

# A Kummer function based Zeta function theory to prove the Riemann Hypothesis and the Goldbach conjecture

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## Prolog

We claim that an alternative Kummer function based Zeta function theory overcomes current challenges of several RH criteria. The probably most elegant RH criterion is the Hilbert-Polya conjecture, which is basically about the existence of a convolution integral representation of the Zeta function, including an appropriately defined domain. The following few pages are about a Kummer function based approach to prove the Hilbert-Polya conjecture, which is basically about a replacement of the exponential function  $g(x) := e^{-x}$  by

$$g^*(x) := \int_x^\infty {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -t\right) \frac{dt}{t} \quad \text{with } -xg^{*'}(x) = {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right),^{(*)}$$

Putting  $\mu(x) := -\int_x^\infty e^{-t} \frac{dt}{t}$  where  $d\mu(x) = e^{-x} \frac{dx}{x}$ , and  $\vartheta(x) := -\int_x^\infty {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -t\right) \frac{dt}{t}$  where  $d\vartheta(x) = {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x}$  this leads to the following Mellin transforms relationship

$$\frac{\Gamma\left(\frac{s}{2}\right)}{1-s} = \frac{1}{1-s} \int_0^\infty x^{s/2} d\mu(x) = \int_0^\infty x^{s/2} d\vartheta(x).$$

Regarding the entire Zeta function

$$\xi(s) := \frac{s}{2} \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s)$$

the concerned Mellin transforms 1ort h underlying Gaussian function  $g(\pi x^2) = e^{-\pi x^2}$  are given by

$$M[e^{-\pi x^2}](s) = \frac{1}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \quad \text{resp.} \quad M[-xg'(x)](s) = \frac{s}{2} \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) = \pi^{-s/2} \Gamma\left(1 + \frac{s}{2}\right).$$

The Poisson summation formula applied to

$$G(x) := \sum_{-\infty}^\infty g(\pi n^2 x^2) = \sum_{-\infty}^\infty e^{-\pi n^2 x^2} \quad , \quad \psi(x^2) := 2 \sum_{n=1}^\infty g(\pi n^2 x^2) = 2 \sum_{n=1}^\infty e^{-\pi n^2 x^2}$$

leads to the "duality" equation  $G(x) = 1 + \psi(x^2) = \frac{1}{x} G\left(\frac{1}{x^2}\right) = \frac{1}{x} \left[1 + \psi\left(\frac{1}{x^2}\right)\right]$ , which is equivalent to the Riemann duality equation  $\xi(s) = \xi(1-s)$  (\*). The underlying formula

$$\Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s) = \int_0^\infty x^{s/2} \psi(x) \frac{dx}{x} = \int_1^\infty \psi(x) [x^{s/2} + x^{(1-s)/2}] \frac{dx}{x} - \frac{1}{s(1-s)} \quad (**)$$

results into the following (Fourier) integral representations

$$\xi(s) = 4 \int_1^\infty \frac{d}{dx} \left[ x^{\frac{3}{2}} \psi'(x) \right] x^{-\frac{1}{4}} \cosh \left[ \frac{1}{2} \left( s - \frac{1}{2} \right) \log x \right] dx, \quad \text{Re}(s) > 1 \quad (**), (***)$$

$$\xi(s) = \xi\left(\frac{1}{2} + it\right) = \sum_{n=0}^\infty a_{2n} \left(s - \frac{1}{2}\right)^{2n} \quad \text{with} \quad a_{2n} := 4 \int_1^\infty x^{-1/4} \frac{\left(\frac{1}{2} \log x\right)^{2n}}{(2n)!} \frac{d}{dx} \left[ x^{3/2} \psi'(x) \right] dx.$$

Riemann stated that this series representation of  $\xi(s)$  as an even function of  $s - \frac{1}{2}$  "converges very rapidly", i.e.  $\xi(s)$  is like a polynomial of infinite degree. Hadamard proved that the rapid decrease of the coefficients  $a_{2n}$  is necessary and sufficient to the validity of Riemann's infinite product formula of  $\xi(s)$ .

(+) The considered Kummer functions are related 1ort h Gaussian function by the following formulas, (Gr1) 3.952,  $(\sin(2xy)) = \sqrt{\pi xy} J_{\frac{1}{2}}(2xy)$

$$x e^{-x^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) = x {}_1F_1\left(1, \frac{3}{2}; -x^2\right) = \int_0^\infty e^{-y^2} \sin(2xy) dy, \quad \sqrt{\pi x} e^{-x^2} {}_1F_1\left(1, \frac{3}{2}; x^2\right) = \sqrt{\pi x} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) = \int_0^\infty e^{-y^2} \sin(2xy) \frac{dy}{y}$$

The functions are the two eigenfunctions 1ort h Differential Whittaker operator  $A[w](z) := zw''(z) + \left(\frac{3}{2} - z\right)w'(z)$  with eigenvalues  $1/2$  and  $1$ , (Gr1) 9.220.

(\*) An equivalent formulation is given by (TiE) (2.1.9)

$$\zeta(s) = \chi(s) \zeta(1-s), \quad \text{where} \quad \chi(s) := \frac{\frac{s}{2} \Gamma\left(\frac{1-s}{2}\right)}{\frac{1-s}{2} \Gamma\left(\frac{s}{2}\right)} = \frac{1}{\chi(1-s)} \quad \text{and} \quad \chi(1-s) = O(|t|^{\sigma-1/2}).$$

(\*\*) (EdH) 1.7: „Because of  $\psi(x)$  decreases more rapidly than any power of  $x$  as  $x \rightarrow \infty$ , the integral in this formula converges for all  $s$ . This gives, therefore, another formula for  $\zeta(s)$  which is „valid for all  $s$ “ other than  $s = 0, 1$ .“

(\*\*\*) resp., as Riemann writes, (EdH) 1.8:  $\Xi(t) := \xi\left(\frac{1}{2} + it\right) = 4 \int_1^\infty x^{-1/4} \frac{d}{dx} \left[ x^{3/2} \psi'(x) \right] \cos\left(\frac{1}{2} \log x\right) dx$

(\*\*\*\*)  $\frac{1}{2} [x^s + x^{1-s}] = \sqrt{x} \left[ \frac{1}{2} (x^{s-\frac{1}{2}} + x^{-(s-\frac{1}{2})}) \right] = \sqrt{x} \cosh \left[ \left(s - \frac{1}{2}\right) \log x \right] = \sqrt{x} \sum_{n=0}^\infty \frac{(\log x)^{2n}}{(2n)!} \left(s - \frac{1}{2}\right)^{2n}$

The Hilbert-Polya conjecture is about the existence of a convolution integral representation of  $\xi(s)$ , including an appropriately defined domain (CaD). There is an only formally valid representation, (EdH) 10.3,

$$\int_0^\infty x^{1-s} G(x) \frac{dx}{x} = \frac{2\xi(s)}{s(s-1)},$$

as the integral  $\int_0^\infty x^{-s} G(x) dx$  does not converge for any  $s$ . This is because of  $G(x) = 1 + \psi(x^2) = \frac{1}{x} \left[ 1 + \psi\left(\frac{1}{x^2}\right) \right]$ , i.e. mathematically speaking, because the constant Fourier term of the Gaussian function  $e^{-a\pi x^2}$  does not vanish.

