \ldots \).

4.10 Show that
\[ \frac{1}{2\pi i} \int_{K} \frac{(q^2 z/a\alpha, q^2 z/b\gamma, q^2 z/c\gamma; q, \gamma z; q) \infty}{(az, bx, cz, \alpha/z; q) \infty} \frac{1 - q^2 / a^2 z}{z} dz \]
\[ = \frac{(a, b, c, a\gamma, q/\gamma, q; q) \infty}{(a, b, c, a, q/\gamma; q) \infty}, \]
where \( q^2 = abc q^2, |\gamma/\alpha| < 1 \), and the contour \( K \) is as defined in §4.9.

4.11 Show that
\[ \frac{1}{2\pi i} \int_{K} \frac{(bq^2 z, qz/\gamma, \gamma z; q) \infty}{(az, bx, \alpha/z; q) \infty} \]
\[ \cdot (qz/\gamma q^m; q) m(a_1 z; q) m, \ldots, (a_r z; q) m, \frac{dz}{z} \]
\[ = \frac{(\gamma/\alpha, aq/\gamma, bq/z; q) \infty}{(ax, q/a, b, ax; q) \infty} \]
\[ \cdot (q/bq^m; q) m(a_1/b; q) m, \ldots (a_r/b; q) m, (bv) m+m+1, \ldots, m, \]
provided \(|\gamma/\alpha| < 1 \), where \( m, m_1, \ldots, m_r, m \) are nonnegative integers, \( q = a\gamma q^m+m+1 \ldots m \), and \( K \) is as defined in §4.9.

4.12 Show that
\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\alpha}{-acs, dgs; q} \infty (a_1 s; q) m, \ldots, (a_r s; q) m, \frac{dz}{z} \]
\[ = \frac{(-a/c d; q) \infty}{(c + d)(-cq/\alpha, q/\alpha; q) \infty} \frac{(a_1/d; q) m, \ldots, (a_r/d; q) m}{(a_1/d; q) m, \ldots, (a_r/d; q) m}. \]
provided $|aq^{-(m_1+\cdots+m_r)}| < 1$, $\text{Re}(c, d, a_1, \cdots, a_r) > 0$, and $m_1, \ldots, m_r$ are nonnegative integers.

4.13 Show that

$$
\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{(acs, -ac^2 s/f, a^2 s/\alpha, a^2 s/\beta; q; \infty)}{(acs, -fs, \alpha s, \beta s; q; \infty)} (1 - ac^2 s^2/q) ds
$$

$$
= (a/q, ac/f, -ac/\alpha, -ac/\beta, -ac/\gamma, -ac/\delta; q; \infty)
\frac{c(q, f/c, -\alpha/c, -\beta/c, -\gamma/c, -\delta/c; q; \infty)}{(q^2/a, ac^2/\alpha \beta, ac^2/\alpha \gamma, ac^2/\beta \delta; q; \infty)}
\frac{(-cq/\alpha, -cq/\beta, -cq/\gamma, -cq/\delta; q; \infty)}{(q^2/a, ac^2/\alpha \beta, ac^2/\alpha \gamma, ac^2/\beta \delta; q; \infty)}
\frac{(ac^2/fq^2, ac/f, -ac^2/f \alpha, -ac^2/f \beta, -ac^2/f \gamma, -ac^2/f \delta; q; \infty)}{(q^2/a, ac^2/\alpha \beta, ac^2/\alpha \gamma, ac^2/\beta \delta; q; \infty)}
\frac{(f^2 q^2/ac^2, ac^2/\alpha \beta, ac^2/\alpha \gamma, ac^2/\beta \delta; q; \infty)}{(f^2 q^2/ac^2, ac^2/\alpha \beta, ac^2/\alpha \gamma, ac^2/\beta \delta; q; \infty)}
\frac{(-f q/\alpha, -f q/\beta, -f q/\gamma, -a^2 c^2 f q/\alpha \beta \gamma \delta; q; \infty)}{(-f q/\alpha, -f q/\beta, -f q/\gamma, -a^2 c^2 f q/\alpha \beta \gamma \delta; q; \infty)}
$$

provided $ac = f q^{n+1}, a^2 c^2 = f a \alpha \beta \gamma \delta q^2$, $\text{Re}(c, f, \alpha, \beta, \gamma, \delta) > 0$, the integrand has only simple poles, and $n = 0, 1, \ldots$.

(For the formulas in Exercises 4.8–4.13, and related formulas, see Gasper [1989c].)

Notes 4

§4.4 Kalnins and Miller [1999b] exploited symmetry (recurrence relation) techniques similar to those used by Nikiforov and Suslov [1986], Nikiforov, Suslov and Uvarov [1985], and Nikiforov and Uvarov [1988] to give another proof of (4.4.3) and of (4.11.1).

§4.6 Contour integrals of the types considered in this section were used by Agarwal [1953c] to give simple proofs of the two-term and three-term transformation formulas for $\phi_7$ series.

§4.12 Sears [1951b] also derived the hypergeometric limit cases of the transformation formulas in this section. Applications of (4.12.1) to some formulas in partition theory are given in M. Jackson [1949].

5

5.1 Notations and definitions

The general bilateral basic hypergeometric series in base $q$ with $r$ numerator and $s$ denominator parameters is defined by

$$
r\psi_s(x) \equiv r\psi_s \left[ \frac{a_1, \ldots, a_r}{b_1, \ldots, b_s}; q, z \right]
= \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} (-1)^{(s-r)n} q^{-r} C_s^2 z^n,
$$

(5.1.1)

In (5.1.1) it is assumed that $q, z$ and the parameters are such that each term of the series is well-defined (i.e., the denominator factors are never zero, $q \neq 0$ if $s < r$, and $z \neq 0$ if negative powers of $z$ occur). Note that a bilateral basic hypergeometric series is a series $\sum_{n=-\infty}^{\infty} v_n$ such that $v_0 = 1$ and $v_{n+1}/v_n$ is a rational function of $q^n$. By applying (1.2.28) to the terms with negative $n$, we obtain that

$$
r\psi_s(x) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} (-1)^{(s-r)n} C_s^2 z^n
+ \sum_{n=1}^{\infty} \frac{(q/b_1, q/b_2, \ldots, q/b_s; q)_n}{(q/a_1, q/a_2, \ldots, q/a_r; q)_n} \left( \frac{b_1 \cdots b_s}{a_1 \cdots a_r} \right)^n
$$

(5.1.2)

Let $R = |b_1 \cdots b_s/a_1 \cdots a_r|$. If $s < r$ and $|q| < 1$, then the first series on the right side of (5.1.2) diverges for $z \neq 0$; if $s < r$ and $|q| > 1$, then the first series converges for $|z| < R$ and the second series converges for all $z \neq 0$. When $r < s$ and $|q| < 1$ the first series converges for all $z$, but the second series converges only when $|z| > R$. If $r < s$ and $|q| > 1$, the second series diverges for all $z \neq 0$. If $r = s$, which is the most important case, and $|q| < 1$, the first series converges when $|z| < 1$ and the second when $|z| > R$; on the other hand, if $|q| > 1$ the second series converges when $|z| > 1$ and the first when $|z| < R$.

We shall assume throughout this chapter that $|q| < 1$, so that the region of convergence of the bilateral series

$$
r\psi_s(x) = \sum_{n=-\infty}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} z^n
= \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} z^n
+ \sum_{n=1}^{\infty} \frac{(q/b_1, q/b_2, \ldots, q/b_s; q)_n}{(q/a_1, q/a_2, \ldots, q/a_r; q)_n} \left( \frac{b_1 \cdots b_s}{a_1 \cdots a_r} \right)^n
$$

(5.1.3)
is the annulus

$$\left| \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1. \quad (5.1.4)$$

When $b_j = q$ for some $j$, the second series on the right sides of (5.1.2) and (5.1.3) vanish and the first series become basic hypergeometric series. If we replace the index of summation $n$ in (5.1.1) by $k + n$, where $k$ is an integer, then it follows that

$$r \psi_3 \left[ \begin{array}{l} a_1, \ldots, a_r \\ b_1, \ldots, b_r \\ q, q \\ z \\ \end{array} \right] = \left( \frac{a_1, \ldots, a_r ; q}{b_1, \ldots, b_r ; q} \right)_k \left[ \frac{(-1)^k (q)_k^2}{(b_1, \ldots, b_r ; q)_k} \right] z^{-r}$$

$$+ r \psi_3 \left[ \begin{array}{l} a_1 q^k, \ldots, a_r q^k \\ b_1 q^k, \ldots, b_r q^k \\ q, q \\ z \end{array} \right]. \quad (5.1.5)$$

When $r$ and $s$ are small we shall frequently use the single-line notation

$$r \psi_3 (a_1, \ldots, a_r ; b_1, \ldots, b_r ; q, z).$$

An $r \psi_3$ series will be called well-poised if $a_1 b_1 = a_2 b_2 = \cdots = a_r b_r$, and very-well-poised if it is well-poised and $a_1 = a_2 = q b_1 = q b_2$.

5.2 Ramanujan’s sum for $\psi_1 (a; b; q, z)$

The bilateral summation formula

$$\psi_1 (a; b; q, z) = \frac{(q, b/a, q/a, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1, \quad (5.2.1)$$

which is an extension of the $q$-binomial formula (1.3.2), was first given by Ramanujan (see Hardy [1940]). In Chapter 2 we saw that this formula follows as a special case of Sears’ $\Phi_2$ summation formula (2.10.12). Andrews [1969, 1970], Hahn [1949b], M. Jackson [1950], Ismail [1977] and Andrews and Askey [1978] published different proofs of (5.2.1). The proof given here is due to Andrews and Askey [1978].

The first step is to regard $\psi_1 (a; b; q, z)$ as a function of $b$, say, $f(b)$. Then

$$f(b) = \psi_1 (a; b; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n + \sum_{n=1}^{\infty} \frac{(q/b; q)_n}{(q/a; q)_n} (b/az)^n \quad (5.2.2)$$

so that, by (5.1.4), the two series are convergent when $|b/a| < |z| < 1$. As a function of $b$, $f(b)$ is clearly analytic for $|b| < \min(1, |az|)$ when $|z| < 1$. Since

$$\psi_1 (a; b; q, z) - \psi_1 (a; b; q, z) = \sum_{n=-\infty}^{\infty} \left( \frac{(a; q)_n}{(b; q)_n} - \frac{aq^n (a; q)_n}{(b; q)_n} \right) z^n$$

$$= \sum_{n=-\infty}^{\infty} \frac{(a; q)_{n+1}}{(b; q)_n} z^n$$

$$= z^{-1}(1 - b/q) \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^{n+1}$$

$$= z^{-1}(1 - b/q) \psi_1 (a; b; q, z),$$

we get

$$f(b) - z^{-1}(1 - b)f(b) = a \psi_1 (a; b; q, z). \quad (5.2.3)$$

However,

$$a \psi_1 (a; b; q, z) = a \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} (z^q)^n$$

$$= a b^{-1} \sum_{n=0}^{\infty} \frac{(a; q)_n (1 - b q^{-n} - 1)}{(b; q)_n} z^n$$

$$= a b^{-1} (1 - b) f(b) + ab^{-1} f(b).$$

Combining (5.2.3) and (5.2.4) gives the functional equation

$$1 - ab^{-1} f(b) = (1 - b)(z^{-1} - ab^{-1})f(b),$$

that is,

$$f(b) = \frac{1 - b/a}{(1 - b)(1 - b/az)} f(bq). \quad (5.2.5)$$

Iterating (5.2.5) $n - 1$ times we get

$$f(b) = \left( \frac{b/a; q}{b, b/az; q} \right) f(bq^n). \quad (5.2.6)$$

Since $f(b)$ is analytic for $|b| < \min(1, |az|)$, by letting $n \to \infty$ we obtain

$$f(b) = \left( \frac{b/a; q}{b, b/az; q} \right) f(0). \quad (5.2.7)$$

However, since

$$f(q) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}$$

by (1.3.2), on setting $b = q$ in (5.2.7) we find that

$$f(0) = \frac{(q, q/az; q)_\infty}{(q/a; q)_\infty} f(q) = \frac{(q, q/az, az; q)_\infty}{(q/a, z; q)_\infty} f(q) \quad (5.2.8)$$

Substituting this in (5.2.7) we obtain formula (5.2.1).
Jacobi’s triple-product identity (1.6.1) is a limit case of Ramanujan’s sum. First replace $a$ and $z$ in (5.2.1) by $a^{-1}$ and $az$, respectively, to obtain
\[
\sum_{n=-\infty}^{\infty} \frac{(a^{-1};q)_n}{(b;q)_n} (az)_n = \frac{(q, ab, z, q/z; q)_{\infty}}{(b, aq, az, b/z; q)_{\infty}},
\]
(5.2.8)
when $|b| < |z| < |a|^{-1}$. Now set $b = 0$, replace $q$ by $q^2$, $z$ by $zq$, and then take $a \to 0$ to get (1.6.1).

### 5.3 Bailey’s sum of a very-well-poised $\phi_6$ series

Bailey [1936] proved that
\[
\phi_6 \left[ \begin{array}{c} \frac{q a^2}{b}, -a^2, a, b, c, \frac{q a^2}{b} \\ a^2, -a, ab, ac, bc, cd, \frac{q a}{b} \end{array} ; \frac{q}{b c d e} \right] = \frac{(aq, ab/bq, ab/bd, ab/cq, ab/cd, ab/cq, ab/cq, ab/cq, a; q)_{\infty}}{(aq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, a; q)_{\infty}}, \tag{5.3.1}
\]
provided $|qa^2/bcde| < 1$. This is an extension of the $\phi_5$ series when one of the parameters $b, c, d, e$ equals $a$. (5.3.1) can be regarded as an extension of the $\phi_5$ summation formula (2.7.1).

There are several known proofs of (5.3.1). Bailey’s proof depends crucially on the identity
\[
\frac{(aq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, a; q)_{\infty}}{a(qa^2, q/a^2, q, ab, bc, cd, a; q)_{\infty}} = \frac{W_2(a^2; ab, ac, ad, ae, af, q^2/abclist) + \text{idem} (a, b, c) = 0}{W_2(a^2; ab, ac, ad, ae, af, q^2/abclist)}
\]
(5.3.2)
when $|q^2/abclist| < 1$, which is easily proved by using the $q$-integral representation (2.10.19) of an $\phi_5$ series (see Exercise 2.15). If we set $c = q/a$, the first and third series in (5.3.2) combine to give, via (5.1.3),
\[
\frac{(aq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, a; q)_{\infty}}{a(qa^2, q/a^2, q, ab, bc, cd, a; q)_{\infty}} = \phi_6 \left[ \begin{array}{c} \frac{q}{b} \\ a, -a, ab, ac, bc, cd, \frac{q}{b} \end{array} ; \frac{q}{bcde} \right],
\]
while, by (2.7.1), the second series reduces to
\[
\phi_6 \left[ \begin{array}{c} b_2, bq, -bq, bd, be, bf \\ a, -a, ab, ac, af \end{array} ; \frac{q}{b \cdot bcde} \right] = \frac{(aq^2, q, df, q/ef, q; q)_{\infty}}{(aq^2, q, df, q/ef, q; q)_{\infty}},
\]
This gives (5.3.1) after we replace $a^2, ab, ad, ae, af$ by $a, b, c, d, e$, respectively, and use the same square root of $a$ everywhere.

Slater and Lakin [1956] gave a proof of (5.3.1) via a Barnes type integral and a second proof via a $q$-difference operator. Andrews [1974a] gave a simpler proof and Askey [1984c] showed that it can be obtained from a simple difference equation. The simplest proof was given by Askey and Ismail [1979] who only used the $\phi_5$ sum (2.7.1) and an argument based on the properties of analytic functions.

Setting $e = a^2$ in (5.3.1), we obtain
\[
\psi_4 \left[ \begin{array}{c} -qa^2, a, bc, d \\ -a, aq/b, aq/c, aq/d, q \end{array} ; \frac{q}{b c d} \right] = \frac{(aq, ab/bq, ab/bd, aq/cd, ab/cq, aq/cq, aq/cq, aq/cq, a; q)_{\infty}}{(aq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, ab/cq, a; q)_{\infty}},
\]
provided $|qa^2/bcde| < 1$. This is an extension of the $\phi_5$ series when one of the parameters $b, c, d, e$ equals $a$. (5.3.1) can be regarded as an extension of the $\phi_5$ summation formula (2.7.1).

### 5.4 A general transformation formula for an $\psi_r$ series

In this section we shall derive a transformation formula for an $\psi_r$ series from those for $\phi_r$ series in Chapter 4. First observe that (5.1.3) gives
\[
\psi_r (a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r; q, z)
\]
\[
= \psi_{r+1} (a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r; q, z) + z^{-1} \prod_{k=1}^{r} \frac{b_k - q}{a_k - q}
\]
\[
\psi_{r+1} (a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r; q, z) = \psi_{r+1} (b_1, b_2, \ldots, b_r; a_1, a_2, \ldots, a_r; q, z) \].
\]
(5.4.1)
In (4.10.11) let us now make the following specialization of the parameters
\[
C = A + 1, \quad D = B + 1, \quad A = B, \quad
\]
\[
c_1 d_1 = c_2 d_2 = \cdots = c_{A+1} d_{A+1} = q,
\]
\[
q d_1 / b_1 = c_1 d_1 / b_{A+1} = c_2 d_2 / b_2 = \cdots = c_{A} d_{A} / b_{A} = \alpha_A,
\]
\[
a_1 d_1 = \beta_1, a_2 d_2 = \beta_2, \ldots, a_4 d_4 = \beta_4,
\]
\[
b_1 \cdots b_{A-1} \cdot d_1 \cdots d_{A+1} = x.
\]
(5.4.2)
Then, combining the pairs of the resulting $\psi_{r+1}$ series in (4.10.11) via (5.4.1), simplifying the coefficients and relabelling the parameters, we obtain Slater’s [1952b, (4)] transformation formula
\[
\psi_r \left[ \begin{array}{c} a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r; q, z \end{array} ; \frac{c_1, c_2, \ldots, c_r}{q, c_1, c_2, \ldots, c_r; q} \right] = \frac{q}{c_1} \left[ \begin{array}{c} q c_1, c_2, \ldots, c_r, q c_1, c_2, \ldots, c_r; q \end{array} ; \frac{c_1, c_2, \ldots, c_r}{q, c_1, c_2, \ldots, c_r; q} \right].
\]
(5.4.3)
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\[ r \psi_r \left[ \frac{q_{a_1}/c_1, q_{a_2}/c_1, \ldots, q_{a_r}/c_1}{gb_1/c_1, gb_2/c_1, \ldots, gb_r/c_1}; q, z \right] + \text{idem } (c_1; c_2, \ldots, c_r), \]

where \( d = a_1 a_2 \cdots a_r/c_1 c_2 \cdots c_r, \) \( \left| \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1. \)

Note that the \( c \)'s are absent in the \( r \psi_r \) series on the left side of (5.4.3). This gives us the freedom to choose the \( c \)'s in any convenient way. For example, if we set \( c_j = q a_j, \) where \( j \) is an integer between 1 and \( r, \) then the \( j \)th series on the right becomes an \( r \psi_{r-1} \) series. So if we set \( c_j = q a_j, j = 1, 2, \ldots, r, \) in (5.4.3), then we get an expansion of an \( r \psi_r \) series in terms of \( r \psi_{r-1} \) series:

\[
\frac{(b_1, b_2, \ldots, b_r, q/a_1, q/a_2, \ldots, q/a_r, z, q/z; q)_\infty}{(q a_1, q a_2, \ldots, q a_r, 1/a_1, 1/a_2, \ldots, 1/a_r; q)_\infty} r \psi_r \left[ \frac{a_1, a_2, \ldots, a_r}{b_1, b_2, \ldots, b_r}; q, z \right] = a_r^{-1} q (q a_1/\beta_1, a_2/\beta_1, \ldots, a_r/\beta_1, b_1/\alpha_1, b_2/\alpha_2, \ldots, b_r/\alpha_r; q)_\infty
\]

\[ r \psi_{r-1} \left[ \frac{q a_1/\beta_1, q a_2/\beta_1, \ldots, q a_r/\beta_1}{q a_1/\alpha_1, q a_2/\alpha_1, \ldots, q a_r/\alpha_r}; q, \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right] + \text{idem } (a_1; a_2, \ldots, a_r), \]

provided \( \left| \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right| < |z| < 1. \)

On the other hand, if we set \( c_j = b_j, j = 1, 2, \ldots, r \) in (5.4.3), then we obtain the expansion formula

\[
\frac{(q/a_1, q/a_2, \ldots, q/a_r, dz, q/dz; q)_\infty}{(q/b_1, q/b_2, \ldots, q/b_r; q)_\infty} r \psi_r \left[ \frac{a_1, a_2, \ldots, a_r}{b_1, b_2, \ldots, b_r}; q, z \right] = q \left( \frac{q a_1/\beta_1, q a_2/\beta_1, \ldots, q a_r/\beta_1}{q b_1/\beta_1, q b_2/\beta_1, \ldots, q b_r/\beta_1}; q \right)_\infty
\]

\[ r \psi_{r-1} \left[ \frac{q a_1/\beta_1, q a_2/\beta_1, \ldots, q a_r/\beta_1}{q b_2/\alpha_1, q b_2/\alpha_2, \ldots, q b_r/\alpha_r}; q, \frac{b_1 \cdots b_r}{a_1 \cdots a_r} \right] + \text{idem } (b_1; b_2, \ldots, b_r), \]

with \( d = a_1 a_2 \cdots a_r/b_1 b_2 \cdots b_r. \)

5.5 A general transformation formula for a very-well-poised \( r \psi_r \) series

Using (4.12.1) and (5.4.1) as in §5.4, we obtain Slater's [1952b] expansion of a well-poised \( 2 \psi_2 \) series in terms of \( r \) other well-poised \( 2 \psi_2 \) series:

\[
\frac{(q/b_1, \ldots, q/b_r, a q/b_1, \ldots, a q/b_r, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, q a^{\frac{1}{2}}, -q a^{\frac{1}{2}}; q)_\infty}{(a, a_1, \ldots, a_r, a q/a_1, \ldots, a q/a_r, q/a, q/a_1, \ldots, q/a_r; q)_\infty} 2 \psi_2 \left[ \frac{a_1 b_1, a_2 b_2, \ldots, a_r b_r}{q a b_1, q a b_2, \ldots, q a b_r}; q, \frac{a^{\frac{1}{2}} q}{b_1 \cdots b_r} \right] = \frac{a_1 a q/b_1, a_2 a q/b_2, \ldots, a_r a q/b_r}{a_1 a q/b_1, a_2 a q/b_2, \ldots, a_r a q/b_r; q}_\infty.
\]

For the very-well-poised case when \( a_1 = b_1 = q a^{\frac{1}{2}}, a_2 = b_2 = -q a^{\frac{1}{2}}, \) the first two terms on the right side vanish and we get

\[
\frac{(q/b_1, \ldots, q/b_r, a q/b_1, \ldots, a q/b_r, a^{\frac{1}{2}}, -a^{\frac{1}{2}}, q a^{\frac{1}{2}}, -q a^{\frac{1}{2}}; q)_\infty}{(a q/a_1, \ldots, a q/a_r; q)_\infty} 2 \psi_2 \left[ \frac{a_1 b_1, a_2 b_2, \ldots, a_r b_r}{a_1 q/b_1, a_2 q/b_2, \ldots, a_r q/b_r}; q, \frac{a^{\frac{1}{2}} q}{b_1 \cdots b_r} \right] + \text{idem } (a_1; a_2, \ldots, a_r).
\]

In particular, for \( r = 3 \) we have

\[
\frac{(q/b_1, \ldots, q/b_3, a q/b_1, \ldots, a q/b_3, q a^{\frac{1}{2}}, -q a^{\frac{1}{2}}, q a^{\frac{1}{2}}, -q a^{\frac{1}{2}}; q)_\infty}{(a q/a_1, a q/a_2, a q/a_3; q)_\infty} 2 \psi_2 \left[ \frac{a_1 b_1, a_2 b_2, a_3 b_3}{a_1 q/b_1, a_2 q/b_2, a_3 q/b_3}; q, \frac{a^{\frac{1}{2}} q}{b_1 b_2 b_3} \right] = \frac{(a q/a_1, a q/a_2, a q/a_3; q)_\infty}{(a q/a_1, a q/a_2, a q/a_3; q)_\infty} 2 \psi_2 \left[ \frac{a_1 b_1, a_2 b_2, a_3 b_3}{a_1 q/b_1, a_2 q/b_2, a_3 q/b_3}; q, \frac{a^{\frac{1}{2}} q}{b_1 b_2 b_3} \right] + \text{idem } (a_1; a_2, a_3).
\]

If we now set \( a_3 = b_3, \) then the \( 2 \psi_2 \) series on the right side becomes a \( 3 \phi_6 \) wi

\[
\frac{(q/b_1, \ldots, q/b_r, a q/b_1, \ldots, a q/b_r, a q/b_1, a q/b_2, a q/b_3; q)_\infty}{(a q/a_1, a q/a_2, a q/a_3; q)_\infty} 2 \psi_2 \left[ \frac{a_1 b_1, a_2 b_2, a_3 b_3}{a_1 q/b_1, a_2 q/b_2, a_3 q/b_3}; q, \frac{a q}{b_1 b_2 b_3} \right] = \frac{(a q/a_1, a q/a_2, a q/a_3; q)_\infty}{(a q/a_1, a q/a_2, a q/a_3; q)_\infty} 2 \psi_2 \left[ \frac{a_1 b_1, a_2 b_2, a_3 b_3}{a_1 q/b_1, a_2 q/b_2, a_3 q/b_3}; q, \frac{a q}{b_1 b_2 b_3} \right] + \text{idem } (a_1; a_2, a_3).
\]
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\[
\psi_{2r}^{2r} \left[ \frac{b_1, \ldots, b_r; a^+ q^r}{aq/b_1, \ldots, aq/b_r; q, b_1 b_2 \cdots b_r} \right] = \psi_{r}^{r} \left[ \frac{a^+ q^{r-1}}{b_1 a, \ldots, b_r a; q, b_1 b_2 \cdots b_r} \right] + \text{idem } (a_1; a_2, \ldots, a_r+1),
\]

(5.5.4)

and

\[
\psi_{2r-1}^{2r-1} \left[ \frac{a^+ q^{r-1}}{b_1, \ldots, b_{r-1} a; q, b_1 b_2 \cdots b_{r-1}} \right] = \psi_{r-1}^{r-1} \left[ \frac{a^+ q^{r-1}}{a b_1, \ldots, a b_{r-1} a; q, b_1 b_2 \cdots b_{r-1}} \right] + \text{idem } (a_1; a_2, \ldots, a_r).\]

\[
\psi_{r}^{r} \left[ \frac{a, a^+ q^{r}}{a, a^+ q^{r}} \right] = \psi_{r}^{r} \left[ \frac{a^+ q^{r}}{a, a^+ q^{r}} \right] + \text{idem } (a_1; a_2, \ldots, a_r).\]

(5.5.5)

5.6 Transformation Formulas for Very-well-posed \( \psi_8 \) and \( \psi_{10} \) Series

In this section we consider two special cases of (5.5.2) that may be regarded as extensions of the transformation formulas for very-well-poised \( \psi_7 \) and \( \psi_9 \) series derived in Chapter 2. First, set \( r = 4 \) in (5.5.2) and replace \( b_1, b_4, b_6, b_7, b_8 \) by \( b, c, d, e, f, g \), respectively, choose \( a_3 = f, a_4 = g \) and simplify to get

\[
\psi_{8} \left[ \frac{aq/b, aq/d, aq/e, q/b, q/c, q/d, q/e; q}{f, g; f/a, g/a, a; q} \right] = \psi_{8} \left[ \frac{a^+ q^2}{a^+ q^2} \right] \left[ \frac{q/a, q/a, q/a, q/a, q/a; q}{q, q, q, q, q} \right] + \text{idem } (q; b; \text{bcdefg}) \]

(5.6.1)

where

\[
\left| \frac{a^2 q^2}{\text{bcdefg}} \right| < 1.
\]

Replacing \( a, b, c, d, e, f, g \) by \( a^2, b, c, d, e, f, g \), respectively, we may rewrite (5.6.1) as

\[
\psi_{8} \left[ \frac{aq/b, aq/c, aq/d, aq/e, q/a, q/c, q/d, q/e; q}{f, g; f/a, g/a, a; q} \right] = \psi_{8} \left[ \frac{a^+ q^2}{a^+ q^2} \right] \left[ \frac{q/a, q/a, q/a, q/a, q/a; q}{q, q, q, q, q} \right] + \text{idem } (q; g),
\]

(5.6.2)

provided \( |q^2/\text{bcdefg}| < 1 \). Note that no \( a \)'s appear in the \( \psi_7 \) series on the right side of (5.6.2). This is essentially the same as eq. (2.2) in M. Jackson [1950a].

For the next special case of (5.5.2) we take \( r = 5 \) and replace \( b_0, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9 \) by \( b, c, d, e, f, g, h, k \), respectively, choose \( a_3 = g, a_4 = h, a_5 = k \), and finally, replace \( a, b, \ldots, k \) by \( a^2, b, a, \ldots, k \) and simplify. This gives

\[
\psi_{8} \left[ \frac{aq/b, aq/c, aq/d, aq/e, q/a, q/c, q/d, q/e; q}{f, g; f/a, g/a, a; q} \right] = \psi_{8} \left[ \frac{a^+ q^2}{a^+ q^2} \right] \left[ \frac{q/a, q/a, q/a, q/a, q/a; q}{q, q, q, q, q} \right] + \text{idem } (a_1, a_2, \ldots, a_r; a).
\]

(5.6.3)

where \( |q^3/\text{bcdefg}| < 1 \).
Exercises 5

5.1 Show that
\[ \sum_{n=0}^{\infty} (-1)^n q^n = \frac{(q; q)_\infty}{(-q; q)_\infty}. \]

5.2 Letting \( c \to \infty \) in (5.3.4) and setting \( a = 1, b = -1 \), show that
\[ 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n} = \left( \frac{(q; q)_\infty}{(-q; q)_\infty} \right)^2. \]

5.3 In (5.4.1) set \( b = a, d = c, e = -1 \) and then let \( a \to 1 \) to show that
\[ 1 + 8 \sum_{n=1}^{\infty} \frac{(-q)^n}{(1 + q^n)^2} = \left( \frac{(q; q)_\infty}{(-q; q)_\infty} \right)^4. \]

See section 8.11 for applications of Exercises 5.1-5.3 to Number Theory.

5.4 Set \( b = c = d = e = -1 \) and then let \( a \to 1 \) in (5.3.1) to obtain
\[ 1 + 16 \sum_{n=1}^{\infty} \frac{q^{2n}(q^2 - q^n - q^{-n})}{(1 + q^n)^2} = \left( \frac{(q; q)_\infty}{(-q; q)_\infty} \right)^8. \]

5.5 Show that
\[ \sum_{n=-\infty}^{\infty} q^{4n^2} z^{2n}(1 + z q^{4n+1}) = (q^2, -zq, -q/z; q^2)_\infty, \quad z \neq 0. \]

5.6 Prove the quintuple product identity
\[ \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} z^{3n}(1 + z q^n) = (q, -z, -q/z; q)_\infty (q z^2, q/z^2; q^2)_\infty, \quad z \neq 0. \]

(See the Notes for this exercise)

5.7 Show that
\[ \sum_{n=-\infty}^{\infty} \frac{(1-q^{10n+4})}{(1-q^{10n+1})(1-q^{10n+3})^2} q^{5n+1} = \frac{q(1-q^2)}{(1-q)^2(1-q^2)^2} \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n^4}. \]

\[ \psi_6 \left[ q^7, -q^7, q, q^3, q^5, q^5 \right]. \]

Deduce that
\[ \sum_{n=0}^{\infty} \left\{ \frac{q^{5n+1}}{(1-q^{5n+1})^2} - \frac{q^{5n+2}}{(1-q^{5n+2})^2} + \frac{q^{5n+3}}{(1-q^{5n+3})^2} + \frac{q^{5n+4}}{(1-q^{5n+4})^2} \right\} = \frac{q^5 (q; q)_\infty^5}{(q; q)_\infty^5}, \quad |q| < 1. \]

See Andrews [1974a] for the above formulas.

5.8 Deduce (5.4.4) directly from (4.5.2).

5.9 Deduce (5.3.1) from (5.4.5) by using (2.7.1).

5.10 Show that
\[ q (e/a, e/b, e/ab, qe/c, qe/d, qe/f, e/q)_\infty \psi_2 \left[ e/c, e/q, e/a, e/b \right] = \text{idem } (e; f) \]
\[ = \frac{(q, q/a, q/b, c/a, c/b, c/e, qf/e, c/q)_\infty}{(e, q/c, e/f, c/ab, q)\infty}. \]

5.11 Show that
\[ \psi_8 \left[ q a^2, -q, c, d, e, f, a q^{-n}, q^{-n}, a^2 q^{2n+2}, \frac{a^2 q^{2n+2}}{cdef} \right] \]
\[ = \frac{(aq, q/a, q/ab, q/e, q/f, q)\infty}{(aq, q/c, q/d, q/e, q/f, q)\infty} \psi_4 \left[ \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{f}, \frac{aq}{q} \right] \]
\[ = \frac{(aq/c, aq/d, aq/e, aq/f, q)\infty}{(aq/c, aq/d, aq/e, aq/f, q)\infty} \psi_4 \left[ \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{f}, \frac{aq}{q} \right], \quad n = 0, 1, \ldots, \]

and deduce the limit cases
\[ \psi_2 \left[ \frac{e, f}{aq/c, aq/d}, \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{ef} \right] \]
\[ = \frac{(aq/c, aq/d, aq/e, aq/f, q)\infty}{(aq, q/c, aq/d, aq/e, aq/f, q)\infty} \psi_4 \left[ \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{f}, \frac{aq}{q} \right] \]
\[ = \frac{(aq/c, aq/d, aq/e, aq/f, q)\infty}{(aq/c, aq/d, aq/e, aq/f, q)\infty} \psi_4 \left[ \frac{aq}{c}, \frac{aq}{d}, \frac{aq}{f}, \frac{aq}{q} \right]. \]

5.12 Using (5.6.2) and (2.11.7), show that
\[ \psi_8 \left[ qa, -qa, ab, ac, ad, ae, af, ag \right] \]
\[ = \frac{(aq, q/a^2, q/a, q/b, q/c, q/d, q/e, q/f, q/g, q/h, q)\infty}{(aq, q/c, q/d, q/e, q/f, q/g, q/h, q)\infty} \psi_4 \left[ q/a, q/b, q/c, q/d, q/e, q/f, q/g, q/h, q/i, q/j \right] \]
\[ = \psi_4 \left[ q/da, q/db, q/dc, q/dd, q/de, q/df, q/dg, q/dh, q/di, q/dj \right] \]
\[ = \psi_4 \left[ q/da, q/db, q/dc, q/dd, q/de, q/df, q/dg, q/dh, q/di, q/dj \right]. \]

(Bailey [1950a])
provided $bcdef = q$ and 

$$
(bf, q/bf, cf, q/cf, df, q/df, ef, q/ef, af, q/af, bf, q/bf, cf, q/cf, df, q/df, ef, q/ef, af, q/af, bf, q/bf, cf, q/cf, df, q/df, ef, q/ef, af, q/af, bf, q/bf) = 1.
$$

Following Gosper [1988b], we may call this the bilateral Jackson formula.

5.13 Deduce from Ex. 5.12 the bilateral $q$-Sasalschütz formula 

$$
\sum_{n=-\infty}^{\infty} \frac{(a, b, c; q)_n}{(d, e, f; q)_n} q^n = \frac{(q, d/a, d/b, d/c, e/a, e/b, e/c, q/f; q)_{\infty}}{(d, e, aq/f, bq/f, cq/f, q/a, q/b, q/c; q)_{\infty}},
$$

provided $def = abcq^2$ and 

$$
(e/a, aq/e, e/b, e/c, e/c, e/c, e/c, e/c; q)_{\infty} = \frac{(f/a, aq/f, f/b, bq/f, f/c, cq/f, q/e; q)_{\infty}}{(c + d)(a, b, -bc/d, -ad/c; q)_{\infty}}.
$$

5.14 Show that 

$$
\int_{c}^{d} \frac{(-qt/c, qt/d; q)_{\infty}}{(-at/c, bt/d; q)_{\infty}} dx = \frac{d(1 - q)}{1 - b} \psi_{1}(q/a; bq; q, -ad/c)
$$

$$
= \frac{cd(1 - q)}{(c + d)(a, b, -bc/d, -ad/c; q)_{\infty}}
$$

when $|ab| < |ad/c| < 1$. 

(Andres and Askey [1981])

5.15 Show that 

$$
\int_{-\infty}^{\infty} \frac{(ct, -dt; q)_{\infty}}{(-at, -bt; q)_{\infty}} dt = \frac{2(1 - q)(c/a, d/b, -c/b, -d/a, ab, q/ab; q)_{\infty}}{(cd/abq, q; q)_{\infty}(2^2/2^2)}
$$

(Askey [1981])

5.16 Show that 

$$
\int_{0}^{\infty} \frac{(aat, a/t, abt, b/t, act, c/t, adt, d/t; q)_{\infty}}{(aq^2, q/ata^2; q)_{\infty}} dt = \frac{(1 - q)(aa, a, ab, b, ac, c, ad, d; q)_{\infty}}{(q/\alpha, q/\alpha; q)_{\infty}}
$$

$$
\psi_{6}\left[q \sqrt{\alpha}, -q \sqrt{\alpha}, q/a, q/b, q/c, q/d; \sqrt{\alpha}, -\sqrt{\alpha}, ba, ab, ac, ad; q, \alpha \cdot 2abcd\right]
$$

$$
= \frac{(1 - q)(qa, aq/b, aq/c, aq/b, qabq, qabq, qabq, qabq; q)_{\infty}}{(a^2abcd/qa^2; q)_{\infty}}
$$

when $|a^2abcd/qa^2| < 1$.

Exercises 5

5.17 Show that 

$$
\int_{0}^{\infty} \frac{(a_1 t, \ldots, a_r t, b_1 t, \ldots, b_s t; q)_{\infty}}{(c_1 t, \ldots, c_r t, d_1 t, \ldots, d_s t; q)_{\infty}} t^{r-s} dt = \frac{(1 - q)(a_1, \ldots, a_r, b_1, \ldots, b_s; q)_{\infty}}{(c_1, \ldots, c_r, d_1, \ldots, d_s; q)_{\infty}}
$$

when $\frac{b_1 \ldots b_s}{d_1 \ldots d_s} q^n < 1 < \frac{c_1 \ldots c_r}{a_1 \ldots a_r} q^n$.

5.18 Derive Bailey’s [1950b] summation formulas:

(i) \( \psi_{3} \left[ b, c, d; q_b, q_c, q_d; q, q_{bcd} \right] = \frac{(q, b/c, q/bd, q/cd; q)_{\infty}}{(q/b, q/c, q/d; q_{bcd}; q)_{\infty}} \)

(ii) \( \psi_{3} \left[ q^2/b, q^2/c, q^2/d; q_{bcd} \right] = \frac{(q^2/b, q^2/c, q^2/d; q_{bcd}; q)_{\infty}}{(q^2/b, q^2/c, q^2/d; q_{bcd}; q)_{\infty}} \)

(iii) \( \psi_{5} \left[ q, b, c, d, e, q_{-n} \right] = \frac{(q, q/bc, q/bd, q/cd; q)_{\infty}}{(q/b, q/c, q/d; q_{bcd}; q)_{\infty}} \)

(iv) \( \psi_{5} \left[ q^2/b, q^2/c, q^2/d, q^2/e, q_{n+1} \right] = \frac{(1 - q)(q^2/b, q^2/c, q^2/d, q^2/e, q_{n+1}; q)_{\infty}}{(q^2/b, q^2/c, q^2/d, q^2/e, q_{n+1}; q)_{\infty}} \)

where $n = 0, 1, \ldots$.