Alternatively to  $g(x) := e^{-x}$  we consider the Kummer function based function  $g^*(x)$  defined by

$$g^*(x) := \int_x^\infty {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -t\right) \frac{dt}{t} \text{ with } -xg^{*'}(x) := {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)$$

leading to a corresponding replacement of  $\psi(x^2)$  by

$$\psi^*(x) := \frac{\sqrt{\pi}}{2} \sum_{n=1}^\infty g^*(n^2 x) = \frac{\sqrt{\pi}}{2} \sum_{n=1}^\infty \int_x^\infty {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -n^2 t\right) \frac{dt}{t}.$$

The corresponding Mellin transform of  $g^*(\pi n^2 x)$  is given by (see Lemma 2 below)

$$M[g^*(\pi n^2 x)]\left(\frac{s}{2}\right) = \frac{1}{n^s} \pi^{-\frac{s}{2}} \frac{2\Gamma\left(\frac{s}{2}\right)}{s(1-s)}.$$

From (SIL) (3.2.26) we recall the formulae ( $Re(z) > 0$ )

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right) = \frac{\sqrt{\pi}}{2} e^z z^{1/4} \int_0^\infty e^{-t} t^{-1/4} J_{1/2}(2\sqrt{zt}) dt \text{ resp. } {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) = e^{x^2} \int_0^\infty e^{-y^2} \sin(2xy) dy.$$

With respect to the related Poisson formula we note that the Fourier transform of the real, odd function  $e^{x^2} \sin(ax) = e^{x^2} \frac{1}{2i} [e^{aix} - e^{-aix}]$  is imaginary and odd. For the corresponding Fourier terms it holds  $a_0 = a_n = 0$ , i.e. the constant Fourier term vanishes (in opposite to the „trouble maker“ constant term of the Gaussian function).

Putting  $\xi^*(s) := \pi^{\frac{1-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \xi^*(1-s)$  one gets the

**Theorem:** In the critical stripe it holds

$$\int_0^\infty x^{s/2} \psi^*(x) \frac{dx}{x} = \frac{\xi^*(s)}{s(1-s)} = \frac{\xi^*(1-s)}{s(1-s)} = \int_0^\infty x^{(1-s)/2} \psi^*(x) \frac{dx}{x}$$

resp.

$$\int_0^\infty x^{\frac{s}{2}} \psi^*(x) \frac{dx}{x} = \frac{1}{2} \left[ \int_0^\infty x^{\frac{s}{2}} \psi^*(x) \frac{dx}{x} + \int_0^\infty x^{\frac{1-s}{2}} \psi^*(x) \frac{dx}{x} \right].$$

**Proof:** From the lemma above one gets

$$\int_0^\infty x^{s/2} \psi^*(x) \frac{dx}{x} = \frac{\sqrt{\pi}}{2} \int_0^\infty x^{s/2} \sum_{n=1}^\infty g^*(n^2 x) \frac{dx}{x} = \frac{\sqrt{\pi}}{2} \pi^{-\frac{s}{2}} \frac{2\Gamma\left(\frac{s}{2}\right)}{s(1-s)} \sum_{n=1}^\infty \frac{1}{n^s} = \pi^{\frac{1-s}{2}} \frac{\Gamma\left(\frac{s}{2}\right)}{s(1-s)} \zeta(s).$$

Applying the duality equation, (TiE) (2.1.9),  $\pi^{\frac{1-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{\frac{s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)$  leads to

$$\int_0^\infty x^{s/2} \psi^*(x) \frac{dx}{x} = \frac{1}{\sqrt{\pi}} \frac{\pi^{\frac{1-s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)}{s(1-s)} = \frac{1}{\sqrt{\pi}} \frac{\pi^{\frac{s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s)}{s(1-s)} = \int_0^\infty x^{(1-s)/2} \psi^*(x) \frac{dx}{x}.$$

The above theorem in combination with the formula

$$\frac{1}{2} \left[ x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \right] = x^{1/4} \left[ \frac{1}{2} \left( x^{-\frac{1}{2}(\frac{1-s}{2})} + x^{\frac{1}{2}(\frac{1-s}{2})} \right) \right] = x^{1/4} \cosh \left[ \frac{1}{2} \left( \frac{1}{2} - s \right) \log x \right] = x^{1/4} \sum_{n=0}^\infty \frac{\left(\frac{1}{2} \log x\right)^{2n}}{(2n)!} \left(\frac{1}{2} - s\right)^{2n}$$

leads to the

**Corollary:** In the critical stripe it holds

$$\frac{\xi^*(s)}{s(1-s)} = \int_0^\infty x^{1/4} \cosh \left[ \frac{1}{2} \left( \frac{1}{2} - s \right) \log x \right] \psi^*(x) \frac{dx}{x}$$

resp.

$$\frac{\xi^*(s)}{s(1-s)} = \sum_{n=0}^\infty a_{2n}^* \left(\frac{1}{2} - s\right)^{2n} \text{ where } a_{2n}^* := \int_0^\infty x^{1/4} \frac{\left(\frac{1}{2} \log x\right)^{2n}}{(2n)!} \psi^*(x) \frac{dx}{x}.$$

**Lemma 1:**

$$\int_0^\infty x^{s-1} {}_1F_1(a, c, -x) dx = \frac{\Gamma(c)}{\Gamma(a)} \Gamma(s) \frac{\Gamma(a-s)}{\Gamma(c-s)}, \quad 0 < \operatorname{Re}(s) < \operatorname{Re}(a).$$

**Proof:** (GrI) 7.612

**Lemma 2:** In the critical stripe it holds

$$M[g^*(\pi n^2 x)]\left(\frac{s}{2}\right) = \frac{1}{n^s} \pi^{-\frac{s}{2}} \frac{2\Gamma(\frac{s}{2})}{s(1-s)}.$$

**Proof:** From Lemma 1 it follows

$$M\left[{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)\right]\left(\frac{s}{2}\right) = \frac{\Gamma(\frac{s}{2})}{s(1-s)}, \quad 0 < \operatorname{Re}(s) < 1.$$

The Mellin transform of  $g^*(x)$  follows from the general rule  $M[h](s) = \frac{1}{s} M[-xh'(x)](s)$ , i.e.

$$M[g^*(x)]\left(\frac{s}{2}\right) = \frac{2}{s} M[-xg^{*'}(x)]\left(\frac{s}{2}\right) = \frac{2}{s} M\left[{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)\right]\left(\frac{s}{2}\right) = \frac{2\Gamma(\frac{s}{2})}{s(1-s)}.$$