5.19 Show that 

(i) \( \sum_{k=-\infty}^{n} (-1)^k \left[ \frac{2n}{n+k} \right]_{q}^3 = \frac{(q, q)_{3n}}{(q, q, q; q)_{n}} \)

and 

(ii) \( \sum_{k=-n-1}^{n} (-1)^k \left[ \frac{2n+1}{n+k+1} \right]_{q}^3 = \frac{(q, q)_{3n+1}}{(q, q, q; q)_{n}} \)

for $n = 0, 1, \ldots$ (Bailey [1950b])

5.20 Derive the $2\psi_{2}$ transformation formulas

(i) \( 2\psi_{2} \left[ \begin{array}{c} a, b \\ c, d \end{array}; q, z \right] = \frac{(az, d/a, c/b, dq/abz; q)_{\infty}}{(z, d, q/b, cd/abz; q)_{\infty}} \)

(ii) \( 2\psi_{2} \left[ \begin{array}{c} a, b \\ c, d \end{array}; q, z \right] = \frac{(az, bz, cq/abz, dq/abz; q)_{\infty}}{(q/a, q/b, c, d; q)_{\infty}} \)
5.21 Verify that
\[
\prod_{\alpha=1}^{d} \frac{a_{\alpha} q / b_{\alpha}}{a_{\alpha} q / b_{\alpha}} = \left(\frac{a_{\alpha} q / b_{\alpha}}{a_{\alpha} q / b_{\alpha}}\right)_{\infty}
\]
\[
=f(a_{\alpha} q / b_{\alpha}) = g(a_{\alpha} q / b_{\alpha}) = h(a_{\alpha} q / b_{\alpha}) = i(a_{\alpha} q / b_{\alpha}) = j(a_{\alpha} q / b_{\alpha}) = k(a_{\alpha} q / b_{\alpha}) = l(a_{\alpha} q / b_{\alpha}) = m(a_{\alpha} q / b_{\alpha}) = n(a_{\alpha} q / b_{\alpha}) = o(a_{\alpha} q / b_{\alpha}) = p(a_{\alpha} q / b_{\alpha}) = q(a_{\alpha} q / b_{\alpha}) = r(a_{\alpha} q / b_{\alpha}) = s(a_{\alpha} q / b_{\alpha}) = t(a_{\alpha} q / b_{\alpha}) = u(a_{\alpha} q / b_{\alpha}) = v(a_{\alpha} q / b_{\alpha}) = w(a_{\alpha} q / b_{\alpha}) = x(a_{\alpha} q / b_{\alpha}) = y(a_{\alpha} q / b_{\alpha}) = z(a_{\alpha} q / b_{\alpha}) = \infty
\]
(Bailey [1950a])

5.22 Extend the above identity to
\[
\prod_{\alpha=1}^{d} \frac{a_{\alpha} q / b_{\alpha}}{a_{\alpha} q / b_{\alpha}} = \left(\frac{a_{\alpha} q / b_{\alpha}}{a_{\alpha} q / b_{\alpha}}\right)_{\infty}
\]
\[
= g(a_{\alpha} q / b_{\alpha}) = h(a_{\alpha} q / b_{\alpha}) = i(a_{\alpha} q / b_{\alpha}) = j(a_{\alpha} q / b_{\alpha}) = k(a_{\alpha} q / b_{\alpha}) = l(a_{\alpha} q / b_{\alpha}) = m(a_{\alpha} q / b_{\alpha}) = n(a_{\alpha} q / b_{\alpha}) = o(a_{\alpha} q / b_{\alpha}) = p(a_{\alpha} q / b_{\alpha}) = q(a_{\alpha} q / b_{\alpha}) = r(a_{\alpha} q / b_{\alpha}) = s(a_{\alpha} q / b_{\alpha}) = t(a_{\alpha} q / b_{\alpha}) = u(a_{\alpha} q / b_{\alpha}) = v(a_{\alpha} q / b_{\alpha}) = w(a_{\alpha} q / b_{\alpha}) = x(a_{\alpha} q / b_{\alpha}) = y(a_{\alpha} q / b_{\alpha}) = z(a_{\alpha} q / b_{\alpha}) = \infty
\]
where \(a_{\alpha} q^2 = bcdefgh\). (Slater [1954a])

5.23 More generally, show that it follows from the general formula for sigma functions in Whittaker and Watson [1965, p. 451, Example 3] that
\[
\sum_{k=1}^{n} \frac{(a_{k} / b_{1}, a_{k} / b_{2}, \ldots, a_{k} / b_{n}; q)_{\infty}}{(a_{k} / a_{1}, a_{k} / a_{2}, \ldots, a_{k} / a_{n}; q)_{\infty}}
\]
\[
= \frac{(qa_{1} / a_{1}, qa_{2} / a_{2}, \ldots, qa_{n} / a_{n}; q)_{\infty}}{(q_{a_{1} / a_{1}, q_{a_{2} / a_{2}}, \ldots, q_{a_{n} / a_{n}}; q)_{\infty}} = 0,
\]
where \(a_{1} a_{2} \cdots a_{n} = b_{1} b_{2} \cdots b_{n}\). (Slater [1954a])

Notes 5

§5.2 Andrews [1979c] used Ramanujan's sum (5.2.1) to prove a continued fraction identity that appeared in Ramanujan's [1988] "lost" notebook. Formal Laurent series and Ramanujan's sum are considered in Askey [1987]. A probabilistic proof of (5.2.1) can be found in Kadell [1987b]. Milne [1986, 1988a, 1989c] derived multidimensional \(U(n)\) generalizations of (5.2.1).

§5.3 Gustafson [1987b, 1989a,b] derived a multilateral generalization of (5.2.1), (5.3.1) and related formulas by employing contour integration and Milne's [1985d,e, 1989a,b] work on \(U(n)\) generalizations of the q-Gauss, q-Saalschütz, and very-well-poised \(\phi_{5}\) summation formulas.

§5.4 M. Jackson [1954] employed (5.4.3) to derive transformation formulas for \(\psi_{3}\) series.
The Askey-Wilson $q$-extension of the beta integral

It should be clear by now that the beta integral and extensions of it that can be evaluated compactly are important. A significant extension of the beta integral was found by Askey and Wilson [1985]. Since it has five degrees of freedom, four free parameters and the parameter $q$ from basic hypergeometric functions, it has enough flexibility to be useful in many situations. This integral is

$$
\int_{-1}^{1} \frac{h(x;1,-1,q^{\frac{1}{2}},-q^{\frac{1}{2}})}{h(x;a,b,c,d)} \frac{dx}{\sqrt{1-x^2}} = \frac{2\pi(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty},
$$

where

$$
\begin{align*}
&h(x; a_1, a_2, \ldots, a_m) = h(x; a_1, a_2, \ldots, a_m; q) \\
&= h(x; a_1)h(x; a_2) \cdots h(x; a_m), \\
&h(x; a) = h(x; a; q) = \prod_{n=0}^{\infty} (1 - 2aq^n + a^2q^{2n}) \\
&= (ae^{i\theta}, ae^{-i\theta}; q)_\infty, \quad x = \cos \theta,
\end{align*}
$$

and

$$
\max (|a|, |b|, |c|, |d|, |q|) < 1.
$$

As in (6.1), we shall use the $h$ notation without the base $q$ displayed when the base is $q$.

Askey and Wilson deduced (6.1) from the contour integral

$$
\frac{1}{2\pi i} \int_{K} \frac{(z^2, z^{-2}; q)_\infty}{(az, az^{-1}, bz, bz^{-1}, cz, cz^{-1}, dz, dz^{-1}; q)_\infty} \frac{dz}{z} = \frac{2(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty},
$$

where the contour $K$ is as defined in §4.9 and the parameters $a, b, c, d$ are no longer restricted by (6.1.3), but by the milder restriction that their pairwise products are not of the form $q^{-j}$, $j = 0, 1, 2, \ldots$. Askey and Wilson's original proof of (6.1.4) required a number of interpolation arguments that had to be removed by continuity and analytic continuation arguments. In their paper they also provided a direct evaluation of the reduced integral

$$
\int_{-1}^{1} \frac{h(x;1,-1)}{h(x;a,b)} \frac{dx}{\sqrt{1-x^2}} = \frac{2\pi(-abq; q)_\infty}{(q,q, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -aq^{\frac{1}{2}}, bq^{\frac{1}{2}}, -bq^{\frac{1}{2}}, ab; q)_\infty}
$$

by using summation formulas for $\psi_1$ and $\psi_2$ series. Simpler proofs of (6.1.1) were subsequently found by Rahman [1984] and Ismail and Stanton [1988].

In the following section we shall give Rahman's proof since it only uses formulas that we have already proved, whereas the Ismail and Stanton proof uses some results for certain orthogonal polynomials which will not be covered until Chapter 7.

We shall conclude this section by showing that the beta integral

$$
\int_{-1}^{1} (1-x)^{a}(1+x)^{b} dx = 2^{a+b+1} \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}
$$

is a limit case of (6.1.5).

Let $0 < q < 1$, $a = q^{a+\frac{1}{2}}$, $b = q^{b+\frac{1}{2}}$ and use the notation

$$
(z; q)_\alpha = \frac{(z; q)_\infty}{(q^\alpha z; q)_\infty}
$$

and the definition (10.1.1) of the $q$-gamma function to express the right side of (6.1.5) in the form

$$
2^{a+b+2} \frac{\Gamma\alpha(\alpha+1)\Gamma\beta(\beta+1)}{\Gamma\alpha+\beta(\alpha+\beta+2)} \frac{\pi}{\Gamma\left(\frac{1}{2}\right)} \frac{\left(-q; q\right)_{\alpha+\beta}(-q^{\frac{1}{2}}; q)_{\alpha+\frac{1}{2}}(-q^{\frac{1}{2}}; q)_{\beta+\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)}
$$

By (10.1.3) this tends to $2^{a+b+1} \Gamma(a+1)\Gamma(b+1)/\Gamma(a+b+2)$ as $q \to 1^-$, since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

For the integrand in (6.1.5) we have

$$
\frac{h(x;1,-1)}{h(x;a,b)} = \left(e^{i\theta}; q\right)_{\alpha+\frac{1}{2}} \left(e^{-i\theta}; q\right)_{\alpha+\frac{1}{2}} \left(-e^{i\theta}; q\right)_{\beta+\frac{1}{2}} \left(-e^{-i\theta}; q\right)_{\beta+\frac{1}{2}}
$$

and hence

$$
\lim_{q \to 1^-} \frac{h(x;1,-1)}{h(x;q^{a+\frac{1}{2}},-q^{b+\frac{1}{2}})} = \left[\left(1 - e^{i\theta}\right)\left(1 - e^{-i\theta}\right)\right]^{\alpha+\frac{1}{2}} \left[\left(1 + e^{i\theta}\right)\left(1 + e^{-i\theta}\right)\right]^{\beta+\frac{1}{2}}
$$

$$
= 2^{a+b+1}(1 - \cos \theta)^{\alpha+\frac{1}{2}}(1 + \cos \theta)^{\beta+\frac{1}{2}},
$$

which shows that (6.1.6) is a limit of (6.1.5).

Formula (6.1.1) is substantially more general than (6.1.5) since it contains two more parameters. It is the freedom provided by these extra parameters which will enable us to prove a number of important results in this and the subsequent chapters.
6.2 Proof of formula (6.1.1)

Denote the integral in (6.1.1) by \( I(a, b, c, d) \). Since \( x = \cos \theta \) is an even function of \( \theta \), one can write

\[
I(a, b, c, d) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{h(x; 1, -1, \sqrt{q}, -\sqrt{q})}{h(x; a, b, c, d)} \, d\theta. \tag{6.2.1}
\]

Let us assume, for the moment, that \( a, b, c, d \) and their pairwise products and quotients are not of the form \( q^{-j} \), \( j = 0, 1, 2, \ldots \). It is easy to check that, by (2.10.18),

\[
h(x; 1)/h(x; a, b) = \frac{(a^{-1}, b^{-1}; q)_\infty}{b(1-q)(q, a/b, q/b, a; q)_\infty} \int_a^b \frac{(u/a, u/b; q)_\infty}{h(x; u)} \, du,
\]

\[
h(x; 1)/h(x; c, d) = \frac{(-c^{-1}, -d^{-1}; q)_\infty}{d(1-q)(q, c/d, dq/cd; q)_\infty} \int_c^d \frac{(v/c, v/d; q)_\infty}{h(x; v)} \, dv,
\]

and

\[
h(x; -q^{1/2})/h(x; u, v) = \frac{q^{1/2}(-q^{1/2}u^{-1} - q^{1/2}v^{-1}; q)_\infty}{v(1-q)(q, u/v, q/v, u, v; q)_\infty} \int_{u^2}^{v^2} \frac{(tq^{1/2}/u, tq^{1/2}/v; q)_\infty}{h(x; t^{1/2})} \, dt.
\]

Also,

\[
\int_{-\pi}^{\pi} \frac{h(x; q^{1/2})}{h(x; t^{1/2})} \, d\theta = \frac{1}{2} \int_{-\pi}^{\pi} \frac{(q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta}; q)_\infty}{(t^{1/2}e^{i\theta}, t^{1/2}e^{-i\theta}; q)_\infty} \, d\theta
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(t^{-1}; q)_k(t^{-1}; q)_\ell}{(q; q)_k(q; q)_\ell} (t^{1/2})^{k+\ell} e^{i(k-\ell)\theta} \, d\theta
\]

\[
= \frac{1}{2} \sum_{k=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(t^{-1}; q)_k(t^{-1}; q)_\ell}{(q; q)_k(q; q)_\ell} (t^{1/2})^{k+\ell} \int_{-\pi}^{\pi} e^{i(k-\ell)\theta} \, d\theta
\]

\[
= \pi \sum_{k=0}^{\infty} \frac{(t^{-1}, t^{-1}; q)_k(q^{2}; q)_k}{(q; q)_k(q^{2}; q)_\infty},
\]

for \( |tq^{1/2}| < 1 \), by (1.5.1). Since

\[
(q^{2}; q)_\infty = (q^{2}, q^{2}; q^{2})_\infty = (tq^{1/2}, -tq^{1/2}, qt, -qt; q)_\infty,
\]

we have

\[
\frac{1}{2} \int_{-\pi}^{\pi} \frac{h(x; q^{1/2})}{h(x; t^{1/2})} \, d\theta = \pi(q^{1/2}; q^{2}; q^{2})_\infty \, (q^{1/2}, -q^{1/2}, -tq^{1/2}; q)_\infty.
\]

Thus

\[
I(a, b, c, d) = \frac{\pi t^{1/2}(a^{-1}, b^{-1}, c^{-1}, d^{-1}; q)_\infty}{bd(1-q)(q, a/b, q/b, a; q)_\infty} \int_a^b \frac{d\theta}{(u/a, u/b, q)_\infty} \int_c^d \frac{d\theta}{(v/c, v/d, -v, -v; q)_\infty} \int_{u^2}^{v^2} \frac{d\theta}{h(x; t^{1/2})}.
\]

by repeated applications of (2.10.18).

Since \( (-1, q^{3/2}, -q^{3/2}; q)_\infty = 2(q^{1/2}, -q^{1/2}, q^{3/2}; q)_\infty = 2 \), which follows from (6.2.6) by setting \( t = 1 \), we get (6.1.1). By analytic continuation, the restrictions on \( a, b, c, d \) mentioned above may be removed.

6.3 Integral representations for very-well-poised \( \phi_7 \) series

Formulas (2.10.18) and (2.10.19) enable us to use the Askey-Wilson \( q \)-beta integral (6.1.1) to derive Riemann integral representations for very-well-poised \( \phi_7 \) series.

Let us first set

\[
w(x; a, b, c, d) = (1 - x^2)^{-1/4} h(x; 1, -1, q^{3/2}, -q^{3/2})
\]

and

\[
J(a, b, c, d, f, g) = \int_{-1}^{1} w(x; a, b, c, d) \frac{h(x; q)}{h(x; f)} \, dx,
\]

where \( \max(|a|, |b|, |c|, |d|, |f|, |g|) < 1 \) and \( g \) is arbitrary. Since, by (2.10.18),

\[
\frac{h(x; g)}{h(x; f)} = \frac{(g/d, g/f; q)_\infty}{f(1-q)(q, d/f, q/f, d, f; q)_\infty} \int_d^f \frac{d\theta}{(q^{2}; q)_\infty h(x; t^{1/2})},
\]

we have

\[
J(a, b, c, d, f, g) = \frac{(g/d, g/f; q)_\infty}{f(1-q)(q, d/f, q/f, d, f; q)_\infty}.
\]
replacing $e^{i\theta}$ by $q^t$. Now let $q \to 1^-$ and use (1.10.1) and (1.10.3) to get the formula
\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\Gamma(a+it)\Gamma(a-it)\Gamma(b+it)\Gamma(b-it)\Gamma(c+it)\Gamma(c-it)}{\Gamma(2it)\Gamma(-2it)} dt
\]
\[
= \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(a+f)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+g)\Gamma(b+g)\Gamma(c+g)\Gamma(d+c)\Gamma(d+f)\Gamma(d+g)}
\]
\[
\cdot \Gamma(a) \cdot \Gamma(b+c+g-1, \frac{1}{2}(a+b+c+g+1), a+b, a+c, b+c, c+g, b+g, a+g, g-d, g-f, a+b+c+d, a+b+c+f, 1),
\]
where $Re(a, b, c, d, f) > 0$.

6.4 Integral representations for very-well-poised $\phi_9$ series

If we set $g = abcdf$ in (6.3.7), then the $_8W_7$ series collapses to one term with value 1 and so we have the formula
\[
\int_{-1}^{1} h(x; 1, -1, q^\frac{1}{2}, -q^\frac{1}{2}, abcdf) \frac{dx}{h(x; a, b, c, d, f)} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_0^1 x^\alpha(1-x)^\beta(1-xx^{-1}) dx,
\]
which was derived in Nassrallah and Rahman [1985], and
\[
\int_{-1}^{1} x^{a-1}(1-x)^{b-1}(1-tx)^{c-1} \cdot \frac{dx}{h(x; a, b, c, d, f)} = \frac{\Gamma(a+b)}{\Gamma(1-t)^{-a-b}} \cdot \Re(a, b) > 0.
\]

Replace $f$ by $f^n$ in (6.4.1), where $n$ is a nonnegative integer, to get
\[
\int_{-1}^{1} v(x; a, b, c, d, f) \frac{dx}{(abcdf e^{i\theta})_n} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{(af, bf, cf, df)_n}{(bcdf, acdf, abdf, abef)_n},
\]
where $v(x; a, b, c, d, f) = (1-x^2)^{-\frac{1}{2}} h(x; 1, -1, q^1, -q^{-1}, abcdf)$

Let $\sigma = abcdf$. If $|z| < 1$, then (6.4.2) gives the formula
\[
\int_{-1}^{1} v(x; a, b, c, d, f) \frac{dx}{(abcdf e^{i\theta})_n} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \frac{(af, bf, cf, df)_n}{(bcdf, acdf, abdf, abef)_n},
\]
where $v(x; a, b, c, d, f) = (1-x^2)^{-\frac{1}{2}} h(x; 1, -1, q^1, -q^{-1}, abcdf)$.
In particular, for \( r = 3 \) and \( z = \sigma^2 / a_1 a_2 a_3 \), we have the formula

\[
\int_{-1}^{1} v(x; a, b, c, d, f) \frac{\psi W(x q^{-1} a_1, a_2, a_3, q e^{i\theta}, q e^{-i\theta}; q, q^2 / a_1 a_2 a_3)}{\sqrt{1 - x^2}} \, dx
\]

where \( \sigma^2 / a_1 a_2 a_3 \) is the ratio of the series, if the series do not terminate.

Let us assume that

\[ a_1 a_2 a_3 = \sigma^2 \]

which ensures that the very-well-poised series on either side of (6.4.5) are balanced. Then, by (2.11.7)

\[
\psi W(x q^{-1} a_1, a_2, a_3, q e^{i\theta}, q e^{-i\theta}; q, q)
\]

and hence

\[
\int_{-1}^{1} \frac{h(x; 1, -1, q; \sigma a_2 / a_3)}{h(x; a, b, c, d, f, q)} \, dx
\]

where \( \sigma = abcdf \) and \( a_1 a_2 a_3 = \sigma^2 \).

Since the integrand on the left side of (6.4.8) is symmetric in \( a, b, c, d, f \), the expression on the right side must have the same property. This provides an alternate proof of Bailey's four-term transformation formula (2.12.9) for very-well-poised \( 10 \phi_9 \) series which are balanced and nonterminating.

If we set \( a_3 = q^{-n} \), \( n = 0, 1, 2, \ldots \), then the coefficient of the second \( 10 \phi_9 \)

on the right side of (6.4.8) vanishes and we obtain

\[
\int_{-1}^{1} \frac{h(x; 1, -1, q; \sigma q^{2n} / a_3)}{h(x; a, b, c, d, f, q)} \, dx
\]

where \( \nu = \sigma q^{-1} \). By applying the iteration of the transformation formula

(2.9.1) given in Exercise 2.19, this can be written in the more symmetric form

\[
\int_{-1}^{1} \frac{\psi W(x; 1, -1, q; \sigma q^{2n} / a_3)}{\sqrt{1 - x^2}} \, dx
\]

where \( \sigma = abcdf \) and \( \tau \) is arbitrary. Similarly, by applying (2.12.9) twice one can rewrite (6.4.8) in the form

\[
\int_{-1}^{1} \frac{h(x; 1, -1, q; \sigma a_2 / a_3)}{h(x; a, b, c, d, f, g)} \, dx
\]

where \( \nu_1 = \lambda / \mu / \nu \) and \( \nu_2 = \lambda / \mu \).
6.5 A quadratic transformation formula for very well-poised balanced $10\phi_9$ series

In (6.4.11) let us set $\mu = -\lambda$, $g = -f$ and $b = -a$ so that $\lambda^2 = -a^2 c df^2$. Then the expression on the right side of (6.4.11) becomes

$$2 \frac{2(\lambda^2 / c, \lambda^2 / d; q)_\infty (\lambda^2 / a_2, \lambda^2 f^2; q^2)_\infty}{(q, 1 - a^2, -f^2, \lambda^2 q; q)_\infty (a^2 c^2, a^2 d^2, \lambda^2 f^2; q^2)_\infty (cd, df, d; q)_\infty}
+ 10 W_9(\lambda^2 q^{-1}; -\lambda^2 q^{-1}, a f, -a f, \lambda / f, -\lambda / f, c, f, df, d; q)_\infty$$

$$+ 10 W_9(\lambda^2 q^{-1}; -\lambda^2 q^{-1}, a f, -a f, \lambda / f, -\lambda / f, c f, d f, d; q)_\infty$$

$$+ 2(\lambda^2 / c, -\lambda^2 / d^2; q)_\infty (\lambda^2 / a_2, \lambda^2 f^2, \lambda^2 f^2; q^2)_\infty (cd, -f, -d f, -d; q)_\infty$$

$$\times 10 W_9(\lambda^2 q^{-1}; -\lambda^2 q^{-1}, a f, -a f, \lambda / f, -\lambda / f, c, d f, d; q)_\infty$$

(6.5.1)

We now turn to the integral on the left side of (6.4.11). Observing that

$$h(x; a, -a) = (a^2 c^2, a^2 d^2, \lambda^2 f^2; q^2)_\infty = h(\xi; a^2, \lambda^2 q^2),$$

(6.5.2)

where $x = \cos \theta$ and $\xi = \cos 2\theta = 2x^2 - 1$, it follows from (2.10.18) that

$$\frac{h(x; \lambda - \lambda)}{h(\xi; \lambda^2 / a^2, \lambda^2 / f^2)} = \left(\frac{\lambda^2 / a^2, \lambda^2 / f^2; q^2}_\infty\right)^{-1}$$

$$\times \int_{a^2}^{\infty} \left(\frac{q^2 u / a^2, q^2 u / f^2, \lambda^2 u q^2; q^2}_\infty\right)^{-1} dq u \int_{x^2}^{1} w(x; c, d, u, -u^2) dx$$

(6.5.3)

Hence the integral on the left side of (6.4.11) can be expressed as

$$\int_{a^2}^{\infty} \frac{(\lambda^2 / a^2, \lambda^2 / f^2; q^2)_\infty}{f^2(1 - q^2)(q^2, a^2 f^2, q^2 f^2/a^2, a^2 f^2; q^2)_\infty}$$

$$\times \int_{a^2}^{\infty} \left(\frac{q^2 u / a^2, q^2 u / f^2, \lambda^2 u q^2; q^2}_\infty\right)^{-1} dq u \int_{x^2}^{1} w(x; c, d, u, -u^2) dx$$

(6.5.4)

by (6.1.1). Since $\lambda^2 = -a^2 c d f^2$, the $q$-integral on the right side of (6.5.4) reduces to

$$\int_{a^2}^{\infty} \frac{(q^2 u / a^2, q^2 u / f^2, -cd qu, -a^2 c d f^2 u; q^2)_\infty}{(c^2 u, d^2 u, -u, -u^2 q^2; q^2)_\infty} d q u$$

$$\times \int_{a^2}^{\infty} \left(\frac{q^2 u / a^2, q^2 u / f^2, \lambda^2 u, cd u, -cd q u; q^2}_\infty\right)^{-1} d q u$$

(6.5.5)

Hence the desired quadratic transformation formula (3.10.3) for the bisasic $\Phi$ series is a balanced $10\phi_9$. For an extension of (3.10.3) when the $\Phi$ series is not balanced, see Nassrallah and Rahman [1986].

6.6 The Askey-Wilson integral when max $(\lambda, |a|, |b|, |c|, |d|) \geq 1$

Our aim in this section is to extend the Askey-Wilson formula (6.1.1) to cases in which $|q| < 1$ and the absolute value of at least one of the parameters is greater than or equal to 1. Since the integral in (6.1.1), which we have already denoted by $I(a, b, c, d)$, is symmetric in $a, b, c, d$, without loss of generality we may assume that $|a| = 1$.

Let us first consider

$$|a| \geq 1 > \max(|b|, |c|, |d|).$$

(6.6.1)

If $a = \pm 1$, then the functions $h(x; \pm 1)$ in the integrand in (6.1.1) cancel and by continuity it follows that

$$I(\pm 1, b, c, d) = \frac{2\pi(\pm bcd; q)_\infty}{(q, \pm b, \pm c, \pm d, bc, bd, cd; q)_\infty}.$$

(6.6.2)

However, if $|a| = 1$ and $a \neq \pm 1$, then $h(x; a) = 0$ for some $x$ in the interval $(-1, 1)$ and so the integral in (6.1.1) does not converge. Similarly, this integral does not converge if $|a|^n = 1$ and $a^n \neq \pm 1$ for some positive integer $n$.

If there is a nonnegative integer $m$ such that

$$|aq^{m+1}| < 1 < |aq^m|$$

(6.6.3)

and if $ab, ac$ and $ad$ are not of the form $q^n$ for any nonnegative integer $n$, then the integral in (6.1.1) converges and we can evaluate it by the following technique.

Observe that, since

$$h(x; a) = (ae^{i\theta}, ae^{-i\theta}; q)_{m+1} h(x; a q^{m+1})$$

$$= a^{2m+2} q^{m+2} h(x; a q^{m+1}, q^{-m}/a) h(x; a),$$

(6.6.4)
The Askey-Wilson q-beta integral

we have

\[ I(a, b, c, d) = a^{-2m-2}q^{-m-m^2} \]
\[ \int_{-1}^{1} \frac{h(x; 1, -1, q^2, q^2, q/a)}{h(x; b/c, d, aq^{-m+1}, q^{-m}/a)} \frac{dx}{(q, q)_\infty} \]
\[ \text{(6.6.5)} \]

where the parameters \( b, c, d, aq^{-m+1}, q^{-m}/a \) in the denominator of the integrand are now all less than 1 in absolute value. By (6.3.8),

\[ I(a, b, c, d) = 2\pi(\frac{ab}{a}, a, q^{-m}/a, q^{-m}/a; q)_\infty \]
\[ = \frac{a^{-2m-2}q^{-m-m^2}}{(bq^{-m}/a, c^{-m}/a, dq^{-m}/a, q^{-m}/a; q)_\infty} \]
\[ \cdot s_{W_T}(bc^{-d^{-1}}, bc, c^{-m}/ad, aq^{m^{-1}}/d, ad, c; q, q). \]
\[ \text{(6.6.6)} \]

The series in (6.6.6) is balanced and so we can apply Bailey's summation formula (2.11.7). After some simplification we find that

\[ I(a, b, c, d) = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} + L_m(a; b, c, d), \]
\[ \text{(6.6.7)} \]

where

\[ L_m(a; b, c, d) = 2\pi(abq/d, dq/d, c/d, aq^{m^{-1}}/d, aq^{m^{-1}}/d, bcdq^{-m}/a; q)_\infty \]
\[ = \frac{a^{-2m-1}d^{-1}q^{-m-m^2}}{(bq^{-m}/a, c^{-m}/a, dq^{-m}/a, q^{-m}/ad; q)_\infty} \]
\[ \cdot s_{W_T}(abcd^{-1}, bc, c^{-m}/ad, aq^{m^{-1}}/d, ad, c; q, q). \]
\[ \text{(6.6.8)} \]

By (2.11.1),

\[ s_{W_T}(abcd^{-1}, bc, c^{-m}/ad, aq^{m^{-1}}/d, ad, c; q, q) \]
\[ = \frac{(abcd/d, q/ad, abq, acq; q)_\infty}{(aq^{m^{-1}}/d, abq, acq; q)_\infty} \]
\[ \cdot s_{W_T}(a^2/bc, abcd, ab, ac, a^2q^{m^{-1}}, q^{-m}; q, q/ad). \]
\[ \text{(6.6.9)} \]

Since \( m \) is a nonnegative integer, the series on the right side of (6.6.9) terminates and hence, by Watson's formula (2.5.1),

\[ s_{W_T}(a^2bc, abcd, ab, ac, a^2q^{m^{-1}}, q^{-m}; q, q/ad) = \frac{(a^2bcq, q; q)_m}{(abq, acq; q)_m} \]
\[ \cdot 4\pi \left[ \begin{array}{c} q^{-m}, a, ab, ac, aq^{-m}/ad, bcdq^{-m}, aq^{-m}/ad; q, q \end{array} \right] \]
\[ = \frac{(a^2bcq, q; q)_m}{(abq, acq, q/bc; q)_m} \]
\[ \cdot \left[ a^2, qa, -aq/b, ab, ac, ad, a^2q^{m^{-1}}, q^{-m}, q, q/ad, \right. \]
\[ \left. \frac{a, a, a, abq, acq, aq/d, q^{-m}, a^2q^{m^{-1}}, q^{-m}q, q}{abcd} \right] \]
\[ = \frac{(a^2bcq, q; q)_m}{(abq, acq, q/bc; q)_m} \]
\[ \cdot \sum_{k=0}^{m} \frac{(a^2q)_k(1-a^2q^{2k})(ab, ac, ad, q/k)}{(q; q)_k(1-a^2)(aq/b, acq, aq/d; q)_k} \frac{q_k}{abcd} \]
\[ \text{(6.6.10)} \]

Using (6.6.10) and (6.6.9) in (6.6.8), we obtain

\[ L_m(a; b, c, d) = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, b/c, d/a, q)_\infty} \]
\[ \cdot \sum_{k=0}^{m} \frac{(a^2q)_k(1-a^2q^{2k})(ab, ac, ad, q/k)}{(q; q)_k(1-a^2)(aq/b, acq, aq/d; q)_k} \frac{q_k}{abcd} \]
\[ \text{(6.6.11)} \]

Hence

\[ \int_{-1}^{1} \frac{h(x; 1, -1, q^2, q^2, q/a)}{h(x; b/c, d, aq^{-m+1}, q^{-m}/a)} \frac{dx}{(q, q)_\infty} \]
\[ = \frac{2\pi(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty} \]
\[ \cdot \sum_{k=0}^{m} \frac{(a^2q)_k(1-a^2q^{2k})(ab, ac, ad, q/k)}{(q; q)_k(1-a^2)(aq/b, acq, aq/d; q)_k} \frac{q_k}{abcd} \]
\[ \text{(6.6.12)} \]

where \( \max(|b|, |c|, |d|, |q|) < 1 \), \( |aq^{m^{-1}}| < 1 < |aq| \) for some nonnegative integer \( m \), and the products \( ab, ac, ad \) are not of the form \( q^{-n} \), \( n = 1, 2, \ldots \). Askey and Wilson [1985] proved this formula by using contour integration. By continuity, formula (6.6.12) also holds if the restriction (6.6.3) is replaced by \( aq^{m^{-1}} = \pm 1 \).

Note that, if one of the products \( ab, ac \) or \( ad \) is of the form \( q^{-n} \) for some nonnegative integer \( n \), the integral in (6.6.12) converges even though the denominator on the right side of (6.6.12) equals zero as does the denominator in the coefficient of the sum in (6.6.12). If we let \( ab \) tend to \( q^{-n} \) then, since \( |b| < 1 \) and \( |aq^{m^{-1}}| < 1 < |aq| \), we must have \( n \leq m \). We may then multiply (6.6.12) by \( 1 - abq^n \) and take the limit \( ab \to q^{-n} \). The result is a terminating \( \phi_2 \) series on the left side and its sum on the right, giving the summation formula (2.4.2).

If \( \max(|c|, |d|, |q|) < 1 \) and there are nonnegative integers \( m \) and \( r \) such that

\[ |aq^{m^{-1}}| < 1 < |aq|, \quad |bq^{-r}| < 1 < |bq^r|, \]
\[ \text{(6.6.13)} \]

then the above technique can be extended to evaluate \( I(a, b, c, d) \) provided the products \( ab, ac, ad, bc, bd \) are not of the form \( q^{-n} \) and \( a/b \neq q^{kn} \) for any nonnegative integer \( n \). Splitting \( h(x; b) \) in the same way as in (6.6.4) and using (6.3.4) we get

\[ I(a, b, c, d) \]
\[ = b^{-2}q^{-2}q^{-r}J(a, b; q^{-r+1}, q^{-r}/b, c, d, q/b) \]
\[ = b^{-2}q^{-2}q^{-r} \frac{(q/bc, q/bd; q)_\infty}{d(1-q)(q/c, d, dq/c, cd; q)_\infty} \]
\[ \cdot \int_{c}^{d} du \frac{(q/bc, q/bd, q/c)}{(q/cd, qd/c, cd; q)_\infty} \int_{-1}^{1} w(x; a, bq^{r+1}, q^{-r}/b, u) \frac{dx}{(q, q)_\infty} \]
\[ \text{(6.6.14)} \]
The Askey-Wilson $q$-beta integral

However, by (6.6.12),

$$
\int_{-1}^{1} w(x; b, qy^{r+1}, q^{-r}/b, u) \, dx = 2\pi(qu; q)_{\infty}^{1} (a, u/bq^{r+1}, uq^{-r}/b; q)_{\infty}
$$

Since (6.6.15),

$$
= \frac{2\pi(abq; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty} - \frac{(a, q)_{k}(1 - a^{2}q^{2k})(a; q)_{k}(au/a; q)_{\infty}}{(a, q)_{k}(1 - a^{2})(au/a; q)_{\infty}}}
$$

we find that

$$I(a, b, c, d) = L_{m}(a; b, c, d) + \frac{2\pi(abq; q)_{\infty}}{(q, b, c, d, q^{-r}/b, q^{-r}/b; q)_{\infty} - \frac{(a, q)_{k}(1 - a^{2}q^{2k})(a; q)_{k}(au/a; q)_{\infty}}{(a, q)_{k}(1 - a^{2})(au/a; q)_{\infty}}}
$$

Substituting (6.6.18) into (6.6.17) and simplifying the coefficients we get

$$I(a, b, c, d) = L_{m}(a; b, c, d) - \frac{2\pi(abcd; q)_{\infty}}{(q, ab, ac, ad, bc, bd, cd; q)_{\infty}}
$$

The expression on the right side of (6.6.19) is the same as that in (6.6.8) with $a, b, c, d, m$ replaced by $b, d, c, a$ and $r$, respectively, and so has the value

$$= \frac{2\pi(b^{2}; q)_{\infty}}{(q, ba, bc, bd, a/b, c/b, d/b; q)_{\infty}}
$$

by (6.6.11). So we find that

$$I(a, b, c, d) = L_{m}(a; b, c, d) - L_{r}(b; c, d, a)
$$

where the parameters satisfy the conditions stated earlier.

It is now clear that we can handle the cases of three or all four of the parameters $a, b, c, d$ exceeding 1 in absolute value in exactly the same way. For example, in the extreme case when $\min\{|a|, |b|, |c|, |d|\} > 1 > |q|$ with

$$|aq^{m-1}| < |aq^{m}|, |bq^{-r+1}| < |bq^{-r}|,
$$

for some nonnegative integers $m, r, s, t$ such that the pairwise products of $a, b, c, d$ are not of the form $q^{-m}$ and the pairwise ratios of $a, b, c, d$ are not of the form $q^n$ for $n = 0, 1, 2, \ldots$, we have the formula

$$I(a, b, c, d) = L_{m}(a; b, c, d) - L_{r}(b; c, d, a) - L_{s}(c; d, a, b)
$$

where $L_{n}(c; d, a, b)$ and $L_{n}(d; a, b, c)$ are the same type of finite series as those in (6.6.11) and (6.6.20), and can be written down by obvious replacement of the parameters.
Exercises 6

6.1 Prove that

\[ \int_0^\pi \frac{\sin^2 \theta \, d\theta}{\prod_{j=1}^{4} \left( 1 - 2a_j \cos \theta + a_j^2 \right)} = \frac{\pi(1 - a_1a_2a_3a_4)}{2 \prod_{1 \leq j < k \leq 4} (1 - a_ja_k)} \]

when \( \max(|a_1|, |a_2|, |a_3|, |a_4|) < 1 \).

6.2 Use the \( q \)-binomial theorem and an appropriate transformation formula for the \( 2\phi_1 \) series to show that

\[ \int_{-1}^1 \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a, b, c, d)}{\sqrt{1 - x^2}} \, dx = \frac{2\pi(q; q)^{\infty}}{(q, -q, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}}, aq^2, -aq^2, -a^2q^{\frac{1}{2}}, -a^2q^{\frac{1}{2}}; q)^{\infty}} \quad |a| < 1. \]

6.3 Prove that

\[ \int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}, q^{2\theta})_\infty}{(qe^{2i\theta}, qe^{-2i\theta}, q^{2\theta})_\infty \cos \theta + 1} \, d\theta = \frac{2\pi(q; q)^{\infty}}{(q, q^2, q^3, q^4; q^{\infty}) \sum_{n=0}^\infty \frac{1 - q}{1 - q^{2n+1}} (q^{2n+1})^n}, \]

where \( 0 < q < 1 \) and \( |xq^{\frac{1}{2}}| < 1 \).

6.4 If \( 0 < q < 1 \) and \( \max(|a|, |b|, |c|, |d|) < 1 \), show that

\[ \int_{-1}^1 \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, a, b, c, d)}{\sqrt{1 - x^2}} \, dx = \frac{2\pi(a, b, c, d, -abcq^{\frac{1}{2}}, -abcdq^{\frac{1}{2}}; q^{\infty})}{(q, ab, ac, ad, bc, bd, cd, -aq^2, -aq^2, -aq^2, -aq^2; q^{\infty})} \cdot sW_7(abcfq^{-1}; ab, ac, bc, -f^{\frac{1}{2}}, -f^{\frac{1}{2}}, f/d, q, -aq^{\frac{1}{2}}). \]

6.5 If \( \max(|a|, |b|, |c|, |d|, |q|) < 1 \), show that

\[ \int_{-1}^1 \frac{h(x; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, -q/c)}{\sqrt{1 - x^2}} \, dx = \frac{2\pi(-q; q)^{\infty}(-a^2, b^2, -q^2/c; q^2)^{\infty}}{(q, q^{\infty})(-a^2, -b^2, a^2b^2, a^2b^2, b^2; q^2)^{\infty}}. \]

6.6 If \( b = a^{-1}, |c| < 1, |d| < 1 \) and \( |aq^{m+1}| < 1 < |aq^n| \) for some nonnegative integer \( m \) such that \( ac, ad \) are not of the form \( q^n, n = 0, 1, 2, \ldots \), show that

\[ I(a, a^{-1}, c, d) + \frac{2\pi(qa^2)^{\frac{1}{2}}}{(q, qa, acd, q/c, adq, d/aq; q^2)^{\infty}} \cdot \sum_{k=0}^{m-1} \frac{(1-a^2q^{2k+2})(acq, adq, qk)(q/cd)^k}{(1-q^{2k+1})(1-a^2q^{2k+1})(aq^2/c, q^2/aq^2/d; q^2)_k} \]

\[ = \frac{2\pi}{(q, qa, ac, ad, c/a, d/a; q^2)^{\infty}} \cdot \sum_{k=0}^{\infty} \frac{1}{a^2 - q^2 + (cd)^{-1} - q^{-1} - ac^{-1} - q^{-1} - ad^{-1} - q^{-1}} q^k, \]

where \( I(a, b, c, d) \) is as defined in (6.2.1). If \( m = 0 \), the series on the left side is to be interpreted as zero.

6.7 Applying (2.12.9) twice deduce from (6.4.8) that

\[ \int_0^\pi \frac{h(r \cos \theta; 1, -1, q^{\frac{1}{2}}, -q^{\frac{1}{2}}, \lambda \mu, \lambda)}{h(r \cos \theta; a_1, a_2, a_3, a_4, a_5, a_6)} \, d\theta = \frac{A(q^2; q)^{\infty}}{(\lambda / \mu; q)^{\infty}} \prod_{j=1}^6 \frac{(\lambda a_j q; q)^{\infty}}{(\mu a_j q; q)^{\infty}} \cdot 10 W_9(\lambda q^{-1}; \lambda \mu, \lambda, \lambda q^{-1}, \lambda, \lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4, \lambda a_5, \lambda a_6; q, q) \]

\[ + A(q^2; q)^{\infty} \prod_{j=1}^6 \frac{(\mu a_j q; q)^{\infty}}{(\lambda q; q)^{\infty}} \cdot 10 W_9(\mu q^{-1}; \lambda \mu, \lambda / q, \lambda / a_1, \lambda / a_2, \lambda / a_3, \lambda / a_4, \lambda / a_5, \lambda / a_6; q, q), \]

where \( \lambda \mu = \prod_{j=1}^6 a_j \), \( \max(|a_1|, |a_2|, \ldots, |a_6|) < 1 \), and

\[ A = \frac{2\pi}{(q, q^2; q)^{\infty}} \prod_{1 \leq j < k \leq 6} (a_j a_k q; q)^{\infty}. \]

(Rahman [1986b, 1988b])

6.8 Prove that

\[ sW_7(abcfq^{-1}; ad, az, bz, cz, ab, abcdq^{-1}, q^{-n}; q, q^{\frac{1}{2}}) \]

\[ = \frac{(q, az, bz / c, cz / b, ab, ab/cb, abz / cz, \mu / q; q)^{\infty}}{2\pi(ab, ac, bc, q/dz, q^{-1-n}/d, abcq^{\infty}, qmu^{2}/dz, aq^{1-n}/d; q^2)^{\infty}} \]

\[ \cdot \sum_{k=0}^{m-1} \frac{(1-a^2q^{2k+2})(acq, adq, qk)(q/cd)^k}{(1-q^{2k+1})(1-a^2q^{2k+1})(aq^2/c, q^2/aq^2/d; q^2)_k} \]

\[ \quad \cdot \sum_{k=0}^{\infty} \frac{1}{a^2 - q^2 + (cd)^{-1} - q^{-1} - ac^{-1} - q^{-1} - ad^{-1} - q^{-1}} q^k, \]

where \( sW_7 \) is defined in (6.4.1).
6.12 Show that
\[
\int_0^\infty \int_0^\infty t_1^{-1} t_2^{-1} \left(-at_1 q^y + y + 2k, -at_2 q^x + x + 2k; q\right)_\infty \\
\cdot t_1^{2k} (t_2 q_1^{-k}/t_1; q)_\infty \cdot d_1 d_2
\]
\[
= \frac{\Gamma_q(x) \Gamma_q(x + k) \Gamma_q(y) \Gamma_q(y + k) \Gamma_q(k + 1) \Gamma_q(2k + 1)}{\Gamma_q(x + y + k) \Gamma_q(x + y + 2k) \Gamma_q(k + 1) \Gamma_q(k + 1) \Gamma_q(2k + 1) \Gamma_q(1 - x - 2k)}
\]
\[
\cdot \Gamma_q(x) \Gamma_q(1 - x) \Gamma_q(x + 2k) \Gamma_q(1 - x - 2k)
\]
when Re \(x > 0\), Re \(y > 0\), \(|q| < 1\), and \(k = 0, 1, \ldots\) (Askey [1980b])

6.13 With \(w(x; a, b, c, d)\) defined as in (6.3.1), prove that
\[
\int_0^1 \int_0^1 w(x; a, q^x, b, bq^y) w(y; a, q^y, b, bq^x) \\
\cdot \left( (q^x e^{i(\theta + \phi)}, q^x e^{(\theta - \phi)}; q)_k \right)^2 dxdy
\]
\[
= \prod_{j=1}^2 \left( \frac{1 - ab(q, abq^{j-1}k+1/2, abq^{j-1}k+1; q)_\infty}{(a^2 b^2 q^{k+1}, q^{k+1} q; q)_\infty} \right)
\]
\[
\cdot \prod_{j=1}^2 \left( \frac{\Gamma_q(x^2 q^{j+1} + 1/2, b^2 q^{j-1}k+1/2, q^{k+1}q; q)_\infty}{\Gamma_q(x^2 q^{j+1} + 1/2, b^2 q^{j-1}k+1/2, q^{k+1}q; q)_\infty} \right)
\]
where \(x = \cos \phi\), \(y = \cos \phi\), \(|q| < 1\), and \(k = 0, 1, \ldots\) (Rahman [1986a])

6.14 Let \(C.T.\) \(f(x)\) denote the constant term in the Laurent expansion of a function \(f(x)\). Prove that if \(j\) and \(k\) are nonnegative integers, then
\[
C.T. (x^2 + 1; q)_j (x^2 + 2; q^2)_k
\]
\[
= \frac{1}{2\pi} \int_0^\pi \left( q e^{2i\theta}, e^{-2i\theta}; q^2, q e^{-4i\theta}, q^2; q^2 \right)_k d\theta
\]
\[
= \frac{(q; q)_j (q^2; q^2)_k (q^2; q^2)_k (q^2; q^2)_k}{(q; q)_j (q^2; q^2)_k (q^2; q^2)_k (q^2; q^2)_k}
\]
when Re \(x > 0\), Re \(y > 0\), \(|q| < 1\), and \(k = 0, 1, \ldots\) (Askey [1982b])

6.15 Verify Ramanujan's identities
\[
\int_0^\infty e^{-x^2 + 2mxz} \left(-aqe^{2kx}, -be^{-2kx}; q\right)_\infty \frac{e^{2mz} \cdot dx}{\sqrt{\pi}(abq; q)_\infty e^{-m^2}}
\]
\[
\int_0^\infty e^{-x^2 + 2mxz} \frac{dz}{\sqrt{\pi}(abq; q)_\infty} = \frac{(ae^{2mk} \sqrt{q}, be^{-2mk} \sqrt{q}; q)_\infty}{(ae^{2ink} \sqrt{q}, be^{-2ink} \sqrt{q}; q)_\infty}
\]

(Assuming \(m, k\) are integers; \(|q| < 1\), \(m, k = 0, 1, \ldots\))
6.16 Derive the q-beta integral formulas

\[
\int_0^\infty \frac{(-t q^h, -q^{a+1}; q)_\infty}{(t, -q/t; q)_\infty} \frac{d_q t}{t} = \Gamma_q(a) \Gamma_q(b) \Gamma_q(a + b)
\]

and

\[
\int_0^\infty \frac{(-t q^h, -q^{a+1}; q)_\infty}{(t, -q/t; q)_\infty} \frac{dt}{t} = -\log q \Gamma_q(a) \Gamma_q(b) \Gamma_q(a + b)
\]

where \(0 < q < 1\), \(\text{Re } a > 0\) and \(\text{Re } b > 0\).

(Askey and Roy [1986], Gasper [1987])

6.17 Extend the above q-beta integral formulas to

\[
\int_0^\infty e^{-t} \frac{(-t q^h, -q^{a+1}; q)_\infty}{(t, -q/t; q)_\infty} \frac{d_q t}{t} = \frac{-q^c, q^{a+c}; q)_\infty}{(-1, -q; q)_\infty} \Gamma_q(a + c) \Gamma_q(b - c)
\]

and

\[
\int_0^\infty e^{-t} \frac{(-t q^h, -q^{a+1}; q)_\infty}{(t, -q/t; q)_\infty} \frac{dt}{t} = \frac{\Gamma(c) \Gamma(1 - c) \Gamma_q(a + c) \Gamma_q(b - c)}{\Gamma_q(c) \Gamma_q(1 - c) \Gamma_q(a + b)}
\]

where \(0 < q < 1\), \(\text{Re } (a + c) > 0\) and \(\text{Re } (b - c) > 0\).

(Ramanujan [1915], Askey and Roy [1986], Gasper [1987])

Notes 6

Ex. 6.7 Setting \(a_1 = -a_2 = a\), \(a_3 = -a_4 = a q^h\), \(a_5 = b\), \(a_6 = c\), \(\lambda = a^2 b q\), \(\mu = a^2 c\) and using Ex. 2.25, Rahman [1988b] evaluated this integral in closed form and found the corresponding systems of biorthogonal rational functions.

Ex. 6.8 This may be regarded as a q-extension of the terminating case of Erdélyi’s [1953, 2.4.3] formula

\[
F(a, b; c; x) = \frac{\Gamma(c)}{\Gamma(c - \mu)} \int_0^1 t^{-\mu - 1}(1 - t)^{c-\mu - 1}(1 - t x)^{\lambda - a - b} dt
\]

where \(|x| < 1\), \(0 < \text{Re } \mu < \text{Re } c\).

Ex. 6.10 Askey [1988b] found the \(\phi_3\) polynomials which are orthogonal with respect to the weight function associated with this integral.

Exercises 6.11–6.13 The double integrals in these exercises are q-analogues of the \(n = 2\) case of Selberg’s [1944] important multivariable extension of the beta integral:

\[
\int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{j-1}(1 - t_i)^{y-1} \prod_{1 \leq i < j \leq n} (t_i - t_j)^2 dt_1 \cdots dt_n
\]

where \(\text{Re } x > 0\), \(\text{Re } y > 0\), \(\text{Re } z > -\min(1/n, \text{Re } x/(n - 1), \text{Re } y/(n - 1))\).

7.1 Orthogonality

Let \( \alpha(x) \) be a non-constant, non-decreasing, real-valued bounded function defined on \(( -\infty, \infty)\) such that its moments

\[
\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \quad n = 0, 1, 2, \ldots, \tag{7.1.1}
\]

are finite. A finite or infinite sequence \( p_0(x), p_1(x), \ldots \) of polynomials, where \( p_n(x) \) is of degree \( n \) in \( x \), is said to be orthogonal with respect to the measure \( d\alpha(x) \) and called an orthogonal system of polynomials if

\[
\int_{-\infty}^{\infty} p_m(x)p_n(x) d\alpha(x) = 0, \quad m \neq n. \tag{7.1.2}
\]

In view of the definition of \( \alpha(x) \) the integrals in (7.1.1) and (7.1.2) exist in the Lebesgue-Stieltjes sense. If \( \alpha(x) \) is absolutely continuous and \( d\alpha(x) = w(x) dx \), then the orthogonality relation reduces to

\[
\int_{-\infty}^{\infty} p_m(x)p_n(x)w(x) dx = 0, \quad m \neq n, \tag{7.1.3}
\]

and the sequence \( \{p_n(x)\} \) is said to be orthogonal with respect to the weight function \( w(x) \).

If \( \alpha(x) \) is a step function (usually taken to be right-continuous) with jumps \( w_j \) at \( x = x_j, j = 0, 1, 2, \ldots \), then (7.1.2) reduces to

\[
\sum_{j} p_m(x_j)p_n(x_j)w_j = 0, \quad m \neq n. \tag{7.1.4}
\]

In this case the polynomials are said to be orthogonal with respect to a jump function and are usually referred to as orthogonal polynomials of a discrete variable.

Every orthogonal system of real valued polynomials \( \{p_n(x)\} \) satisfies a three-term recurrence relation of the form

\[
x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x) \tag{7.1.5}
\]

with \( p_0(x) \equiv 0, p_0(x) \equiv 1 \), where \( A_n, B_n, C_n \) are real and \( A_n C_{n+1} > 0 \). Conversely, if (7.1.5) holds for a sequence of polynomials \( \{p_n(x)\} \) such that \( p_0(x) \equiv 0, p_0(x) \equiv 1 \) and \( A_n, B_n, C_n \) are real with \( A_n C_{n+1} > 0 \), then there exists a positive measure \( d\alpha(x) \) such that

\[
\int_{-\infty}^{\infty} p_m(x)p_n(x) d\alpha(x) = \begin{cases} 0, & m \neq n, \\ v_n^{-1} \int_{-\infty}^{\infty} d\alpha(x), & m = n. \end{cases} \tag{7.1.6}
\]

where

\[
v_n = \prod_{k=1}^{n} \frac{A_{k-1}}{C_k}, \quad v_0 = 1. \tag{7.1.7}
\]

If \( \{p_n(x)\} = \{p_n(x)\}_{n=0}^{\infty} \) and \( A_n C_{n+1} > 0 \) for \( n = 0, 1, 2, \ldots \), then the measure has infinitely many points of support, (7.1.5) holds for \( n = 0, 1, 2, \ldots \), and (7.1.6) holds for \( m, n = 0, 1, 2, \ldots \). If \( \{p_n(x)\} = \{p_n(x)\}_{n=0}^{N} \) and \( A_n C_{n+1} > 0 \) for \( n = 0, 1, 2, \ldots, N-1 \), where \( N \) is a fixed positive integer, then the measure can be taken to have support on \( N + 1 \) points \( x_0, x_1, \ldots, x_N \); (7.1.5) holds for \( n = 0, 1, \ldots, N-1 \), and (7.1.6) holds for \( m, n = 0, 1, 2, \ldots, N \).

This characterization theorem of orthogonal polynomials is usually attributed to Favard [1935], but it appeared earlier in published works of Perron [1929], Wintner [1929] and Stone [1932]. For a detailed discussion of this theorem see, for example, Atkinson [1964], Chihara [1978], Freud [1974] and Szegö [1975].

In the finite discrete case the recurrence relation (7.1.5) is a discrete analogue of a Sturm-Liouville two-point boundary-value problem with boundary conditions \( p_{-1}(x) = 0, p_{N+1}(x) = 0 \). If \( x_0, x_1, \ldots, x_N \) are the zeros of \( p_{N+1}(x) \), which can be easily proved to be real and distinct (see e.g., Atkinson [1964] for a complete proof), then the orthogonality relation (7.1.6) can be written in the form

\[
\sum_{j=0}^{N} p_m(x_j)p_n(x_j)w_j = v_n^{-1} \sum_{j=0}^{N} w_j \delta_{m,n}, \tag{7.1.8}
\]

\( m, n = 0, 1, \ldots, N \), where \( w_j \) is the positive jump at \( x_j \) and \( v_n \) is as defined in (7.1.7). The dual orthogonality relation

\[
\sum_{n=0}^{N} p_n(x_j)p_n(x_k)v_n = w_j^{-1} \sum_{n=0}^{N} w_n \delta_{j,k}, \quad j, k = 0, 1, \ldots, N, \tag{7.1.9}
\]

follows from the fact that a matrix that is orthogonal by rows is also orthogonal by columns. It can be shown that

\[
w_j = (A_N v_N)^2 p_{N+1}(x_j)^{-1} \sum_{n=0}^{N} w_n, \quad j = 0, 1, \ldots, N, \tag{7.1.10}
\]

where the prime indicates the first derivative.

In general, the measure in (7.1.6) is not unique and, given a recurrence relation, it may not be possible to find an explicit formula for \( \alpha(x) \). Even though the classical orthogonal polynomials, which include the Jacobi polynomials

\[
P_n(x, \beta(x)) = \frac{(\alpha + 1)_n}{n!} F_1 \left( -n, n + \alpha + \beta + 1; \frac{1-x}{2} \right), \tag{7.1.11}
\]

and the ultraspherical polynomials

\[
C_n^{(\lambda)}(x) = \frac{(2\lambda)_n}{(\lambda + \frac{1}{2})_n} P_n^{(\lambda - \frac{1}{2}, \lambda + \frac{1}{2})}(x) = \sum_{k=0}^{n} \frac{(\lambda)_k (\lambda)_n - k}{k! (n-k)!} e^{i(n-2k)x}, \quad x = \cos \theta, \tag{7.1.12}
\]

where
are orthogonal with respect to unique measures (see Szegő [1975]), it is not easy to discover these measures from the corresponding recurrence relations (see e.g., Askey and Ismail [1984]). However, for a wide class of discrete orthogonal polynomials it is possible to use the recurrence relation (7.1.5) and the formulas (7.1.8)–(7.1.10) to compute the jumps \( w_j \) and hence the measure. We shall illustrate this in the next section by considering the \( q \)-Racah polynomials (Askey and Wilson [1979]).

### 7.2 The finite discrete case: the \( q \)-Racah polynomials and some special cases

Suppose \( \{ p_n(x) \} \) is a finite discrete orthogonal polynomial sequence which satisfies a three-term recurrence relation of the form (7.1.5) and the orthogonality relations (7.1.8) and (7.1.9) with the weights \( w_j \) and the normalization constants \( v_n \) given by (7.1.10) and (7.1.7), respectively. We shall now assume, without any loss of generality, that \( p_n(x_0) = 1 \) for \( n = 0, 1, \ldots, N \). This enables us to rewrite (7.1.5) in the form

\[
(x - x_0)p_n(x) = A_n [p_{n+1}(x) - p_n(x)] - C_n [p_n(x) - p_{n-1}(x)],
\]

(7.2.1)

where \( n = 0, 1, \ldots, N \). Setting \( j = k = 0 \) in (7.1.9) we find that

\[
\sum_{n=0}^{N} v_n = w_0^{-1} \sum_{n=0}^{N} w_n.
\]

(7.2.2)

It is clear that in order to obtain solutions of (7.2.1) which are representable in terms of basic series it would be helpful if \( v_n \) and \( \sum_{n=0}^{N} v_n \) were equal to quotients of products of \( q \)-shifted factorials. Therefore, with the \( \phi_5 \) sum (2.4.2) in mind, let us take

\[
v_n = \frac{(abq;q)_n}{(q;q)_n} \frac{((aq, c), (bdq, q)_n)}{(aq, c, bdq, q)_n} \frac{(cdq)^{-n}}{(q^{-1};q)_n}
\]

\[
= \prod_{k=1}^{n} \frac{(1 - abq^k)(1 - abq^{2k+1})(1 - adq^k)(1 - adq^{k+1})}{(1 - q^{-k})(1 - abq^{2k-1})(1 - bdq^k)(1 - bdq^{k+1})(1 - adq^k/d)(cdq)^k}
\]

(7.2.3)

where \( bdq = q^{-N}, 0 < q < 1 \), so that

\[
\sum_{n=0}^{N} v_n = \phi_5 \left[ \frac{abq, q(abq)^q - q(abq)^q, aq, c, q^{-N}}{(abq), (aq, c), \phi_5(q^{-2};q^2)} \left( -\frac{bq^{-N}}{c} \right) \right]
\]

\[
= \left( \frac{abq^2, b/c; q)_N}{(bq, abq/c; q)_N} \right) \phi_5
\]

(7.2.4)

where it is assumed that \( a, b, c, d \) are such that \( v_n > 0 \) for \( n = 0, 1, \ldots, N \). In view of (7.1.7) we can take

\[
A_{k-1} = \frac{(1 - abq^k)(1 - adq^k)(1 - bdq^k)}{(1 - abq^{2k+1})} r_k,
\]

(7.2.5)

The finite discrete case

\[
C_k = \frac{cdq(1 - q^k)(1 - bq^k)(1 - abq^k/c)(1 - adq^k/d)}{(1 - abq^{2k+1})} r_k,
\]

(7.2.6)

where \( \{ r_k \}_{k=1}^{N} \) is an arbitrary sequence with \( r_k \neq 0, 1 \leq k \leq N \). Since \( C_0 = \frac{1}{-aq} \) and \( A_0 = (1 - aq)(1 - cq)(1 - bdq) r_1 \), we have from the \( n = 0 \) case of (7.2.1) that

\[
p_1(x_j) = 1 - \frac{(1 - q^{-1})(x_j - x_0)q_{-1}^{-1}}{(1 - q)(1 - aq)(1 - cq)(1 - bdq)}.
\]

(7.2.7)

This suggests that we should look for a basic series representation of \( p_n(x_j) \) whose \( (k + 1) \)-th term has \( (q, a, b, q) \) as its denominator, in turn suggests considering a terminating \( \phi_5 \) series. In view of the product \( (-n)_{a+b+1} \) in the numerator of the \( (k + 1) \)-th term in the hypergeometric series representation of \( p_n(x_j) \) in (7.1.7) and the dual orthogonality relation (7.1.8) and (7.1.9) required for \( p_n(x_j) \) it is natural to look for a \( \phi_5 \) series whose \( (k + 1) \)-th term has the numerator

\[
(q^{-n}, q^{n+a+b+1}, q^{-j}, q^{j+n+k+1}; q)_k q^k.
\]

Replacing \( q^n, q^a, q^b, q^d \) by \( a, b, c, d \), respectively, we are then led to consider the \( \phi_5 \) series of the form

\[
4 \phi_3 \left[ \frac{q^{-n}, a, b, q^{-1}, q^{-j}, cdq^j}{aq, cq, bdq}; q, q \right]
\]

(7.2.8)

Observing that \( (q^{-j}, cdq^j; q)_k \) is a polynomial of degree \( k \) in the variable \( q^{-j} + cdq^j \), we find that if we take

\[
J = q^{-j} + cdq^j
\]

(7.2.9)

then \( x_j - x_0 = -(1 - q^{-j})(1 - cdq^j) \), and so (7.2.7) is satisfied with

\[
r_k = (1 - abq^{-2k+1}).
\]

(7.2.10)

Then \( A_k C_{k+1} > 0 \) for \( 0 \leq k \leq N - 1 \) if, for example, \( a, b, c, d \) are real, \( d = b^{-1} q^{-N-1}, b < 0, \max(|aq|, |bq|, |cq|, |ab/c|) < 1 \) and \( 0 < q < 1 \).

We shall now verify that

\[
p_n(x_j) = 4 \phi_3 \left[ \frac{q^{-n}, a, b, q^{-1}, q^{-j}, cdq^j}{aq, cq, bdq}; q, q \right]
\]

(7.2.11)

satisfies (7.2.1) with \( x = x_j \). A straightforward calculation gives

\[
p_n(x_j) - p_{n-1}(x_j) = \left[ q^{-n-1}(1 - abq^{2n}) (1 - q^{-j})(1 - cdq^j) \right]
\]

\[
\left[ (1 - aq)(1 - cq)(1 - bdq) \right]
\]

\[
\cdot 4 \phi_3 \left[ \frac{q^{-n-1}, a, b, q^{-1}, cdq^{j+1}}{aq^2, cq^2, bdq^2}; q, q \right].
\]

(7.2.12)
\[ 0 \leq n \leq N. \text{ So we need to verify that} \]
\[ 4\phi_3 \left[ q^{-n}, abq^{-n+1}, q^{-j}, cdq^{j+1} \right. \]
\[ \left. aq, cq, bdq \right] ; q, q \]
\[ = \frac{A_n q^{-n} (1 - abq^{2n+2})}{(1 - aq)(1 - cq)(1 - bdq)} 4\phi_3 \left[ q^{-n}, abq^{-n+2}, q^{-j}, cdq^{j+2} \right. \]
\[ \left. aq^2, cq^2, bdq^2 \right] ; q, q \]
\[ - \frac{C_n q^{-n} (1 - abq^{2n})}{(1 - aq)(1 - cq)(1 - bdq)} 4\phi_3 \left[ q^{-n}, abq^{-n+1}, q^{-j}, cdq^{j+2} \right. \]
\[ \left. aq^2, cq^2, bdq^2 \right] ; q, q , \]
\[ (7.2.13) \]

where \( A_n \) and \( C_n \) are given by (7.2.5) and (7.2.6) with \( r_k \) as defined in (7.2.10). Use of (2.10.4) on both sides of (7.2.13) reduces the problem to verifying that

\[ 4\phi_3 \left[ q^{-n}, abq^{n+1}, bq^{-j}/c, bdq^{j+1} \right. \]
\[ \left. bq, abq/c, bdq \right] ; q, q \]
\[ = \frac{(1 - abq^{n+1})(1 - bdq^{n+1})}{(1 - abq^{2n+1})(1 - bdq)} 4\phi_3 \left[ q^{-n}, abq^{n+2}, bq^{-j}/c, bdq^{j+1} \right. \]
\[ \left. bq, abq/c, bdq^2 \right] ; q, q \]
\[ - \frac{(1 - q^n)(1 - aq^n/d)}{(1 - abq^{2n+1})(1 - bdq)} bdq \]
\[ - 4\phi_3 \left[ q^{-n}, abq^{n+1}, bq^{-j}/c, bdq^{j+1} \right. \]
\[ \left. bq, abq/c, bdq^2 \right] ; q, q , \]
\[ (7.2.14) \]

which follows immediately from the fact that

\[ \frac{(q^{-n}; q)_k (abq^{n+1}; q)_k (1 - bdq^{n+1})}{(1 - abq^{2n+1})(1 - bdq^{k+1})} \]
\[ \left. (q^{-n}; q)_k+1 (abq^{n+1}; q)_k+1 (1 - aq^n/d) bdq^{n+1} \right) = (q^{-n}, abq^{n+1}; q)_k . \]

The verification that (7.2.11) satisfies the boundary condition \( p_{N+1}(x_j) = 0 \) for \( 0 \leq j \leq N \) is left as an exercise (Ex. 7.3).

Note that the \( 4\phi_3 \) series in (7.2.11) remains unchanged if we switch \( n, a, b, c \), respectively, with \( j, c, d \). This implies that the polynomials \( p_n(x_j) \) are self-dual in the sense that they are of degree \( n \) in \( x_j = q^{-j} + cdq^{j+1} \) and of degree \( j \) in \( y_n = q^{-n} + abq^{n+1} \) and that the weights \( w_j \) are obtained from the \( v_n \) in (7.2.3) by replacing \( n, a, b, c, d, j, c, d, a, b \), respectively, i.e.,

\[ w_j = \frac{(cdq; q)_j (1 - cdq^{2j+1}) (cq, ac, bdq; q)}{(q; q)_j (1 - cdq) (aq, cdq/a, cq/b; q)} (abq)^{-j} . \]
\[ (7.2.15) \]

\[ 0 \leq j \leq N. \text{ Thus} \{ p_n(x_j) \}_{n=0}^{N} \text{ is orthogonal with respect to the weights} \ w_j, \text{ while} \{ p_j(y_n) \}_{j=0}^{N} \text{ is orthogonal with respect to the weights} \ v_n \text{ (see Ex. 7.4). The calculations have so far been done with the assumption that} bdq = q^{-N}, \text{ but the same results will hold if we assume that one of} aq, cq \text{ and} bdq \text{ is } q^{-N}. \]
7.3 The infinite discrete case: the little and big q-Jacobi polynomials

As a q-analogue of the Jacobi polynomials (7.1.11), Hahn [1949a] (also see Andrews and Askey [1977]) introduced the polynomials

\[ p_n(x; a, b; q) = 2 \phi_1(q^{-n}, abq^{n+1}; aq; x, x). \]  

(7.3.1)

It can be easily verified that

\[ \lim_{q \to 1} p_n \left( \frac{1 - x}{2}; q^n, q^a; q \right) = \frac{P_n^{(a, b)}(x)}{P_n^{(a, b)}(1)}. \]  

(7.3.2)

He proved that

\[ \sum_{n=0}^{\infty} p_n(q^n; a, b; q)p_n(q^n; a, b; q) \frac{(bq; q)_n}{(q; q)_n} \frac{(aq)_n}{(aq; q)_n} = \frac{\delta_{m,n}}{h_n(a, b; q)}, \]  

(7.3.3)

where \( 0 < q, aq < 1 \) and

\[ h_n(a, b; q) = \frac{(ab; q)_n(1 - abq^{2n+1})(aq; q)_n}{(q; q)_n(1 - abq)(aq; q)_n(abq; q)_n}. \]  

(7.3.4)

Observe that (7.3.3) and (7.3.4) also follow from (7.2.22) when we replace \( x \) by \( N - x \) and then let \( N \to \infty \). To prove (7.3.3), assume, as we may, that \( 0 \leq m \leq n \), and observe that

\[ \sum_{n=0}^{\infty} \frac{(bq; q)_n}{(q; q)_n} \frac{(aq)_n}{(aq; q)_n} p_m(q^n; a, b; q) = \sum_{j=0}^{m} \frac{(q^{-m}, abq^{m+1}; q)_j}{(q, aq; q)_j} q^j. \]

(7.3.10)

which has the property that

\[ \lim_{q \to 1} P_n(x; a, b; q) = \frac{P_n^{(a, b)}(x)}{P_n^{(a, b)}(1)}, \]  

(7.3.11)

where \( \gamma \) is real. In view of the third free parameter in (7.3.10) they called the \( P_n(x; a, b, c; q) \) the big \( q \)-Jacobi and the \( p_n(x; a, b; q) \) the little \( q \)-Jacobi polynomials.

We shall now prove that the big \( q \)-Jacobi polynomials satisfy the orthogonality relation

\[ \int_{c}^{a} P_m(x; a, b, c; q)P_n(x; a, b, c; q) \frac{(x/a, x/c; q)_\infty}{(x, bx/c; q)_\infty} \frac{dx}{q} \]

(7.3.12)

where

\[ h_n(a, b, c; q) = M^{-1}(abq; q)_n(1 - abq^{2n+1})(aq; q)_n(abq; q)_n \frac{q^2}{(q; q)_n(1 - abq)(aq; q)_n(abq; q)_n}. \]  

(7.3.13)
and
\[
M = \int_{c q}^{a q} \frac{(x/a, x/c; q)_\infty}{(x, b x/c; q)_\infty} d_q x \\
= (a q (1 - q))(q, c/a, a q/c, a b q^2; q)_\infty \\
(a q, b q, c q, a b q/c; q)_\infty
\] (7.3.14)

by (2.10.20). Since
\[
\int_{c q}^{a q} \frac{(x/a, x/c; q)_\infty}{(x, b x/c; q)_\infty} d_q x \\
= \int_{c q}^{a q} \frac{(x/a, x/c; q)_\infty}{(c x^k, b x^2/c; q)_\infty} d_q x \\
= M \frac{(b q, a b q/c; q)_j(a q, c q, a q)_k}{(a b q^2; q)_{j+k}},
\]

the left side of (7.3.12) becomes
\[
M \frac{(b q, a b q/c; q)_m(a c/b)_m}{(a q, c q; q)_m} \\
\sum_{j=0}^{m} \sum_{k=0}^{n} \frac{(q^{-m}, a b q^{m+1}; q)_j}{(a q, q q; q)_{j+k}} \frac{(q^{n-n}, a b q^{n+1}; q)_k}{(a b q^2; q)_{j+k}} q^{j+k},
\] (7.3.15)

where we used (7.3.10) and the observation that, by (3.2.2) and (3.2.5),
\[
P_m(x; a, b, c; q) \\
= \frac{(b q, a b q/c; q)_m(a c/b)_m}{(a q, c q; q)_m} \Phi_2(q^{-m}, a b q^{m+1}, b x/c; q, q),
\] (7.3.16)

Assume that 0 ≤ m ≤ n. Since, by (1.5.3)
\[
\sum_{j=0}^{m} \frac{(q^{-m}, a b q^{m+1}; q)_j}{(a q, q q; q)_{j+k}} q^j = \frac{(q^{n-n}, a b q^{n+1}; q)_k}{(a b q^2; q)_{j+k}},
\]

the double sum in (7.3.15) equals
\[
\frac{(q^{n-n}, a b q^{n+1}; q)_k}{(a b q^2; q)_{2m}} \sum_{k=0}^{m} \frac{(q^{n-n}, a b q^{n+1}; q)_k}{(a b q^{2m+2}; q)_{2n}} q^k
\]
\[
= \frac{(q^{n-n}, a b q^{n+1}; q)_k}{(a b q^2; q)_{2n}} \delta_{m,n}.
\] (7.3.17)

Substituting this into (7.3.15), we obtain (7.3.12).

7.4 An absolutely continuous measure: the continuous
q-ultraspherical polynomials
In this and the following section we shall give two important examples of orthogonal polynomials which are orthogonal with respect to an absolutely continuous measure dα(x) = w(x)dx.

In his work of the 1890's, in which he discovered the now-famous Rogers-Ramanujan identities, Rogers [1893b, 1894, 1895] introduced a set of orthogonal polynomials that are representable in terms of basic hypergeometric series and have the ultraspherical polynomials (7.1.12) as limits when q → 1. Following Askey and Ismail [1983], we shall call these polynomials the continuous q-ultraspherical polynomials and define them by the generating function
\[
\frac{(\beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty}{(e^{i\theta}, e^{-i\theta}; q)_\infty} = \sum_{n=0}^{\infty} C_n(x; \beta|q) e^{i(n-2k)\theta}
\] (7.4.1)

where x = cos θ, 0 ≤ θ ≤ π and max(|x|, |t|) < 1. Using the q-binomial theorem, it follows from (7.4.1) that
\[
C_n(x; \beta|q) = \sum_{k=0}^{n} \frac{(\beta; q)_k(\beta; q)_{n-k} e^{i(n-2k)\theta}}{(q; q)_k(q; q)_{n-k}}
\]
\[
= \frac{(\beta; q)_n e^{i\theta}}{(q; q)_n} 2 \Phi_1(q^{-n}, \beta, \beta^{-1} q^{1-n}; q, q \beta^{-1} e^{-2i\theta}).
\] (7.4.2)

Note that
\[
\lim_{q \to 1} C_n(x; q) = \sum_{k=0}^{n} \frac{(\lambda)_k(\lambda n-k) e^{i(n-2k)\theta}}{k! (n-k)!}
\]
\[
= C_n(x).
\] (7.4.3)

Before considering the orthogonality relation for C_n(x; β|q), we shall first derive some important formulas for these polynomials. For 0 < θ < π, |β| < 1, set a = qβ−1 e^{iθ}, b = qβ−1, c = qe^{iθ} and z = β^2 q^n in (3.3.5) to obtain
\[
2 \Phi_1(q^{-n}, \beta, \beta^{-1} q^{1-n}; q, q \beta^{-1} e^{-2i\theta})
\]
\[
= \frac{(\beta q^n, \beta e^{-2i\theta}; q)_\infty}{(q^n+1, e^{-2i\theta}; q)_\infty} 2 \Phi_1(q^{-1}, \beta^{-1} q^{-1} e^{i\theta}; q e^{i\theta}; q, \beta^2 q^n)
\]
\[
+ \frac{(\beta, \beta q^{-1} e^{i\theta}, \beta^{-1} q^{-1} e^{-2i\theta}; q)_\infty}{(q^{n+1}, e^{2i\theta}, q \beta^{-1} e^{-2i\theta}, \beta^{-1} q^{1-n}; q)_\infty}
\]
\[
2 \Phi_1(q^{n-1}, \beta^{-1} e^{-2i\theta}; q e^{-2i\theta}; q, \beta^n q^n).
\] (7.4.4)

Then, application of the transformation formula (1.4.3) to the two \Phi_1 series on the right side of (7.4.4) gives
\[
C_n(x; \beta|q)
\]
\[
= \frac{(\beta; q)_\infty W_{\beta^{-1}}(x|q) (\beta^2; q)_n}{(\beta^2; q)_\infty (q; q)_n}
\]
\[
\times \left\{ \frac{(e^{i\theta}; q)_\infty e^{i\theta}}{(\beta e^{i\theta}; q)_\infty} 2 \Phi_1(\beta, \beta e^{i\theta}; q e^{i\theta}; q, q^{n+1})
\right.
\]
\[
+ \frac{(e^{-i\theta}; q)_\infty e^{-i\theta}}{(\beta e^{-i\theta}; q)_\infty} 2 \Phi_1(\beta, \beta e^{-i\theta}; q e^{-i\theta}; q, q^{n+1}) \right\}.
\] (7.4.5)
where
\[ W_\theta(x|q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}, \quad x = \cos \theta. \]  
(7.4.6)

Rewriting the right side of (7.4.5) as a q-integral we obtain the formula
\[
C_n(x; \beta|q) = \frac{2i \sin \theta}{(\beta, \beta|q)_\infty} \frac{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \int_{e^{2i\theta}}^{e^{-2i\theta}} (\beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty u^n du,
\]  
(7.4.7)

which was found by Rahman and Verma [1986a].

Now use (4.1.1) to obtain from (7.4.5) that
\[
C_n(x; \beta|q) = \frac{(\beta, \beta; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \frac{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \int_{e^{2i\theta}}^{e^{-2i\theta}} (\beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty u^n du \sin(n + 2k + 1) \theta,
\]  
(7.4.8)

where \(0 < \theta < \pi, |\beta| < 1\) and
\[
b(k, n; \beta) = \frac{(\beta, \beta; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \frac{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \sin(n + 2k + 1) \theta.
\]  
(7.4.9)

The series on the right side of (7.4.8) is absolutely convergent if \(|\beta| < 1\). For \(|x| < 1, |q| < 1\) and large \(n\) it is clear from (7.4.5) that the leading term in the asymptotic expansion of \(C_n(\cos \theta; \beta|q)\) is given by
\[
C_n(\cos \theta; \beta|q) \approx \frac{(\beta, \beta; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \frac{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \int_{e^{2i\theta}}^{e^{-2i\theta}} (\beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty u^n du \sin(n \theta - \alpha),
\]  
(7.4.10)

where
\[
A(z) = \frac{(\beta^2; q)_\infty}{(z^2; q)_\infty} \quad \text{and} \quad \alpha = \arg A(e^{i\theta}).
\]  
(7.4.11)

For further results on the asymptotics of \(C_n(x; \beta|q)\), see Askey and Ismail [1980] and Rahman and Verma [1986a].

If we use (3.5.4) to express \(2\phi_1(q^{-n}, \beta; q^{-1} q^{-n}; q)\) as a terminating very-well-poised \(8\phi_7\) series in base \(q^{1/2}\) and then apply (2.5.1), we obtain
\[
2\phi_1(q^{-n}, \beta; q^{-1} q^{-n}; q) = (q^{-1} q^{1-n}; q)_n (-q^{1-n}/2 \beta^{-1}; q^{1/2})_n (\beta^{-1} q^{1-n}; q^{1/2})_n
\]  
\[
\cdot 4\phi_3 \left[ q^{-n/2}, \beta^{1/2} q^{-n/2}, -\beta^{-1/2} q^{-n/2}, q^{1/2}, \beta^{1/2} q^{1/2} ; q^{1/2}, q^{1/2} \right]
\]  
\[
= (\beta^2; q)_n \beta^{-n/2} e^{-in\theta} 4\phi_3 \left[ q^{-n/2}, \beta^{1/2} q^{-n/2}, -\beta^{-1/2} q^{-n/2}, -\beta^{1/2} q^{1/2} ; q^{1/2}, q^{1/2} \right]
\]  
(7.4.12)

by (2.10.4). However, by (3.10.13),
\[
4\phi_3 \left[ q^{-n/2}, \beta^{1/2} q^{-n/2}, -\beta^{-1/2} q^{-n/2}, -\beta^{1/2} q^{1/2} ; q^{1/2}, q^{1/2} \right]
\]  
\[
= 4\phi_3 \left[ q^{-n/2}, \beta^{1/2} q^{-n/2}, -\beta^{-1/2} q^{-n/2}, -\beta^{1/2} q^{1/2} ; q^{1/2}, q^{1/2} \right]
\]  
(7.4.13)

and hence, from (7.4.2), (7.4.12) and (7.4.13), we have
\[
C_n(\cos \theta; \beta|q) = \frac{(\beta^2; q)_n}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \frac{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty}{(\beta^2, q^n \beta^2, q^{n+1}; q)_\infty} \int_{e^{2i\theta}}^{e^{-2i\theta}} (\beta e^{i\theta}, \beta e^{-i\theta}; q)_\infty u^n du \sin(n \theta - \alpha),
\]  
(7.4.14)

Since \(W_\theta(\cos \theta|q) = |A(e^{i\theta})|^{-2}\) for real \(\beta\), it follows from Theorem 40 in Nevai [1979] and the asymptotic formula (7.4.10) that the polynomials \(C_n(\cos \theta; \beta|q)\) are orthogonal on \([0, \pi]\) with respect to the measure \(W_\beta(\cos \theta|q)d\theta\), \(-1 < \beta < 1\). One can also guess the weight function by setting \(\beta = q^n\) and comparing the generating function (7.4.1) and the expansion (7.4.8) with the \(q \to 1\) limit cases and the weight function \((1 - e^{2i\theta})^\lambda (1 - e^{2i\theta})^\lambda\) for the ultra-spherical polynomials \(C_n^L(\cos \theta)\).

We shall now give a direct proof of the orthogonality relation
\[
\int_0^\pi C_m(\cos \theta; \beta|q) C_n(\cos \theta; \beta|q) W_\beta(\cos \theta|q) d\theta = \frac{\delta_{mn}}{h_n(\beta|q)},
\]  
(7.4.15)

where \(|q| < 1, |\beta| < 1\) and
\[
h_n(\beta|q) = \frac{(q; \beta^2)_\infty (q; \beta q^n)_\infty (1 - \beta q^n)}{2\pi (\beta, \beta; q)_\infty (\beta^2, q^n \beta^2, q^{n+1}; q)_\infty}.
\]  
(7.4.16)

As we shall see in the next section, (7.4.15) can be proved by using (7.4.14) and the Askey-Wilson \(q\)-beta integral (6.1.1); but here we shall give a direct proof by using (7.4.2), (1.9.10) and (1.9.11), as in Gasper [1981b], to evaluate the integral. Since the integrand in (7.4.15) is even in \(\theta\), it suffices to prove that
\[
\int_{-\pi}^{\pi} C_m(\cos \theta; \beta|q) C_n(\cos \theta; \beta|q) W_\beta(\cos \theta|q) d\theta = \frac{2\delta_{mn}}{h_n(\beta|q)},
\]  
(7.4.17)

when \(0 \leq m \leq n\). We first show that, for any integer \(k\),
\[
\int_{-\pi}^{\pi} e^{ik\theta} W_\beta(\cos \theta|q) d\theta = \begin{cases} 0, & \text{if } k \text{ is odd}, \\ c_{k/2}(\beta|q), & \text{if } k \text{ is even}, \end{cases}
\]  
(7.4.18)

where
\[
c_{k}(\beta|q) = \frac{2\pi (\beta, \beta; q)_\infty (\beta^{-1}; q)_k (1 + q^2) \beta^k}{(q, \beta^2; q)_\infty (\beta q; q)_k}.
\]  
(7.4.19)
the q-binomial theorem,
\[\int_{-\pi}^{\pi} e^{ik\theta} W_\beta(\cos \theta | q) \, d\theta = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\beta^{-1}; q)_r (\beta^{-1}; q)_s}{(q;q)_r (q;q)_s} \beta^{r+s} \int_{-\pi}^{\pi} e^{ik(2r+2s)\theta} \, d\theta,\]
which equals zero when \(k\) is odd and equals
\[2\pi \sum_{s=0}^{\infty} \frac{(\beta^{-1}; q)_s (\beta^{-1}; q)_{s+j}}{(q;q)_s (q;q)_{s+j}} \beta^{s+j+2s} \]
\[= 2\pi \beta \beta q^{j+1} \phi_1 \left( \beta^{-1}, \beta^{-1} q^j; q, q^j \right) \]
when \(k = 2j, j = 0, 1, \ldots\). By (1.4.5), the above \(\phi_1\) series equals
\[
\frac{(\beta, \beta q^{-1}; q)_\infty}{(q, q^{-1}; q)_\infty} \frac{(\beta^{-1} q^{-1}; q)_{j+1}}{(\beta^{-1} q^{-1}; q)_\infty} \phi_1 \left( \beta^{-1} q^{-1}, \beta^{-1} q; q, q^{-1} \right) \]
\[= \frac{(\beta, \beta q; q)_\infty}{(q, \beta q; q)_\infty} \left( 1 + q^j \right).\]
From this and the fact that \(W_\beta(\cos \theta | q)\) is symmetric in \(\theta\), so that we can handle negative \(k\)'s, we get (7.4.18). Hence, from (7.4.2),
\[\int_{-\pi}^{\pi} e^{ik\theta} C_n(\cos \theta; \beta | q) W_\beta(\cos \theta | q) \, d\theta = \int_{-\pi}^{\pi} e^{ik\theta} C_n(\cos \theta; \beta | q) W_\beta(\cos \theta | q) \, d\theta = 0 \quad (7.4.20)\]
equals zero when \(n - k\) is odd and equals
\[
\frac{2\pi (\beta, \beta q; q)_\infty}{(q, \beta^2 q; q)_\infty} \frac{(\beta^{-1}; q)_{j+1}}{(\beta^{-1}; q)_\infty} \beta^j \phi_3 \left[ q^{-n}, \beta, \beta^{-1} q^{-j}, \beta^{-1} q^{-n-j} \right] \]
\[= \frac{2\pi (\beta, \beta q; q)_\infty}{(q, \beta^2 q; q)_\infty} \frac{(\beta^{-1}; q)_{j+1}}{(\beta^{-1}; q)_\infty} \beta^j \phi_3 \left[ q^{-n}, \beta, \beta^{-1} q^{-j}, \beta^{-1} q^{-n-j} \right] \]
when \(n - k = 2j\) is even. From (1.9.11) it follows that this \(\phi_3\) series and hence the integral (7.4.20) are equal to zero when \(n > |k|\). Hence (7.4.17) holds when \(m \neq n\). If \(k = \pm 0\), then (1.9.10) gives
\[\int_{-\pi}^{\pi} e^{ik\theta} C_n(\cos \theta; \beta | q) W_\beta(\cos \theta | q) \, d\theta = \frac{2\pi (\beta, \beta q; q)_\infty}{(q, \beta^2 q; q)_\infty} \frac{(\beta^{-1}; q)_{j+1}}{(\beta^{-1}; q)_\infty} \beta^j \phi_3 \left[ q^{-n}, \beta, \beta^{-1} q^{-j}, \beta^{-1} q^{-n-j} \right] \]
from which it follows that (7.4.17) also holds when \(m = n\).