Substituting the integration variable results into

$$M[g^*(\pi n^2 x)]\left(\frac{s}{2}\right) = \frac{2}{s} M[-xg^{*'}(x)]\left(\frac{s}{2}\right) = \frac{2}{s} M\left[{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -\pi n^2 x\right)\right]\left(\frac{s}{2}\right) = \frac{1}{n^s} \pi^{-\frac{s}{2}} \frac{2\Gamma(\frac{s}{2})}{s(1-s)}.$$

Using the series  $-\log(1-x) = \sum_{n=1}^\infty \frac{1}{n} x^n$  puts the Euler product formula in the form

$$\log \zeta(s) = \sum_p \sum_n \frac{1}{n} p^{-ns} = \sum_p \sum_n \frac{1}{n} (p^n)^{-s}.$$

In combination with Riemann's density function  $J(x) := \frac{1}{2} \left[ \sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n} \right]$  this results into the Stieltjes integral representation

$$\log \zeta(s) = \int_0^\infty x^{-s} dJ(x) = s \int_0^\infty x^{-s} J(x) \frac{dx}{x}, \quad (\operatorname{Re}(s) > 1),$$

providing the Riemann density function representation, (EdH) 1.13,

$$J(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log \zeta(a+iy)}{a+iy} x^{a+iy} dy, \quad a > 1$$

and the related Riemann error function (derived from the formula,  $\log \zeta(s) = \log \xi(s) - \log \left[ \Gamma(1 + \frac{s}{2})(s-1)\pi^{-s/2} \right]$ )

$$\int_x^\infty \frac{dt}{t(t^2-1) \log t} = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log \Gamma(1 + \frac{s}{2})}{s} \right] x^s ds \cong -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \Gamma(1 + \frac{s}{2}) x^s \frac{ds}{s}.$$

The following formula

$$\int_0^\infty x^n \left[ \sum_{k=0}^\infty \varphi(k) \frac{(-x)^k}{k!} \right] \frac{dx}{x} = \Gamma(n) \varphi(-n) \quad n > 0,$$

from Ramanujan's notebook, „of which he was especially fond and made conditional use“, (BeB) chapter 4, leads to

$$\frac{1}{2} M[g^*(x)](n) = \frac{1}{2} M \left[ \int_x^\infty {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -t\right) \frac{dt}{t} \right](n) \sim \int_0^\infty x^n \left[ \sum_{k=1}^\infty \frac{1}{2k} \frac{1}{2k+1} \frac{(-x)^k}{k!} \right] \frac{dx}{x} = \frac{\Gamma(n)}{2n(2n-1)}.$$

The series representation of

$$g^*(x) = -\vartheta(x) = \int_x^\infty {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -t\right) \frac{dt}{t} \sim \sum_{k=1}^\infty \frac{1}{2k} \frac{1}{2k+1} \frac{(-x)^k}{k!}$$

may be used alternatively to  $\sum_{n=1}^\infty \frac{1}{n} x^n$  to consider the Stieltjes integral representation

$$\log \zeta(s) = \sum_p \sum_n \frac{1}{n} \frac{1}{n+1/2} p^{-ns} = \int_0^\infty x^{-s} dJ^*(x) dx.$$

When applying the Fourier theorem to conclude

$$J^*(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \zeta(s) x^s \frac{ds}{s} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\log \zeta(a+iy)}{a+iy} x^{\frac{1}{2}+iy} dy$$

it is now possible to put  $a = 1/2$ .

The relationship between the two densities  $J(x)$  and  $J^*(x)$  is accompanied by the formula

$$\vartheta(x) - \mu(x) = {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)$$

and the corresponding Mellin transform formulas.

The class of sine-type functions  $S_0$  was introduced in calculus by Levin, (LeB1), playing an important role in non-harmonic analysis with specific interests on those sine-type functions, which were not Fourier-Stieltjes transforms. The link to the considered Kummer functions is provided by the generalized classes  $S_{\alpha}$ , (SeA). In case  $\frac{\Gamma(a)}{\Gamma(c-a)}$  is defined the function

$$F(z) := e^{-i\pi z} {}_1F_1(a, c; 2\pi iz)$$

belongs to the class  $S_{Re(a)}$ . The special case  $Re(c) = 1 + Re(a)$  provides additional interesting properties, e.g. for  $Re(s) > -1, \beta > 0$  it holds (Gri) 3.761, lemma 4 and p. 15 below,

$$\int_0^1 x^s e^{2\pi i \beta x} \frac{dx}{x} = \int_1^{\infty} y^{-s} e^{2\pi i \frac{\beta}{y}} \frac{dy}{y} = \frac{1}{s} {}_1F_1(s, s+1; 2\pi i \beta) = \frac{e^{-1}}{2\pi i \beta} \left\{ \sum_{k=0}^{\infty} (-1)^k (1-s)_k (2\pi i \beta)^{-k} + O(|2\pi i \beta|^{-n-1}) \right\}.$$

With respect to the Beurling RH criterion, (BeA1), we note that the Mellin transforms of the  $\{e^{2\pi i \beta x}\}$  system is given by, (Gri) 3.761

$$\int_0^{\infty} x^{s-1} e^{2\pi i \beta x} dx = \int_0^1 x^{s-1} e^{2\pi i \beta x} dx + \int_1^{\infty} x^{s-1} e^{2\pi i \beta x} dx = \int_0^{\infty} y^{-s} e^{2\pi i \frac{\beta}{y}} \frac{dy}{y} = \frac{\Gamma(s)}{(2\pi \beta)^s} e^{i\frac{\pi}{2}s}.$$

The proposed Kummer functions

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) = \sum_{k=0}^{\infty} \frac{1}{2k+1} \frac{x^k}{k!} \quad (*) \quad , \quad {}_1F_1\left(1, \frac{3}{2}; x\right) = e^x {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)$$

with

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) \sim \frac{1}{2} \frac{e^x}{x} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(\frac{1}{x}\right)^k \quad \text{resp.} \quad {}_1F_1\left(1, \frac{3}{2}; \log x\right) \sim \frac{1}{2} \frac{x}{\log x} \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)_k \left(\frac{1}{\log x}\right)^k$$

are linked to the  $erf$  – function and the Dawson function  $F(x) := e^{-x^2} \int_0^x e^{t^2} dt = \sum_{k=0}^{\infty} (-1)^k \frac{2^k x^{2k+1}}{1 \cdot 3 \cdot \dots \cdot (2k+1)}$  by the following formulas

$$\begin{aligned} erf(x) &= x {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) = x e^{-x^2} {}_1F_1\left(1, \frac{3}{2}; x^2\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k! (2k+1)} \\ F(x) &= x {}_1F_1\left(1, \frac{3}{2}; -x^2\right) = x e^{-x^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) = \int_0^{\infty} e^{-t^2} \sin(2xt) dt. \end{aligned}$$

The Mellin transforms and asymptotics of the concerned Kummer functions are given by (lemma 1 & 4)