### 7.5 The Askey-Wilson polynomials

In view of the \(\phi_3\) series representation (7.4.14) for the continuous q-ultraspherical polynomials it is natural to consider the more general polynomials
\[r_n(x) = \phi_3 \left[ q^{-n}, \alpha q^n, \beta e^{i\theta}, \beta e^{-i\theta}; q, q \right] \quad (7.5.1)\]
which are of degree \(n\) in \(x = \cos \theta\), and to try to determine the values of \(\alpha, \beta, \gamma, \delta, \epsilon\) for which these polynomials are orthogonal. Because terminating balanced \(\phi_3\) series can be transformed to other balanced \(\phi_3\) series and to very-well-poised \(\phi_7\) series which satisfy three-term-transformation formulas (see, e.g., (7.2.13), (2.11.1)), Exercise 2.15 and the three-term recurrence relation for the \(\phi_3\) polynomials, one is led to consider balanced \(\phi_3\) series. From Sears' transformation formula (2.10.4) it follows that if we set \(\alpha = abcdq^{-n}, \beta = a, \gamma = ab, \delta = ac\) and \(\epsilon = ad\), then the polynomials
\[p_n(x) = p_n(x; a, b, c, d | q) \quad (7.5.2)\]
are symmetric in \(a, b, c, d\). In addition, for real \(\theta\) these polynomials are analytic functions of \(a, b, c, d \) and are, in view of the coefficient \((ab, ac, ad; q)_n a^{-n}\), real-valued when \(a, b, c, d\) are real or, if complex, occur in conjugate pairs.

Askey and Wilson [1985] introduced these polynomials as q-analogues of the \(\phi_3\) polynomials of Wilson [1978, 1980]. Since they derived the orthogonality relation, three-term recurrence relation, difference equation and other properties of \(p_n(x; a, b, c, d | q)\), these polynomials are now called the Askey-Wilson polynomials.

Since the three-term recurrence relation (7.2.1) for the \(\phi_3\) polynomials continues to hold without the restriction \(bdq = q^{-n}\), by translating it into the notation for \(p_n(x; a, b, c, d | q)\) we find, as in Askey and Wilson [1985], that the recurrence relation for these polynomials can be written in the form
\[2p_{n+1}(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \quad n \geq 0, \] (7.5.3)
with \(p_{-1}(x) = 0, p_0(x) = 1\), where
\[A_n = \frac{1 - abcdq^{-n}}{(1 - abcq^{-n}) (1 - abdq^{-n})}, \]
\[C_n = \frac{(1 - q^n)(1 - abq^{-n})(1 - acq^{-n})(1 - adq^{-n})}{(1 - abdq^{-2n})(1 - abdq^{-2n})}, \]
and
\[B_n = a + a^{-n} - A_n a^{-n} (1 - abq^{-n})(1 - acq^{-n})(1 - adq^{-n}), \]
\[- C_n a/(1 - abq^{-n})(1 - acq^{-n})(1 - adq^{-n}). \] (7.5.6)
It is clear that \(A_n, B_n, C_n\) are real if \(a, b, c, d\) are real or, if complex, occur in conjugate pairs. Also \(A_nC_{n+1} > 0, n = 0, 1, \ldots\), if the pairwise products of \(a, b, c, d\) are less than 1 in absolute value. So by Favard's theorem, there exists a measure \(da(x)\) with respect to which \(p_n(x; a, b, c, d | q)\) are orthogonal.

In order to determine this measure let us assume that \(max(|a|, |b|, |c|, |d|, |q|) < 1\). Then, by (2.5.1),
\[p_n(\cos \theta; a, b, c, d | q)\]
and, by (2.11.1),
\[
\begin{aligned}
\vartheta W_7(abx, q^{-1}; a, b, c, d, q^{-1}; q) &= (ab, ac, bc, d; q)_{\infty} \frac{e^{i\theta}}{(a, b, c, d; q)_{\infty}}
\end{aligned}
\]

where \(0 < \theta < \pi\). Hence
\[
\begin{aligned}
p_n(x; a, b, c, d; q) &= (bc, bd, cd; q)_{\infty} \{ Q_n(x; e^{i\theta}; a, b, c, d; q) + Q_n(x; e^{-i\theta}; a, b, c, d; q) \}
\end{aligned}
\]

where
\[
\begin{aligned}
Q_n(x; a, b, c, d; q) &= \frac{(abcdq^n, bdcdq^n, bcq^{n+1}, cz^{n+1}, a/z, b/z, c/z, d/z; q)_{\infty}}{(bc, bd, cd, abq^n, acq^n, q^{n+1}, bcq^{n+1}, z^{-2}; q)_{\infty}}
\end{aligned}
\]

It is clear from (7.5.10) that
\[
Q_n(x; a, b, c, d; q) \sim z^n B(z^{-1})/(bc, bd, cd; q)_{\infty},
\]

where
\[
B(z) = (az, bz, cz, dz; q)_{\infty}/(z^2; q)_{\infty}
\]
as \(n \to \infty\), uniformly for \(z, a, b, c, d\) in compact sets avoiding the poles \(z^2 = q^{-k}\), \(k = 0, 1, \ldots\). Using (7.5.9) we find that
\[
\begin{aligned}
p_n(x; a, b, c, d; q) &\sim e^{-i\theta} B(e^{-i\theta}) + e^{i\theta} B(e^{i\theta}) \\
&= 2B(e^{i\theta}) |\cos(n\theta - \beta)|,
\end{aligned}
\]

where \(\beta = \arg B(e^{i\theta})\) and \(0 < \theta < \pi\) (see Rahman [1986c]). Then
\[
\begin{aligned}
|B(e^{i\theta})|^2 &= \sin^2 \theta \{w(x; a, b, c, d; q)\}_{\infty},
\end{aligned}
\]

where, in order to be consistent with the \(p_n(x; a, b, c, d; q)\) notation, we have used \(w(x; a, b, c, d; q)\) to denote the weight function \(w(x; a, b, c, d)\) defined in (6.3.1). It follows from Theorem 40 in Nevai [1979] that the polynomials

\[
p_n(x; a, b, c, d, q)\]

are orthogonal on \([-1, 1]\) with respect to the measure \(w(x; a, b, c, d, q) dx\) when \(\max(|a|, |b|, |c|, |d|, |q|) < 1\).

We shall now give a direct proof of the orthogonality relation
\[
\begin{aligned}
\int_{-1}^{1} p_n(x)p_n(x) w(x) \, dx &= \frac{\delta_{mn}}{h_n},
\end{aligned}
\]

where \(w(x) \equiv w(x; a, b, c, d; q)\) and
\[
\begin{aligned}
h_n &\equiv h_n(a, b, c, d; q)
\end{aligned}
\]

with
\[
\begin{aligned}
\kappa(a, b, c, d; q) &= \int_{-1}^{1} w(x; a, b, c, d; q) \, dx \\
&= \frac{2\pi(abcdq^{-1}; q)_{\infty}}{(q; q)_{\infty}(1 - abcdq^{-2}) (ab, ac, ad, bc, bd, cd; q)_{\infty}}.
\end{aligned}
\]

by (6.1.1). First observe that, by (7.5.17),
\[
\begin{aligned}
\int_{-1}^{1} (ae^{i\theta}, ae^{-i\theta}; q)_{\infty} (be^{i\theta}, be^{-i\theta}; q)_{\infty} w(x; a, b, c, d; q) \, dx
\end{aligned}
\]

\[
= \int_{-1}^{1} w(x; aq^{k}, bq^{k}, c, d; q) \, dx
\]

\[
= \kappa(aq^{k}, bq^{k}, c, d; q).
\]

By using (7.5.2) and the fact that \(p_n(x; a, b, c, d; q) = p_n(x; a, b, c, d; q)\) we find that the left side of (7.5.15) equals

\[
\begin{aligned}
\kappa(a, b, c, d; q) (ab, ac, ad, bd; q)_{\infty} \sum_{j=0}^{m} \frac{(q^{-m}, abcdq^{-m}; q)_{\infty} 3\phi_2(q^{-m}, abcdq^{-m}; q)}{(q; abcdq^{-m}; q)_{\infty}}.
\end{aligned}
\]

Assuming that \(0 \leq n \leq m\) and using the \(q\)-Saalschütz formula to sum the \(3\phi_2\) series, the sum over \(j\) in (7.19) gives

\[
\begin{aligned}
\int_{-1}^{1} (aq^{k}, bq^{k}, c, d; q)_{\infty} (ae^{i\theta}, ae^{-i\theta}; q)_{\infty} (be^{i\theta}, be^{-i\theta}; q)_{\infty} w(x; a, b, c, d; q) \, dx
\end{aligned}
\]

\[
= \int_{-1}^{1} w(x; aq^{k}, bq^{k}, c, d; q) \, dx
\]

\[
= \kappa(aq^{k}, bq^{k}, c, d; q).
\]

Combining (7.19) and (7.20) completes the proof of (7.5.15).
Askey and Wilson proved a more general orthogonality relation by using contour integration. They showed that if \(|q| < 1\) and the pairwise products and quotients of \(a, b, c, d\) are not of the form \(q^{-k}\), \(k = 0, 1, \ldots\), then

\[
\int_{-1}^{1} p_m(x)p_n(x)w(x)\,dx + 2\pi \sum_k p_m(x_k)p_n(x_k)w_k = \frac{\delta_{m,n}}{h_n(a, b, c, d; q)},
\]

where \(x_k\) are the points \(\frac{1}{2} (f q^k + f^{-1} q^{-k})\) with \(f\) equal to any of the parameters \(a, b, c, d\) whose absolute value is greater than one, the sum is over the \(k\) with \(|f q^k| > 1\), and

\[
w_k \equiv w_k(a, b, c, d; q) = \frac{(a^2; q)_\infty}{(q, ab, ac, ad, b/a, c/a, d/a; q)_\infty} \cdot \frac{(a^2; q)_k(1-a^2 q^{2k}) (ab, ac, ad; q)_k}{(q; q)_k(1-a^2)(aq/b, aq/c, aq/d; q)_k},
\]

when \(x_k = \frac{1}{2} (aq^k + a^{-1} q^{-k})\). For a proof and complete discussion, see Askey and Wilson [1985]. Also, see Ex. 7.31.

In order to get a \(q\)-analogue of Jacobi polynomials, Askey and Wilson set

\[
a = q^{(2\alpha+1)/4}, b = q^{(2\alpha+3)/4}, c = -q^{(2\beta+1)/4}, d = -q^{(2\beta+3)/4},
\]

and defined the continuous \(q\)-Jacobi polynomials by

\[
P_n^{(\alpha, \beta)}(x|q) = \frac{\left(q^{\alpha+1}; q\right)_n}{\left(q; q\right)_n} \cdot 4^{\alpha+\beta} \cdot q^{-\alpha} \cdot q^{\alpha+\beta+1} \cdot q^{(2\alpha+1)/4} e^{i\theta} \cdot q^{(2\beta+1)/4} e^{-i\theta} \cdot \Phi_3\left[q^{-n}, q^{\alpha+\beta+1}, q^{(2\alpha+1)/4} e^{i\theta}, q^{(2\beta+1)/4} e^{-i\theta}, q^{\alpha+\beta+1}/2, 1-q^{\alpha+\beta+1}/2; q, q\right].
\]

On the other hand, Rahman [1981] found it convenient to work with an apparently different \(q\)-analogue, namely,

\[
P_n^{(\alpha, \beta)}(x|q) = \frac{\left(q^{\alpha+1}, -q^{\beta+1}; q\right)_n}{\left(q, q^{-1}; q\right)_n} \cdot 4^{\alpha+\beta} \cdot q^{-\alpha} \cdot q^{\alpha+\beta+1} \cdot q^{(2\alpha+1)/4} e^{i\theta} \cdot q^{(2\beta+1)/4} e^{-i\theta} \cdot \Phi_3\left[q^{-n}, q^{\alpha+\beta+1}, q^{(2\alpha+1)/4} e^{i\theta}, q^{(2\beta+1)/4} e^{-i\theta}, q^{\alpha+\beta+1}, -q; q, q\right].
\]

However, as Askey and Wilson pointed out, these two \(q\)-analogues are not really different since, by the quadratic transformation (3.10.13),

\[
P_n^{(\alpha, \beta)}(x|q) = \frac{\left(-q; q\right)_n}{\left(q^{\alpha+\beta+1}; q\right)_n} \cdot q^{-n} P_n^{(\alpha, \beta)}(x|q).
\]

Note that

\[
\lim_{q \to 1} P_n^{(\alpha, \beta)}(x|q) = \lim_{q \to 1} P_n^{(\alpha, \beta)}(x; q) = P_n^{(\alpha, \beta)}(x).
\]

The orthogonality relations for these \(q\)-analogues are

\[
\int_0^\pi P_m^{(\alpha, \beta)}(\cos \theta|q) P_n^{(\alpha, \beta)}(\cos \theta|q) w(\theta; q^{1/2}) \,d\theta = \frac{\delta_{m,n}}{a_n(\alpha, \beta; q)},
\]

where

\[
w(\theta; q) = \left(\frac{e^{i\theta}, -e^{-i\theta}; q}_\infty}{(q^\alpha e^{i\theta}, -q^\alpha e^{i\theta}; q)_\infty}\right)^2,
\]

\[
a_n(\alpha, \beta; q) = \frac{(q, q^{\alpha+1}, q^{\beta+1}, q^{\alpha+\beta+1}+1/2, q^{\alpha+\beta+2}/q)_\infty}{2\pi (q^{\alpha+\beta+2}/q, q^{\alpha+\beta+3}/q)_\infty} \cdot \left(\frac{1-q^{2n+\alpha+\beta+1}}{1-q^{\alpha+\beta+1}}\right) \left(q^{\alpha+\beta+1}, q^{\alpha+\beta+1}, -q^{\alpha+\beta+1}/q\right)_n q^{-n(2\alpha+1)/2} \cdot \Phi_3\left[q^{-n}, q^{\alpha+\beta+1}, q^{\alpha+\beta+1}, -q^{\alpha+\beta+1}, -q^{\alpha+\beta+1}, -q^{-1}; q\right].
\]

From (7.4.14), (7.5.24) and (7.5.25) it is obvious that

\[
C_n(\cos \theta; q^\lambda|q) = \frac{(q^{\lambda^2}, q^{-\lambda} q^{-\lambda-1/2} P_{n-\lambda}^{(\lambda+1/2, \lambda-1/2)}(\cos \theta|q)}{(q^{\lambda+1/2}; q)_n q^{-n/2} P_{n-\lambda}^{(\lambda+1/2, \lambda-1/2)}(\cos \theta; q^{1/2})}.
\]

It can also be shown that

\[
C_{2n}(x; q^\lambda|q) = \frac{(q^{\lambda}, q^{-\lambda} q^{-\lambda-1/2} P_{n-\lambda}^{(\lambda-1/2, \lambda+1/2)}(2x^2 - 1); q)}{(q^{1/2}, q^{-1/2} q^{-1}; q)_n q^{-n/2} P_{n-\lambda}^{(\lambda-1/2, \lambda+1/2)}(2x^2 - 1); q)}.
\]

which are \(q\)-analogues of the quadratic transformations

\[
C_{2n}^\lambda(x) = \frac{(\lambda)_n}{(1/2)_n} P_{n-\lambda}^{(\lambda-1/2, \lambda+1/2)}(2x^2 - 1).
\]
and
\[ C_{2n+1}(x) = x \left( \frac{\lambda}{2} \right)_{n+1} P_{n}^{(\lambda-\frac{1}{2}, \frac{1}{2})}(x^2 - 1), \] (7.5.38)
respectively. To prove (7.5.35), observe that from (7.4.2)
\[ C_{2n}(\cos \theta; q^{\lambda}|q) = \frac{(q^\lambda;q)_{2n}}{(q;q)_{2n}} 2\phi_1 \left( q^{-2n}; q^{1-\lambda-2n}; q, q^{1-\lambda}; e^{2i \theta} \right), \]
and hence, by the Sears-Carlitz formula (Ex. 2.26),
\[ C_{2n}(\cos \theta; q^{\lambda}|q) = \frac{(q^\lambda;q)_{2n}}{(q;q)_{2n}} \left( q^{\frac{1}{2}} e^{-2i \theta}, q^{\frac{1}{2}} e^{2i \theta}; q \right)_{n} e^{2i n \theta}, \]
Reversing the \( \phi_3 \) series, we obtain
\[ C_{2n}(\cos \theta; q^{\lambda}|q) = \frac{(q^\lambda, q^{\frac{1}{2}} + q^{-\frac{1}{2}}; q)_{n} q^{-n/2}}{(q, q^{\frac{1}{2}}; q)_{n}} 4\phi_3 \left[ q^{-n}, q^{n+\lambda}, q^{\frac{1}{2}} e^{2i \theta}, q^{\frac{1}{2}} e^{-2i \theta}; q, q \right]. \] (7.5.39)
This, together with (7.5.26), yields (7.5.35). The proof of (7.5.36) is left as an exercise.

Following Askey and Wilson [1985] we shall now obtain another interesting special case of the Askey-Wilson polynomials. First note that the orthogonality relation (7.5.15) can be written in the form
\[ \int_{-\pi}^{\pi} p_{m}(\cos \theta; a, b, c, d|q) p_{n}(\cos \theta; a, b, c, d|q) w(\cos \theta; a, b, c, d|q) \sin \theta \, d\theta = \frac{\delta_{m,n}}{h_{n}(a, b, c, d|q)}. \] (7.5.41)
Replace \( \theta \) by \( \theta + \phi \), \( a \) by \( ae^{i \phi} \), \( b \) by \( ae^{-i \phi} \) and then set \( c = be^{i \phi}, d = be^{-i \phi} \) to find by periodicity that if \(-1 < a, b < 1 \) or if \( a = b = 1 \) and \( |a| < 1 \), then
\[ \int_{-\pi}^{\pi} p_{m}(\cos(\theta + \phi); a, b|q) p_{n}(\cos(\theta + \phi); a, b|q) W(\theta) \, d\theta = \frac{\delta_{m,n}}{\rho_n(a, b|q)}, \] (7.5.42)
where
\[ \rho_n(a, b|q) = \frac{p_n(\cos(\theta + \phi); a, b|q) = (a^2, ab, abe^{2i \phi}, q_n(\alpha e^{i \phi})^{-n})}{q^{-n}, a^2 q^{-n+1}, a e^{2i \phi} + i \theta, a^{-i \theta}; q, q}. \] (7.5.43)
\[ W(\theta) = \frac{(e^{2i(\theta+\phi)}, q)_{\infty}}{(ae^{i \phi}, be^{i \phi}, ae^{(\theta+2\phi)}, be^{(\theta+2\phi)}; q)_{\infty}}^{2}, \] (7.5.44)
\[ \rho_n(a, b|q) = \frac{(q, a^2, b^2, ab, abe^{2i \phi}, abe^{-2i \phi}; q)_{\infty}}{4\pi(a^2 b^2 |q)_{\infty} (1 - a^2 b^2 q^{2n-1})(a^2 b^2 q^{-1}; q)_{n} (1 - a^2 b^2 q^{-1}) (a^2 b^2 q^{-1}; q)_{n}}. \] (7.5.45)
The recurrence relation for these polynomials is
\[ 2 \cos(\theta + \phi) p_n(\cos(\theta + \phi); a, b|q) = A_n p_{n+1}(\cos(\theta + \phi); a, b|q) + B_n p_{n-1}(\cos(\theta + \phi); a, b|q), \] (7.5.46)
for \( n = 0, 1, \ldots \), where \( p_{-1}(x; a, b|q) = 0 \),
\[ A_n = \frac{1 - a^2 b^2 q^{-n+1}}{(1 - a^2 b^2 q^{-2n-1})(1 - a^2 b^2 q^{2n-1})}, \] (7.5.47)
\[ C_n = \frac{(1 - q^n)(1 - a^2 q^{-n+1})(1 - b^2 q^{-n+1})(1 - abq^{-n+1})(1 - abq^{-n+1})}{(1 - a^2 b^2 q^{-2n-1})(1 - a^2 b^2 q^{2n-1})}, \] (7.5.48)
and
\[ B_n = a e^{i \phi} + a^{-1} e^{-i \phi} - A_n a^{-1} e^{-i \phi}(1 - a^2 q^n)(1 - abq^n)(1 - abq^{-n+1}) \] (7.5.49)
\[ - C_n a e^{i \phi}/(1 - a^2 q^{-n+1})(1 - a^2 b^2 q^{-n+1})(1 - abq^{-n+1}). \]
If we set \( a = q^x, b = q^b, \theta = x \log q \) in (7.5.42) and then let \( q \to 1 \) we obtain
\[ \int_{-\infty}^{\infty} p_{m}(x; \alpha, \beta)p_{n}(x; \alpha, \beta)\Gamma(\alpha + ix)\Gamma(\beta + ix)\, dx = 0, \quad m \neq n, \] (7.5.50)
where
\[ p_{n}(x; \alpha, \beta) = z^n F_2 \left[ \begin{array}{c} -n, n+2\alpha+2\beta-1, \alpha-i \beta \end{array} ; a \beta, \alpha \beta \right]. \] (7.5.51)

### 7.6 Connection coefficients

Suppose \( f_n(x) \) and \( g_n(x), n = 0, 1, \ldots, \) are polynomials of exact degree \( n \) in \( x \). Sometimes it is of interest to express one of these sequences as a linear combination of the polynomials in the other sequence, say,
\[ g_n(x) = \sum_{k=0}^{n} c_{k,n} f_k(x). \] (7.6.1)
The numbers \( c_{k,n} \) are called the connection coefficients. If the polynomials \( f_n(x) \) happen to be orthogonal on an interval \( I \) with respect to a measure
A particular interesting problem is to determine the conditions under which the connection coefficients are nonnegative for particular systems of orthogonal polynomials. Formula (7.6.1) is sometimes called a projection formula when all of the coefficients are nonnegative. See the applications to positive definite functions, isometric embeddings of metric spaces, and inequalities in Askey [1970, 1975], Askey and Gasper [1971], Gangolli [1967] and Gasper [1975a]. As an illustration we shall consider the coefficients $c_{k,n}$ in the relation

$$p_n(x; a, b, c, d(q)) = \sum_{k=0}^{n} c_{k,n} p_k(x; a, b, c, d(q)).$$

Askey and Wilson [1985] showed that

$$c_{k,n} = \frac{(a; q)_n (\alpha \beta \gamma \delta q^{-1}; q)_k}{(\alpha; q)_k (\beta \gamma \delta q^{-1}; q)_n} \left[ \frac{q^{k-n} \alpha \beta \gamma \delta a^{n+k-1}}{(\alpha \beta \gamma \delta q^{-1}; q)_k (\beta \gamma \delta q^{-1}; q)_n} \right] q^{k-n}.$$  

To prove (7.6.3), temporarily assume that $\max(|a|, |b|, |c|, |d|, |q|) < 1$ and observe that, by orthogonality,

$$b_{k,j} = \int_{-1}^{1} w(x; a, b, c, d) p_k(x; a, b, c, d(q))(d\theta, de^{i\theta}; q)_j \, dx$$

vanishes if $j < k$, and that

$$b_{k,j} = \frac{\alpha (a, b, c, d(q) (ab, ac, ad, q) q^{-k}}{(ab, bc, q) j \Phi \left[ q^{k-n}, (ab, c, d(q) \right]}$$

is given for $j \geq k$. Since

$$p_n(x; a, b, c, d(q)) = \frac{(a; q)_n (\alpha \beta \gamma \delta q^{-1}; q)_n}{(\alpha; q)_n (\beta \gamma \delta q^{-1}; q)_n} \sum_{j=0}^{n} \frac{(q^{-n} \alpha \beta \gamma \delta q^{-1}; q)_j}{(q, \alpha \beta \gamma \delta; q)_j} q^{j},$$

we find that

$$c_{k,n} = \frac{h_k(a, b, c, d(q)) \int_{-1}^{1} w(x; a, b, c, d(q)) p_k(x; a, b, c, d(q)) p_n(x; a, b, c, d(q)) \, dx}{h_k(a, b, c, d(q))(\alpha \beta \gamma \delta; q)_n (\alpha \beta \gamma \delta q^{-1}; q)_n} \sum_{j=0}^{n} \frac{(q^{-n} \alpha \beta \gamma \delta q^{-1}; q)_j}{(q, \alpha \beta \gamma \delta; q)_j} q^{j} b_{k,j}$$

and hence (7.6.3) follows from (7.5.16), (7.5.17) and (7.6.5).
7.7 A difference equation and a Rodrigues-type formula for the Askey-Wilson polynomials

The polynomials $p_n(x; a, b, c, d | q)$, unlike the Jacobi polynomials, do not satisfy a differential equation; but, as Askey and Wilson [1985] showed, they satisfy a second-order difference equation. Define

$$E^+_q f(q^{1/2} e^{i\theta}) = f (q^{1/2} e^{i\theta}),$$

$$\delta_q f (e^{i\theta}) = (E^+_q - E^-_q) f (e^{i\theta}),$$

and

$$D_q f(x) = \frac{\delta_q f(x)}{\delta x}, \quad x = \cos \theta.$$  \hspace{1cm} (7.7.1, 7.7.2, 7.7.3)

Clearly,

$$\delta_q (e^{i\theta}) = \frac{1}{2} (q^{1/2} - q^{-1/2}) (e^{i\theta} - e^{-i\theta}) = -i q^{-1/2} (1 - q) \sin \theta,$$  \hspace{1cm} (7.7.4)

and

$$\delta_q (q^{1/2} e^{i\theta}; q)_n = 2 a i q^{-1/2} (1 - q^n) \sin \theta (aq^{1/2} e^{i\theta}; aq^{1/2} e^{-i\theta}; q)_{n-1},$$

so that

$$D_q (q^{1/2} e^{i\theta}; q)_n = -\frac{2a (1 - q^n)}{(1 - q)} (aq^{1/2} e^{i\theta}; aq^{1/2} e^{-i\theta}; q)_{n-1},$$  \hspace{1cm} (7.7.5, 7.7.6)

which implies that the divided difference operator $D_q$ plays the same role for $(q^{1/2} e^{i\theta}; q)_n$ as $d/d\tau$ does for $x^n$. When $q \to 1$, formula (7.7.6) becomes

$$\frac{d}{d\tau} (1 - 2ax + a^2)^n = -2an (1 - 2ax + a^2)^{n-1}.$$  \hspace{1cm} (7.7.7)

Generally, for a differentiable function $f(x)$ we have

$$\lim_{q \to 1} D_q f(x) = \frac{d}{dx} f(x).$$

Following Askey and Wilson [1985], we shall now use the operator $D_q$ and the recurrence relation (7.7.3) to derive a Rodrigues-type formula for $p_n(x; a, b, c, d | q)$. First note that by (7.7.2) and (7.7.5)

$$\delta_q p_n(x; a, b, c, d | q) = -2 ai q^{-n/2} \sin \theta (1 - q^n) (1 - q^{n-1} abcd) p_{n-1}(x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2} | q).$$  \hspace{1cm} (7.7.8)

If we define

$$A(\theta) = \frac{(1 - ae^{i\theta})(1 - be^{i\theta})(1 - ce^{i\theta})(1 - de^{i\theta})}{(1 - e^{2i\theta})(1 - q^{2i\theta})},$$

and

$$r_n(e^{i\theta}) = 4q^3 \left[ q^{-n}, abcd q^{n-1}, ae^{i\theta}, ae^{-i\theta}; ab, ac, ad \right] ; q, q,$$

then the recurrence relation (7.7.3) can be written as

$$q^{-n} (1 - q^n) (1 - abcd q^{n-1}) r_n(e^{i\theta}) = A(-\theta) \left[ r_n(q^{-1} e^{i\theta}) - r_n(e^{i\theta}) \right] + A(\theta) \left[ r_n(q e^{i\theta}) - r_n(e^{i\theta}) \right].$$  \hspace{1cm} (7.7.9)

Also, setting

$$V(e^{i\theta}) = \frac{(e^{2i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}, de^{i\theta}; q)_\infty},$$

and

$$W(e^{i\theta}) = W(e^{i\theta}; a, b, c, d | q) = V(e^{i\theta}) V(e^{-i\theta})$$

we find that

$$q^{-n/2} W(e^{i\theta}) = \delta_q \left[ \left( E^+_q V(e^{i\theta}) \right) \left( E^-_q V(e^{-i\theta}) \right) \{ \delta_q p_n(x) \} \right].$$  \hspace{1cm} (7.7.10)

Combining (7.7.7) and (7.7.10) we have

$$q^{-n/2} W(e^{i\theta}) = \delta_q \left[ \left( E^+_q V(e^{i\theta}) \right) \left( E^-_q V(e^{-i\theta}) \right) \left( e^{i\theta} - e^{-i\theta} \right) \cdot p_{n-1}(x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2} | q) \right].$$  \hspace{1cm} (7.7.11)

Since

$$\left( e^{i\theta} - e^{-i\theta} \right) \left( e^{i\theta}; q^k \right) \left( q^{2i\theta}; q^k \right)_\infty \hspace{1cm}$$

$$= \frac{h(\cos \theta; aq^{1/2}; bq^{1/2}; cq^{1/2}; dq^{1/2})}{h(\cos \theta; aq^{1/2}; bq^{1/2}; cq^{1/2}; dq^{1/2})} W(e^{i\theta}; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2} | q) = \frac{1}{e^{i\theta} - e^{-i\theta}},$$

(7.7.14) can be written in a slightly better form

$$q^{-n/2} W(e^{i\theta}; a, b, c, d | q) = \delta_q \left[ \left( e^{i\theta} - e^{-i\theta} \right)^{-1} W(e^{i\theta}; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2} | q) \right].$$

(7.7.15)

Observing that,

$$\sqrt{1 - x^2} w(x; a, b, c, d | q) = W(e^{i\theta}; a, b, c, d | q)$$

we find by iterating (7.7.15) that

$$\sqrt{1 - x^2} w(x; a, b, c, d | q) p_n(x; a, b, c, d | q) = (-1)^n \left( \frac{1 - q}{2} \right)^k q^{nk/2 - k(k+1)/4}$$

$$\cdot D_{\theta}^{k} \left[ \sqrt{1 - x^2} w(x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2} | q) p_{n-k}(x; aq^{1/2}, bq^{1/2}, cq^{1/2}, dq^{1/2} | q) \right]$$

$$= (-1)^n \left( \frac{1 - q}{2} \right)^n q^{(n^2-n)/4} D_{\theta}^{n} \left[ \sqrt{1 - x^2} w(x; aq^{n/2}, bq^{n/2}, cq^{n/2}, dq^{n/2} | q) \right].$$

(7.7.16)

This gives a Rodrigues-type formula for the Askey-Wilson polynomials.
By combining (7.7.7) and (7.7.15) it can be easily seen that the polynomials $p_n(x) = p_n(x; a, b, c, d; q)$ satisfy the second-order difference equation

$$
D_q \left[ \sqrt{1 - 2w(x; a, b, c, d; q)}D_q p_n(x) \right] + \lambda_n \sqrt{1 - 2w(x; a, b, c, d; q)} p_n(x) = 0,
$$

(7.7.17)

where

$$
\lambda_n = -4a(1 - q^{-n})(1 - abcdq^{n-1})(1 - q)^{-2}.
$$

(7.7.18)

**Exercises 7**

7.1 If $\{p_n(x)\}$ is an orthogonal system of polynomials on $(-\infty, \infty)$ with respect to a positive measure $da(x)$ that has infinitely many points of support, prove that they satisfy a three-term recurrence relation of the form

$$
x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x), \quad n \geq 0,
$$

with $p_{-1}(x) = 0$, $p_0(x) = 1$, where $A_n, B_n, C_n$ are real and $A_n C_{n+1} > 0$ for $n \geq 0$.

7.2 Let $p_0(x), p_1(x), \ldots, p_N(x)$ be a system of polynomials that satisfies a three-term recurrence relation

$$
x p_n(x) = A_n p_{n+1}(x) + B_n p_n(x) + C_n p_{n-1}(x),
$$

$n = 0, 1, \ldots, N$, where $p_{-1}(x) = 0, p_0(x) = 1$. Prove the Christoffel-Darboux formula

$$
(x - y) \sum_{j=0}^{n} p_j(x) p_j(y) v_j = A_n v_n \left[ p_{n+1}(x) p_n(y) - p_n(x) p_{n+1}(y) \right],
$$

$0 \leq n \leq N$, where $v_0 = 1$ and $v_n C_n = v_{n-1} A_{n-1}, 1 \leq n \leq N$. Deduce that

$$
\sum_{j=0}^{n} p_j^2(x) v_j = A_n v_n \left[ p'_{n+1}(x) p_n(x) - p'_n(x) p_{n+1}(x) \right]
$$

and hence

$$
\sum_{j=0}^{N} p_j^2(x_k) v_j = A_N v_N p_N(x_k) p'_{N+1}(x_k)
$$

if $x_k$ is a zero of $p_{N+1}(x)$.

7.3 Show that when $n = N$, the recurrence relation (7.2.1) reduces to

$$
(1 - q^{-a}) (1 - c a d q^{n+1}) p_N(x) = C_N \left[ p_{N}(x_j) - p_{N-1}(x_j) \right],
$$

where $p_n(x_j)$ is given by (7.2.11), $x_j$ by (7.2.9), and $A_n$ and $C_n$ by (7.2.5) and (7.2.6). Hence show that (7.2.1) holds with $x = x_j, j = 0, 1, \ldots, N$. (Askey and Wilson [1979])

7.4 If $p_n(x_j), v_n$ and $w_j$ are defined by (7.2.11), (7.2.3) and (7.2.5), respectively, prove directly (i.e. without the use of Favard’s theorem) that

$$
\sum_{j=0}^{N} p_m(x_j) p_n(x_j) w_j = v_n^{-1} \sum_{j=0}^{N} w_j \delta_{m,j},
$$

and

$$
\sum_{n=0}^{N} p_m(x_j) p_n(x_j) v_n = w_j^{-1} \sum_{n=0}^{N} w_n \delta_{j,n}.
$$

[Hint: First transform one of the polynomials, say $p_n(x)$, to be a multiple of the $\phi_3$ series on the left side of (7.2.14)].

7.5 Let one of $a, b, c, d$ be a nonnegative integer power of $q^{-1}$ and let

$$
\phi(a, b) = \phi_3 \left[ a, b, c, d; e, f, g ; q, q \right],
$$

where $efg = abcdq$. Prove the following contiguous relation

$$
A\phi(aq^{-1}, bq) + B\phi(a, b) + C\phi(aq, bq^{-1}) = 0,
$$

where

$$
A = b(1 - b)(aq - b)(a - e)(a - f)(a - g),
$$

$$
B = ab(a - bq)(a - b)(q - 1)(1 - d) - b(1 - b)(aq - b)(a - e)(a - f)(a - g),
$$

$$
C = -a(1 - a)(a - bq)(e - b)(f - b)(g - b).
$$

(Askey and Wilson [1979])

7.6 Determine the conditions that $a, b, c, d$ must satisfy so that $A_n C_{n+1} > 0$ for $0 \leq n \leq N - 1$, where $A_n$ and $C_n$ are as defined in (7.2.5) and (7.2.6) and one of $aq, c, bdq$ is $q^{-N}, N$ a nonnegative integer.

7.7 Prove (7.2.22) directly by using the appropriate transformation and summation formulas derived in Chapters 1-3.

7.8 (i) Prove that the $q$-Krawtchouk polynomials

$$
K_n(x; a, N; q) = 3\phi_2 \left[ q^{-n}, x; -a^{-1}q^n ; q, q \right]
$$

satisfy the orthogonality relation

$$
\sum_{x=0}^{N} K_m(q^{-x}; a, N; q) K_n(q^{-x}; a, N; q) \frac{(q^{-N}; q)_x}{(q; q)_x} (-a)^x
$$

$$
= (q^{-a}; q)_N q^{-N}(q^{-1})^N \left( q_n(1 + a^{-1})(-a^{-1}q^{N+1}; q)_n \right) \delta_{m,n}
$$

$$
+ (-aq^{N+1} - aq^{n+1}) \delta_{m,n},
$$

and find their three-term recurrence relation. (Stanton [1980b])
(ii) Let
\[ K_n(x; a, N|q) = 2\phi_1(q^{-n}; x; q^{-N}; q, aq^{N+1}) \]
be another family of q-Krawtchouk polynomials. Prove that they satisfy the orthogonality relation
\[ \sum_{n=0}^{N} K_m(q^{-z}; a, N|q) K_n(q^{-z}; a, N|q) (aq; q)_N (q; q)_N (q^{-N}; q)_N (1 - q^{-z}) (1 - q^{-z}) = \frac{(q, aq; q)_N (q; q)_N (q^{-N}; q)_N (1 - q^{-z}) (1 - q^{-z})}{(q, q; q)_N} \delta_{m,n}. \]

7.9 Prove that
\[ x p_n(x) = A_n [x p_{n+1}(x) - p_n(x)] - C_n [p_n(x) - x p_{n-1}(x)], \quad n \geq 0, \]
where \( p_n(x) = p_n(x; a, b; q) \) are the little q-Jacobi polynomials and
\[ A_n = \frac{-q^n (1 - aq^{n+1}) (1 - abq^{n+1})}{(1 - abq^{n+2}) (1 - abq^{n+2})}, \quad C_n = \frac{(1 - q^n) (1 - bq^n) (-aq^n)}{(1 - abq^{n+2}) (1 - abq^{n+2})}. \]

7.10 Prove that
\[ (x - 1) P_n(x) = A_n [P_{n+1}(x) - P_n(x)] + C_n [P_n(x) - P_{n-1}(x)], \quad n \geq 0, \]
where \( P_n(x) = P_n(x; a, b, c; q) \) are the big q-Jacobi polynomials and
\[ A_n = \frac{(1 - aq^{n+1}) (1 - cq^{n+1}) (1 - abq^{n+1})}{(1 - abq^{n+2}) (1 - abq^{n+2})}, \quad C_n = \frac{(1 - q^n) (1 - bq^n) (-abcq^n)}{(1 - abq^{n+2}) (1 - abq^{n+2})}. \]

7.11 The affine q-Krawtchouk polynomials are defined by
\[ K_n^{Aff}(x; a, N; q) = 3\phi_2 \left[ q^{-n}, x, 0 \mid aq, q^{-N}, q, q \right], \quad 0 < aq < 1. \]
Prove that they satisfy the orthogonality relation
\[ \sum_{n=0}^{N} K_m^{Aff}(q^{-z}; a, N; q) K_n^{Aff}(q^{-z}; a, N; q) (aq, q^{-N}; q)_x \left( -q^{N-1} \right) \frac{z}{a} q^{-\left(\frac{z}{2}\right)} = \frac{(aq, q^{-N}; q)_N (1 - q^{-z}) (1 - q^{-z})}{(q; q)_N} \delta_{m,n}, \]
where
\[ h_n = \begin{cases} \frac{(aq, q^{-N}; q)_n (1 - q^{-z}) (1 - q^{-z})}{(q; q)_n}, & m, n = 0, 1, \ldots, N, \\ \frac{(aq, q^{-N}; q)_n (1 - q^{-z}) (1 - q^{-z})}{(q; q)_n}, & \text{otherwise}. \end{cases} \]
(Delsarte [1976a, b], Dunkl [1977])

7.12 The q-Meixner polynomials are defined by
\[ M_n(x; a, c; q) = 2\phi_1(q^{-n}, x; aq; q, -q^{n+1}/c), \]
with \( 0 < aq < 1 \) and \( c > 0 \). Show that they satisfy the orthogonality relation
\[ \sum_{n=0}^{\infty} M_m(q^{-z}; a, c; q) M_n(q^{-z}; a, c; q) \frac{(aq; q)_x}{(q - acq; q)_x} q^{-\left(\frac{z}{2}\right)} = \frac{\delta_{m,n}}{h_n}, \]
where
\[ h_n = \frac{(aq, q; q)_\infty (aq, q)_n}{(c; q)_\infty (q, q)_\infty (1 - q^{-c}) (1 - q^{-c}) q^{-n}}. \]
(When \( a = q^{-r-1} \), the q-Meixner polynomials reduce to the q-Krawtchouk polynomials considered in Koornwinder [1989c].)

7.13 The q-Charlier polynomials are defined by
\[ c_n(x; a; q) = 2\phi_1(q^{-n}, x; 0; q, -q^{n+1}/a). \]
Show that
\[ \sum_{m=0}^{\infty} c_m(q^{-z}; a; q) c_n(q^{-z}; a; q) \frac{a^m}{(q; q)_m} q^{-\left(\frac{z}{2}\right)} = \frac{1}{(a; q)_\infty (q, q)_\infty (aq, q^{-1}; q)_n}. \]

7.14 Show that, for \( x = \cos \theta \),
\[ C_n(x; q; q) = \frac{\sin(n + 1) \theta}{\sin \theta} = U_n(x), \quad n \geq 0 \]
and
\[ \lim_{\theta \to 1} \frac{1 - q^n}{2(1 - \beta)} C_n(x; \beta; q) = \cos n \theta = T_n(x), \quad n \geq 1, \]
where \( T_n(x) \) and \( U_n(x) \) are the Chebichef polynomials of the first and second kind, respectively.

7.15 Verify that formulas (7.6.2) and (7.6.14) follow from the q-analogue of the Fields-Wimp formula (3.7.9).

7.16 Let \( x = \cos \theta, |\theta| < 1, |q| < 1 \). Show that
\[ \sum_{n=0}^{\infty} C_n(x; \beta; q) (\beta; q)_n \theta^n = \frac{2i \sin \theta}{(1 - q) W_3(x|q) (q, \beta^2; q)_\infty} \int_0^{e^{-i\theta}} \frac{\left( q e^{i\theta}, q e^{-i\theta}, q \right)_\infty}{\left( q e^{i\theta}, q e^{-i\theta}, q \right)_\infty} d_q u \]
and deduce that
\[ (i) \sum_{n=0}^{\infty} C_n(x; \beta; q) \frac{\theta^n}{(\beta^2; q)_n} = (te^{-i\theta}; q)_\infty 2\phi_1 \left( \beta, \beta e^{-2i\theta}, \beta^2; q, te^{i\theta} \right). \]
\[ (ii) \sum_{n=0}^{\infty} C_n(x; \beta; q) \frac{(\beta)^n}{(\beta^2; q)_n} = (te^{-i\theta}; q)_\infty 2\phi_1 \left( \beta, \beta e^{2i\theta}, \beta^2; q, -te^{-i\theta} \right). \]
7.17 Using (1.8.1), or otherwise, prove that
\[ C_n(0; \beta | q) = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (-1)^{n/2} \left( \frac{\beta^{2} q^{n}}{q^{2} - q} \right)^{n/2}, & \text{if } n \text{ is even.} \end{cases} \]

7.18 If \(-1 < q, \beta < 1\), show that
\[ |C_n(x; \beta | q)| \leq C_n(1; \beta | q). \]

7.19 Derive the recurrence relation
\[ 2x C_n(x; \beta | q) = \frac{1 - q^{n+1}}{1 - \beta q^{n}} C_{n+1}(x; \beta | q) + \frac{1 - \beta^{2} q^{n-1}}{1 - \beta q^{n}} C_{n-1}(x; \beta | q), \quad n \geq 0, \]
with \( C_{-1}(x; \beta | q) = 0, \ C_0(x; \beta | q) = 1. \)

7.20 Prove that
\[ \int_{0}^{\pi} C_n(\cos \theta; \beta | q) \cos(n + 2k) \theta W_\beta(\cos \theta | q) \, d\theta = \frac{\pi (\beta, \beta | q)_{\infty}}{(q, \beta^2; q)_{\infty}} \frac{\beta^k}{(q^{n+2k}, \beta | q)_{n+k}(1 - q^{n+2k})}, \quad n, k \geq 0, \]
where \( W_\beta(x | q) \) is defined in (7.4.6).

7.21 Using (7.4.15) and (7.6.14) prove that
\[ \frac{h(x; \gamma | q)}{h(x; a | q)} C_n(x; \beta | q) = \frac{(\gamma^2, \beta, \beta | q)_{\infty}}{(\gamma, \gamma | q, \beta; q)_{\infty}} \sum_{k=0}^{\infty} d_{k,n} C_{n+2k}(x; \gamma | q), \]
where \( h(x; a | q) \) is as defined in (6.1.2) and
\[ d_{k,n} = \frac{\beta^k (\gamma / \beta | q)_{n+k}(\beta^2 | q)_{n} (\gamma | q)_{n+k}}{(\gamma | q)^{n+k} (\beta | q)_{n+k}} \frac{1 - \gamma q^{n+k}}{1 - q^{n+k}}. \]
(Askey and Ismail [1983])

7.22 Prove that the continuous \( q\)-Hermite polynomials defined in Ex. 1.28 satisfy the orthogonality relation
\[ \int_{0}^{\pi} H_n(\cos \theta | q) H_m(\cos \theta | q) \left( (e^{2i\theta} | q)_{\infty} \right)^2 \, d\theta = \frac{2\pi (q | q)_{\infty}}{(q, q)_{\infty}} \delta_{n,m}, \]

7.23 Setting
\[ C_n(x; \beta | q) = \frac{(\beta^2, q | q)_{n}}{(q | q)_{n}} c_n(x; \beta | q) \]
in Ex. 7.19, show that
\[ 2x(1 - \beta q^n)c_n(x; \beta | q) = (1 - \beta^2 q^n)c_{n+1}(x; \beta | q) + (1 - q^n)c_{n-1}(x; \beta | q), \]
for \( n \geq 0 \), with \( c_{-1}(x; \beta | q) = 0, \ c_0(x; \beta | q) = 1 \). Now set \( \beta = s^{k} \) and \( q = s \omega_k \), where \( \omega_k = \exp(2\pi i/k) \) is a \( k \)-th root of unity, divide the above recurrence relation by \( 1 - s \omega_k^n \) and take the limit as \( s \to 1 \) to show that the limiting polynomials, \( c_n^\lambda(x; k) \), called the sieved ultraspherical polynomials of the first kind, satisfy the recurrence relation
\[ 2x c_n^\lambda(x; k) = c_{n+1}^\lambda(x; k) + c_{n-1}^\lambda(x; k), \quad n \neq mk, \]
\[ 2x(m + \lambda) c_n^\lambda(x; k) = (m + 2\lambda) c_{n+1}^\lambda(x; k) + mc_{n-1}^\lambda(x; k) \]
where \( c_0^\lambda(x; k) = 1 \) and \( c_1^\lambda(x; k) = x \).
(Al-Salam, Allaway and Askey [1984b])

7.24 Rewrite the orthogonality relation (7.4.15) in terms of the sieved orthogonal polynomial \( c_n(x; \beta | q) \) defined in Ex. 7.23 and set \( \beta = s^{k} \) and \( q = s \omega_k \). By carefully taking the limits of the \( q \)-shifted factorials prove that
\[ \int_{-1}^{1} c_n^\lambda(x; k) c_n(x; k) w(x) \, dx = \frac{\delta_{n,m}}{h_n}, \]
where
\[ w(x) = 2^{2k(k-1)}(1 - x^2)^{-\frac{1}{2}} \prod_{j=0}^{k-1} |x^2 - \cos^2(\pi j/k)|^\lambda \]
and
\[ h_n = \frac{\Gamma(\lambda + 1)}{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})} \frac{(\lambda + 1)_{[n/k]} (2\lambda)_{[n/k]}}{(\lambda + \frac{1}{2})_{[n/k]}}, \]
where the roof and floor functions are defined by
\[ [a] = \text{smallest integer greater than or equal to } a, \]
\[ [a] = \text{largest integer less than or equal to } a. \]
(Al-Salam, Allaway and Askey [1984b])

7.25 The sieved ultraspherical polynomials of the second kind are defined by
\[ B_n^\lambda(x; k) = \lim_{s \to 1} C_n(x; s^{k+1} \omega_k | s \omega_k), \quad \omega_k = \exp(2\pi i/k). \]
Show that \( B_n^\lambda(x; k) \) satisfies the recurrence relation
\[ 2x B_n^\lambda(x; k) = B_{n+1}^\lambda(x; k) + B_{n-1}^\lambda(x; k), \quad n \neq mk, \]
\[ 2x(m + \lambda) B_{n+1}^\lambda(x; k) = m B_{n+1}^\lambda(x; k) + (m + 2\lambda) B_{n-1}^\lambda(x; k), \]
where \( B_0^\lambda(x; k) = 1; \ B_1^\lambda(x; k) = 2xk \geq 2; \ B_1^\lambda(x; 1) = 2(\lambda + 1)x. \) Show also that \( B_n^\lambda(x; k) \) satisfies the orthogonality relation
\[ \int_{-1}^{1} B_m^\lambda(x; k) B_n^\lambda(x; k) w(x) \, dx = \frac{\delta_{n,m}}{h_n}, \]
where
\[ w(x) = 2^{2k(k-1)}(1 - x^2)^{-\frac{1}{2}} \prod_{j=0}^{k-1} |x^2 - \cos^2(\pi j/k)|^\lambda \]
and
\[ h_n = \frac{2\Gamma(\lambda + 1)}{\Gamma(\frac{1}{2}) \Gamma(\lambda + \frac{1}{2})} \frac{(\lambda + 1)_{[n/k]} (2\lambda + 1)_{[n/k]}}{(\lambda + \frac{1}{2})_{[n/k]}}. \]
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7.26 Using (2.5.1) show that

\[ p_n(x; a, b, c, d|q) = \frac{(ab, ac, bc, q;q)_n}{(abcdq^{-1}; q)_n} \]

\[ \cdot \sum_{k=0}^{n} \frac{(abcq^{-1}e^{i\theta}; q)_k (1 - abcde^{i\theta})^{2k-1}}{(q;q)_k (1 - abc^{-1}e^{i\theta})(bc, ac, ab; q)_k} \]

\[ \cdot \frac{(abcdq^{-1})_n}{(abcde^{i\theta}; q)_n} \cdot \frac{(de^{-i\theta}; q)_n}{(q; q)_n} \cdot e^{(n-2k)i\theta}. \]

Deduce that the polynomials

\[ p_n(x) = \lim_{q \to 1} p_n(x; a, b, c, d|q) \]

are given by

\[ p_0(x) = 1 = U_0(x), \]

\[ p_1(x) = (1 - s_4)U_1(x) + (s_3 - s_1)U_0(x), \]

\[ p_2(x) = U_2(x) - s_1U_1(x) + (s_2 - s_4)U_0(x), \]

\[ p_n(x) = \sum_{k=0}^{n} (-1)^k s_k U_n(x), \quad n \geq 3, \]

where

\[ s_0 = 1, \quad s_1 = a + b + c + d, \quad s_2 = ab + ac + ad + bc + bd + cd, \]

\[ s_3 = abc + abd + acd + bcd, \quad s_4 = abcd, \]

and \( U_n(\cos \theta) = \sin(n+1)\theta / \sin \theta, U_1(x) = 0. \) When max\(|a, |b, |c, |d| < 1\) show that these polynomials satisfy the orthogonality relation

\[ \int_{-1}^{1} \frac{p_m(x)p_n(x)(1 - x^2)^{1/2}}{(1 - 2ax + a^2)(1 - 2bx + b^2)(1 - 2cx + c^2)(1 - 2dx + d^2)} dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases} \]

(Askey and Wilson [1985])

7.27 Prove that

(i) \( p_n(\cos \theta; q, -q, q^4, -q^4|q) = \frac{(q^{n+2}; q)_n}{(q^{n+2}; q)_n} \frac{\sin(n+1)\theta}{\sin \theta} \),

(ii) \( p_n(\cos \theta; 1, -1, q^4, -q^4|q) = 2(q^n; q)_n \cos n\theta, \quad n \geq 1, \)

(iii) \( p_n(\cos \theta; q, -1, q^4, -q^4|q) = \frac{(q^{n+1}; q)_n}{(q^{n+1}; q)_n} \frac{\sin(n+1/2)\theta}{\sin 1/2} \),

(iv) \( p_n(\cos \theta; 1, q^4, -q^4|q) = \frac{(q^{n+1}; q)_n}{(q^{n+1}; q)_n} \frac{\cos(n+1/2)\theta}{\cos 1/2} \).

Exercises 7

7.28 Use the orthogonality relations (7.5.28) and (7.5.29) to prove the quadratic transformation formula (7.5.26).

7.29 Verify the orthogonality relations (7.5.28) and (7.5.29).

7.30 Verify formula (7.5.36).

7.31 Suppose that \( a, b, c, d \) are complex parameters with max\(|a, |b, |c, |d| < 1\) such that \( |aq^{N+1}| < |aq|^N \), where \( N \) is a nonnegative integer. Use (6.6.12) to prove that

\[ \int_{-1}^{1} p_m(x)p_n(x)w(x; a, b, c, d|q) dx + 2\pi \sum_{k=0}^{N} p_m(x_k)p_n(x_k)w_k \]

\[ = \delta_{m,n} \frac{\delta_{m,n}}{h_n(a, b, c, d|q)}, \]

where \( p_n(x) = p_n(x; a, b, c, d|q), x_k = \frac{1}{2} (aq^k + a^{-1}q^{-k}) \) and \( w_k \) is given by (7.5.22).

7.32 Prove that

(i) \( P_n^{(a, b)}(-x; q) = (-1)^n P_n^{(b, a)}(x; q), \)

(ii) \( P_n^{(a, b)}(x; q) = (-1)^n P_n^{(a, b)}(x|q). \)

7.33 Using (7.4.1), (7.4.7) and (2.11.2) prove that

\[ \int_{-1}^{1} \frac{p_m(x)p_n(x)(1 - x^2)^{1/2}}{(1 - 2ax + a^2)(1 - 2bx + b^2)(1 - 2cx + c^2)(1 - 2dx + d^2)} dx = \begin{cases} 0, & m \neq n, \\ 1, & m = n. \end{cases} \]

(Askey and Wilson [1985])

7.34 Show that

\[ p_n(\cos \theta; a, b, c, d|q) = A^{-1}(\theta)(ab, ac, bc, q)_n \]

\[ \int_{e^{i\theta}/d}^{e^{-i\theta}/d} \frac{(du/e^{i\theta}, du/e^{-i\theta}, abcdq/u, q)_\infty}{(dau/q, dbu/q, dcv/q, q)_\infty} \frac{(q/u; q)_n}{(abcdq/u; q)_n} \frac{du}{q} d_4u, \]

where

\[ A(\theta) = -i(q(1 - q)/2d)(q, ab, ac, bc; q)_\infty h(\cos \theta; d)w(\cos \theta; a, b, c, d|q). \]
Hence show that
\[
\sum_{n=0}^{\infty} \frac{(a^2c^2;q)_n}{(q;q)_n(a^2c^2,\Psi_0c, \Psi_0c_1^2, \Psi_0c_2^2; q)_n} t^n
\]
\[
= \frac{(at, ae^{-i\theta}, -ae^{-i\theta}; q)_\infty}{(-ct, -ae^{i\theta}, te^{-i\theta}; q)_\infty}
\cdot \left\{ \begin{align*}
&-a^2 ctq^{-1} - ac, -a/c, -ctq^{-\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta}; q, -ctq^2 \ \text{if} \\
&\text{for (Gasper and Rahman [1986])}
\end{align*} \right.
\]

7.35 Show that
\[
\int_1^\infty \frac{\psi(x; a, b, \mu, \nu; q)(a; q)\mu}{(q; q)^{\mu} \cdot \psi(x; a, b, c, d; q)\psi(x; a, b, c, d; q)}
\]
\[
= \frac{2\pi(\mu^2; q)_\infty}{(q, q, a, \mu, \mu^{-1}, q^{-1}; q)_\infty}
\cdot \left\{ \begin{align*}
&-a^2 ctq^{-1} - ac, -a/c, -ctq^{-\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta}; q, -ctq^2 \ \text{if} \\
&\text{for (Gasper and Rahman [1986])}
\end{align*} \right.
\]
\[
\text{where } \max(|a|, |b|, |\mu|) < 1.
\]

7.36 Show that if for \(|q| < 1\) we define
\[
(a; q)_\nu = \frac{(a; q)_\infty}{(aq^\nu; q)_\infty}
\]
\[
\text{where } \nu \text{ is a complex number and the principal value of } q^\nu \text{ is taken, then (7.7.6) extends to}
\]
\[
D_q \left( ae^{i\theta}, ae^{-i\theta}; q \right)_\nu = \frac{-2a(1-q^\nu)}{1-q} \left( aq^\frac{1}{2} e^{i\theta}, aq^\frac{1}{2} e^{-i\theta}; q \right)_{\nu-1}
\]

7.37 Let \(n = 1, 2, \ldots, r\), \(x = \cos \theta\), and
\[
U_n(x) = A_{n,r} \left( q^{x+1} e^{i\theta}, q^{x+1} e^{-i\theta}; q \right)_n
\]
\[
\cdot \left\{ \begin{align*}
&-a^2 ctq^{-1} - ac, -a/c, -ctq^{-\frac{1}{2}}, ae^{i\theta}, ae^{-i\theta}; q, -ctq^2 \ \text{if} \\
&\text{for (Gasper and Rahman [1986])}
\end{align*} \right.
\]
\[
\text{with}
\]
\[
A_{n,r} = \frac{(q; q)_n}{(q; q)_r - (q^{x+1}, q^{x+1}, q^{x+1}, q^{x+1}, q^{x+1}, q^{x+1}, q^{x+1}, q^{x+1}; q)_n + q^{\frac{1}{2} \sum_{r+1}^n (2r+1)-r}}
\]
\[
\text{Show that } U_n(x) \text{ satisfies the } q \text{-differential equation}
\]
\[
D_q [(q^{x+1} e^{i\theta}, q^{x+1} e^{-i\theta}; q)_n U_n(x)]
\]
\[
= \frac{-1 - q^{x+2}}{1 - q^{x+1}} \left( q^{x+1} e^{i\theta}, q^{x+1} e^{-i\theta}; q \right)_{-2n-2\nu-1}
\]
\[
\cdot D_q [(q^{-x-\nu} e^{i\theta}, q^{-x-\nu} e^{-i\theta}; q)_{n+2\nu+1} U_{n+1}(x)].
\]
\[
\text{(Gasper [1989b])}
\]

7.38 Show that the discrete \(q\)–Hermite polynomials
\[
H_n(x; q) = \sum_{k=0}^{n/2} \frac{(q; q)_n}{(q^2; q)_k(q; q)_{n-2k}} (-1)^k q^{k(k-1)} x^{n-2k}
\]
satisfy the recurrence relation
\[
H_{n+1}(x; q) = xH_n(x; q) - q^{n-1}(1 - q^n)H_{n-1}(x; q), \quad n \geq 1,
\]
and the orthogonality relation
\[
\int_{-1}^1 H_m(x; q) H_n(x; q) d\psi(x) = \frac{(q^2; q)_\infty}{2} \delta_{m,n},
\]
where \(\psi(x)\) is a step function with jumps
\[
\frac{|x|}{2} (q^2; q)_\infty
\]
at the points \(x = \pm q^k, j = 0, 1, 2, \ldots\).
\[
\text{(Al-Salam and Carlitz [1965], Al-Salam and Ismail [1988])}
\]

7.39 Let \(a < 0\) and \(0 < q < 1\). Show that
\[
\int_{-\infty}^{\infty} U_m^{(a)}(x; q) U_n^{(a)}(x; q) d\alpha^{(a)}(x)
\]
\[
= (1 - a)(-a)^n q^{(n)} b_{m,n},
\]
where
\[
U_n^{(a)}(x; q) = (-a)^n q^{(n)} \cdot \phi_1 (q^{-x-1}; 0, q x/a)
\]
and \(\alpha^{(a)}(x)\) is a step function with jumps
\[
\frac{q^k}{(aq; q)_\infty (q^2; q)_\infty}
\]
at the points \(x = q^k, k = 0, 1, \ldots\). Verify that when \(a = -1\) this orthogonality relation reduces to the orthogonality relation for the discrete \(q\)-Hermite polynomials in Ex.7.38.
\[
\text{(Al-Salam and Carlitz [1957, 1965], Chihara [1978, (10.7)]) use the (3.3.5))}
\]

7.40 Show that if
\[
h_n(x; q) = \sum_{k=0}^{n} \frac{(q; q)_k}{(q; q)_{n-k}} x^k
\]
then
\[
h_n^2(x; q) - h_{n+1}(x; q) h_{n-1}(x; q)
\]
\[
= (1 - q)(q; q)_{n-1} \sum_{k=0}^{n} \frac{h_{n-k}^2(x; q) q^{n-k} x^k}{(q; q)_{n-k}}, \quad n \geq 1.
\]
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- Deduce that the polynomials \( h_n(x; q) \) satisfy the Turán-type inequality
  \[
  h_n^2(x; q) - h_{n+1}(x; q) h_{n-1}(x; q) \geq 0
  \]
  for \( x \geq 0 \) when \( 0 < q < 1 \) and \( n = 1, 2, \ldots \).
  (Carlitz [1957b])

7.41 Derive the addition formula

\[
p_n(q^x; 1, 1; q) p_y(q^z; q^n; 0; q) = p_n(q^{x+y}; 1, 1; q) p_n(q^y; 1, 1; q) p_y(q^z; q^n; 0; q)
+ \sum_{k=1}^n \frac{(q; q)_n (q^{x+y-k}; q) p_{n-k}(q^x; q; q) p_{n-k}(q^y; q; q) p_{n-k}(q^z; q; q)}{(q; q)_{n-k} (q; q)_{k-1} (q; q)_k} \]

(7.44)

where \( x, y, z, n = 0, 1, \ldots \), and \( p_n(t; a, b; q) \) is the little \( q \)-Jacobi polynomial defined in Ex. 1.32.
(Koornwinder [1989b])

7.42 Derive the product formula

\[
p_n(q^x; 1, 1; q) p_n(q^y; 1, 1; q) = (1 - q) \sum_{z=0}^\infty p_n(q^{x+y}; 1, 1; q) K(q^{x+y}, q^z, q; q^z) q^z
\]

where \( x, y, z, n = 0, 1, \ldots \), and \( p_n(t; a, b; q) \) is the little \( q \)-Jacobi polynomial, and

\[
K(q^x, q^y, q^z; q) = \frac{(q^{x+y}; q)_\infty (q^{x+y+z}; q)_\infty}{(q; q^x)_\infty (q; q^y)_\infty (q; q^z)_\infty}
\cdot \left\{ \frac{3\phi_2(q^{-x}, q^{-y}, q^{-z}; 0, 0; q, q)}{2} \right\}
\]

(Koornwinder [1989b])

7.43 The \( q \)-Laguerre polynomials are defined by

\[
L_n^\alpha(x; q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} \phi_1(q^{-\alpha}; q^{\alpha+1}; q; -x q^{\alpha+1} + 1)
\]

Show that if \( \alpha > -1 \) then these polynomials satisfy the orthogonality relation

\[
(i) \quad \int_0^\infty L_m^\alpha(x; q) L_n^\alpha(x; q) \frac{x^\alpha dx}{(-1-q)x; q)_\infty} = \frac{\Gamma(\alpha+1) \Gamma(-\alpha) (q^{\alpha+1}; q)_n}{\Gamma(-\alpha)(q; q)_n q^n} \delta_{m,n}
\]

and the discrete orthogonality relation

\[
(ii) \quad \sum_{k=-\infty}^\infty L_k^\alpha(q^k; q) L_n^\alpha(q^k; q) \frac{q^{\alpha+1} - c(1-q)}{(-c(1-q) q^k; q)_\infty} = A(q^{\alpha+1}; q)_n \delta_{m,n, n}
\]

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\[
A = \frac{(q, -c(1-q) q^{\alpha+1} - 1/cq^n (1-q); q)_\infty}{(q^{\alpha+1}, -c(1-q), -q/c(1-q); q)_\infty}
\]

(Moak [1981])

7.44 Let

\[
v(y; a_1, a_2, a_3, a_4, a_5|q) = \frac{h(y; 1, -1, q^{1/2}, -q^{1/2}, a_1 a_2 a_3 a_4 a_5) (1-y^2)^{-1/2}}{h(y; a_1, a_2, a_3, a_4, a_5)}
\]

and

\[
g(a_1, a_2, a_3, a_4, a_5|q) = \int_{-1}^1 v(y; a_1, a_2, a_3, a_4, a_5|q) dy.
\]

Show that

\[
\int_{-1}^1 v(y; a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}; q) \frac{(abc\mu e^{i\theta}, abc\mu e^{-i\theta}; q)_n}{(ab^2 e^{i\theta}, abc\mu^2 e^{-i\theta}; q)_n} \cdot p_n(y; a, b, c, d|q) dy
\]

\[
= g(a, b, c, \mu e^{i\theta}, \mu e^{-i\theta}; q) \frac{(ab, ac, bc; q)_n}{(ab^2, acc\mu^2, bc^2; q)_n} \mu^n \cdot p_n(x; a\mu, b\mu, c\mu, d\mu^{-1}|q), \quad x = \cos \theta,
\]

where \( p_n(x; a, b, c, d|q) \) are the Askey-Wilson polynomials defined in (7.5.2) and \( \max(|a|, |b|, |c|, |\mu|, |q|) < 1 \).
(Rahman [1988a])

Notes 7

7.1. See also Atakishiyev and Suslov [1987a,b], Nikiforov and Uvarov [1988], and Szegő [1968, 1982].

7.2 Andrews and Bressoud [1984] used the concept of a crossing number to provide a combinatorial interpretation of the \( q \)-Hahn polynomials. Koelink and Koornwinder [1989] showed that the \( q \)-Hahn and dual \( q \)-Hahn polynomials admit a quantum group theoretic interpretation, analogous to an interpretation of (dual) Hahn polynomials in terms of Clebsch-Gordan coefficients for \( SU(2) \). For how Clebsch-Gordan coefficients arise in quantum mechanics, see Biedenharn and Louck [1981a,b]. L. Chihara [1987] considered the locations of zeros of \( q \)-Racah polynomials and employed her results to prove non-existence of perfect codes and tight designs in the classical association schemes. For the relationship between orthogonal polynomials and association schemes, see Bannai and Ito [1984], L. Chihara and Stanton [1986], Delsarte [1976b], and Leonard [1982].

7.3 Al-Salam and Ismail [1977] constructed a family of reproducing kernels (bilinear formulas) for the little \( q \)-Jacobi polynomials. In Al-Salam and Ismail [1983] they considered a related family of orthogonal polynomials associated with the Rogers-Ramanujan continued fraction. A biorthogonal extension...
of the little $q$-Jacobi polynomials is studied in Al-Salam and Verma [1983a]. When $b = 0$ the little $q$-Jacobi polynomials reduce (after changing variables and renormalizing) to the Wall polynomials

$$W_n(x; b, q) = (-1)^n(b; q)_n q^{n(n+1)/2} \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] (b; q)_j \frac{q^j}{(-q^{-1}x)^j}$$

and to the generalized Stieltjes-Wigert polynomials

$$S_n(x; p, q) = (-1)^n q^{-n(2n+1)/2} (p; q)_n \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] q^j (q^{-1/2}x)^j (p; q)_j,$$

which are $q$-analogues of the Laguerre polynomials that are different from those considered in Ex. 7.43. Since the Hamburger and Stieltjes moment problems corresponding to these polynomials are both indeterminate, there are infinitely many nonequivalent measures on $[0, \infty)$ for which these polynomials are orthogonal. See Chihara [1978, Chapter VI], [1985a, 1971, 1979, 1982, 1985], Al-Salam and Verma [1982b], L. Chihara and T.S. Chihara [1987], and Shohat and Tamarkin [1950].

§7.4 An integral of the product of two continuous $q$-ultraspherical polynomials and a $q$-ultraspherical function of the second kind is evaluated in Askey, Koornwinder and Rahman [1986]. Al-Salam, Allaway and Askey [1984a] gave a characterization of the continuous $q$-ultraspherical polynomials as orthogonal polynomial solutions of certain integral equations. Askey [1989b] showed that the polynomials $C_n(iz; \beta|q)$, $0 \leq n \leq N$, are orthogonal on the real line with respect to a positive measure when $0 < q < 1$ and $\beta > q^{-N}$.

§7.5 Asymptotic formulas and generating functions for the Askey-Wilson polynomials and their special cases are derived in Ismail and Wilson [1982] and Ismail [1986c]. Kalnins and Miller [1989a] employed symmetry techniques to give an elementary proof of the orthogonality relation for the Askey-Wilson polynomials.

§7.6 For additional results on connection coefficients (and the corresponding projection formulas), see Andrews [1979a], Gasper [1974, 1975a].

Ex. 7.8 Stanton [1981b] showed that the $q$-Krawtchouk polynomials $K_n(x; a, N; q)$ are spherical functions for three different Chevalley groups over finite fields and derived three addition theorems for these polynomials by decomposing the irreducible representations with respect to maximal parabolic subgroups. In Koornwinder [1989c] it is shown that the orthogonality relation for the $q$-Krawtchouk polynomials $K_n(x; a, N|q)$ expresses the fact that the matrix representations of the quantum group $SU_q(2)$ are unitary

Ex. 7.11 The affine $q$-Krawtchouk polynomials are the eigenvalues of the association schemes of bilinear, alternating, symmetric and hermitian forms over a finite field (see Carlitz and Hodges [1955], Delsarte [1978], Delsarte and Goethals [1975], and Stanton [1981a, b, 1984]). L. Chihara and Stanton [1987] showed that the zeros of the affine $q$-Krawtchouk polynomials are never zero at integral values of $x$, and they gave some interlacing theorems for the zeros of $q$-Krawtchouk polynomials.

Ex. 7.22 Askey [1989b] proved that the polynomials $H_n(iz|q)$ are orthogonal on the real line with respect to a positive measure when $q > 1$.

Exercises 7.23–7.25 Other sieved orthogonal polynomials are considered in Al-Salam and Chihara [1985], Askey [1984b], Askey and Shukla [1989], Charris and Ismail [1986, 1987], and Ismail [1985a, 1986a,b].

Exercises 7.38–7.40 For additional material on $q$-analogues of Hermite polynomials, see Allaway [1980], Al-Salam and Chihara [1976], Al-Salam and Ismail [1988], Carlitz [1963b, 1972], Chihara [1986a, 1982, 1985], Dehesa [1979], Désarménien [1982], Ismail [1985b], Ismail, Stanton and Viennot [1987], Lubinsky and Saff [1987], and Szegő [1926].

Ex. 7.41 Rahman [1989a] gave a simple proof for this addition formula. For derivations of the addition formulas for Jacobi polynomials, see Koornwinder [1974a,b] and Laine [1982].

Ex. 7.43 See also Cigler [1981] and Pastro [1985]. In view of the two different orthogonality relations for the $q$-Laguerre polynomials, it follows that there are infinitely many measures for which these polynomials are orthogonal. The Stieltjes-Wigert polynomials (see Chihara [1978, pp. 172-174], Szegő [1975, p. 35] and the above Notes for §7.3)

$$s_n(x) = (-1)^n q^{n(n+1)/4} (q; q)_n \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] q^j (-q^{1/2}x)^j,$$

which are orthogonal with respect to the log normal weight function

$$w(x) = k\pi^{-1} \exp(-k^2 \log^2 x), \quad 0 < x < \infty,$$

where $q = \exp[-(2k^2)^{-1}]$ and $k > 0$, are a limit case of the $q$-Laguerre polynomials. Askey [1986] gave the orthogonality relation for these polynomials (with a slightly different definition) that follows as a limit case of the first orthogonality relation in this exercise. Al-Salam and Verma [1983b,c] studied a pair of biorthogonal sets of polynomials, called the $q$-Konhauser polynomials, which were suggested by the $q$-Laguerre polynomials.
8.1 Introduction

In this chapter we derive some formulas that are related to products of the \( q \)-orthogonal polynomials introduced in the previous chapter and use these formulas to obtain \( q \)-analogues of various product formulas, Poisson kernels and linearization formulas for ultraspherical and Jacobi polynomials. The method in Gasper and Rahman [1984] originates with the observation that since

\[
(q^{-x},aq^x ; q)_j = \prod_{k=0}^{j-1} (1 - q^k(q^{-x} + aq^x) + aq^{2k})
\]

is a polynomial of degree \( j \) in powers of \( q^{-x} + aq^x \), there must exist an expansion of the form

\[
(q^{-x},aq^x ; q)_j (q^{-x},aq^x ; q)_k = \sum_{m=0}^{j+k} A_m(j, k, a; q) (q^{-x},aq^x ; q)_m.
\]

Since, for \( k \geq j \),

\[
\phi_2(q^{-j},q^{k-j},aq^{k+j};aq^j,q^{1+k-j};q,q) = \frac{(q^{-x},aq^x ; q)_j}{(q^{-y},aq^y ; q)_j} \tag{8.1.1}
\]

by the \( q \)-Saalschütz formula (1.7.2), it is easy to verify that

\[
(q^{-x},aq^x ; q)_j (q^{-x},aq^x ; q)_k = (q; q)_j (q; q)_k (a; q)_j + k
\]

\[
\sum_{m=\max(j,k)}^{j+k} \frac{(q^{-x},aq^x ; q)_m q^{(j+1)+k+1} m(j+k)(-1)^{j+k+m}}{(q; q)_m (q; q)_{m-j}(q; q)_{m-k}(q; q)_{j+k-m}}. \tag{8.1.2}
\]

This linearizes the product on the left side and forms the basis for the product formulas derived in the following section.

Suppose \( \{B_j\}_{j=0}^{\infty} \) and \( \{C_j\}_{j=0}^{\infty} \) are arbitrary complex sequences and \( b, c \) are complex numbers such that \( (b; q)_k, (c; q)_k \) do not vanish for \( k = 1, 2, \ldots \). Then, setting

\[
F_n = \sum_{j=0}^{n} \frac{(q^{-n},aq^n ; q)_j B_j C_k}{(q, b; q)_j} \sum_{k=0}^{n} \frac{(q^{-n},aq^n ; q)_k}{(q, c; q)_k} \tag{8.1.3}
\]

we find by using (8.1.2) that

\[
F_n = \sum_{m=0}^{n} \frac{(q^{-n},aq^n ; q)_m C_k q^{k^2-2nk}}{(q, c; q)_k (q, b; q)_{m-k}}.
\]
Then, replacing \( j \) by \( j - k \) in the sum on the right side of (8.2.4), we obtain

\[
4\phi_3 \left[ q^{-m}, aq^n, b_1, b_2 \middle| q \right] 4\phi_3 \left[ q^{-m}, aq^n, c_1, c_2 \middle| b_3, b_3, qab_2b_2/bb_3, q \right] = \sum_{m=0}^{n} \sum_{j=0}^{m} \left( \frac{bq_{q^{-m-1}/a}}{b_3} \right)^j \left( \frac{bq_{q^{-m-1}/a}}{bb_3} \right) \left( \frac{bq_{q^{-m-1}/a}}{bb_3} \right)
\]

\[
= \sum_{m=0}^{n} \sum_{j=0}^{m} \left( \frac{bq_{q^{-m-1}/a}}{b_3} \right)^j \left( \frac{bq_{q^{-m-1}/a}}{bb_3} \right) \left( \frac{bq_{q^{-m-1}/a}}{bb_3} \right)
\]

Note that the \( 5\phi_4 \) series in (8.2.5) is balanced and, in the special case \( c = aq/b \) and \( c_3 = aq/b_3 \), becomes a \( 3\phi_2 \) which is summable by (1.7.2). Thus, we obtain the formula

\[
4\phi_3 \left[ q^{-m}, aq^n, b_1, b_2 \middle| q \right] 4\phi_3 \left[ q^{-m}, aq^n, c_1, c_2 \middle| b_3, b_3, qab_2b_2/bb_3, q \right] = \sum_{m=0}^{n} \sum_{j=0}^{m} \left( \frac{bq_{q^{-m-1}/a}}{b_3} \right)^j \left( \frac{bq_{q^{-m-1}/a}}{bb_3} \right) \left( \frac{bq_{q^{-m-1}/a}}{bb_3} \right)
\]

where \( \lambda = bb_3q_{q^{-m-1}/a} \). This formula is a \( q \)-analogue of Bailey’s [1933] product formula

\[
2F_1(-n, a + n; b; x) 2F_1(-n, a + n; 1 + a - b; y) = F_4(-n, a + n + b, 1 + a - b; x(1 - y), y(1 - x)),
\]

where

\[
F_4(a; b; c; d; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a)_m(b)_n}{(c)_m(d)_n} x^m y^n.
\]

However, even though (8.2.6) is valid only when the series on both sides terminate, (8.2.7) holds whether or not \( n \) is a nonnegative integer, subject to the absolute convergence of the two \( 2F_1 \) series on the left and the \( F_4 \) series on the right.

Application of Sears’ transformation formula (2.10.4) enables us to transform one or both of the \( 4\phi_3 \) series on the left side of (8.2.6) and derive a number of equivalent forms. Two particularly interesting ones are

\[
4\phi_3 \left[ q^{-n}, aq^n, b_1, b_2 \middle| q \right] 4\phi_3 \left[ q^{-n}, aq^n, b_3c_1/q, b_3c_2/q \middle| aq \right]
\]
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\[
\text{bb}_3c_1/aq, \quad aq^m, \quad q^{-m}, \quad qab_1b_2/bb_3, \quad b_1b_3c_1q^{-m}/a, \quad b_1b_3c_1q^m; q, q \],
\]
(8.2.12)

where \( \mu = b_1b_3c_1q^{-1} \). This provides a \( q \)-analogue of Bateman’s [1932, p. 392] product formula

\[
2F_1(-n, a + n; b; x) \quad 2F_1(-n, a + n; b; y) = (-1)^n \left( \frac{1}{a - b} \right) \sum_{k=0}^{n} \frac{(-n)_k(a + n)_k}{k!(a - b)_k} (1 - x - y)^k \cdot 2F_1(-k, a + k; b; -xy/(1 - x - y)).
\]
(8.2.13)

8.3 Product formulas for \( q \)-Racah and Askey-Wilson polynomials

Let us replace the parameters \( a, b, b_1, b_2, c_1, c_2 \) in (8.2.9) by \( abq, aq, q^{-m}, cq^{q-N}, bcq, c^{-1}q^{-y}, q^{q-N} \), respectively, to obtain the following product formula for the \( q \)-Racah polynomials introduced in §7.2:

\[
W_n(x; a, b, c, N; q) \quad W_n(y; a, b, c, N; q) = \frac{(abq, qc; \infty; q)_n}{(abq, bc; \infty; q)_n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, q^{q-N}; q)_m}{(q, bq, q^{-c}; q)_n, q^{-1}q^{-x}, q^{-1}q^{-y}; q)_m} \cdot q^{-m} \quad q^{-n} \quad q^{q-N} \]
(8.3.1)

where

\[
W_n(x; a, b, c, N; q) = 4 \phi_3 \left[ q^{n}, abq^{n+1}, q^{-x}, q^{q-N}; q \right. \quad q^{q-N}, bcq; q, q \right]
\]
(8.3.2)
is the \( q \)-Racah polynomial defined in (7.2.17). This is a Watson-type formula. Two additional Watson-type formulas are given in Ex. 8.1.

Letting \( c \to 0 \) in (8.3.1) gives a product formula for the \( q \)-Hahn polynomials defined in (7.2.21):

\[
Q_n(x; a, b, c, N; q) = (-aq^nq^c) \frac{(bq; q)_n}{(aq; q)_n} \sum_{m=0}^{n} \frac{(q^{-m}, abq^{m+1}, q^{q-N}; q)_m}{(q, bq, q^{-N}; q)_m} \cdot q^{2n+3}.
\]
(8.3.3)

To obtain a Watson-type product formula for the Askey-Wilson polynomials defined in (7.5.2) we replace \( a, b, b_1, b_2, c_1, c_2 \) in (8.2.9) by \( abcdq^{-1}, ab, ac, de^{-i\theta}, de^{-i\theta}, de^{-i\theta}, \) respectively, where \( x = \cos \theta, y = \cos \phi \).

This gives

\[
p_n(x; a, b, c, d; q) \quad p_n(y; a, b, c, d; q) = (ab, ac, ad, ab, bd, cd, q)_n (ad)^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, abcdq^{-1}; q)_m}{(q, ad, ad, bd, cd, da^{-1}; q)_m} \cdot q^{-m} \quad q^{-n} \quad q^{-1} \quad q^{-1} \quad q^{-1}; q, q \]
(8.3.4)

where \( b = q^3 \) and \( d = cq^2 \), the 10\( \phi_9 \) series in (8.3.4) becomes balanced and hence can be transformed to another balanced 10\( \phi_9 \) via (2.9.1). This leads to a Bateman-type product formula

\[
p_n(x; a, q^3, c, q^3; q) \quad p_n(y; a, q^3, c, q^3; q) = (a^2q^4, ac, ac^3q^3; q)_n (acq^3)^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, a^2c^2q^6, acq^3; q)_m}{(q, c^2q^3, acq^3, acq^3; q)_m} \cdot q^{-m} \quad q^{-n} \quad q^{-1} \quad q^{-1} \quad q^{-1}; q, q \]
(8.3.5)

where \( \nu = a^2c^2q^{-\frac{1}{2}} \). In fact, if we replace \( a \) and \( c \) by \( q^{(2n+1)/4} \) and \( -q^{(2n+1)/4} \), respectively, then this gives a Bateman-type product formula for the continuous \( q \)-Jacobi polynomials (7.5.24) which, on letting \( q \to 1 \), gives Bateman’s [1932] product formula for the Jacobi polynomials:

\[
\frac{P_n^{(a, b)}(x)}{P_n^{(a, b)}(1)} \quad \frac{P_n^{(a, b)}(y)}{P_n^{(a, b)}(1)} = (1 - n)(\beta + 1)n \sum_{k=0}^{\infty} \frac{(-n)(n + \alpha + \beta + 1)_k}{k!} \left( \frac{x + y}{2} \right)^k.
\]
(8.3.6)

which is equivalent to (8.2.13).