**Lemma 3:**

$$i) \quad \int_0^{\infty} x^{\frac{s}{2}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x} = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s}, \quad 0 < Re(s) < 1,$$

$$ii) \quad \int_0^{\infty} x^{\frac{s}{2}} {}_1F_1\left(1, \frac{3}{2}; -x\right) \frac{dx}{x} = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \tan\left(\frac{\pi}{2}s\right), \quad 0 < Re(s) < 1,$$

$$\text{because of } \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \frac{\Gamma\left(\frac{1+s}{2}\right) \cos\left(\frac{\pi}{2}s\right)}{\Gamma\left(\frac{1+s}{2}\right) \sin\left(\frac{\pi}{2}s\right)} = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \frac{\Gamma\left(\frac{s}{2}\right) \sin\left(\frac{\pi}{2}s\right)}{\Gamma\left(\frac{1+s}{2}\right) \cos\left(\frac{\pi}{2}s\right)} = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \tan\left(\frac{\pi}{2}s\right)$$

$$iii) \quad {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) = e^{x^2} \int_0^{\infty} e^{-y^2} \sin(2xy) dy \quad (\text{SIL}) \quad (3.2.26)$$

$$iv) \quad {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) = -{}_1F_1'\left(-\frac{1}{2}, \frac{1}{2}; x\right) \sim \frac{e^x}{x} \sum_{k=0}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)_k x^{-k} = \frac{e^x}{x} \left[ \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)_k \left(\frac{1}{x}\right)^k \right], \quad (\text{SIL}) \quad 4.2.$$

**Note:** Lemm 3 grants an alternative approximation function to the „number of primes less than  $x$  function“  $\pi(x)$  ( $N \geq 1$ ),

$$\pi(x) \sim \frac{x}{\log x} \sim \int_2^x \frac{dt}{\log t} \sim {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log x\right) \sim \frac{x}{\log x} \left[ \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{2} \left(\frac{1}{2}\right)_k \left(\frac{1}{\log x}\right)^k \right] \sim \frac{x}{\log x} \left[ \frac{1}{2} + \sum_{k=1}^N \frac{1}{2} \left(\frac{1}{2}\right)_k \left(\frac{1}{\log x}\right)^k \right].$$

From the general rules  $M[(xh)'(x)](s) = (1-s)M[h(x)](s)$  and  $M[-xh'(x)](s) = sM[h](s)$  it follows

$$M\left[\frac{d}{dx}(-x {}_1F_1\left(1, \frac{3}{2}; -x\right))\right]\left(\frac{s}{2}\right) = \frac{1}{2}(s-1)\Gamma\left(\frac{s}{2}\right) \tan\left(\frac{\pi}{2}s\right)$$

resp.

$$4M\left[\frac{d}{dx}(-x^2 {}_1F_1\left(1, \frac{3}{2}; -\pi x^2\right))\right](s) = \pi^{-s/2}(s-1)\Gamma\left(\frac{s}{2}\right) \frac{1}{\pi} \tan\left(\frac{\pi}{2}s\right).$$

The asymptotics  $\tan\left(\frac{\pi}{2}s\right) \sim \frac{\pi}{2}s$  provides the link to the entire Zeta function  $\xi(s) := \frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\pi^{-s/2}\zeta(s)$ , while the entire function  $\xi^{**}(s) := \pi^{(1-s)/2}(s-1)\zeta(s)\Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right)$  is governed by the same set of critical zeros than  $\xi(s)$ .

The Riemann density function is given by

$$J(x) = Li(x) - \sum_{Im(\rho)>0} [Li(x^\rho) + Li(x^{1-\rho})] + \sum_{n=1}^{\infty} \int_x^{\infty} \frac{t^{-2n}}{\log t} dt + \log \frac{1}{2},$$

where

$$\sum_{Im(\rho)>0} Li(x^\rho) + Li(x^{1-\rho}) = \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right), \sum_{n=1}^{\infty} \int_x^{\infty} \frac{t^{-2n}}{\log t} dt = \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}, \text{ and } \log \frac{1}{2} = \log \xi(0).$$

By differentiation one finds, (EdH) 1.18,  $dJ = \left[ \left(1 - \frac{1}{\sqrt{x}} \sum_{Re(\alpha)>0} 2\cos(\alpha \cdot \log x)\right) \frac{dx}{\log x} - \frac{1}{x(x^2-1)} \frac{dx}{\log x} \right]$ , resp. in a distributional sense, (LaG)  $\langle \psi \rangle' = \langle 1 \rangle' - \frac{1}{\sqrt{x}} \sum_{Re(\alpha)>0} (2\cos(\alpha \cdot \log x)) - \left\langle \frac{1}{x(x^2-1)} \right\rangle$ , where  $\alpha$  ranges over all values such that  $\rho = \frac{1}{2} + i\alpha$ , resp.  $\alpha = -i(\rho - \frac{1}{2})$  where  $\rho$  ranges over the roots, so that

$$x^{\rho-1} + x^{-\rho} = x^{-1/2} [x^{i\alpha} + x^{-i\alpha}] = 2x^{-1/2} \cos(\alpha \cdot \log x).$$

Riemann's suggested (best known) approximation to  $\pi(x)$  (denoting the number of primes less than  $x$ ) is given by

$$\pi(x) \sim Li(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n} Li(x^{1/n}).$$

Regarding the considered Kummer function we note the possible split (see lemma 4 below)

$$Li(x) = Li^{(1)}(x) + Li^{(2)}(x) := \int_{-\infty}^{\log x} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; t\right) \frac{dt}{t} + {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log x\right).$$

With respect to the above proposed „circular“ adequate domain of the zeta function the correspondingly modified von Mangoldt formula  $\left(\frac{\zeta'(s)}{\zeta(s)}\right) = \log(2\pi)$

$$\psi_0^*(x) := x - \left[ \sum_{Im(\rho)>0} \frac{x^\rho}{\rho} + \sum_{n \geq 1} \frac{x^{-2n}}{-2n} \right] - \log(2\pi)$$

enables estimates in the form

$$|J^*(x) - Li(x) - \log 2| \leq \left| \sum_{Im(\rho)>0} Li^{(1)}(x^\rho) + \sum_{n=1}^{\infty} Li^{(2)}(x^{-2n}) \right|.$$

The Chebyshev point measure  $d\psi$  assigns the weight  $\frac{1}{n} \log(p^n)$  to prime powers  $p^n$ , and the weight 0 to all other points, i.e.  $\psi(x) = \sum_{p^n < x} \log p$ . The corresponding Riemann point measure is given by

$$J(x) = \frac{1}{2} \left[ \sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n} \right].$$

The counterparts to the split  $Li(x) = Li^{(1)}(x) + Li^{(2)}(x)$  can be derived by the split (see lemma 4 below)

$$p^n = {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; n \log p\right) + \frac{2}{3} n \log p \cdot {}_1F_1\left(\frac{3}{2}, \frac{5}{2}; n \log p\right).$$

Alternatively to the divergent series  $\sum_p \frac{1}{p}$  (e.g. (InA)), it is proposed to replace the series by  $\sum_p {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log \frac{1}{p}\right)$ . For the related asymptotics of  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -\log x\right)$  we refer to (SeA).