For terminating series there is really no difference between the Watson formula (8.2.11) and the Bailey formula (8.2.7) once one can be transformed into the other in a trivial way. However, for the continuous \( q \)-ultraspherical polynomials given in (7.4.14), there is an interesting Bailey-type product formula that can be obtained from (8.2.6) by replacing \( a, b, b_1, b_2, b_3, c_1, c_2 \) by...
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\[ a^4, a^2q^4, ae^{i\theta}, ae^{-i\phi}, -a^2q^4, ae^{i\phi}, \text{ and } ae^{-i\phi}, \text{ respectively:} \]

\[ 4\phi_3 \left[ \frac{q^n, a^4q^n, ae^{i\theta}, ae^{-i\phi}}{a^4q^4, -a^2q^4, -a^2} ; q, q \right] 4\phi_3 \left[ \frac{q^n, a^4q^n, ae^{i\phi}, ae^{-i\phi}}{a^4q^4, -a^2q^4, -a^2} ; q, q \right] \]

\[ = \sum_{m=0}^{n} \left( \frac{q-n, a^4q^n, ae^{i\theta}, ae^{-i\phi}}{a^4q^4, -a^2q^4, -a^2} ; q, q \right) \]

\[ \cdot \frac{1}{m(m-1)} \left( \begin{array}{c}
q^{-m}, q(-q^{-m}) \frac{1}{2}, -q(-q^{-m}) \frac{1}{2}, q^{-1-m}/a^2, -q^{-1-m}/a^2, q^{-m},
\frac{1}{2}(-q^{-m}) \frac{1}{2}, -(-q^{-m}) \frac{1}{2}, a^2q^4, a^2q^4, -q,
-ae^{i\theta}, -ae^{-i\phi},
-ae^{i\phi}, -ae^{-i\phi},
-ae^{i\theta} - ae^{-i\phi}, -ae^{i\phi} - ae^{-i\phi}, q^{-1-m}/a^2, q^{-1-m}/a^2, q^{-m}, q^4 \end{array} \right). \]

(8.3.7)

For further information about product formulas see Rahman [1982] and Gasper and Rahman [1984].

8.4 A product formula in integral form for the continuous $q$-ultraspherical polynomials

As an application of the Bateman-type product formula (8.3.5) for the Askey-Wilson polynomials we shall now derive a product formula for the continuous $q$-ultraspherical polynomials in the integral form

\[ C_n(x; \beta; q)C_n(y; \beta; q) = \frac{1}{(q; q)_n} \int_{-1}^{1} K(x, y; \beta; q)C_n(x; \beta; q)\, dz, \]

(8.4.1)

where

\[ K(x, y; \beta; q) = \frac{(\beta^2; q)_\infty}{2\pi(\beta^2; q)_\infty} |(\beta e^{i\theta}; \beta e^{i\phi}; q)_\infty|^2 \]

\[ \cdot w(x; \beta^\frac{1}{2} e^{i\theta} + i\phi, \beta^\frac{1}{2} e^{-i\theta} - i\phi, \beta^\frac{1}{2} e^{i\theta} - i\phi, \beta^\frac{1}{2} e^{-i\theta} + i\phi) \]

(8.4.2)

with $w(z; a, b, c, d)$ defined as in (6.3.1) and $x = \cos \theta, y = \cos \phi$.

First, we set $c = -a$ in (8.3.5) and rewrite it in the form

\[ r_n(x; a, q^\frac{1}{2}, -a, -aq^\frac{1}{2}) = \frac{1}{1 + a^2} \left( q^{-1} \right)^n \]

\[ \sum_{m=0}^{n} \left( \frac{q^n, a^4q^n, a^2q^\frac{1}{2} e^{i\theta} - i\phi, -a^2q^\frac{1}{2} e^{i\theta} - i\phi, -aq^\frac{1}{2} e^{-i\phi}}{a^2q^4, -a^2q^4, -a^2q^4, -a^2q^4, q^4} ; q, q \right) \]

\[ \cdot 10W_0 \left( -a^3q^{-\frac{3}{2}} e^{i\phi}; \beta e^{-i\phi}, -ae^{i\phi}, ae^{i\phi}, ae^{i\phi}, ae^{-i\phi}, a^4q^m, q^{-m} ; q, q \right), \]

(8.4.3)

where

\[ r_n(x; a, b, c, d) = 4\phi_3 \left[ \frac{q^n, abcdq^{n-1}, ae^{i\theta}, ae^{-i\phi}}{ab, ac, ad} ; q, q \right] \]

(8.4.4)

The key step now is to use the $d = -(aq)^\frac{1}{2}$ case of Bailey's transformation formula (2.8.3) to transform the balanced $10\phi_9$ series in (8.4.3) to a balanced $4\phi_3$ series:

\[ 10W_0 \left( -a^3q^{-\frac{3}{2}} e^{i\phi}; ae^{i\phi}, -ae^{i\phi}, ae^{i\phi}, aq^\frac{1}{2} e^{i\phi}, ae^{i\theta}, ae^{-i\phi}, a^4q^m, q^{-m} ; q, q \right) \]

\[ = \frac{1}{a^4, -aq^\frac{1}{2} e^{-i\phi}; q} \]

\[ \cdot 4\phi_3 \left[ \frac{a^2e^{2i\phi}, -a^2e^{2i\phi}, -a^2e^{2i\phi}, -a^2e^{2i\phi}, q^{-m}}{a^2, q^2, q^2, q^2, q^2} ; q, q \right]. \]

(8.4.5)

So (8.4.3) reduces to

\[ r_n(x; a, q^\frac{1}{2}, -a, -aq^\frac{1}{2}) = \frac{1}{1 + a^2} \left( q^{-1} \right)^n \sum_{m=0}^{n} \left( \frac{q^n, a^4q^n, a^2q^\frac{1}{2} e^{i\theta} - i\phi, -a^2q^\frac{1}{2} e^{i\theta} - i\phi, -aq^\frac{1}{2} e^{-i\phi}}{a^2q^4, -a^2q^4, -a^2q^4, -a^2q^4, q^4} ; q, q \right) \]

\[ \cdot 4\phi_3 \left[ \frac{a^2e^{2i\phi}, -a^2e^{2i\phi}, -a^2e^{2i\phi}, -a^2e^{2i\phi}, q^{-m}}{a^2, q^2, q^2, q^2, q^2} ; q, q \right]. \]

(8.4.6)

Transforming this $4\phi_3$ series by Sears' transformation formula (2.10.4), we obtain a further reduction

\[ r_n(x; a, q^\frac{1}{2}, -a, -aq^\frac{1}{2}) = \frac{1}{1 + a^2} \left( q^{-1} \right)^n \sum_{m=0}^{n} \left( \frac{q^n, a^4q^n, a^2q^\frac{1}{2} e^{i\theta} - i\phi, -a^2q^\frac{1}{2} e^{i\theta} - i\phi, -aq^\frac{1}{2} e^{-i\phi}}{a^2q^4, -a^2q^4, -a^2q^4, -a^2q^4, q^4} ; q, q \right) \]

\[ \cdot 4\phi_3 \left[ \frac{q^{-m}, a^2, a^2e^{2i\phi}, a^2e^{2i\phi}}{a^4, -a^2q^\frac{1}{2} e^{-i\phi}, -q^{-1-m}e^{i\phi} - i\phi ; q, q} \right]. \]

(8.4.7)

Now observe that, by (6.1.1),

\[ \int_{-1}^{1} w(x; ae^{i\theta} - i\phi, ae^{i\theta} - i\phi, ae^{i\theta} - i\phi, ae^{i\theta} - i\phi) (ae^{i\theta} - i\phi + i\psi, ae^{i\theta} - i\phi - i\psi) ; q) \, dz \]

\[ = \int_{-1}^{1} w(z; a^2q^\frac{1}{2} e^{-i\phi}, ae^{i\theta} - i\phi, ae^{i\theta} - i\phi, ae^{i\theta} - i\phi) \, dz \]

\[ = \frac{2\pi(a^4; q)_\infty}{(a^2q^4; q)_\infty^2} \left( \frac{a^2, a^2e^{2i\phi}, a^2e^{2i\phi}}{a^4, -a^2q^\frac{1}{2} e^{-i\phi}, -q^{-1-m}e^{i\phi} - i\phi ; q, q} \right) \]

(8.4.8)

where $|a| < 1$ and $z = \cos \psi$. Hence

\[ 4\phi_3 \left[ \frac{q^{-m}, a^2, a^2e^{2i\phi}, a^2e^{2i\phi}}{a^4, -a^2q^\frac{1}{2} e^{-i\phi}, -q^{-1-m}e^{i\phi} - i\phi ; q, q} \right]. \]
Further Applications

\[
\begin{align*}
\frac{(q, a^2, a^2; q)_{\infty} \left( a^2 e^{2i\phi}, a^2 e^{-2i\phi}; q \right)_{\infty}}{2\pi(a^2; q)_{\infty}} \\
\int_{-1}^{1} w(z; ae^{i\phi-\theta}, ae^{i\phi+\theta}, ae^{-i\phi-\theta}, ae^{-i\phi+\theta}) \\
\cdot 3\phi_2 \left[ q^{-m}, ae^{i\phi-\theta}, ae^{i\phi+\theta}, ae^{-i\phi-\theta}, ae^{-i\phi+\theta} \right] \left[ -a^2 q^{\frac{1}{2}} e^{i\theta-\phi}, -q^{\frac{1}{2}} e^{-i\theta+\phi}, q^{-\frac{1}{2}} e^{-i\theta-\phi} \right] \left[ q^{-m} e^{i\phi-\theta}, -q^{\frac{1}{2}} e^{i\phi+\theta}, q^{m} e^{-i\phi+\theta} \right] \left[ -a^2 q^{\frac{1}{2}} e^{i\theta+\phi}, -q^{\frac{1}{2}} e^{-i\theta-\phi}, q^{-\frac{1}{2}} e^{-i\theta+\phi} \right] \left[ q^{m} e^{i\phi+\theta}, -q^{\frac{1}{2}} e^{i\phi-\theta}, q^{-m} e^{-i\phi-\theta} \right] \left[ -a^2 q^{\frac{1}{2}} e^{i\theta-\phi}, -q^{\frac{1}{2}} e^{-i\theta+\phi}, q^{-\frac{1}{2}} e^{-i\theta-\phi} \right] \right] dz \\
= \frac{(q, a^2, a^2; q)_{\infty} \left( a^2 e^{2i\phi}, a^2 e^{-2i\phi}; q \right)_{\infty}}{2\pi(a^2; q)_{\infty}} \\
\int_{-1}^{1} w(z; ae^{i\phi-\theta}, ae^{i\phi+\theta}, ae^{-i\phi-\theta}, ae^{-i\phi+\theta}) \\
\cdot 3\phi_2 \left[ q^{-m}, ae^{i\phi-\theta}, ae^{i\phi+\theta}, ae^{-i\phi-\theta}, ae^{-i\phi+\theta} \right] \left[ -a^2 q^{\frac{1}{2}} e^{i\theta-\phi}, -q^{\frac{1}{2}} e^{-i\theta+\phi}, q^{-\frac{1}{2}} e^{-i\theta-\phi} \right] \left[ q^{-m} e^{i\phi-\theta}, -q^{\frac{1}{2}} e^{i\phi+\theta}, q^{m} e^{-i\phi+\theta} \right] \left[ -a^2 q^{\frac{1}{2}} e^{i\theta+\phi}, -q^{\frac{1}{2}} e^{-i\theta-\phi}, q^{-\frac{1}{2}} e^{-i\theta+\phi} \right] \left[ q^{m} e^{i\phi+\theta}, -q^{\frac{1}{2}} e^{i\phi-\theta}, q^{-m} e^{-i\phi-\theta} \right] \left[ -a^2 q^{\frac{1}{2}} e^{i\theta-\phi}, -q^{\frac{1}{2}} e^{-i\theta+\phi}, q^{-\frac{1}{2}} e^{-i\theta-\phi} \right] \right] dz.
\end{align*}
\]

Substituting (8.4.9) into (8.4.7) and using (2.10.4) we finally obtain

\[
\begin{align*}
\frac{r_n(x; a, a^2 q^{\frac{1}{2}}, -a, -a^2 q^{\frac{1}{2}}; q) \left( r_n(x; a, a, a^2 q^{\frac{1}{2}}, -a, -a^2 q^{\frac{1}{2}}; q) \right)}{\Gamma(n+1)} = \int_{-1}^{1} K(x, y, z; a, a^2 q^{\frac{1}{2}}, -a, -a^2 q^{\frac{1}{2}}; q) \left( r_n(x; a, a, a^2 q^{\frac{1}{2}}, -a, -a^2 q^{\frac{1}{2}}; q) \right) dz.
\end{align*}
\]

This yields (8.4.1) if we replace \( a \) by \( \beta^\frac{1}{2} \) and use (7.4.14). By setting \( \beta = q^4 \) in (8.4.1) and taking the limit \( q \to 1 \), Rahman and Verma [1986b] showed that (8.4.1) tends to Gegenbauer's [1874] product formula

\[
\frac{C_n^\lambda(x) C_n^\lambda(y)}{C_n^\lambda(1) C_n^\lambda(1)} = \int_{-1}^{1} K(x, y, z;\lambda, \lambda, \lambda, \lambda; q) \left( C_n^\lambda(x) \right) dz,
\]

where

\[
K(x, y, z) = \frac{\Gamma(\lambda + \frac{1}{2})}{\Gamma(\lambda)} \left( 1 - x^2 - y^2 - z^2 + 2xyz \right)^{\lambda-\frac{1}{2}},
\]

according as \( 1 - x^2 - y^2 - z^2 + 2xyz \) is positive or negative.

Rahman and Verma [1986b] were also able to derive an addition formula for the continuous \( q \)-ultraspherical polynomials corresponding to the product formula (8.4.1). This is left as an exercise (Ex. 8.11).

For an extension of (8.4.10) to the continuous \( q \)-Jacobi polynomials \( r_n(x; a, a q^{\frac{1}{2}}, b, -b q^{\frac{1}{2}}; q) \), see Rahman [1986d].

8.5 Rogers’ linearization formula for the continuous \( q \)-ultraspherical polynomials

Rogers [1985] used an induction argument to prove the linearization formula

\[
C_m(x; \beta|q) C_n(x; \beta|q) = \sum_{k=0}^{\min(m,n)} \frac{(q, q)_m q^{-m-k} (\beta^2; q)_n q^{-n-k} (\beta; q)_k}{(q, q)_n q^{-n-k} (\beta; q)_m q^{-m-k} (\beta^2; q)_k} C_{m+n-k}(x; \beta|q) \left( \frac{1-\beta q^{m+n-k}}{1-\beta} \right).
\]

Different proofs of (8.5.1) have been given by Bressoud [1981d], Rahman [1981] and Gasper [1985]. We shall give Gasper’s proof since it appears to be the simplest.

We use (7.4.2) for \( C_n(x; \beta|q) \) and, via Heine’s transformation formula (1.4.3),

\[
C_m(x; \beta|q) = \frac{(\beta e^{-2i\theta}; q)_\infty}{(q, \beta e^{-i\theta}; q)_\infty} \left( \beta q^{-m} \right)^{\frac{1}{2}} \left( \beta q^{-m+1}; q \right)^{\frac{1}{2}}
\]

\[
\cdot \phi_2 \left( q^{-1}; \beta^{-1}, q^{-1-m}; q, \beta^{-1} q^{-1-m}; q, \beta e^{-2i\theta} \right),
\]

where \( x = \cos \theta \). Then, temporarily assuming that \( q < |\beta| < 1 \), we have

\[
C_m(x; \beta|q) C_n(x; \beta|q) = A_{m,n} \sum_{r=0}^{\infty} \left( q^{n-r}, \beta, q^r \right) \left( q^{-1} e^{-2i\theta} \right)^r
\]

\[
\cdot \phi_2 \left( q^{-1}; \beta^{-1}, q^{-1-m}; q, \beta^{-1} q^{-1-m}; q, \beta e^{-2i\theta} \right).
\]

The crucial point here is that the \( \phi_3 \) series in (8.5.3) is balanced and so, by (2.5.1),

\[
4 \phi_3 \left[ q^{-k, q^{-n}, \beta, \beta q^{m-k}} \right] = \left( \beta^{-2} q^{-1-m-n}, \beta^{-1} q^{-1-m-n} \right) \left( \beta^{-1} q^{-1-m-n}, \beta^{-2} q^{-1-m-n} \right)
\]

\[
\cdot 8 W_7 \left( \beta^{-1} q^{-m-n}, \beta^{-2} q^{-1+k-m-n}, q^{-m-n}, q^{-k}, \beta e^{-1} \right).
\]
Substituting this into (8.5.3) and interchanging the order of summation, we obtain
\[
C_m(x; \beta|q)C_n(x; \beta|q) = A_{m,n}
\]
\[
\sum_{k=0}^{\min(m,n)} \frac{(\beta - 1)_{q^{-m-n}} q_k (1 - \beta^{-1} q^{2k-m-n}) (\beta q^m, q^n; q)_k}{(q; q)_k (1 - \beta^{-1} q^{m-n}) (1 - \beta^{-1} q^{2k-m-n}) (\beta q^{1-m-n}, \beta^{-1} q^{1-m-n}; q)_k} \cdot 2\psi^2_k(q^\beta, q^{-1} q^{1+2k-m-n}, q^{-1} q^{1+2k-m-n}; q, \beta e^{-2i\theta})
\]
which gives (8.5.1) by using (8.5.2) and observing that both sides of (8.5.1) are polynomials in \(x\). Notice that the linearization coefficients in (8.5.1) are nonnegative when \(-1 < \beta < 1\) and \(-1 < q < 1\).

For an extension of the linearization formula to the continuous \(q\)-Jacobi polynomials, see Rahman [1981].

### 8.6 The Poisson kernel for \(C_n(x; \beta|q)\)

For a system of orthogonal polynomials \(\{p_n(x)\}\) which satisfies an orthogonality relation of the form (7.1.6), the bilinear generating function
\[
K_t(x, y) = \sum_{n=0}^{\infty} h_n p_n(x) p_n(y) t^n
\]
is called a Poisson kernel for these polynomials provided that \(h_n = c_n\) for some constant \(c > 0\). The Poisson kernel for the continuous \(q\)-ultraspherical polynomials is defined by
\[
K_t(x, y; \beta|q) = \sum_{n=0}^{\infty} \frac{(q; q)_n (1 - \beta^{-1})}{(\beta^2; q)_n (1 - \beta)} C_n(x; \beta|q)C_n(y; \beta|q) t^n
\]
where \(|t| < 1\).

Gasper and Rahman [1983] used (8.5.1) to show that
\[
K_t(x, y; \beta|q) = \frac{(\beta^2, q^t; q)_\infty}{(\beta^2, \beta^t; q)_\infty} \left(\frac{\beta e^{i\theta} t^n, \beta e^{-i\theta} t^n; q}_{(\beta e^{i\theta} t^n, \beta e^{i\theta} t^n; q)_\infty}\right)^2
\]
where \(x = \cos \theta, y = \cos \phi\) and \(\max(|q|, |t|, |\beta|) < 1\). They also computed a closely related kernel
\[
L_t(x, y; \beta|q) = \sum_{n=0}^{\infty} \frac{(q; q)_n}{(\beta^2, q)_n} C_n(x; \beta|q)C_n(y; \beta|q) t^n
\]
alternative derivations of (8.6.3) and (8.6.4) were given by Rahman and Verma [1986]. In view of the product formula (8.4.1), however, one can now give a simpler proof. Let us assume, for the moment, that \(|t\beta^{-\frac{1}{2}}| < 1\) and \(|\beta| < 1\). Then, by (7.4.1) and (8.4.1), we find that
\[
L_t(x, y; \beta|q) = \int_{-1}^{1} K(x, y, z; \beta|q) \frac{t \beta^{\frac{1}{2}} e^{i\psi}, t \beta^{\frac{1}{2}} e^{-i\psi}; q}{(t \beta^{\frac{1}{2}} e^{i\psi}, t \beta^{\frac{1}{2}} e^{-i\psi}; q)_\infty}^2 dz
\]
\[
= \frac{(q, \beta, \beta; q)_\infty}{(q, \beta, \beta; q)_\infty} \left(\frac{(\beta e^{i\theta} t^n, \beta e^{i\theta} t^n; q)}{(\beta e^{i\theta} t^n, \beta e^{i\theta} t^n; q)_\infty}\right)^2
\]
\[
= \frac{2 \pi}{2 \pi} \int_{0}^{\pi} \frac{h(\cos \psi; 1, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}, t \beta^{\frac{1}{2}})}{h(\cos \psi; \beta^{\frac{1}{2}} e^{i\theta} + t, \beta^{\frac{1}{2}} e^{i\theta} - t, \beta^{\frac{1}{2}} e^{i\theta} - t, t \beta^{\frac{1}{2}})} d\psi
\]
\[
= \frac{2 \pi (\beta, t^2; q)_\infty}{(q, \beta, \beta^t; q)_\infty} \left(\frac{(\beta e^{i\theta} t^n, \beta e^{i\theta} t^n; q)}{(\beta e^{i\theta} t^n, \beta e^{i\theta} t^n; q)_\infty}\right)^2
\]
\[
= \int_{0}^{\pi} \frac{h(\cos \psi; 1, -q^{\frac{1}{2}}, -q^{\frac{1}{2}}, t \beta^{\frac{1}{2}})}{h(\cos \psi; \beta^{\frac{1}{2}} e^{i\theta} + t, \beta^{\frac{1}{2}} e^{i\theta} - t, \beta^{\frac{1}{2}} e^{i\theta} - t, t \beta^{\frac{1}{2}})} d\psi
\]
Formula (8.6.4) follows immediately from (8.6.5) and (8.6.6).

It is slightly more complicated to compute (8.6.1). Consider the generating function
\[
G_t(x) = \sum_{n=0}^{\infty} \frac{1 - \beta^n}{1 - \beta} C_n(x; \beta|q) t^n
\]
\[
= \frac{(\beta e^{i\phi}, \beta e^{-i\phi}; q)_\infty}{(1 - \beta)(te^{i\phi}, te^{-i\phi}; q)_\infty} \cdot \frac{\beta}{1 - \beta} \left(\frac{(\beta e^{i\phi} q^{-t}, \beta e^{-i\phi} q^{-t}; q)}{(te^{i\phi}, te^{-i\phi}; q)_\infty}\right)_\infty
\]
\[
= (1 - \beta^2) \left(\frac{\beta e^{i\phi} q^{-t}, \beta e^{-i\phi} q^{-t}; q}_\infty}{(te^{i\phi}, te^{-i\phi}; q)_\infty}\right)_\infty
\]
Then

\[
\begin{align*}
K_t(x,y;\beta;q) &= \int_{t^{-1}} \int_{z^{-1}} K(x,y,z;\beta;q) \, G_{t^{\beta-1}}(z) \, dz \\
&= \frac{(1-t^2)(q,\beta,\beta;\infty)}{2\pi(\beta^2;\infty)} \left( \beta e^{i\theta}, \beta e^{i\theta}; q \right)_{\infty}^2 \\
&= \int_0^{\pi} \int_0^{\pi} h(\cos \psi; 1, -1, q^{1/2}, -q^{-1/2}, q^t \beta^{1/2}) \, d\psi.
\end{align*}
\]  

(8.6.8)

This gives (8.6.3) via (7.4.1) and an application of (2.10.1).

By analytic continuation, formulas (8.6.3) and (8.6.4) hold when \( \max(|q|, |t|, |\beta|) < 1 \).

Even though it is clear from (8.6.3) and (8.6.4) that these kernels are positive when \(-1 < q, t < 1 \) and \(-1 \leq x, y \leq 1 \) if \( 0 \leq \beta < 1 \), it is not clear what happens when \(-1 < \beta < 0 \) since both \( \phi_\gamma \) series in (8.6.3) and (8.6.4) become alternating series. It is shown in Garper and Rahman [1983a] that the Poisson kernel \( K_t(x,y;\beta;q) \) is also positive for \(-1 < t < 1 \) when \(-1 < q < \beta < 0 \) and when \( 2^{3/2} - 3 \leq \beta < 0 \), \(-1 < q \leq 0 \).

For the nonnegativity of the Poisson kernel for the continuous \( q \)-Jacobi polynomials, see Garper and Rahman [1986].

### 8.7 Poisson kernels for the \( q \)-Racah polynomials

For the \( q \)-Racah polynomials

\[
W_n(x; q) \equiv W_n(x; a, b, c, N; q)
\]

we shall give conditions under which the Poisson kernel

\[
\sum_{n=0}^{\infty} h_n(q) W_n(x; q) W_n(y; q) q^n, \quad 0 \leq t < 1,
\]

(8.7.1)

and the so-called discrete Poisson kernel

\[
\sum_{n=0}^{z} \frac{(-z; q)_n h_n(q) W_n(x; q) W_n(y; q)}{(-N; q)_n}
\]

(8.7.2)

\( z = 0, 1, \ldots, N \), are nonnegative for \( x, y = 0, 1, \ldots, N \).

Let us first consider a more general bilinear sum

\[
P_t(x, y) \equiv P_t(x, y; a, b, c, \alpha, \gamma, K, M, N; q)
\]

\[
= \sum_{n=0}^{z} \frac{(q^{-z}; q)_n p_n(a, b, c, N; q) W_n(x; a, b, c, N; q)}{(q^{-K}; q)_n} \cdot W_n(y; \alpha, ab\alpha^{-1}, \gamma, M; q),
\]

(8.7.3)

where \( z = 0, 1, \ldots, \min(K, N) \) and \( N \leq M \). If \( \alpha = a, \gamma = c \) and \( M = N \), then (8.7.3) reduces to (8.7.2) when \( K = N \), and it has the Poisson kernel (8.7.6) as a limit case.

From the product formula (8.2.5) it follows that

\[
W_n(x; a, b, c, N; q) W_n(y; \alpha, ab\alpha^{-1}, \gamma, M; q)
\]

\[
= (aq, q^{\alpha}; q)_n q^n \sum_{r=0}^{n} \sum_{s=0}^{n-r} \frac{(q^{-z}, cq^{-N}, q^{\gamma}; q)_r (q^{-z}, cq^{-N}, q^{\gamma}; q)_s}{(aq, b; q)_n (aq, q^{-M}; q)_r}
\]

\[
\left( q^{-z}, c^{-1}; q^{-z}; q)_s 1 - cq^{z-s} \right) \cdot \left( q, bq, ac^{-1}; q; q \right)_s 1 - cq^{z-s} \cdot \sum_{s=0}^{r+s} A_{r,s} B_{r,s},
\]

(8.7.2)

where

\[
A_{r,s} = 5 \phi_4 \begin{bmatrix} q^{-r}, q^{-s}, cq^{-r+s-M}, aq^{r+s-1}, b; q \end{bmatrix}
\]

(8.7.7)

Using (8.7.5) in (8.7.4) and changing the order of summation, we find that

\[
P_t(x, y) = \sum_{r=0}^{z} \sum_{s=0}^{n-r} \frac{(q^{-z}; q)_r (aq^2, c^{-1}; q)_s (aq^2, q^{-M}; q)_s}{(aq^2, c^{-1}; q)_s (aq^2, q^{-M}; q)_s}
\]

\[
\left( q^{-z}, cq^{-N}, q^{\gamma}; q^0 \right) (q^{-z}, cq^{-N}, q^0; q)_s 1 - cq^{z-s} \cdot \left( q, bq, ac^{-1}; q; q \right)_s 1 - cq^{z-s} \cdot \left( -1 \right)^{r+s} q^{-1} \cdot (2n-rs+1)/2 \cdot \sum_{r=0}^{r+s} A_{r,s} B_{r,s},
\]

(8.7.6)

where

\[
B_{r,s} = 5 \phi_4 \begin{bmatrix} \lambda, q^{-1} \lambda, q^{-1} \lambda, cq^{-N}, q^{r+s-1}, \lambda^{-1} \lambda^{-1} \lambda; q, q^{-N-rs} \end{bmatrix}
\]

(8.7.7)

with \( \lambda = abq^{2r+2s+1} \). We shall now show that when \( K = N \),

\[
B_{r,s} = \frac{(q^{-N-r}; q)_r (q^{-r-s}; q)_r}{(aq^{N-r+s+2}, q^{r+s-K}; q)_r} 2 \phi_1 \left( q^{r+s-z}, q^{r+s-z}, q^{r+s-z}, q^{N-r-s}; q, abq^{N+r+s+2} \right).
\]

(8.7.8)

To prove (8.7.8) it suffices to show that

\[
4 \phi_3 \begin{bmatrix} a, b, c, q^{-1} a, q^{-1} b, q^{-1} c; q \end{bmatrix}
\]

\[
= \frac{(wb/aq, w/b; q)_\infty}{(w/aq, w; q)_\infty} 2 \phi_1 \left( b, bq; wb/a; q, w/b \right)
\]

(8.7.9)

whether or not \( b \) is a negative integer power of \( q \), provided the series on both sides converge. Since \( 1 - qa^{2k} = 1 - q^k + q^k (1 - aq^k) \), the left side of (8.7.9) equals

\[
2 \phi_1 (aq, b; wq, w/a) + w(1-b) \cdot aq(1-w) 2 \phi_1 (aq, bq; wq, w/aq)
\]

(8.7.7)

\[
= \frac{(wb/a, w/b; q)_\infty}{(w/a, w; q)_\infty} 2 \phi_1 (b, bq; wb/a; q, w/b)
\]
Further Applications

\[
\begin{align*}
&+ \frac{w(1 - b)(wb/a, w/b; q)_\infty}{aq(w/aq, w; q)_\infty} 2\phi_1(b, bq; wb/a; q, w/b) \\
&= \frac{(wb/aq, w/b; q)_\infty}{(w/aq, w; q)_\infty} 2\phi_1(b, bq; wb/a; q, w/b)
\end{align*}
\]  
(8.7.10)

by (1.4.5). Also, for \( \alpha = \alpha \) the \( 5\phi_4 \) series in (8.7.5) reduces to a \( 4\phi_3 \) series which, by (2.10.4), equals

\[
\left[ \frac{(q^M; q)_\infty}{(q^{-M}; q)_\infty} \right] \frac{(b, bq^r + \gamma - M)_\infty}{(bq^r + \gamma; q)_\infty} 2\phi_3 \left( \begin{array}{c} q^{-s}, q^{-y}, b^{-1}q^{-s}, cq^{-1}q^{M+1-y-s} \\
q^{-1}q^{-1}q^{M+1-y} \end{array} \right; \begin{array}{c} q, q, q^{-1}q^{-1}q^{M+1-y} \\
q^{-1}q^{-1}q^{M+1} \end{array} )
\]  
(8.7.11)

From (8.7.6), (8.7.8) and (8.7.11) it follows that

\[
P_2(x, y; a, b, c, a, \gamma, N, M, N; q) \geq 0
\]  
(8.7.12)

for \( x = 0, 1, \ldots, N, y = 0, 1, \ldots, M, z = 0, 1, \ldots, N \) when \( 0 < q < 1, 0 < \alpha < 1, 0 \leq bq < 1, 0 < c < \alpha q N \) and \( cq \leq \gamma < \alpha q^{M-1} \leq q^{N-1} \). Hence the discrete Poisson kernel (8.7.2) is nonnegative for \( x, y, z = 0, 1, \ldots, N \) when \( 0 < q < 1, 0 < \alpha q < 1, 0 \leq bq < 1 \) and \( 0 < c < \alpha q N \).

If in (8.7.3) we write the sum with \( N \) as the upper limit of summation, replace \( (q^{-s}; q)_n \) by \( (tq^{-K}; q)_n \) and let \( K \to \infty \), it follows from (8.7.6) that

\[
L_t(x, y; a, b, c, a, \gamma, M, N; q)
\]  
(8.7.13)

for \( x = 0, 1, \ldots, N, y = 0, 1, \ldots, M \) with \( A_{r,s} \) defined in (8.7.5) and

\[
2\phi_1 \left( \begin{array}{c} \lambda, q\lambda^\frac{1}{2} - q\lambda^\frac{1}{2} \lambda^{-1}q^{r+s-N} \\
\lambda^{-1}, 0, abq^{N+r+s+1}; q, tq^{N-r-s} \\
\end{array} \right; \begin{array}{c} q, q \\
q^{-1}q^{-1}q^{M+1} \end{array} 
\]  
(8.7.14)

where \( \lambda = abq^{2r+2s+1} \). However, by Ex. 2.2,

\[
2\phi_1 \left( \begin{array}{c} \lambda, q\lambda^\frac{1}{2} - q\lambda^\frac{1}{2} \lambda^{-1}q^{r+s-N} \\
\lambda^{-1}, 0, abq^{N+r+s+1}; q, t; q, abq^{2r+2s+1} \\
\end{array} \right; \begin{array}{c} q, q \\
q^{-1}q^{-1}q^{M+1} \end{array} 
\]  
(8.7.15)

for \( \max(|tb|, |\lambda q|) < 1 \). Use of this in (8.7.14) yields

\[
C_{r,s} = (t, abq^{2r+2s+2}; q)_N^{r-s} 2\phi_1 \left( \begin{array}{c} q^{N-r-s}, tq^{N-r-s}; q, abq^{2r+2s+2} \\
\end{array} \right; \begin{array}{c} q, q \\
q^{-1}q^{-1}q^{M+1} \end{array} 
\]  
(8.7.16)

from which it is obvious that \( C_{r,s} \geq 0 \) for \( 0 \leq t < 1, \gamma + s \leq N \) when \( 0 \leq abq^2 < 1 \). Combining this with our previous observation that \( A_{r,s} \) equals the expression in (8.7.11) when \( \alpha = a \), it follows from (8.7.13) that

\[
L_t(x, y; a, b, c, a, \gamma, M, N; q) > 0
\]  
(8.7.17)

for \( x = 0, 1, \ldots, N, y = 0, 1, \ldots, M, 0 \leq t < 1 \) when \( 0 < q < 1, 0 \leq q < qN \) and \( cq \leq \gamma < q^{M-1} \leq q^{N-1} \). In particular, the Poisson kernel (8.7.1) is positive for \( x, y = 0, 1, \ldots, N, 0 \leq t < 1 \) when \( 0 < q < 1, 0 < qN < 1, 0 \leq bq < 1 \) and \( 0 < c < \alpha q N \).

For further details on the nonnegative bilinear sums of discrete orthogonal polynomials, see Gasper and Rahman [1984] and Ramanujan [1982].

8.8 q-Analogues of Clausen’s formula

Clausen’s [1828] formula

\[
\left\{ \begin{array}{c} 2F1 \\
a, b \\
a + b + \frac{1}{2} \\
\frac{z}{2} \\
\end{array} \right\}^2 = 3F2 \left[ \begin{array}{c} 2a, 2b, a + b \\
2a, 2b, a + b + \frac{1}{2} \\
\frac{z}{2} \\
\end{array} \right],
\]  
(8.8.1)

where \( |z| < 1 \), provides a rare example of the square of a hypergeometric series that is expressible as a hypergeometric series. Ramanujan’s [1927, pp. 23-39] rapidly convergent series representations of \( 1/\pi \), which have been used to compute \( \pi \) to millions of decimal digits, are based on special cases of (8.8.1); see the Chudnovskys’ [1988] survey paper. Clausen’s formula was used in Askey and Gasper [1976] to prove that

\[
3F2 \left[ \begin{array}{c} -n, n + \alpha + 2, \frac{1}{2}(\alpha + 1) \\
\frac{1}{2}, \frac{1}{2} \\
\frac{1}{2} \end{array} \right; \begin{array}{c} \alpha + 1, \frac{1}{2}(\alpha + 3) \\
\frac{1}{2} \end{array} 
\]  
(8.8.2)

when \( \alpha > -2, -1 \leq x \leq 1, n = 0, 1, \ldots, which was then used to prove the positivity of certain important kernels involving sums of Jacobi polynomials; see Askey [1975] and the extensions in Gasper [1975a, 1977]. The special cases \( \alpha = 2, 4, 6, \ldots \) of (8.8.2) turned out to be the inequalities of Branges [1985] needed to complete the last step in his celebrated proof of the Bieberbach conjecture. In this section we consider \( q \)-analogues of (8.8.1).

Jackson [1940, 1941] derived the product formula given in Ex. 3.11 and additional proofs of it have been given by Singh [1959], Nassrallah [1982], and Jain and Srivastava [1986]. But, unfortunately, the left side of it is not a square and so Jackson’s formula cannot be used to write certain basic hypergeometric series as sums of squares as was done with Clausen’s formula in Askey and Gasper [1976] to prove (8.8.2).

In order to obtain a \( q \)-analogue of Clausen’s formula which expressed the square of a basic hypergeometric series as a basic hypergeometric series, the authors derived the formula

\[
\left\{ \begin{array}{c} 4\phi_3 \\
a, b, abz, ab/z \\
a^2b^2, abq^2, -abq^2, -ab \\
-; q, q \\
\end{array} \right\}^{2} = 4\phi_4 \left[ \begin{array}{c} a^2b^2, abq^2, -abq^2, -ab \\
a^2b^2, abq^2, -abq^2, -ab \\
\end{array} \right; \begin{array}{c} a^2b^2, abq^2, -abq^2, -ab \\
a^2b^2, abq^2, -abq^2, -ab \\
\end{array} 
\]  
(8.8.3)
which holds when the series terminate. See Gasper [1989b], where it was pointed out that there are several ways of proving (8.8.3), such as using the Rogers’ linearization formula (8.5.1), the product formula in §8.2, or the Rahman and Verma integral (8.4.10).

In this section we derive a nonterminating \( q \)-analogue of Clausen’s formula which reduces to (8.8.3) when it terminates. The key to the discovery of this formula is the observation that the proof of Rogers’ linearization formula given in §§5 is independent of the fact that the parameter \( n \) in the \( 2 \phi_1 \) series in (7.4.2) is a nonnegative integer. In view of (7.4.2) let

\[
f(z) = 2 \phi_1(\alpha, \beta; q, z q / \beta),
\]

which reduces to the \( 2 \phi_1 \) series in (7.4.2) when \( \alpha = q^{-n} \) and \( z = e^{2 \pi i \theta} \). Temporarily assume that \( |q| < |\beta| < 1 \) and \( |z| \leq 1 \). From Heine’s transformation (1.4.3),

\[
f(z) = (\frac{\beta z; q}{z q / \beta; q}) \phi_1(\alpha q^2 / \beta, q / \beta ; q, \beta z).
\]

Hence, if we multiply the two \( 2 \phi_1 \) series in (8.8.4) and (8.8.5) and collect the coefficients of \( z^k \), we get

\[
f^2(z) = \frac{(\beta z; q \infty)}{(z q / \beta; q \infty)} \sum_{k=0}^{\infty} A_k \frac{(\alpha q^2 / \beta, q / \beta; q)_k (\beta z)_k}{(q, \alpha q / \beta; q)_k} = \sum_{k=0}^{\infty} q^k A_k \frac{(\alpha q^2 / \beta, q / \beta; q)_k (\beta z)_k}{(q, \alpha q / \beta; q)_k},
\]

where

\[
A_k = 4 \phi_3 \left[ \frac{q^{-k}, \beta, \beta q^{-k} / \alpha, \alpha, \alpha^2 q^{k+1} / \beta^2, q^{-k} / \beta, q / \beta}{\beta q^{-k}, \beta q^{2k+1} / \beta; q, q} \right] (8.8.7)
\]

is a terminating balanced series. As in (8.8.5) we now apply (2.5.1) to the \( 4 \phi_3 \) series in (8.8.7) to obtain that

\[
A_k = \frac{(\alpha q / \beta, \alpha^2 q^2 / \beta^2; q)_\infty}{(\alpha q / \beta, \alpha^2 q^2 / \beta^2; q)_\infty} \Phi_7(\alpha^2 / \beta, \alpha, \beta, \alpha^2 q^{k+1} / \beta^2, q^{-k} / \beta, q / \beta).
\]

Using (8.8.8) in (8.8.6) and changing the order of summation we get the formula

\[
f^2(z) = \frac{(\beta z; q \infty)}{(z q / \beta; q \infty)} \sum_{k=0}^{\infty} \left( 1 - \frac{\alpha^2 q^2 / \beta^2}{1 - \alpha^2 q^2 / \beta^2} \right) \left( \frac{(\alpha q / \beta, \alpha^2 q / \beta^2; q)_k}{(q, \alpha q / \beta; q)_k} \right)
\]

\[
(\alpha^2 q^2 / \beta^2; q)_k \frac{z q}{\beta} \frac{1}{(\alpha q / \beta; q)_k} \frac{(\alpha q^2 / \beta, q / \beta; q)_k}{(\alpha q / \beta; q)_k} (8.8.9)
\]

Observe that since the \( 2 \phi_1 \) series in (8.8.9) is well-poised we may transform it by applying the quadratic transformation formula (3.4.7) to express it as an \( \Phi_7 \) series and then apply (2.10.10) to get the transformation formula

\[
2 \phi_1 \left[ \frac{\alpha^2 q^{2k+1} / \beta^2, q / \beta}{\alpha^2 q^{2k+1} / \beta; q, \beta z} \right] = \frac{(\alpha q / \beta; q)_\infty}{(\beta z / \alpha q; q)_\infty} \frac{(\alpha q / \beta)_k (\beta z / \alpha q)_k}{(q, \alpha q / \beta, q \alpha z / \beta; q)_k}.
\]

\[
4 \phi_3 \left[ \frac{q^{-k}, \beta, \beta q^{-k} / \alpha, \alpha, \alpha^2 q^{k+1} / \beta^2, q^{-k} / \beta, q / \beta}{\beta q^{-k}, \beta q^{2k+1} / \beta; q, q} \right],
\]

\[
(8.8.10)
\]

We can now substitute (8.8.10) into (8.8.9) and change the orders of summation to find that

\[
f^2(z) = (\frac{\beta z, q^2 z / \beta, q}{z q / \beta, q; q, q}) \sum_{m=0}^{\infty} (\frac{\alpha q z / \beta, \alpha q^2 / \beta, q^{2m}}{q, \alpha q z, \alpha q^2 / \beta, q^{2m}}) \frac{(\alpha q / \beta, \alpha q^2 / \beta, q / \beta; q)_m}{(q, \alpha q, \alpha q z / \beta; q)_m} \frac{z q}{\beta} \frac{1}{(\beta z / \alpha q; q)_m} \frac{(\beta z / \alpha q, \alpha q z / \beta; q)_m}{(q, \alpha q z, \alpha q^2 / \beta, q^{2m})} \frac{(\alpha q / \beta, \alpha q^2 / \beta, q / \beta; q)_m}{(q, \alpha q, \alpha q z / \beta; q)_m}.
\]

\[
6 \Phi_5(\alpha^2 / \beta, \alpha, \beta, \alpha^2 q / \beta, q, q z^m / \beta; q, q) \frac{z q}{\beta} \frac{1}{(\beta z / \alpha q; q)_m} \frac{(\beta z / \alpha q, \alpha q z / \beta; q)_m}{(q, \alpha q z, \alpha q^2 / \beta, q^{2m})} \frac{(\alpha q / \beta, \alpha q^2 / \beta, q / \beta; q)_m}{(q, \alpha q, \alpha q z / \beta; q)_m}.
\]

\[
(8.8.11)
\]

Summing the above \( \Phi_5 \) series by means of (2.7.1), we obtain the formula

\[
(2 \phi_1(\alpha, \beta; q, z q / \beta)) = \frac{(\beta z, q^2 z / \beta, q)}{z q / \beta, q; q, q} \sum_{m=0}^{\infty} \frac{(\alpha q z / \beta, \alpha q^2 / \beta, q^{2m}}{q, \alpha q z, \alpha q^2 / \beta, q^{2m}} \frac{(\alpha q / \beta, \alpha q^2 / \beta, q / \beta; q)_m}{(q, \alpha q, \alpha q z / \beta; q)_m} \frac{z q}{\beta} \frac{1}{(\beta z / \alpha q; q)_m} \frac{(\beta z / \alpha q, \alpha q z / \beta; q)_m}{(q, \alpha q z, \alpha q^2 / \beta, q^{2m})} \frac{(\alpha q / \beta, \alpha q^2 / \beta, q / \beta; q)_m}{(q, \alpha q, \alpha q z / \beta; q)_m}.
\]

\[
(8.8.12)
\]

which gives the square of a well-poised \( 2 \phi_1 \) series as the sum of the two balanced \( \Phi_5 \) series. By analytic continuation, (8.8.12) holds when \( |q| < 1 \) and \( |z| < 1 \).