Putting  $T(x) := \sum_{n=1}^{\infty} \psi\left(\frac{x}{n}\right)$  we note the formula for the Chebyshev density function, (PrK) p. 231, (EdH),

$$\psi_0(x) = \frac{1}{2}[\psi(x+0) + \psi(x-0)] = x - \left[ \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n \geq 1} \frac{x^{-2n}}{-2n} \right] - \frac{\zeta'(0)}{\zeta(0)} = \sum_{n=1}^{\infty} \mu(n) T\left(\frac{x}{n}\right), \quad x \geq 2,$$

i.e.  $\psi_0(x)$  is built from the trivial and non-trivial zeros of the Zeta function.

**Remark:** If the RH is true, then  $Im(\rho) \leq \frac{1}{2}$ , resp.  $\left| \frac{x^{\rho}}{\rho} \right| \leq \frac{x^{1/2}}{\rho}$ , and therefore  $|\pi(x) - Li(x)| \leq 3\sqrt{x} \log x$ .

The Riemann error term  $\int_x^{\infty} \frac{dt}{t(1-t^2) \log t}$ , reflects the trivial zeros of the Zeta function; it is derived from the formula, (EdH) 1.16,

$$\Gamma\left(1 + \frac{s}{2}\right) = \prod \left(1 - \frac{s}{-2n}\right)^{-1} \left(1 + \frac{1}{n}\right)^{s/2} \quad \text{resp.} \quad \frac{1}{\Gamma\left(1 + \frac{s}{2}\right)} = \prod \left(1 + \frac{s}{2n}\right) \left(1 + \frac{1}{n}\right)^{-s/2} = e^{\gamma s/2} \prod \left(1 + \frac{s}{2n}\right) e^{-s/(2n)}.$$

With lemma 4 below this leads to the following alternative representations

$$\frac{{}_1F_1\left(\frac{13}{2}; s\right)}{\Gamma\left(1 + \frac{s}{2}\right)} = e^{s\left(\frac{\gamma}{2} + \frac{1}{3}\right)} \prod \left(1 + \frac{s}{2n}\right) \left(1 - \frac{s}{\alpha_n}\right) e^{s\left(\frac{1}{\alpha_n} - \frac{1}{2n}\right)}$$

$$\Gamma\left(1 + \frac{s}{2}\right) = {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \frac{s}{2}\right) e^{-\frac{s}{2}\left(\gamma + \frac{1}{3}\right)} \prod \left[\left(1 - \frac{s}{(-2n)}\right) \left(1 - \frac{s}{2\alpha_n}\right)\right]^{-1} e^{-\frac{s}{2}\left(\frac{1}{\alpha_n} - \frac{1}{n}\right)}, \quad \text{Re}(2\alpha_n) > 1.$$

**Remark:** With respect to the below proposed replacement of  $\xi(s) := \frac{s}{2} \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s) \rightarrow \xi^{**}(s)$  resp. the proposed „circular model“, we note that  $-\xi(s)$  is bounded by  $O(e^{|s|^a})$  for any  $a > 1$ , as there exists a constant  $c > 0$  such that for every  $s \in \mathbb{C}$

$$\left| s \Gamma\left(\frac{s}{2}\right) (1-s) \pi^{-s/2} \zeta(s) \right| < c e^{\frac{1}{2}|s-\frac{1}{2}| \log^+ \left(|s-\frac{1}{2}|\right)}, \quad \log^+(x) := \max\{0, \log x\},$$

and is therefore a holomorphic integral function of order 1, (PaS) chapter 3.

For the considered Kummer function we summarize the following formulae / properties

**Lemma 4:**

- i) The infinite set of zeros  $\alpha_n$  of  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; s\right)$  has only imaginary zeros  $\alpha_n$  fulfilling, (SeA),

$$n - 1 < \frac{|\operatorname{Im}(\alpha_n)| - \pi}{2\pi} < n - \frac{1}{2} < \frac{|\operatorname{Im}(\alpha_n)|}{2\pi} < n, \operatorname{Re}(\alpha_n) > 1/2.$$

With respect to the below we mention that the mapping  $z = 1 - \frac{1}{s} \leftrightarrow s = \frac{1}{1-z}$  takes the right half plane  $\operatorname{Re}(s) > 1/2$  to the interior of the unit circle in the complex  $z$ -plane., i.e. all zeros of  ${}_1F_1\left(\frac{1}{2}; \frac{3}{2}; z\right)$  lie in the interior of the unit circle. With respect to CaD), (CaD1) and the underlying Polya theorem we note that  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; is\right)$  has only real zeros.

ii)  $M\left[{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)\right]\left(\frac{s}{2}\right) = \frac{\Gamma\left(\frac{s}{2}\right)}{(1-s)}, M\left[-x {}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; -x\right)\right]\left(\frac{s}{2}\right) = \frac{\Gamma\left(1+\frac{s}{2}\right)}{(1-s)}, 0 < \operatorname{Re}(s) < 1, \text{ (Grl) 7.612}$

i)  $\int_0^\infty {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) \cos(2xy) dx = \frac{1}{4} \int_0^\infty e^{-(1+t)y^2} \frac{dt}{1+t} = \frac{1}{4} \int_1^\infty e^{-xy^2} \frac{dx}{x}, \text{ (Grl) 7.642, 9.210}$

ii)  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; s\right) = e^{\frac{s}{3}} \prod\left(1 - \frac{s}{\alpha_n}\right) e^{\frac{s}{\alpha_n}}$  with  $\sum_{n=1}^\infty \frac{1}{\alpha_n^{1+\varepsilon}} < \infty$  for  $\varepsilon > 0$ , (BuH) p.184

iii)  $\int_0^\infty e^{-sx} \left[x^{1/4} e^{-x/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right)\right]^2 dx = \int_0^\infty e^{-sx} M_{\frac{1}{4}}^2(x) \frac{dx}{x} = \frac{\sqrt{\pi}}{2} \frac{1}{\sqrt{1+s}} \arcsin\left(\frac{1}{s}\right), \operatorname{Re}(s) > 1,$

where  $M_{\frac{1}{4}}^2(z) = \left(\frac{z}{2}\right)^{3/2} \int_{-1}^1 (1+t)^{-1/2} e^{\left(\frac{z}{2}\right)t} dt, \text{ (Grl) 7.622, 9.121, 9.220, 9.221}$

iv)  $F_n(s) := \frac{{}_1F_1\left(\frac{1+\frac{3}{2}s}{2}, s\right)}{{}_1F_1\left(\frac{1+\frac{3}{2}\alpha_n}{2}, \alpha_n\right)(s-\alpha_n)} = \frac{2\alpha_n}{s-\alpha_n} e^{-\alpha_n} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; s\right) = \frac{2\alpha_n}{s-\alpha_n} e^{-\alpha_n} e^{\frac{s}{3}} \prod\left(1 - \frac{s}{\alpha_k}\right) e^{\frac{s}{\alpha_k}}$

i.e.  $F_n(s) = \frac{{}_1F_1\left(\frac{1+\frac{3}{2}s}{2}, s\right)}{{}_1F_1\left(\frac{1+\frac{3}{2}\alpha_n}{2}, \alpha_n\right)(s-\alpha_n)} = -2e^{-\frac{2}{3}s} \prod\left(\frac{\alpha_k - s}{\alpha_n - s}\right) e^{\frac{s}{\alpha_k}} e^{-(\alpha_n - s)}.$

v)  ${}_1F_1(a, c, z) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{\pm a\pi i} z^{-a} \left\{ \sum_{k=0}^n \frac{(-1)^k (a)_k (1+a-c)_k}{k!} z^{-k} + O(|z|^{-n-1}) \right\}.$   
 $+ \frac{\Gamma(c)}{\Gamma(a)} e^{-(c-a)} z^{-(c-a)} \left\{ \sum_{k=0}^n \frac{(-1)^k (1-a)_k (c-a)_k}{k!} z^{-k} + O(|z|^{-n-1}) \right\} \text{ (LeN) 9.12}$

vi)  ${}_1F_1(a, a+1; x) \sim a \frac{e^x}{x} \sum_{k=0}^\infty \left(\frac{1}{2}\right)_k x^{-k}, \text{ (SIL) 4.2}$

vii)  $F_a(z) := {}_1F_1(a, a+1, z) \sim \frac{1}{\Gamma(a)} \frac{e^z}{z}, z \rightarrow \infty, ph(z) = 0, \text{ (OIF) p.257}$

viii)  $F_a^{(n)}(z) = \frac{a}{a+n} F_{a+n}(z), \frac{1}{2} \frac{d}{dx} F_a^2(x) = \frac{a}{a+1} F_a(x) \cdot F_{a+1}(x) \sim \frac{1}{a+1} \frac{1}{\Gamma^2(a)} \left(\frac{e^x}{x}\right)^2.$

With respect to lemma 4i) above we note that for the zeros  $z_n$  of the function  $\Psi(x)$  it holds, (Gil)

$$n - \frac{1}{2} < -z_n < n.$$

The periodical Hilbert space  $L_2(-\pi, \pi)$  resp. the corresponding generalized Hilbert scale framework  $H_\alpha^\# \cong l_\alpha^\alpha, \alpha \in R$ , is built on the  $2\pi$ -periodic Hilbert space  $L_2^\#(\Gamma)$  with  $\Gamma := S^1(R^2)$ , i.e.  $\Gamma$  is the boundary of the unit circle sphere. For  $u \in L_2^\#(\Gamma)$  and for real  $\beta \in R, n \in Z$  the Fourier coefficients  $u_n := \frac{1}{2\pi} \oint u(x) e^{inx} dx$  enable the definition of the norms  $\|u\|_\beta^2 := \sum_{n=-\infty}^\infty |n|^{2\beta} |u_n|^2.$

The relationship between the Dirichlet series

$$f(s) := \sum_1^\infty a_n e^{-s \log n} \quad g(s) := \sum_1^\infty b_n e^{-s \log n}$$

and the Hilbert space  $H_{-1/2}^\# \cong L_2^{-1/2}$  on the critical line is given by ([LaE] §227, Satz 40):

$$((f, g))_{-1/2} := \lim_{\omega \rightarrow \infty} \frac{1}{2\omega} \int_{-\omega}^\omega f\left(\frac{1}{2} + it\right) g\left(\frac{1}{2} - it\right) dt = \sum_1^\infty \frac{1}{n} a_n b_n.$$

The cardinal series theory is an extension of the Dirichlet series theory. Property vi) puts the spot on non-harmonic Fourier series theory and the related „generalized“ trigonometric moment problem in  $L_2(-\pi, \pi)$  given by, (YoR) p.124,

$$(*) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) e^{i\lambda_n t} dt = c_n.$$

In case  $\{e^{i\lambda_n t}\}$  is a Riesz basis its moment space is  $l^2$  and (\*) admits a solution whenever  $c_n$  is square-summable, and for these sequences only. If  $\{g_n\}$  is the unique sequence in  $L_2(-\pi, \pi)$  biorthogonal to  $\{e^{i\lambda_n t}\}$ , then the unique solution to (\*) is given by the norm-convergent series  $\varphi(t) = \sum_{n=-\infty}^{\infty} c_n g_n(t)$ .

The replacement of  $\xi(s) \rightarrow \xi^{**}(s)$  with

$$\xi^{**}(s) := \pi^{\frac{1-s}{2}} (s-1) \zeta(s) \Gamma\left(\frac{s}{2}\right) \tan\left(\frac{\pi}{2}s\right) = \xi^*(s)(s-1) \tan\left(\frac{\pi}{2}s\right)$$

where

$$\xi^{**}(s) \frac{\tan\left(\frac{\pi}{2}(1-s)\right)}{1-s} = \xi^{**}(1-s) \frac{\tan\left(\frac{\pi}{2}s\right)}{s}$$

leads to a corresponding replacement of the Riemann error function in the following form

$$\int_x^{\infty} \frac{dt}{t(1-t^2) \log t} = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \Gamma\left(1 + \frac{s}{2}\right) x^s \frac{ds}{s} \rightarrow \frac{1}{2\pi i} \left[ \int_{1/2-i\infty}^{1/2+i\infty} \log \Gamma\left(\frac{s}{2}\right) + \log \left(\tan\left(\frac{\pi}{2}s\right)\right) \right] x^s \frac{ds}{s}.$$

Putting  $2h_n := \sum_{k=1}^n \frac{2}{2k-1} = 2H_{2n} - H_n$  resp.  $c_n := \frac{2h_n}{n}$ , where  $H_n = \sum_{k=1}^n \frac{1}{k}$  denote the harmonic numbers, a Fourier series based analysis is supported by the following two lemma 5 and lemma 6:

**Lemma 5:**

- i)  $\int_0^1 \log \left(\tan\left(\frac{\pi}{2}x\right)\right) \cos(k\pi x) dx = \begin{cases} -1/k & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$
- ii)  $\frac{\pi}{2} \log \left(\tan\left(\frac{\pi}{2}x\right)\right) = -\sum \frac{2h_n}{n} \sin(2\pi n x) = -\sum c_n \sin(2\pi n x)$
- iii)  $\frac{\pi}{2} \log \left(\tan\left(\frac{\pi}{2}x\right)\right) \in L_2^\#(0,1)$ , and therefore, because of  $\sum_{n=1}^{\infty} c_n^2 < \infty$ ,  
 $\log' \left(\tan\left(\frac{\pi}{2}x\right)\right) = \frac{\pi}{\sin(\pi x)} \in H_{-1}^\#(0,1)$
- iv)  $\log \left(\cot\left(\frac{\pi}{2}x\right)\right) = 2 \sum \frac{\cos((2k-1)\pi x)}{(2k-1)}$ ,  $\log \left(\tan\left(\frac{\pi}{2}x\right)\right) = -2 \sum \frac{\cos((2k+1)\pi x)}{(2k+1)}$ ,  $0 < x < 1$ .