To derive (8.8.3) from (8.8.12), observe that if \( \alpha = q^{-n} \), \( n = 0, 1, \ldots \), then \( (\alpha q)_\infty = 0 \) and (8.8.12) gives

\[
f^2(z) = (\frac{\beta z, q^2 z / \beta, q}{z q / \beta, q; q, q}) \Phi_5(\alpha^2 / \beta, \alpha, \beta, \alpha^2 q / \beta, q, q z^m / \beta; q, q) \frac{z q}{\beta} \frac{1}{(\beta z / \alpha q; q)_m} \frac{(\beta z / \alpha q, \alpha q z / \beta; q)_m}{(q, \alpha q z, \alpha q^2 / \beta, q^{2m})} \frac{(\alpha q / \beta, \alpha q^2 / \beta, q / \beta; q)_m}{(q, \alpha q, \alpha q z / \beta; q)_m}.
\]

\[
(8.8.13)
\]

by reversing the order of summation. Since

\[
f(z) = (\frac{\beta^2 / \beta, q z / \beta}{z \beta}) \Phi_3(\alpha q^2 / \beta, (\beta z / \beta), (\beta z / \beta) z^2; q, q) \frac{z q}{\beta} \frac{1}{(\beta z / \alpha q; q)_m} \frac{(\beta z / \alpha q, \alpha q z / \beta; q)_m}{(q, \alpha q z, \alpha q^2 / \beta, q^{2m})} \frac{(\alpha q / \beta, \alpha q^2 / \beta, q / \beta; q)_m}{(q, \alpha q, \alpha q z / \beta; q)_m}.
\]

\[
(8.8.14)
\]

by (7.4.14). It follows from (8.8.13) that

\[
\{ \Phi_3(\alpha q^2, (\beta z / \beta), (\beta z / \beta) z^2; q, q) \}^2.
\]
Further Applications

\[ z_\phi(x) [q^n, \beta^2, \beta z, \beta z/\beta ; q] \]

(8.8.15)

for \( n = 0, 1, \ldots \), which is formula (8.8.3) written in an equivalent form.

Now note that

\[ 2\phi_1(\alpha, \beta; \alpha q/\beta; q, \alpha q/\beta) \]

\[ = \frac{(z(\alpha q)^{\beta}, -z(\alpha q)^{\beta}, \alpha q^2/\beta, -z\alpha q^{\beta}/\beta; q)_{\infty}}{(z^2q^{\beta}, -zq^{\beta}, \alpha q^{\beta}/\beta, -z\alpha q^{\beta}/\beta; q)_{\infty}} \cdot zW_\gamma(-az/\beta, a^{\beta}; a^{\beta}, -a^{\beta}; a^{\beta}, -zq, -zq). \]

(8.8.16)

by (3.4.7) and (2.10.1), and set \( a = \alpha^{\beta}/\beta \), \( b = (\alpha q)^{\beta} \) to obtain from (8.8.12) the following q-analogue of Clausen's formula:

\[ \begin{array}{c}
\left\{ \frac{(z^2q^{\beta}, -zq^{\beta}; q^2)_{\infty}}{(z^{\beta}q^{\beta}, z^{\beta}q^{\beta}; q^2)_{\infty}} \cdot zW_\gamma(-azq^{\beta}; a, -a, b, -b, -z; q, -qz) \right. \\
\left. = \frac{(az^{\beta}/b, bsz^{\beta}/a; q)_{\infty}}{(az^{\beta}/b, azq^{\beta}; q)_{\infty}} \cdot z\phi_4 \left[ \frac{a^2, b^2, ab, -ab^{\beta} + (az^{\beta}/b, bsz^{\beta}/a, -azq^{\beta}; q, -qz, zq, z^2q^{\beta}, azq^{\beta}; z^{\beta}/ab} {abq^{\beta}, abq^{\beta}, abq^{\beta}; q_{\infty}} \right] \right. \\
\end{array} \]

(8.8.17)

where \(|q| < 1 \) and \(|zq| < 1 \).

To see that (8.8.17) is a nonterminating q-analogue of Clausen's formula, it suffices to replace \( a \) by \( q^a, b \) by \( q^b \) and let \( q \to 1 \); then the left side and the first term on the right side of (8.8.17) tend to the left and right sides of (8.8.1) with \( z \) replaced by \(-4z(1-z)^{-2} \) and so, by (8.8.1), the second term on the right side of (8.8.17) must tend to zero.

It is shown in Gasper and Rahman [1998b] that the nonterminating extension (3.4.1) of the Sears-Carlitz quadratic transformation can be used in place of (8.8.10) to derive the product formula

\[ 2\phi_1(ab, b; c, q, z) \}

\[ = \frac{(az, abz/c, q; q)_{\infty}}{(z, bz/c, q)_{\infty}} \cdot 2\phi_5 \left[ a, c/b, (ac/b)^{\frac{1}{2}}, -(ac/b)^{\frac{1}{2}}, (acq/b)^{\frac{1}{2}}, -(acq/b)^{\frac{1}{2}} ; q, q \right. \\
\left. + (acq/b, abz/c, q; q)_{\infty} \right. \\
\left. = \frac{z, abz/c, z(ab/c)^{\frac{1}{2}}, -(z(ab/c)^{\frac{1}{2}}, z(abq/c)^{\frac{1}{2}}, -(z(abq/c)^{\frac{1}{2}} ; q, q} \right. \}

(8.8.18)

where \(|z| < 1\) and \(|q| < 1\). This formula reduces to (8.8.12) when \( a = \alpha, b = \beta, c = \alpha q/\beta \) and \( z \) is replaced by \( zq/\beta \).

Nonnegative basic hypergeometric series

By applying various transformation formulas to the \( 2\phi_1 \) series in (8.8.12) and (8.8.18), these formulas can be written in many equivalent forms. For instance, by replacing \( b \) in (8.8.18) by \( c/b \) and applying (1.5.4) we obtain

\[ 2\phi_2(a, b; c, q, c/b) \]

\[ = \frac{(z, az/b, q; q)_{\infty}}{(az, z/b, q)_{\infty}} \cdot 2\phi_5 \left[ a, b, (ab)^{\frac{1}{2}}, -(ab)^{\frac{1}{2}}, (aq/b)^{\frac{1}{2}}, -(aq/b)^{\frac{1}{2}} ; q, q \right. \\
\left. + (aqb/c, c/b, qz/c, zq; q)_{\infty} \right. \\
\left. = \frac{z, abz, z(ab/c)^{\frac{1}{2}}, -(z(ab/c)^{\frac{1}{2}}, z(abq/c)^{\frac{1}{2}}, -(z(abq/c)^{\frac{1}{2}} ; q, q} \right. \}

(8.8.19)

where \( |q|, |aqz/c|, |cz/b| < 1 \). If we replace \( a, b, z \) in (8.8.19) by \( q^a, b^b \), \( z/(z-1) \), respectively, and let \( q \to 1^{-} \), we obtain the Ramanujan [1957, Vol. 2] and Bailey [1933, 1935a] product formula

\[ 2F_1(a, b; c, z) \]

\[ = 4F_3(a, b, (a+b)/2, (a+b+1)/2; c, a+b, a+b+1; 4z(1-z)), \]

(8.8.20)

where \(|z| < 1 \) and \(|4z(1-z)| < 1 \). This is an extension of Clausen's formula in the sense that by replacing \( a, b, c, 4z(1-z) \) in (8.8.20) by \( 2a, 2b, a+b+\frac{1}{2}, z \), respectively, and using the quadratic transformation (Erdélyi [1953, 2.11 (2)])

\[ 2F_1(2a, 2b; a+b+\frac{1}{2}, z) = 2F_1(a, b; a+b+\frac{1}{2}, 4z(1-z)), \]

(8.8.21)

we get (8.8.1). See Askey [1989d].

8.9 Nonnegative basic hypergeometric series

Our main aim in this section is to show how the terminating \( q \)-Clausen formula (8.8.3) can be used to derive \( q \)-analogues of the Askey-Gasper inequalities (8.8.2) and of the nonnegative hypergeometric series in Gasper [1975a, Equations (8.19), (8.20), (8.22)].

As in Gasper [1989b], let us set

\[ \gamma = q^{2s}, a_1 = q^n, a_2 = q^{n+1}, a_3 = q^n, b_1 = q^{2s}, b_2 = -q^b, \]

\[ c_1 = q^{-n}, c_2 = q^{n+s}, d_1 = q^{s}(a+1) = -d_2, e_1 = q^{s+1} = -e_3, x = q, w = 1, \]

in the \( r = 3, s = t = u = k = 2 \) case of (3.7.9) to obtain the expansion

\[ 2\phi_4 \left[ q_1^{n}, q_1^{n+1}, q_1^{n+s}, q_1^{s+1} ; q_1^{s+1} ; q_1^2, q_1^{s+1} \right] \]

(8.8.19)

\[ = \sum_{j=0}^{n} \left( q_1^{n}, q_1^{n+s}, q_1^{s+1}, q_1^{s+1} ; q_1^{s+1} ; q_1^{s+1} \right) j \cdot \left( q_1^{s+1} \right) ; q_1^2 \]
Further Applications

\[4 \phi_3 \left[ q^{-n}, q^{i+n+a}, q^{i+b+\frac{1}{2}}, -q^{i+b+\frac{1}{2}}; q_{2j+2b+1}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)} ; q, q \right] \]
\[5 \phi_4 \left[ q^{-j}, q^{j+2b}, q^{j+b} e^{i\theta}, q^{j} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{b} ; q, q \right] \] (8.9.1)

where, as through this section, \( n = 0, 1, \ldots \). By Ex. 2.8 the \( 4 \phi_3 \) series in (8.9.1) equals zero when \( n-j \) is odd and equals

\[\left( q^{n-2b}, q^2 \right)_k \]
\[\frac{(q^{2n-4k+a+1}, q^{2n-4k+2b+2}, q^2)_k}{(q^{2n-4k+a+1}, q^{2n-4k+2b+2}, q^2)_k} q^{2k(n-2k+b+1/2)} \]

when \( n-j = 2k \) and \( k = 0, 1, \ldots \). Hence, using (8.8.3) to write the \( 5 \phi_4 \) as the square of a \( 4 \phi_3 \) series, we have

\[5 \phi_4 \left[ q^{-n}, q^{n+a}, q^{b} e^{i\theta}, q^{b} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{b} ; q, q \right] \]
\[= \sum_{k=0}^{[n/2]} (-1)^k (q^{-n}, q^{n+a}, q^{b} e^{i\theta}, q^{b} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{b} ; q, q) \]
\[\cdot \left( q^{2k-n}, q^{2k-2b}, q^{2k} e^{i\theta}, q^{2k} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{b} ; q, q \right)^2 \] (8.9.2)

Since \((-1)^n(q^{-n}; q)_{n-2k} \geq 0\), it is clear from (8.9.2) that

\[5 \phi_4 \left[ q^{-n}, q^{n+a}, q^{b} e^{i\theta}, q^{b} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{b} ; q, q \right] \geq 0 \] (8.9.3)

when \( a \geq 2b \geq -1 \) and \( 0 < q < 1 \). By setting \( x = \cos \theta \) and letting \( q \to 1^- \), it follows from (8.9.3) that

\[3 F_2 \left[ -n, n+a+b, 1-x^2; 2b, \frac{1}{2}(a+1) \right] \geq 0, \quad -1 \leq x \leq 1, \] (8.9.4)

when \( a \geq 2b \geq -1 \) which shows that (8.9.3) is a \( q \)-analogue of (8.9.4). When \( a = a+2b = \frac{1}{2} \), (8.9.4) reduces to (8.8.2). Special cases of (8.9.4) were used by de Branges [1986] in his work on coefficient estimates for Riemann mapping functions.

Another \( q \)-analogue of (8.9.4) can be derived by using (8.9.1), (8.8.3) and Ex. 2.8 to obtain

\[6 \phi_5 \left[ q^{-n}, q^{n+a}, q^{b}, q^{\frac{1}{2} a} e^{i\theta}, q^{\frac{1}{2} a} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2} a} ; q, q \right] \]
\[= \sum_{j=0}^{\infty} \frac{(q^{-n}, q^{n+a}, q^{b} e^{i\theta}, q^{b} e^{-i\theta}; q)_{j+1}}{(q, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{b} ; q, q)_{j}} \]
\[\cdot 5 \phi_4 \left[ q^{j-n}, q^{j+n+a}, q^{j+b}, q^{j+\frac{1}{2} a} e^{i\theta}, q^{j+\frac{1}{2} a} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2} a} ; q, q \right] \] (8.9.5)

which shows that

\[6 \phi_5 \left[ q^{-n}, q^{n+a}, q^{b}, q^{\frac{1}{2} a} e^{i\theta}, q^{\frac{1}{2} a} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2}(a+1)}, -q^{\frac{1}{2} a} ; q, q \right] \geq 0 \] (8.9.6)

when \( a \geq 2b > -1 \) and \( 0 < q < 1 \).

The expansions (8.12) and (8.17) in Gasper [1975a] are special cases of the \( q \to 1^- \) limit cases of (8.9.2) and (8.9.5), respectively, when (7.4.14) and Gasper [1975a, 8.10] are used. A \( q \)-analogue of the expansion (Gasper [1975a, 8.98b])

\[3 F_2 \left[ -n, n+a, b, 1-x^2; 2b, \frac{1}{2}(a+1) \right] \]
\[= \sum_{j=0}^{n} \frac{n!(n+a+2, b+\frac{1}{2}(a+1))}{j!(n-j)!} (1-y^2) \]
\[\cdot \left[ (\frac{1}{2}(a+1), \frac{1}{2}(a+1), 1) \right] \]
\[= C_j \left( \frac{1}{2}(a+1) \right) C_{n-j} \left( \frac{1}{2} \right)^2 \] (8.9.7)

is easily derived by employing (3.7.9), (8.8.3) and (7.4.14) to obtain

\[7 \phi_6 \left[ q^{-n}, q^{n+a+2}, q^{\frac{1}{2}(a+1)} e^{i\theta}, q^{\frac{1}{2}(a+1)} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+3)}, -q^{\frac{1}{2}(a+3)}, -q^{\frac{1}{2} a} ; q, q \right] \]
\[= \sum_{j=0}^{n} \frac{(q^{-n}, q^{n+a+2}, q^{\frac{1}{2}(a+3)} e^{i\theta}, q^{\frac{1}{2}(a+3)} e^{-i\theta}; q_{2b}, q^{\frac{1}{2}(a+3)}, -q^{\frac{1}{2}(a+3)}, -q^{\frac{1}{2} a} ; q, q)_{j+1}}{(q, q^{\frac{1}{2}(a+3)}, -q^{\frac{1}{2}(a+3)}, -q^{\frac{1}{2} a} ; q, q)_{j}} \]
\[\cdot C_j \left( \cos \theta, \frac{1}{2}(a+1) \right) C_{n-j} \left( \cos \theta, \frac{1}{2}(a+2) \right) \] (8.9.8)

which is obviously nonnegative for real \( \theta \) and \( \tau \) when \( a > -2 \).

If we proceed as in (8.9.5), but use the \( q \)-Saalschütz summation formula (1.7.2) instead of Ex. 2.8, we find that
Applications in the theory of partitions of positive integers

\[ n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n, \]

(8.10.2)

\( p(0) = 1 \) and, for a positive integer \( n \), \( p(n) \) is the number of partitions of \( n \) into parts \( \leq n \). In the partition (8.10.2) of \( n \) there are \( k_m \) \( m \)'s and hence \( 0 \leq k_m \leq n/m, 1 \leq m \leq n \). For small values of \( n \), \( p(n) \) can be calculated quite easily, but the number increases very rapidly. For example, \( p(3) = 3 \), \( p(4) = 5 \), \( p(5) = 7 \), but \( p(243) = 133978259348888 \). Hardy and Ramanujan [1918] found the following asymptotic formula for large \( n \):

\[ p(n) \sim \frac{1}{4n\sqrt{3}} \exp \left[ \pi \left( \frac{2n}{3} \right)^{1/2} \right]. \]

(8.10.3)

Also of interest are the enumerations of partitions of a positive integer \( n \) into parts restricted in certain ways such as:

(i) \( p_{tn}(n) \), the number of partitions of \( n \) into parts \( \leq N \), which is given by the generating function

\[ (q; q)_N^{-1} = \sum_{n=0}^{\infty} p_{tn}(n) q^n, \]

(8.10.4)

where \( n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_N \cdot N \) has \( k_m \) \( m \)'s;

(ii) \( p_e(n) \), the number of partitions of an even integer \( n \) into even parts, generated by

\[ (q^2; q^2)_\infty^{-1} = \prod_{k=0}^{\infty} (1 - q^{2k+2})^{-1} \]

\[ = \sum_{n=0}^{\infty} p_e(n) q^n; \]

(8.10.5)

(iii) \( p_{\text{dist}}(n) \), the number of partitions of \( n \) into distinct positive integers, generated by

\[ (-q; q)_\infty = \sum_{k_2=0}^{1} \cdots \sum_{k_2=0}^{1} q^{k_1+1+k_2+2} \]

\[ = \sum_{n=0}^{\infty} p_{\text{dist}}(n) q^n, \]

(8.10.6)

where \( n = k_1 \cdot 1 + k_2 \cdot 2 + \cdots + k_n \cdot n \), \( 0 \leq k_i \leq 1 \), \( 1 \leq i \leq n \), and

(iv) \( p_0(n) \), the number of partitions of \( n \) into odd parts, generated by

\[ (q; q^2)_\infty^{-1} = \prod_{k=0}^{\infty} (1 - q^{2k+1})^{-1} \]

\[ = \sum_{n=0}^{\infty} p_0(n) q^n. \]

(8.10.7)

Euler's partition identity

\[ p_{\text{dist}}(n) = p_0(n) \]  

(8.10.8)
follows from (8.10.6), (8.10.7) and the fact that

\[ (q; q)_\infty = \left( q^\frac{1}{2}, -q^\frac{1}{2}; q \right)_\infty = (q; q^2)_\infty^{-1}. \]  

(8.10.9)

Other combinatorial identities of this type can be discovered from q-series identities similar to, but perhaps somewhat more complicated than, (8.10.9). For example, let us consider Euler’s [1748] identity involving the pentagonal numbers \( n(3n \pm 1)/2 \):

\[ (q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2(3n+1)/2} \]

\[ = 1 + \sum_{n=1}^{\infty} (-1)^n q^{n^2(3n+1)/2} + \sum_{n=1}^{\infty} (-1)^n q^{n^2(3n-1)/2}, \]  

(8.10.10)

which is given in Ex. 2.18. A formal power series expansion gives

\[ (q; q)_\infty = \prod_{k=0}^{\infty} (1 - q^{k+1}) \]

\[ = \sum_{k_1 = 0}^{1} \cdots \sum_{k_r = 0}^{1} (-1)^{k_1 + k_2 + \cdots + k_r} q^{k_1 + k_2 + \cdots}, \]  

(8.10.11)

which differs from the multiple series in (8.10.6) only in the factor \((-1)^{k_1 + k_2 + \cdots}\). This factor is ±1 according as the partition has an even or odd number of parts. Denoting these numbers by \( p_{\text{even}}(n) \) and \( p_{\text{odd}}(n) \), respectively, we find that

\[ (q; q)_\infty = 1 + \sum_{n=1}^{\infty} [p_{\text{even}}(n) - p_{\text{odd}}(n)] q^n. \]  

(8.10.12)

From (8.10.10) and (8.10.12) it follows that

\[ p_{\text{even}}(n) - p_{\text{odd}}(n) = \begin{cases} (-1)^k & \text{for } n = k(3k \pm 1)/2, \\ 0 & \text{otherwise}. \end{cases} \]  

(8.10.13)

Thus Euler’s identity (8.10.10) expresses the important property that a positive integer \( n \) which is not a pentagonal number of the form \( k(3k \pm 1)/2 \) can be partitioned as often into an even number of parts as into an odd number of parts. However, if \( n = k(3k \pm 1)/2, k = 1, 2, \ldots \), then \( p_{\text{even}}(n) \) exceeds \( p_{\text{odd}}(n) \) by \((-1)^k\). See Hardy and Wright [1979], Rademacher [1973] and Andrews [1976, 1983] for related results.

Two of the most celebrated identities in combinatorial analysis are the so-called Rogers-Ramanujan identities (2.7.3) and (2.7.4) which, for the purposes of the present discussion, we rewrite in the following form:

\[ 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \prod_{k=0}^{\infty} \frac{1}{(1 - q^{5k+1})(1 - q^{5k+4})}. \]  

(8.10.14)

It is clear that the infinite product on the right side of (8.10.14) enumerates the partitions of a positive integer \( n \) into parts of the form \( 5k + 1 \) and \( 5k + 4 \), while that on the right side of (8.10.15) enumerates partitions of \( n \) into parts of the form \( 5k + 2 \) and \( 5k + 3, k = 0, 1, \ldots \).

Following Hardy and Wright [1979] we shall now give combinatorial interpretations of the left sides of the above identities. Since

\[ k^2 = 1 + 3 + 5 + \cdots + (2k - 1), \]

we can exhibit this square in a graph of \( k \) rows of dots, each row having 2 more dots than the lower one. We then take any partition of \( n - k^2 \) into at most \( k \) parts with the parts in descending order, marked with \( \times \)'s and placed at the ends of the rows of dots to obtain a partition of \( n \) into parts with minimal difference 2. For example, when \( k = 5 \) and \( n = 32 \) \( = 5^2 + 7 \) we add 4 \( \times \)'s to the top row, 1 each to the 2nd, 3rd and 4th rows, counted from above. This gives the partition \( 32 = 13 + 8 + 6 + 4 + 1 \) displayed in the graph below.

\[ \cdots \times \times \times \times \times \times \times \times \times \times \times \]

\[ \cdots \times \times \times \times \times \times \times \times \times \times \times \]

\[ \cdots \times \times \times \times \times \times \times \times \times \times \times \]

\[ \cdots \times \times \times \times \times \times \times \times \times \times \times \]

The identity (8.10.14) states that the number of partitions of \( n \) in which the differences between parts are at least 2 is equal to the number of partitions of \( n \) into parts congruent to 1 or 4 (mod 5).

Observing that

\[ k(k + 1) = 2 + 4 + 6 + \cdots + 2k, \]

a similar interpretation can be given to the left side of (8.10.15). Since the first number in the above sum is 2, one deduces that the partitions of \( n \) into parts not less than 2 and with minimal difference 2 are equinumerous with the partitions of \( n \) into parts congruent to 2 or 3 (mod 5).

For more applications of basic hypergeometric series to partition theory, see Andrews [1976-1988], Fine [1948, 1988], Andrews and Askey [1977], and Andrews, Dyson and Hickerson [1988]. Additional results on Rogers-Ramanujan type identities are given in Slater [1951, 1952a], Jain and Verma [1980, 1982] and Andrews [1975a, 1984a,b,c,d].

8.11 Representations of positive integers as sums of squares

One of the most interesting problems in number theory is the representations of positive integers as sums of squares of integers. Fermat proved that all primes of the form \( 4n + 1 \) can be uniquely expressed as the sum of 2 squares.
Lagrange showed in 1770 that all positive integers can be represented by sums of 4 squares and that this number is minimal. Earlier in the same year Waring posed the general problem of representing a positive integer as a sum of a fixed number of nonnegative $k$-th powers of integers (positive, negative, or zero) with order taken into account and stated without proof that every integer is the sum of 4 squares, of 9 cubes, of 19 biquadrates, and so on. More than 100 years later Hilbert [1909] proved that all positive integers are representable by $s$ $k$-th powers where $s = s(k)$ depends only on $k$. For an historical account of the Waring problem, see Dickson [1920], Grosswald [1985], and Hua [1982].

To illustrate the usefulness of basic hypergeometric series in the study of such representations we shall restrict ourselves to the simplest cases: sums of 2 and sums of 4 squares, where it is understood, for example, that $n = x_1^2 + x_2^2 = y_1^2 + y_2^2$ are two different representations of $n$ as a sum of 2 squares if $x_1 \neq y_1$ or $x_2 \neq y_2$.

Let $r_2k(n)$ be the number of different representations of $n$ as a sum of $2k$ squares, $k = 1, 2, \ldots$. We will show by basic hypergeometric series techniques that, for $n \geq 1$,

\begin{align*}
  r_2(n) &= 4(d_1(n) - d_3(n)), \quad (8.11.1) \\
  r_4(n) &= 8 \sum_{d | n, 4 \mid d} d, \quad (8.11.2)
\end{align*}

where $d_i(n)$, $i = 1, 3$, is the number of (positive) divisors of $n$ congruent to $i$ (mod 4) and the summation in (8.11.2) indicates the sum over all divisors of $n$ not divisible by 4. The numbers 4 and 8 in (8.11.1) and (8.11.2), respectively, reflect the fact that $r_2(1) = 4$ since $1 = 0^2 + (\pm 1)^2 = (\pm 1)^2 + 0^2$, and $r_2(1) = 8$ since $1 = 0^2 + 0^2 + 0^2 + (\pm 1)^2 = 0^2 + 0^2 + (\pm 1)^2 + 0^2 = 0^2 + (\pm 1)^2 + 0^2 + 0^2 = (\pm 1)^2 + 0^2 + 0^2 + 0^2$. Both of these results were proved by Jacobi by means of the theory of elliptic functions, but the proofs below are based, as in Andrews [1974a], on the formulas stated in Ex. 5.1, 5.2, and 5.3. Combining Ex. 5.1 and 5.2 we have

\begin{equation}
  \left[ \sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} \right]^2 = 1 + 4 \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n}. \quad (8.11.3)
\end{equation}

However, the bilateral sum on the left side is clearly a generating function of $(-1)^n r_2(n)$ and so it suffices to prove that

\begin{equation}
  \sum_{n=1}^{\infty} [d_1(n) - d_3(n)](-q)^n = \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n}. \quad (8.11.4)
\end{equation}

By splitting into odd and even parts and then by formal series manipulations, we find that

\begin{align*}
  \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1 + q^n} &= \sum_{r=1}^{\infty} q^{(2r+1)} - \sum_{m=0}^{\infty} q^{(m+1)(2m+1)} \\
  &= \sum_{r=1}^{\infty} \sum_{m=r}^{\infty} (-1)^{m+r} q^{(2m+1)r} + \sum_{r=0}^{\infty} \sum_{m=r+1}^{\infty} (-1)^{m+r} q^{(2m+1)r}
\end{align*}

which completes the proof of (8.11.4).

To prove (8.11.2) we first replace $q$ by $-q$ in Ex. 5.1 and 5.3 and find that

\begin{align*}
  \sum_{n=0}^{\infty} r_4(n) q^n &= \left( \sum_{n=-\infty}^{\infty} q^{n^2} \right)^4 \\
  &= 1 + 8 \sum_{n=1}^{\infty} \frac{q^n}{1 + (-q)^n} \\
  &= 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1 + (-q)^n}, \quad (8.11.6)
\end{align*}

where the last line is obtained from the previous one by expanding $(1 + (-q)^n)^{-2}$ and interchanging the order of summation. Now,

\begin{align*}
  \sum_{n=1}^{\infty} \frac{nq^n}{1 + (-q)^n} &= \sum_{n \geq 2} \frac{nq^n}{1 - q^n} + \sum_{n \geq 2} \frac{nq^n}{1 + q^n} \\
  &= \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} + \sum_{n \geq 2} \frac{nq^n}{1 - q^{2n}} \\
  &= \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - \sum_{n \geq 2} \frac{2nq^{2n}}{1 - q^{2n}} \\
  &= \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n} - \sum_{n=1}^{\infty} \frac{4nq^{4n}}{1 - q^{4n}} \\
  &= \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}. \quad (8.11.7)
\end{align*}

Thus,

\begin{equation}
  \sum_{n=0}^{\infty} r_4(n) q^n = 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}. \quad (8.11.8)
\end{equation}
Further Applications

Since \( r_4(0) = 1 \), this immediately leads to (8.11.2). For a direct proof of (8.11.8) based only on Jacobi’s triple product identity (1.6.1), see Hirschhorn [1987].

Ex. 8.1 and 8.1 can be employed in a similar manner to show that

\[
r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^2, \quad n \geq 1.
\]

(8.11.9)

See Andrews [1974a]. Some remarks on other applications are given in Notes 8.

Exercises 8

8.1 Prove for the little \( q \)-Jacobi polynomials \( p_n(x; q, b; q) \) defined in (7.3.1) that

(i)

\[
p_n(x; a, b; q)p_n(y; a, b; q) = (-aq)^n q^{\binom{n}{2}} \frac{(bq; q)_n}{(aq; q)_n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, x^{-1}, y^{-1}; q)_m}{(q, bq; q)_m} \left( \frac{xy^{1-m}}{a} \right)^m \sum_{k=0}^{m} \frac{(q^{-m}, b^{-1}a^{-m}; q)_k}{(q, aq, xq^{1-m}, yq^{1-m}; q)_k} (abxy)^k q^{k+1}.
\]

(ii)

\[
p_n(x; a, b; q)p_n(y; a, b; q) = (-aq)^n q^{-\binom{n}{2}} \frac{(bq; q)_n}{(aq; q)_n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}; q)_m}{(q, bq; q)_m} (-byq^{-2})^{m} \frac{(m)}{(2)} \sum_{k=0}^{m} \frac{(q^{-m}, abq^{m+1}; q)k}{(q, aq; q)_k} (xz)^k 2\phi_1(k^{-m}, bxq; 0, q, b^{-1}y^{-1}).
\]

8.2 Derive the following product formula for the big \( q \)-Jacobi polynomials defined in (7.3.10):

\[
P_n(x; a, b; q)P_n(y; a, b; q) = (-aq)^n q^{\binom{n}{2}} \frac{(bq; q)_n}{(aq; q)_n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, cqx^{-1}, cqy^{-1}; q)_m}{(q, bq, cq; q)_m} \left( \frac{xy}{a} \right)^m \cdot 4\psi_3 \left[ \begin{array}{c}
q^{-m}, b^{-1}q^{-m}, x, y \\
aq, xc^{-1}, q^{-1}m, yc^{-1}q^{-1}m; q, abqc^{-2} \end{array} \right].
\]

8.3 Prove that

\[
\sum_{n=0}^{\infty} \frac{(aq, bq; q)_n(1 - abq^{n+1})}{(q, bq; q)_n(1 - abq)} \left( \frac{t}{aq} \right)^n p_n(qx; a, b; q)p_n(qy; a, b; q) = (t, abq^2; q)_\infty \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(q^{-x}, q^{-y}; q)_s a^{-s}}{(q, bq; q)_s(q, aq; q)_r} \theta_4(q^{x+y}(r+s)) \cdot 2\phi_1(0, 0; q, q, abq^{2r+2s+2}; q^{-2rs-s^2}t^{r+s}),
\]

and that this gives the positivity of the Poisson kernel for the little \( q \)-Jacobi polynomials for \( x, y = 0, 1, \ldots, 0 \leq t < 1 \) when \( 0 < q < 1 < aq < 1 \) and \( 0 < bq < 1 \).

8.4 Prove for the \( q \)-Hahn polynomials that

(i)

\[
Q_n(x; a, b, N; q)Q_n(y; a, b, N; q) = \frac{(bq, abq^{N+2}; q)_n}{(aq, q^{-N}; q)_n} (bq^{N+1})^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, bq^{N+1}; q)_m}{(q, bq, abq^{N+1}; q)_m} \sum_{k=0}^{m} \frac{(q^{-m}, b^{-1}q^{-m}; q)_k}{(q, aq, bq^{N-m}; q)_k} (abxy)^k q^{k+1}.
\]

(ii)

\[
Q_n(x; a, b, N; q)Q_n(y; a, b, N; q) = \frac{(bq, abq^{N+2}; q)_n}{(aq, q^{-N}; q)_n} (bq^{N+1})^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, bq^{N+1}; q)_m}{(q, bq, abq^{N+1}; q)_m} \sum_{k=0}^{m} \frac{(q^{-m}, b^{-1}q^{-m}; q)_k}{(q, aq, bq^{N-m}; q)_k} (abxy)^k q^{k+1}.
\]

8.5 Prove for the \( q \)-Racah polynomials that

(i)

\[
W_n(x; a, b, c, N; q)W_n(y; a, b, c, N; q) = \frac{(bq, abq^{N+2}; q)_n}{(aq, q^{-N}; q)_n} (bq^{N+1})^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, bq^{N+1}; q)_m}{(q, bq, abq^{N+1}; q)_m} \sum_{k=0}^{m} \frac{(bq^{N-x}; q)_k}{(bq^{N-x}; q)_k} q^{k+1}.
\]

(ii)

\[
W_n(x; a, b, c, N; q)W_n(y; a, b, c, N; q) = \frac{(bq, abq^{N+2}; q)_n}{(aq, q^{-N}; q)_n} (bq^{N+1})^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, bq^{N+1}; q)_m}{(q, bq, abq^{N+1}; q)_m} \sum_{k=0}^{m} \frac{(bq^{N-x}; q)_k}{(bq^{N-x}; q)_k} q^{k+1}.
\]
Further Applications

\[ W_n(x; a, b, c, N; q) W_n(y; a, b, c, N; q) = \frac{(aq^{-1} abq^{N+2}; q)_n}{(bcq, q^{-N}; q)_n} (ac^{-1} q^{N+1})^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, ac^{-1} q^{-N-x+1}, ac^{-1} q^{-y+1}; q)_m}{(q, abq^{N+2}, aqc^{-1}, ac^{-1} q^{N+1}; q)_m} \cdot \frac{(aq^{x+1}, aqc^{y+1}; q)_m}{(aq, aq; q)_m} q^m \cdot 10W_9 (ca^{-1} q^{-N-m}, ca^{-1} q^{-m}, a^{-1} b^{-1} q^{-N-m-1}; q_m^{q^{-x}}, c^{-1} q^{-x}, q^{-y}, c q^{y-N}, q, qba^{-1}) \cdot q_m^{q^{-y}}, q^{-x}, c q^{y-N}, q, qba^{-1}) \cdot 8.6 \text{ Show that} \]

(i) \[ W_n(x; a, b, c, N; q) W_n(y; a, b, c, M; q) = \frac{(bq, aqc^{-1}; q)_n}{(aq, bcq; q)_n} c_n \sum_{m=0}^{n} \frac{(q^{-n}, abq^{n+1}, q^{-N}, q^{M}; q)_m}{(q, bq, aqc^{-1}, c^{-1} q; q)_m} \frac{(a^{-1} q^{-x}, a^{-1} q^{-y}; q)_m}{(q, q^{-N}, q^{M}; q)_m} \cdot 10W_9 (cq^{m-1} c^{-1} q^{-m}, b^{-1} q^{-m}, q^{-y}, c q^{x-N}, q^{-x}, q^{M}; q, abq^{M+N+3}) \]

where \( n = 0, 1, \ldots, \min(M, N), x = 0, 1, \ldots, N, \) and \( y = 0, 1, \ldots, M; \)

(ii) \[ W_n(x; a, b, c, N; q) W_n(y; a, b, c, N; q) = \frac{(aq^{-1}, a^2 q^{N+2}; q)_n}{(aq, aq^{-1}; q)_n} (ac^{-1} q^{N+1})^{-n} \sum_{m=0}^{n} \frac{(q^{-n}, a^2 q^{n+1}, aqc^{-1}; q)_m}{(q, aq^{N+2}, aq; q)_m} \frac{(a^2 q^{-x}, aq^{-x+y}; q)_m}{(ac^{-1}, aq^{-x}; q)_m} \cdot 10W_9 (aq^{-x}, a^2 q^{m+1}, q^{-m}, abq^{-N+z}, c^{-1} q^{-z}, c q^{y-N}; q, q^{N}; q) \]

8.7 For the \( q \)-Hahn polynomials prove that

(i) \[ \sum_{n=0}^{\infty} \frac{(abq, aq, q^{-z}; q)_n}{(aq, bq, abq^{N+2}; q)_n} (-aq)^{-n} q^{Nn}(\frac{(-q)}{q})^n \cdot Q_n(x; a, b, N, q) W_n(y; a, b, M, q) = 0 \]

\[ = \sum_{n=0}^{\infty} \frac{(q^{-x}, q^{-y}; q)_r x \cdot (abq^2, q)_{x+y+2} (q^{-x}, q^{-y}; q)_r (q^{-N}, q^{-N}; q, aqc^{-1}, c^{-1} x; q)_m}{(q^{-x}, q^{-y}; q)_r (abq^{N+2}, q)_r (aq, aq; q)_m} \cdot 10W_9 (aq^{N^{-x}}, q^{N^{-x}} q^{N^{-x}} (2N-r-s+1)/2 (z+y+1) \)

8.8 Prove that

\[ \sum_{n=0}^{\infty} \frac{(aq, abq, bcq; q)_n (-c)^{-n}}{(aq, bq, aqc^{-1}; q)_n} \frac{(abq^{2+n+1}; q)_n}{(abq^{2+n+1}; q)_n} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot 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8.10 Use (7.4.14) to show that (8.6.3) is equivalent to the formula
\[
(C_n(\cos \theta; \beta|q))^2 = (\beta^2, \beta^2; q)_n \beta^{-n} \sum_{\ell=0}^n \phi_{\ell} \left[ q^{-n}, \beta^2 q^\alpha, \beta, \beta\tau^{2\theta}, \beta e^{2\theta}; q, q \right],
\]
where \(n = 0, 1, \ldots\).