**Proof:** for i)-iii) see (EIL); for iv) see (Grl) 1.442.

Following the notations of (KuE) we put

$$A_0 := \frac{1}{2} \log(2\pi) \quad , \quad A_k := \frac{1}{4k},$$

$$B_1 := \frac{1}{2\pi} [\gamma + \log(2\pi)] \quad , \quad B_k := B_1 + \frac{1}{2\pi} \frac{\log(k)}{k}.$$

**Lemma 6:**

- i) For  $0 < x < 1$  it holds  $\log \Gamma(x) = A_0 + 2 \sum_{k=1}^{\infty} A_k \cos(2\pi k x) + 2 \sum_{k=1}^{\infty} B_k \sin(2\pi k x)$
- ii)  $\sum_{k=1}^n \log \Gamma\left(x + \frac{k-1}{n}\right) - \log \Gamma(nx) = \frac{1}{2}(n-1) \log(2\pi) + \frac{1}{2}(1-2nx) \log(n)$

**Proof:** see (KuE) resp. (BeB) 8, Entry 17 iv), resp. (WhE) XII and (Grl) 8.335.



**Corollary:** For  $0 < x < 1$  it holds

$$\log \Gamma(x) + \frac{\pi}{2} \log \left( \tan \left( \frac{\pi}{2} x \right) \right) = A_0 + 2 \sum_{k=1}^{\infty} A_k \cos(2\pi kx) + 2 \sum_{k=1}^{\infty} \left[ B_k - \frac{2h_k}{k} \right] \sin(2\pi kx),$$

with

$$B_1 - \frac{2h_1}{1} = \frac{1}{2\pi} [\gamma + \log(2\pi) - 4\pi], \quad B_k - \frac{2h_k}{k} = B_1 + \frac{1}{2\pi} \left[ \frac{\log k - \left[ \sum_{k=1}^n \frac{4\pi}{2k-1} \right]}{k} \right],$$

resp.

$$\log \Gamma \left( \frac{x}{2} \right) + \frac{\pi}{2} \log \left( \tan \left( \frac{\pi}{2} x \right) \right) = A_0 + \sum_{k=1}^{\infty} 2A_k \cos(\pi kx) + \sum_{k=1}^{\infty} 2B_k \sin(\pi kx) - \sum \frac{2h_k}{k} \sin(2\pi kx).$$

The link between the considered Hilbert spaces and the Banach space of continuous functions is given by the Sobolev embedding theorem. In case of space dimension  $n = 1$  it states that the Hilbert space  $H_{\frac{1}{2}+\varepsilon} = H_{-\frac{1}{2}-\varepsilon}^*$  is continuously embedded into  $C^0$ .

**Remark:** Lemma 6 goes back to E. Kummer (KuE). It is a special case ( $f(x) := \log \Gamma(x)$ ) of the general property of a Fourier series  $f(x) = A_0 + 2 \sum_{k=1}^{\infty} A_k \cos(2\pi kx) + 2 \sum_{k=1}^{\infty} B_k \sin(2\pi kx)$ , ( $0 < x < 1$ ) in relationship to corresponding Fourier series in the form

$$F(x) := A_0 + 2 \sum_{k=1}^{\infty} A_{2k} \cos(2\pi kx) + 2 \sum_{k=1}^{\infty} B_{2k} \sin(2\pi kx), \quad 0 < x < \frac{1}{n}.$$

It holds, (KuE),

$$(*) \quad \sum_{k=1}^{\infty} f \left( x + \frac{k-1}{n} \right) = nF(nx).$$

In the special case  $f(x) := \log \Gamma(x)$  the property  $\Gamma(1+x) = x\Gamma(x)$  ensures its validity for  $0 < x < 1$ .

From (KuE) we quote:

*„The type of formula (\*) namely occurs in the theory of transcendental numbers. It is only interesting for analysis, if there is another relationship between the both functions,  $f(x)$  and  $F(x)$ “.*

In other words, Kummer's „contribution to the theory of the function  $\Gamma(x) = \int_0^{\infty} e^{-v} v^{x-1} dv$ “ is also applicable to the function  $f_1(x) := \frac{\pi}{2} \log \left( \tan \left( \frac{\pi}{2} x \right) \right)$ , but not only in the context of the alternatively proposed Zeta function theory.

The Fourier series approach is also applicable in the context of the theory of irrational and transcendental numbers, e.g. ((BaA), (LaS), (ScT), (ShA)). We note that the „*deus ex machina*“ method from Lindemann's proof of the irrationality of  $\pi$  enables also a simple proof of the transcendence of  $e$ , (HeG). Hilbert's related proof is basically based on an appropriately defined rational function  $f(t)$  and its related Hermite integral  $F(c) = e^c \int_c^{\infty} e^{-x} f(x) dx$  (HeG).

The Hilbert (Gamma function & integral calculus related) power series 2-parts approach, (HeG) VIII), is about a number theoretical and an analytical part, i.e.

- i) part 1: If  $f(x) = \sum a_k (x-c)^k$  then  $F(x) = \sum a_k k!$
- ii) part 2: an appropriate estimation of  $e^c F(0) - F(c)$ .

**Remark:** Formal differentiation of lemma 6 i) resp. the corollary results into the divergent series

$$\gamma = -\log' \Gamma(1) = -2 \sum_{k=1}^{\infty} [2\pi k B_1 + \log k], \quad \text{where } B_1 := \frac{1}{2\pi} [\gamma + \log(2\pi)]$$

resp.

$$\log' \Gamma \left( \frac{x}{2} \right) + \frac{\pi}{2} \log' \left( \tan \left( \frac{\pi}{4} x \right) \right) = -\frac{\pi}{2} \sum_{k=1}^{\infty} \sin(\pi kx) + \sum_{k=1}^{\infty} [k[\gamma + \log(2\pi)] + \log k - 2\pi h_k] \cos(\pi kx).$$

**Remark:** The Fourier coefficient function  $\log(k)$  is typical for number theoretical problems, especially in the context of the harmonic numbers, but it is inappropriate for a harmonic Fourier series based analysis. However, the odd number based numbers  $h_n := \sum_{k=1}^n \frac{1}{2k-1}$  resp.  $c_n := \frac{2h_n}{n}$  are appropriate for a Hilbert space based analysis, whereby the link to the zeta function theory is given by, (EIL),  $\sum \frac{h_n}{n} e^{-nx} = \frac{1}{4} \log^2 \left( \tanh \frac{x}{4} \right)$ .

The standard tool trying to prove the tertiary and binary Goldbach conjecture is about the Hardy-Littlewood circle method. It is about a dissection of the circle  $x = e^{2\pi i\alpha}$ , or rather a smaller concentric circle, into „Farey arcs“. The „major arcs“, or „basic intervals“, provide the main term in the asymptotic formula for the number of representations. Their treatment does not give rise to any very serious difficulties compared to the problems presented by the „minor arcs“, or „supplementary intervals“. The latter ones are analyzed by estimates of the Weyl (trigonometrical) sums

$$S(x) := \sum_n e^{2\pi i n x}$$

without taking any (Goldbach) problem relevant information into account. We note that an asymptotic behavior in the form  $O(N^{\frac{1}{2}+\varepsilon})$  of the Farey series is equivalent to the Riemann Hypothesis (LaE5).