8.11 In view of the product formula (8.4.10) and Gegenbauer's [1874, 1893] addition formula for ultraspherical polynomials (see also Erdélyi [1953, 10.9 (34)] and Szegő [1975, p. 98]) it is natural to look for an expansion of the form
\[
p_n(z; a, aq^{1/2}, -a, -aq^{1/2}) = \sum_{k=0}^n A_k(n, \theta, \phi) p_k(z; a, aq^{1/2}, -a, -aq^{1/2}) q_
\]
where \(p_n(z; a, b, c, d|q)\) is the Askey-Wilson polynomial. By multiplying both sides by
\[
w(z; a, aq^{1/2}, -a, -aq^{1/2})\]
and then integrating over \(z\) from \(-1\) to \(1\), show that
\[
A_m(n, \theta, \phi) = \frac{(q; q)_n (a^q a^{q-1} a^{2q^{1/2}}, -a^2 q^{1/2}, -a^2 q^{1/2}) n \rightarrow m}{(q; q)_m (q; q)_n m \rightarrow (a^q a^{q-1} a^{2q^{1/2}}, -a^2 q^{1/2}, -a^2 q^{1/2}) q_n}.
\]

8.12 Prove the inverse of the linearization formula (8.5.1), namely,
\[
C_{m+n}(x; \beta|q) = \sum_{k=0}^{\min(m, n)} b(k, m, n) C_{m-k}(x; \beta|q) C_{n-k}(x; \beta|q),
\]
where
\[
b(k, m, n) = \frac{(q; q)_m (q; q)_n (\beta; q)_{m+n}}{(q; q)_m (\beta; q)_n (q; q)_{m+n}} \frac{(\beta^{-2} q^{1-n}, \beta; q)_{m+n}}{(q; q)_k (1 - \beta^{-2} q^{2k-n}) (\beta^2 q^{-1})^k}.
\]

8.13 Give alternate derivations of (8.6.3) and (8.6.4) by using the \(q\)-integral representation (7.4.7) of \(C_n(x; \beta|q)\) and the \(q\)-integral formula (2.10.19) for an \(\phi_7\) series.

8.14 By equating the coefficients of \(e^{(m+n-2k)\theta}\) on both sides of the linearization formula (8.5.1) show that
\[
4\Phi_3 \left[ \begin{array}{c}
q^{-k}, q^{-m}, \beta, \beta q^{n-k} \\
\beta^{-1} q^{1-k}, \beta^{-1} q^{1-m}, \beta^{-1} q^{1-n}
\end{array} ; q, q^2 \right] = \frac{(q-m-n, \beta^{-1} q^{-1-n} ; q_k)}{(q-m-n, \beta^{-1} q^{-1-n} ; q_k)}
\]
\[
q^{-k}, \beta^{-1} q^{1-k}, \beta^{-1} q^{1-m}, \beta^{-1} q^{1-n}, \beta^{-1} q^{-1-n}
\]
\[
q^{-k}, \beta^{-1} q^{1-k}, \beta^{-1} q^{1-m}, \beta^{-1} q^{1-n} ; q, q^2 ; \beta
\]
for \(k = 0, 1, \ldots, n\).

8.15 Show that, by analytic continuation, it follows from Ex. 8.14 that
\[
4\Phi_3 \left[ \begin{array}{c}
a, b, c, d
\end{array} ; q, q^2 \right]
\]
\[
= \frac{(a, b q^{1/2}, -a, -b q^{1/2}) q^2}{(a, b q^{1/2}, -a, -b q^{1/2}) q^2}
\]
\[
\times \frac{12 W_1(\beta c d ; (\beta c q d)^{1/2}, -(\beta c q d)^{1/2}, q(\beta c q d)^{1/2}, -q(\beta c q d)^{1/2})}{a b, a, c, b, c, q, a, q},
\]
where at least one of \(a, b, c\) is of the form \(q^{-n}, n = 0, 1, \ldots\).

8.16 From Ex. 8.15 deduce that
\[
10 W_0 \left[ \begin{array}{c}
a q^{1/2}, -a q^{1/2}, -a q^2 / b, a q^2 / c, b, c, a, q / a
\end{array} \right] = \frac{(a, b, c, d q)_{\infty}}{(a, b, c, d q)_{\infty}} \Phi_3 \left[ \begin{array}{c}
a, b, c, d q \end{array} ; b q / a, c q / a, b c q / a^2 \right],
\]
where one of \(a, b, c\) is of the form \(q^{-n}, n = 0, 1, \ldots\).
Further Applications

8.17 Prove that
\[
\left\{ \phi_3 \left[ a^2, b^2, abz, abz/z, \frac{a^2 b^2 q}{-ab, -abq : q^2} \right] \right\}^2 = \phi_4 \left[ a^2, b^2, ab, abz, abz/z \frac{a^2 b^2 q, -ab, -abq : q}{a^2 b^2 q, -ab, -abq : q} \right]
\]
and
\[
\left\{ \phi_3 \left[ a^2, b^2, abz, abz/z \frac{abq, -abq, -a^2 b^2 q}{a^2 b^2 q, -ab, -abq : q} \right] \right\}^2 = \phi_4 \left[ a^2, b^2, a^2 b^2, a^2 b^2 z^2 \frac{a^2 b^2 q}{-ab, -abq : q^2} \right]
\]
when the series terminate.
(Gasper [1989b])

8.18 With the notation of Ex. 7.37 prove that
\[
D_q \left[ \left( q^{1-v} e^{i\theta}, q^{1-v} e^{-i\theta} ; q \right)_{2n} \right] U_n(\cos \theta) \leq 0
\]
when \( \nu > -\frac{1}{2}, \lambda \geq 0, \theta \) is real and \( n = 1, \ldots, r. \quad \circ < a < \frac{1}{2} \)
(Gasper [1989b])

Notes 8


§8.6 and 8.7 The nonnegativity of other Poisson kernels and their applications to probability theory and other fields are considered in Beckmann [1973] and Gasper [1973, 1975a,b, 1976, 1977].


§8.11 Mordell [1917] considered the representation of numbers as the sum of \( 2r \) squares. Another proof of (8.11.1) is given in Hirschhorn [1985].
Basic Hypergeometric Series

\[
(aq^n; q)_n = \frac{(q/a; q)_n}{(q/ q)_n} (-a)^{n-k} q^{kn} q^{(n)}.
\]

(1.14)

\[
(aq^{2n}; q)_n = \frac{(q/a; q^2)_n}{(q/ q)_n} (-a)^n q^{2n} q^{(n)}.
\]

(1.15)

\[
(aq^{kn}; q)_n = \frac{(q/a; q)_n}{(q/ q)_n} (-a)^n q^{kn} q^{(n)}.
\]

(1.16)

\[
(aq^n; q)_{n+k} = (a; q)_n (aq^n; q)_{k}.
\]

(1.17)

\[
(aq^n; q)_k = \frac{(a; q)_{k} q^{kn} q^{(n)}}{(a; q)_n}.
\]

(1.18)

\[
(aq^n; q)_n = \frac{(a; q)_{n+k}}{(a; q)_n}.
\]

(1.19)

\[
(aq^n; q)_{n-k} = \frac{(a; q)_{n}}{(a; q)_n}.
\]

(1.20)

\[
(aq^{nk}; q)_{n-k} = \frac{(a; q)_{n} q^{an} q^{(n)}}{(a; q)_n}.
\]

(1.21)

\[
(aq^{i}q^j; q)_{n-k} = \frac{(a; q)_{j} q^{an} q^{(n)} q^{(j-1)}}{(a; q)_n}.
\]

(1.22)

\[
(a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n.
\]

(1.23)

\[
(a_1, a_2, \ldots, a_k; q)_n = (a_1; q)_n \cdots (a_k; q)_n.
\]

(1.24)

\[
(a_1, a_2, \ldots, a_k; q)_n = (a_1, a_2, \ldots, a_k; q)_n.
\]

(1.25)

\[
(a_1, a_2, \ldots, a_k; q)_n = (a_1, a_2, \ldots, a_k; q)_n.
\]

(1.26)

and, in general,

\[
(a; q)_n = (a, aq, \ldots, aq^{n-1}; q)_n.
\]

(1.27)

\[
(a^2, q^2)_n = (a, -a; q)_n.
\]

(1.28)

\[
(a^3, q^3)_n = (a, a\omega, a\omega^2; q)_n, \quad \omega = e^{2\pi i/3}.
\]

(1.29)

and, in general,

\[
(a^k, q^k)_n = (a, a\omega_k, \ldots, a\omega_k^{k-1}; q)_n, \quad \omega_k = e^{2\pi i/k}.
\]

(1.30)

\[
(qa^k, qa^k; q)_n = (aq^2; q^2)_n = 1 - aq^{-2n}.
\]

(1.31)

\[
(qa^k, qa^k, qa^2; q)_n = (aq^3; q^3)_n = 1 - aq^{-3n}.
\]

(1.32)

and, in general,

\[
(qa^k, qa^k, \ldots, qa^k, q^k; q)_n = (aq^kn; q^k)_n = 1 - aq^{-kn}.
\]

(1.33)

where \( \omega = e^{2\pi i/3} \) and \( \omega_k = e^{2\pi i/k} \).

\[
\lim_{q \to 1^{-}} (qz^2; q)_\infty = (1 - z)^{-\alpha}, \quad |z| < 1.
\]

(1.34)

Appendix I

\[
q\text{-Gamma function:}
\]

\[
\Gamma_q(x) = \begin{cases} 
\frac{(q; q)_\infty}{(q^{1}; q)_\infty} (1 - q)^{1-x}, & 0 < q < 1, \\
\frac{(q^{-1}; q)_\infty}{(q^{-1}; q)_\infty} (q - 1)^{1-x} q^{(x)} & q > 1.
\end{cases}
\]

(1.35)

\[
\lim_{q \to 1^{-}} \Gamma_q(x) = \Gamma(x).
\]

(1.36)

\[
\Gamma_q(2x)q^x = \frac{1}{2} = \Gamma_q(x)q^x \left( x + \frac{1}{2} \right) (1 + q)^{2x-1}.
\]

(1.37)

\[
\Gamma_q(nx)R_q \left( \frac{1}{n} \right) \Gamma_q \left( \frac{2}{n} \right) \cdots \Gamma_q \left( \frac{n - 1}{n} \right) = (1 + q + \ldots + q^{n-1})^{n-1} \Gamma_q(x)R_q \left( x + \frac{1}{n} \right) \cdots R_q \left( x + \frac{n - 1}{n} \right),
\]

(1.38)

with \( r = q^n \).

\[
q\text{-Binomial coefficient:}
\]

\[
\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \left[ \begin{array}{c} n \\ n - k \end{array} \right] = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}
\]

(1.39)

and, for \( |q| < 1 \) and complex \( \alpha \) and \( \beta \),

\[
\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_q = \frac{(q^{\alpha+1}; q)_{\infty}}{(q; q^\alpha+1; q)_\infty},
\]

(1.40)

\[
\left[ \begin{array}{c} \alpha \\ \beta \end{array} \right]_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\Gamma_q(\alpha+\beta+1)}.
\]

(1.41)

\[
\left[ \begin{array}{c} \alpha \\ k \end{array} \right]_q = \left[ \begin{array}{c} q^{\alpha+1}; q \end{array} \right]_q \left[ \begin{array}{c} \alpha \end{array} \right]_q
\]

(1.42)

\[
\left[ \begin{array}{c} k + \alpha \\ k \end{array} \right]_q = \left[ \begin{array}{c} q^{\alpha+1}; q \end{array} \right]_q
\]

(1.43)

\[
\left[ \begin{array}{c} \alpha + k - 1 \\ k \end{array} \right]_q = \left[ \begin{array}{c} q^{\alpha+1}; q \end{array} \right]_q
\]

(1.44)

\[
\left[ \begin{array}{c} \alpha + 1 \\ k \end{array} \right]_q = \left[ \begin{array}{c} \alpha \end{array} \right]_q q^k + \left[ \begin{array}{c} \alpha \\ k - 1 \end{array} \right]_q = \left[ \begin{array}{c} \alpha \end{array} \right]_q + \left[ \begin{array}{c} \alpha \\ k - 1 \end{array} \right]_q q^{\alpha+1-k}.
\]

(1.45)
Appendix II

SELECTED SUMMATION FORMULAS

Sums of basic hypergeometric series:

The two q-exponential functions,
\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q^2)\,q^n} = \frac{1}{(1-x/q^2)} \quad |x| < 1,
\]

\[
E_q(x) = \sum_{n=0}^{\infty} \frac{q^n x^n}{(q^2)^n (q^n)} = \frac{(-x)}{(1-x/q^2)}.
\]

The q-binomial theorem,
\[
\binom{n}{k}_q = \frac{[n]!}{[k]![n-k]!}_q = \frac{x^n}{(1-x/q^2)} \quad |x| < 1,
\]

or, when \(a = q^{-n}\), where, as elsewhere in this appendix, \(n\) denotes a nonnegative integer,
\[
\binom{n}{-k}_q = \frac{(aq^n)}{(1-q^n x)} = \frac{q^k}{[k]!}_q \quad |x| < 1, \quad |q| < 1.
\]

The sum of a \(\phi_1\) series,
\[
\phi_1(1; a; q, c/a) = \frac{(c/a; q)_\infty}{(c; q)_\infty}.
\]

The q-Vandermonde (q-Chu-Vandermonde) sums,
\[
\phi_1(1; a, q^{-n}; q, c) = \frac{(c/a; q)_n}{(c; q)_n} a^n.
\]

and, reversing the order of summation,
\[
\phi_1(1; a, q^{-n}; q, c/q^n/a) = \frac{(c/a; q)_n}{(c; q)_n} a^n.
\]

The q-Gauss sum,
\[
\phi_1(1; a, b; q, c/ab) = \frac{(c/a, c/ab; q)_\infty}{(c, c/ab; q)_\infty}.
\]

The q-Kummer (Bailey-Daum) sum,
\[
\phi_1(1; a, b; q, ab/b; q, q/b) = \frac{\left(-q^{-1}; q\right)_\infty}{\left(-q^{-1}; q\right)_\infty} \frac{(aq, aq^2/b^2; q^2)_\infty}{(q^{-1}; q)_\infty}.
\]

A q-analogue of Bailey's \(\phi_2\) sum,
\[
\phi_2(1; a, b; q/a, q/b; q, q/b) = \frac{(a, b; q)_\infty}{(a, b; q)_\infty}.
\]

A q-analogue of Gauss' \(\phi_2\) sum,
\[
\phi_2(1; a, b; q/a, q/b; q, q/b) = \frac{(a,b,q^{-1}; q)_\infty}{(a,b,q^{-1}; q)_\infty}.
\]

The q-Saalschütz (q-Pfaff-Saalschütz) sum,
\[
\phi_2(1; a, b, q^{-n}; q, q) = \frac{(c/a, c/b; q)_n}{(c/a; q)_n}.
\]

The q-Dixon sum,
\[
\phi_2(1; a, b, q^{-n}; q, q) = \frac{(aq, q/b; q)_\infty}{(aq, q/b; q)_\infty}.
\]

or, when \(c = q^{-n}\),
\[
\phi_2(1; a, b, q^{-n}; q, q) = \frac{(aq, q/b; q)_\infty}{(aq, q/b; q)_\infty}.
\]

Jackson's terminating q-analogue of Dixon's sum,
\[
\frac{1}{\phi_2(1; a, b, q^{-n}; q, q)} = \frac{(q^{-1}; q)_\infty}{(q^{-1}; q)_\infty}.
\]

A q-analogue of Watson's \(\phi_2\) sum,
\[
\frac{1}{\phi_2(1; a, b, q^{-n}; q, q)} = \frac{(aq, q/b; q)_\infty}{(aq, q/b; q)_\infty}.
\]

where \(\lambda = -c(ab/q; q)_\infty\); and Andrews' terminating q-analogue,
\[
\phi_2(1; a, b, q^{-n}; q, q) = \frac{(aq, q/b; q)_\infty}{(aq, q/b; q)_\infty}.
\]

A q-analogue of Whipple's \(\phi_2\) sum,
\[
\phi_2(1; a, b, q^{-n}; q, q) = \frac{(aq, q/b; q)_\infty}{(aq, q/b; q)_\infty}.
\]

A q-analogue of Andrews' \(\phi_2\) sum,
\[
\phi_2(1; a, b, q^{-n}; q, q) = \frac{(aq, q/b; q)_\infty}{(aq, q/b; q)_\infty}.
\]

A q-analogue of Whipple's \(\phi_2\) sum,
\[
\phi_2(1; a, b, q^{-n}; q, q) = \frac{(aq, q/b; q)_\infty}{(aq, q/b; q)_\infty}.
\]
The sum of a very-well-poised $\phi_5$ series,

$$\phi_5\left[\begin{array}{c}
a, aq^4, -aq^4, b, c, d, e; \frac{aq}{bc}
\end{array}\right]$$

$$= \frac{(aq, aq/b, aq/bc, aq/bd, aq/cd; q)_\infty}{(aq/b, aq/c, aq/d, aq/bcd; q)_\infty}$$  \hspace{1cm} \text{(II.20)}

or, when $d = q^{-n}$,

$$\phi_5\left[\begin{array}{c}
a, aq^4, -aq^4, b, c, d, e, q^{-n}; \frac{aq^{n+1}}{bc}
\end{array}\right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}$$  \hspace{1cm} \text{(II.21)}

Jackson's $q$-analogue of Dougall's $\mathbf{\gamma F_0}$ sum,

$$\phi_7\left[\begin{array}{c}
a, aq^4, -aq^4, b, c, d, e, q^{-n}; \frac{aq^{n+1}}{bc}
\end{array}\right]$$

$$= \frac{(aq, aq/bc, aq/bd, aq/cd; q)_n}{(aq/b, aq/c, aq/d, aq/bcd; q)_n}$$  \hspace{1cm} \text{(II.22)}

where $a^2q = bcdeq^{-n}$.

A nonterminating form of the $q$-Vandermonde sum,

$$\phi_1(a, b; c, q; q) + \frac{(q/c, a, b; q)_\infty}{(c/q, aq/c, bq/c; q)_\infty}$$

$$\cdot \phi_1(aq/c, bq/c; q^2/c; q; q) = \frac{(q/c, abq/c; q)_\infty}{(aq/c, bq/c; q)_\infty}$$  \hspace{1cm} \text{(II.23)}

A nonterminating form of the $q$-Saalschütz sum,

$$\phi_2\left[\begin{array}{c}
a, b, c, e, f; q, q
\end{array}\right] + \frac{(q/e, a, b, c, qf/e; q)_\infty}{(e/q, aq/e, bq/e, cq/e, f; q)_\infty}$$

$$\cdot \phi_2\left[\begin{array}{c}
aq/e, bq/e, cq/e, q^2/e, qf/e; q, q
\end{array}\right] = \frac{(aq/e, bq/e, cq/e, f; q)_\infty}{(aq/e, bq/e, cq/e, f; q)_\infty}$$  \hspace{1cm} \text{(II.24)}

where $ef = abq$.

Bailey's nonterminating extension of Jackson's $\phi_7$ sum,

$$\phi_7\left[\begin{array}{c}
a, aq^4, -aq^4, b, c, d, e, f; q, q
\end{array}\right]$$

$$= \frac{b}{a} \frac{(aq/b, aq/c, aq/d, aq/e, aq/f; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f; q)_\infty}$$

$$\cdot \phi_7\left[\begin{array}{c}
b^2a, qba^{-1}, -qba^{-1}, b, bc/a, ba/c/a, bf/a; q, q
\end{array}\right]$$

$$= \frac{(aq/b, aq/c, aq/d, aq/e, aq/f; q)_\infty}{(aq/c, aq/d, aq/e, aq/f; q)_\infty}$$  \hspace{1cm} \text{(II.25)}

where $qa^2 = bcdef$.

$q$-Analogue of the Carlsson-Minton sums,

$$\phi_1(a, b, q^m; b, q^m; q, a^{-1}q^{1-(m_1+\ldots+m_r)}; q, a^{-1}q^{1-(m_1+\ldots+m_r)}; q)$$

$$= \frac{(aq/b, a; q)_\infty}{(aq/b, b; q)_\infty}$$  \hspace{1cm} \text{(II.26)}

and

$$\phi_1(a, b_1q^{m_1}, \ldots, b_rq^{m_r}; q, a^{-1}q^{1-(m_1+\ldots+m_r)}; q) = 0,$$  \hspace{1cm} \text{(II.27)}

where $m_1, \ldots, m_r$ are arbitrary nonnegative integers.

Sums of bilateral basic series:

Jacobi's triple product,

$$\sum_{k=-\infty}^{\infty} q^k z^k = \left(q^2, -qz, -q/z; q^2\right)_\infty.$$  \hspace{1cm} \text{(II.28)}

Ramanujan's sum,

$$\psi_1(a; b, q, z) = \frac{(q, b/a, az, q/a z, q)_\infty}{(b, q/a, z, b/az; q)_\infty}.$$  \hspace{1cm} \text{(II.29)}

The sum of a well-poised $\psi_2$ series,

$$\psi_2(b, c; q, aq/bc, aq/c, q, -aq/bc)$$

$$= \frac{(aq/bc; q)_\infty(aq/bc^2; q)_\infty}{(aq/b, aq/c, bq/c, q/abc; q)_\infty}$$  \hspace{1cm} \text{(II.30)}

Bailey's sum of a well-poised $\psi_3$ series,

$$\psi_3\left[\begin{array}{c}
b, c, d; q, q
\end{array}\right]$$

$$= \frac{(q, q/bc, q/bd, q/cd; q)_\infty}{(q/b, q/c, q/d, q/bcd; q)_\infty}$$  \hspace{1cm} \text{(II.31)}

A basic bilateral analogue of Dixon's sum,

$$\psi_4\left[\begin{array}{c}
-qa^4, b, c, d; q, qa^3
\end{array}\right]$$

$$= \frac{(aq, aq/b, aq/c, aq/d; q)_\infty}{(aq/b, aq/c, aq/d, q/abc; q)_\infty}$$  \hspace{1cm} \text{(II.32)}

The sum of a very-well-poised $\psi_6$ series,

$$\psi_6\left[\begin{array}{c}
a, a^3, b, c, d, e; q, qa^2
\end{array}\right]$$

$$= \frac{(aq/b, aq/c, aq/d, aq/e, q/abc; q)_\infty}{(aq/b, aq/c, aq/d, aq/e, q/abc; q)_\infty}$$  \hspace{1cm} \text{(II.33)}
Bibasic sums:

Gasper’s indefinite bibasic sum,
\[
\sum_{k=0}^{n} \frac{1 - ap^k q^k}{1 - a} \frac{(a;p)_k(c;q)_k}{(q;q)_k(ap/c;p)_k} = \frac{(ap;p)_n(c;q)_n}{(q;q)_n(ap/c;p)_n} (c - n).
\] (II.34)

An extension of (II.34),
\[
\sum_{k=0}^{n} \frac{(1 - ap^k q^k)(1 - bp^k q^{-k})}{(1 - a)(1 - b)} \frac{(a;b;p)_k(c,a/bc;q)_k}{(q,aq/b;q)_k(ap/c,bcp;p)_k} q^k
\]
\[
= \frac{(ap,bp;p)_n(cq,aq/bc;q)_n}{(q,aq/b;q)_n(ap/c,bcp;p)_n}
\] (II.35)

and, more generally,
\[
\sum_{k=-m}^{n} \frac{(1 - adp^k q^k)(1 - bp^k / dq^k)}{(1 - ad)(1 - b/d)} \frac{(a;b;p)_k(c,ad^2/bc;q)_k}{(dq,adq/b;q)_k(ap/c,bcp/d;p)_k} q^k
\]
\[
= \frac{(1 - a)(1 - b)(1 - c)(1 - ad^2/bc)}{d(1 - ad)(1 - b/d)(1 - c/d)(1 - ad/bc)} \left( \frac{(ap,bp;p)_n(cq,ad^2q/bc;q)_n}{(aq,bcq/ad^2;q)_n(ap/c,bcp/d;p)_n} \right)
\]
\[
\frac{1}{a,c,bc/a^2d^2;q}_{m+1} \frac{1}{1/d,b/ad;q}_{m+1}
\] (II.36)

where \(m\) is an integer or \(+\infty\).

An extension of the formula for the \(n\)-th \(q\)-difference of \((ap^k;q)_n\),
\[
\left( \frac{1 - a}{1 - q} \right) \left( \frac{1 - b}{1 - q} \right) \sum_{k=0}^{n} \frac{(ap^k,bp^{-k};q)_{n-1} (1 - ap^k/b)}{(p;p)_k(p;p)_{n-k} (ap^k/b;p)_{n+1}} \frac{(-1)^k p^k}{q^k} = \delta_{n,0}.
\] (II.37)

Appendix III

SELECTED TRANSFORMATION FORMULAS

Heine’s transformations of \(2\phi_1\) series:
\[
2\phi_1(a, b; c; q, z) = \frac{(b, az; q)_\infty}{(c, z; q)_\infty} 2\phi_1(c/b, z; az; q, b)
\] (III.1)
\[
= \frac{(c/b, bz; q)_\infty}{(c, z; q)_\infty} 2\phi_1(abz/c, b; bz, q, c/b)
\] (III.2)
\[
= \frac{(abz/c; q)_\infty}{(z; q)_\infty} 2\phi_1(c/a, c/b; c, abz/c).
\] (III.3)

Jackson’s transformations of \(2\phi_1, 3\phi_2\) and \(3\phi_3\) series:
\[
2\phi_1(a, b; c; q, z) = \frac{(az, q)_\infty}{(z; q)_\infty} 2\phi_2(a, c/b; c, az; q, bz)
\] (III.4)
\[
= \frac{(abz/c; q)_\infty}{(bz/c; q)_\infty} 3\phi_2[a, c/b, 0]
\] (III.5)
\[
q = q^{-n}; m = c, i, \ldots
\]

Transformations of terminating \(2\phi_1\) series:
\[
2\phi_1(q^{-n}, b; c; q, z) = \frac{(c/b; q)_n}{(c, q)_n} \left( \frac{bz}{q} \right)^n
\]
\[
\cdot 3\phi_2[q^{-n}, q/z, c^{-1}q^{1-n}; bc^{-1}q^{1-n}, 0; q, q]
\] (III.6)
\[
= (c/b; q)_n 3\phi_2[a, dz/c, 0; q, q]
\] (III.7)
\[
= (c/b; q)_n b^n 3\phi_1[p^{-n}, q, q/c, z]
\]
\[
(c, q)_n \cdot bz^{-1}/c; q, z
\] (III.8)

where, as elsewhere in this appendix, \(n\) denotes a non-negative integer.

Transformations of \(3\phi_2\) series:
\[
3\phi_2 \left[ \begin{array}{c} a, b, c \\ d, e, f \\ q, \frac{de}{abc} \end{array} \right]
\]
\[
= \frac{(e/a, de/abc; q)_\infty}{(e, de/abc; q)_\infty} 3\phi_2 \left[ \begin{array}{c} a, d/b, d/c \\ d, e, f \\ q, \frac{e}{a} \end{array} \right]
\] (III.9)
\[
= \frac{(b, de/abc; q)_\infty}{(d, e, de/abc; q)_\infty} 3\phi_2 \left[ \begin{array}{c} d/b, e/b, de/abc \\ d, e, de/abc \\ q, b \end{array} \right]
\] (III.10)
\[
3\phi_2 \left[ \begin{array}{c} q^{-n}, b, c \\ d, e \\ q, q \end{array} \right]
\]
\[
= \frac{(de/abc; q)_n}{(e, q)_n} \left( \frac{de}{d} \right)^n 3\phi_2 \left[ \begin{array}{c} q^{-n}, d/b, d/c \\ d, e, de/abc \\ q, q \end{array} \right]
\] (III.11)
where \( def = abcq^{-n} \) and \( \sigma = ef/aq \).

Another transformation of a terminating balanced \( 4\phi_3 \) series to a very-well-poised \( 8\phi_7 \) series,
\[
4\phi_3 \left[ \begin{array}{c} q^{-n}, a, b, c \\ d, e, f \\ q, q \end{array} \right] = \frac{(aq/b, aq/c, bq/f, q/f; q/f; q)_\infty}{(aq/a, bq/f, f, q/f; q)_\infty}
\]

\( 8\phi_7 \left[ \begin{array}{c} \mu, -\mu, \mu, \mu, \mu, \mu, \mu, \mu \\ a, b, c, d, e, f, q \end{array} \right] = \frac{(aq/b, b, c, d, e, f, q; q)_\infty}{(aq/b, aq/c, b, c, d, e, f; q)_\infty}
\]

provided that \( a = q^{-n} \). See (III.35) for a nonterminating case.

**Sears' transformations of terminating balanced \( 4\phi_3 \) series:**
\[
4\phi_3 \left[ \begin{array}{c} q^{-n}, a, b, c \\ d, e, f \\ q, q \end{array} \right] = \frac{(e/f, a; q)_n}{(e; q)_n} a_n 4\phi_3 \left[ \begin{array}{c} q^{-n}, a, d/b, d/c \\ d, q^{-n}/e, q^{-n}/f \\ q, q \end{array} \right]
\]

where \( def = abcq^{-n} \).

**Watson's transformation formulas:**
\[
s\phi_7 \left[ \begin{array}{c} a, qa^4, -qa^4, b, c, d, e, f \\ a^4, -a^4, qa/b, qa/c, qa/d, qa/e, qa/f; q \end{array} \right] = \frac{(aq, aq/de, aq/v, aq/e/f; q)_\infty}{(aq/a, aq/e, aq/f, aq/d, q; q)_\infty} 4\phi_3 \left[ \begin{array}{c} q, a^2, q^2 \\ d, e, f \\ q, q \end{array} \right]
\]

provided both series terminate. A non-terminating \( q \)-analogue of Clausen's formula is given in (8.8.17).

**Transformations of very-well-poised \( 8\phi_7 \) series:**
\[
s\phi_7 \left[ \begin{array}{c} a, qa^4, -qa^4, b, c, d, e, f \\ a^4, -a^4, qa/b, qa/c, qa/d, qa/e, qa/f; q \end{array} \right] = \frac{(aq, aq/e, aq/c, aq/d, q; q)_\infty}{(aq/a, aq/e, aq/f, aq/c, q; q)_\infty}
\]

or, equivalently,
\[
4\phi_3 \left[ \begin{array}{c} q^{-n}, a, b, c \\ d, e, f \\ q, q \end{array} \right] = \frac{(d/b, d/c; q)_n}{(d, d/b, d/c; q)_n} 4\phi_3 \left[ \begin{array}{c} q^{-n}, a, d/b, d/c \\ d, e, f \\ q, q \end{array} \right]
\]

\[
s\phi_7 \left[ \begin{array}{c} \sigma, q\sigma^4, -q\sigma^4, f/a, e/a, b, c, q^{-n} \\ \sigma^4, -\sigma^4, a, f, ef/ab, ef/ac, efq^n/a; q \end{array} \right] = \frac{efq^n/a}{bc}
\]

where \( \lambda = qa^2/bcd \) and \( \mu = q^2a^3/b^2cdef \).
Transformations of a nearly-poised $\phi_4$ series:

\[
\Phi_4 \left[ \begin{array}{ccccc}
a & b & c & d & \frac{q^n}{\lambda^2 q} \\
aq/b, aq/c, aq/d, a^2q^{-n}/\lambda^2; q, q^d \\
\end{array} \right] \\
= (aq/a, \lambda^2 q/aq; q)_n \Phi_4 \left[ \begin{array}{ccccc}
aq/b, aq/c, aq/d, a^2q^{-n}/\lambda^2; q, q^d \\
\lambda^2 & -\lambda^2 & b/a & -b/a & \lambda^2/a \\
\end{array} \right] \\
+ (aq/a, \lambda^2 q/aq; q)_n \Phi_4 \left[ \begin{array}{ccccc}
aq/b, aq/c, aq/d, a^2q^{-n}/\lambda^2; q, q^d \\
\lambda^2 & -\lambda^2 & b/a & -b/a & \lambda^2/a \\
\end{array} \right]
\]

(III.25)

\[
\Phi_4 \left[ \begin{array}{ccccc}
a & b & c & d & \frac{e}{\mu^2} \\
q^{-n}/b, q^{-n}/c, q^{-n}/d, eq^{-2n}/\mu^2; q, q^e \\
\end{array} \right] \\
= (\mu^2 q^{-n}/e, \mu e; q)_n \Phi_4 \left[ \begin{array}{ccccc}
a & b & c & d & \frac{e}{\mu^2} \\
q^{-n}/b, q^{-n}/c, q^{-n}/d, eq^{-2n}/\mu^2; q, q^e \\
\end{array} \right]
\]

(III.26)

where $\lambda = qa^2/bcd$ and $\mu = q^{-2n}/bcd$.

**Transformation of a nearly-poised $\phi_6$ series:**

\[
\Phi_6 \left[ \begin{array}{cccccc}
a & b & c & d & e & \frac{f}{g} \\
\end{array} \right] \\
= (\lambda/aq, \lambda^2 /aq; q)_n \Phi_6 \left[ \begin{array}{cccccc}
a & b & c & d & e & \frac{f}{g} \\
\end{array} \right] \\
+ (\lambda/aq, \lambda^2 /aq; q)_n \Phi_6 \left[ \begin{array}{cccccc}
a & b & c & d & e & \frac{f}{g} \\
\end{array} \right]
\]

(III.27)

where $\lambda = qa^2/bcd$.

**Bailey's $10\phi_9$ transformation formula:**

\[
\Phi_9 \left[ \begin{array}{cccccc}
a & b & c & d & e & \frac{f}{g} \\
aq/b, aq/c, aq/d, eq/f, efq^{-n}/\lambda; q, q^e \\
\end{array} \right] \\
= (aq/a, \lambda q/aq; q)_n \Phi_9 \left[ \begin{array}{cccccc}
a & b & c & d & e & \frac{f}{g} \\
aq/b, aq/c, aq/d, eq/f, efq^{-n}/\lambda; q, q^e \\
\end{array} \right]
\]

(III.28)

where $\lambda = qa^2/bcd$.

**Appendix III**

Transformations of $r+2\phi_{r+1}$ series:

\[
r+2\phi_{r+1} \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
bq^{-n}, \frac{a^{-1}q^{-n}}{bq^{-n}}; q, q^{-n} \\
\end{array} \right] \\
= (bq^{-n}, \frac{a^{-1}q^{-n}}{bq^{-n}}; q, q^{-n}) \Phi_{r+1} \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
bq^{-n}, \frac{a^{-1}q^{-n}}{bq^{-n}}; q, q^{-n} \\
\end{array} \right]
\]

(III.29)

and

\[
r+2\phi_{r+1} \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
bq^{-n}, \frac{a^{-1}q^{-n}}{bq^{-n}}; q, q^{-n} \\
\end{array} \right] \\
= (bq^{-n}, \frac{a^{-1}q^{-n}}{bq^{-n}}; q, q^{-n}) \Phi_{r+1} \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
bq^{-n}, \frac{a^{-1}q^{-n}}{bq^{-n}}; q, q^{-n} \\
\end{array} \right]
\]

(III.30)

where $m, m_1, \ldots, m_r$ are arbitrary nonnegative integers.

**Three-term transformation formulas:**

\[
2\phi_1(a, b; c, q, z) = (\frac{abz/c, q; q}{az/c, q; q})_\infty 2\phi_1(c/a, cq/abz; cq/azq, bq/c) \\
- (bq/c, c/a, az/c, q^2; az/c, q^2)_\infty 2\phi_1(aq/c, bq/c, q^2/c, q, z). \quad (III.31)
\]

\[
2\phi_1(a, b; c, q, z) = (\frac{b/a, az/c, q^2; az/c, q^2}{b/a, az/c, q^2; az/c, q^2})_\infty 2\phi_1(aq/c, bq/c, q^2/c, q, z) + (a/c, b/a, z/c, q^2/c, q^2)_\infty 2\phi_1(bq/c, bq/a, q, cq/abz) \quad (III.32)
\]

\[
3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
= (e/b, e/c, cq/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
- (g/d, e/bq/c, b/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
+ (d/q, e/bq/c, b/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
= (e/b, e/c, cq/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
- (g/d, e/bq/c, b/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
+ (d/q, e/bq/c, b/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
\]

(III.33)

\[
3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
= (e/b, e/c, cq/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
- (g/d, e/bq/c, b/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
+ (d/q, e/bq/c, b/a, d/q; d/q)_\infty 3\phi_2 \left[ \begin{array}{cccccccc}
a & b & c & d & e & \frac{f}{g} & \frac{h}{j} & \frac{k}{l} \\
\end{array} \right] \\
\]

(III.34)
Basic Hypergeometric Series

\[ \phi_2 \left[ \frac{a}{aq/b}, \frac{c}{aq/c}; q, \frac{aqz}{bc} \right] = \frac{(az; q)^\infty}{(z; q)_\infty} \phi_4 \left[ \frac{a^\frac{1}{4}, -a^\frac{1}{4}, (aq)^\frac{1}{4}, -(aq)^\frac{1}{4}, aq/bc, aq/z}{aq/b, aq/c, ax, q/z, q} \right] \\
+ \frac{(a, aq/bc, aqz/b, aqz/c; q)^\infty}{(aq/b, aq/c, aqz/b, aqz/c; z^{-1}; q)_\infty} \phi_4 \left[ \frac{za^\frac{1}{4}, -za^\frac{1}{4}, (aq)^\frac{1}{4}, -(aq)^\frac{1}{4}, aqz/bc, aqz/b, aqz/c, ax, aqz^2}{za^\frac{1}{4}, -za^\frac{1}{4}, (aq)^\frac{1}{4}, -(aq)^\frac{1}{4}, aqz/bc, aqz/b, aqz/c, ax, aqz^2; q} \right]. \tag{III.35} \]

Appendix III

\[ \phi_7 \left[ \frac{a, aq^\frac{1}{4}, -aq^\frac{1}{4}, b, c, d, e, f, a^2q^2}{aq/b, aq/c, aq/d, aq/e, aq/f; q^4, bcd\def g} \right] \\
= \frac{(aq, aq/de, aq/df, aq/ef; q)^\infty}{(aq/d, aq/e, aq/f, aq/def; q)_{\infty}} \phi_3 \left[ \frac{aq/bc, d, e, f, a^2q^2}{aq/b, aq/c, def/a; q} \right] \\
+ \frac{(aq, aq/bc, d, e, f, a^2q^2/def, a^2q^2/cddef; q)^\infty}{(aq/b, aq/c, aq/d, aq/e, aq/f, a^2q^2/def, def/qaq; q)_{\infty}} \phi_3 \left[ \frac{aq/de, aq/ef, a^2q^2/cddef, a^2q^2/def; q}{aq^2/def, a^2q^2/def, a^2q^2/cddef; q} \right]. \tag{III.36} \]

Transformation of an \( \psi_8 \):

\[ (aq/b, aq/c, aq/d, aq/e, q/ab, q/ac, q/ad, q/ae; q)_{\infty} \]

\( \psi_8 \left[ \frac{q, qa}{aq/b, aq/c, aq/d, aq/e, aq/f, aq/g; q^2, bcd\def g} \right] \)

Bailey's four-term \( 10\phi_9 \) transformation:

\[ \phi_9 \left[ \frac{b^2/q, qa, -qa}{b^2/q, qa, -qa, b, c, d, e, f, g, h} \right] \\
\phi_9 \left[ \frac{a, aq^\frac{1}{4}, -aq^\frac{1}{4}, b, c, d, e, f, a^2q^2}{aq/b, aq/c, aq/d, aq/e, aq/f, aq/g, aq/h; q^4, bcd\def g} \right] \\
\phi_9 \left[ \frac{b^2/q, qa, -qa, b, c, d, e, f, a^2q^2}{b^2/q, qa, -qa, b, c, d, e, f, a^2q^2; q^4, bcd\def g} \right] \\
\phi_9 \left[ \frac{b^2/q, qa, -qa}{b^2/q, qa, -qa, b, c, d, e, f, a^2q^2, bcd\def g; q^4} \right]
\]

\[ \phi_9 \left[ \frac{b^2/q, qa, -qa}{b^2/q, qa, -qa, b, c, d, e, f, a^2q^2, bcd\def g; q^4} \right]
\]

Transformations of \( 10\psi_{10} \):

\[ (aq/b, aq/c, aq/d, aq/e, q/ab, q/ac, q/ad, q/ae, q/af; q)_{\infty} \]

\[ (ag/ah, ak/g/a, h/a, k/a, qa^2, q/a^2; q)_{\infty} \]

\( \psi_{10} \left[ \frac{a, a, qa, -qa}{a, a, qa, -qa, b, c, d, e, f, a, g, h, ka} \right] \\
\psi_{10} \left[ \frac{a, a, qa, -qa}{a, a, qa, -qa, b, c, d, e, f, a, g, h, ka; q^2, bcd\def g, h, k} \right] \\
\]

where \( a^2q^2 = bcd\def g \) and \( \lambda = qa^2/cd\def g \).

\[ (aq/b, aq/c, aq/d, aq/e, q/ab, q/ac, q/ad, q/ae, q/af, q/ah, q/ak; q)_{\infty} \]

\[ (ag/ah, ak/g/a, h/a, k/a, qa^2, q/a^2, q/ab, q/ac, q/ad, q/ae, q/af, q/ah, q/ak; q^2, bcd\def g, h, k, qa^2; q^2)_{\infty} \]

\[ \psi_{10} \left[ \frac{a, a, qa, -qa}{a, a, qa, -qa, b, c, d, e, f, a, g, h, ka; q^2, bcd\def g, h, k} \right] \\
\]

\[ (aq/b, aq/c, aq/d, aq/e, q/ab, q/ac, q/ad, q/ae, q/af, q/ah, q/ak; q^2, bcd\def g, h, k, qa^2; q^2)_{\infty} \]
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\[ = \frac{(q, q/bg, q/cg, q/dg, q/eg, q/fg, q/hg, q/gg/b, qg/c, qg/d, qg/e, qg/f; q)_{\infty}}{(gh, gk, h/g, ag, q/aa, g/aq, qag, qag^2; q)_{\infty}} \]

\[ \cdot 10^{99} \left[ \frac{q^2}{g, gg/b, gg/c, gg/d, gg/e, gg/f, gg/h, gg/k} \right] \]

\[ + \text{idem} (g; h, k). \] (III.40)

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Note: For a complete list of W. N. Bailey's publications, see his obituary notice Slater [1963].


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### Notes

- The page discusses various aspects of hypergeometric series, including summation formulas, transformations, and identities.
- It covers a range of topics such as q-series, theta functions, and transformations, with specific references to well-poised and nearly-poised series.
- The document is a comprehensive resource for advanced mathematical research in hypergeometric series.