The Hardy-Littlewood circle method is concerned with the set of the winding numbers of the unit circle, related to the zeros of the Weyl sum components, which are the basis functions  $e^{2\pi i n x}$ . The method itself is about Fourier analysis over  $Z$ , which acts on the circle  $R/Z$ . The analyzed functions are complex-valued power series

$$f(x) = \sum_0^\infty a_n z^n \quad , \quad |z| < 1.$$

The key principle of the circle method is the fact, that for  $N$  being an integer it holds

$$\int_0^1 e^{2\pi i N \alpha} d\alpha = \begin{cases} 1 & \text{if } N = 0 \\ 0 & \text{otherwise} \end{cases}$$

which can be reformulated in the form

**Lemma 7:** Let  $f(x) = \sum_0^\infty a_n x^n$  with  $|x| < 1$ , then for  $0 < r < 1$  it holds

$$r^n a_n = \int_0^1 f(re^{2\pi i t}) e^{-2\pi i n t} dt.$$

The alternatively proposed to be analyzed functions are „functions“ with domain  $\Gamma = S^1(R^2)$ , which are elements of an appropriately defined distributional, periodical Hilbert space  $H_\alpha^\#(0,1)$ . The corresponding alternative fundamental principle is given by

$$-n f_n(x) = \frac{1}{2} \int_0^1 \frac{f_n(y)}{\sin^2(\pi(x-y))} dy = \frac{1}{2} \int_0^1 \cot(\pi(x-y)) f_n'(y) dy$$

with  $f_n(y) := a_n \cos 2\pi n y + b_n \sin 2\pi n y$ .

The right side of the above formula describes the Hilbert transform of  $f_n'(x)$ . Therefore, the corresponding integral operator with domain  $L_2^\#(0,1) = H_0^\#(0,1)$  is well defined with respect to the weak  $H_{-1/2}^\#(0,1)$  (inner product induced) topology.

The proposed *truly* circle method (not a rather smaller concentric circle within the unit disc) is about the  $H_{-1/2}^\#(0,1)$  framework in combination with the alternatively considered zeros of the Kummer function  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right)$ . Its imaginary parts  $\omega_n$  enable the definition of sequences like  $\lambda_n^{(1)} = \omega_n$  resp.  $\lambda_n^{(2)} = \frac{1}{2}(\omega_n + \omega_{n+1})$ , and the average  $\frac{1}{2}(\lambda_n^{(1)} + \lambda_n^{(2)})$ .

The pair  $(\lambda_n^{(1)}, \lambda_n^{(2)})$  can be interpreted as a pair of semi-winding numbers for the left and the right unit semi-circle, while at the same point in time both sequences do have Snirelman density  $\frac{1}{2}$ .

The average  $\frac{1}{2}(\lambda_n^{(1)} + \lambda_n^{(2)})$  provides the link to Kadec's  $\frac{1}{4}$ -theorem.

The distributional Hilbert space framework  $H_\alpha^\#$  comes along with generalized harmonic Fourier series theory. The proposed *truly* circle method is based on the theory of nonharmonic Fourier series, allowing to replace the integer (winding number)  $n$  of the Hardy-Littlewood „nearly“ circle method) by appropriate sequences  $\lambda_n^*$  accompanied by the Riesz basis concept as central element of in the nonharmonic Fourier series theory.

Therefore, the proposed *truly* circle method provides an appropriate tool for additive number theory, especially for the proof of the binary Goldbach problem.

### A brief overview of nonharmonic Fourier series (YoR)

A basis for a Hilbert space is a Riesz basis, if it is equivalent to an orthogonal basis, that is, if it is obtained from an orthogonal basis by means of a bounded invertible operator.

A number of important characteristic properties of Riesz bases for a separable Hilbert space  $H$  are the following equivalent statements:

- The sequence  $\{f_n\}$  forms a Riesz basis
- There is an equivalent inner product on  $H$ , with respect to which the sequence  $\{f_n\}$  becomes an orthogonal basis for  $H$
- The sequence  $\{f_n\}$  is complete in  $H$ , and there exist positive constants  $A$  and  $B$  such that for an arbitrary positive integer  $n$  and arbitrary scalars  $c_1, c_2, \dots, c_n$  one has

$$A \sum_{i=1}^n c_i^2 \leq \|\sum_{i=1}^n c_i f_i\|^2 \leq B \sum_{i=1}^n c_i^2$$

- The sequence  $\{f_n\}$  is complete in  $H$ , and its Gram matrix  $((f_i, f_l))_{i,j=1}^{\infty}$  generates a bounded invertible operator on  $\{l^2\}$
- The sequence  $\{f_n\}$  is complete in  $H$  and possesses a complete biorthogonal sequence  $\{g_n\}$  such that

$$\sum_{n=1}^{\infty} |(f, f_n)|^2 < \infty \text{ and } \sum_{n=1}^{\infty} |(f, g_n)|^2 < \infty \text{ for every } f \text{ in } H$$

- The sequence  $\{f_n\}$  is both, a Bessel basis and a Hilbert basis
- The sequence  $\{f_n\}$  is an exact frame, (YoR) p. 157.

The totality of all entire functions of exponential type at most  $\pi$  that are square integrable on the real axis is known as the Paley-Wiener (separable Hilbert) space  $\mathbf{P}$ , equipped with the inner product  $(f, g) = \int_{-\infty}^{\infty} f(x)g(x)dx$ , which is isometrically isomorph to  $L_2(-\pi, \pi)$ . The isomorphism between  $\mathbf{P}$  and  $L_2(-\pi, \pi)$  has far-reaching consequences, e.g. regarding known properties of  $L_2(-\pi, \pi)$  which can be transformed easily into nontrivial assertions about  $\mathbf{P}$ .

If  $f$  belongs to  $\mathbf{P}$  and has the representation  $f(z) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) e^{izt} dt$ , with  $\varphi \in L_2(-\pi, \pi)$ , then Planchel's theorem shows that  $\|f\|^2 = \|\varphi\|^2$ . By taking the Fourier transform of  $e^{int}$  ( $n = 0, \pm 1, \pm 2, \pm \dots$ ), we see that the set of functions  $\left\{ \frac{\sin \pi(z-n)}{\pi(z-n)} \right\}$  forms an orthogonal basis for  $\mathbf{P}$ . Accordingly every function  $f$  in  $\mathbf{P}$  has an unique expansion of the form

$$f(z) = \sum_{-\infty}^{\infty} c_n \frac{\sin \pi(z-n)}{\pi(z-n)} \text{ with } \sum_{-\infty}^{\infty} |c_n|^2 < \infty.$$

Every function  $f \in \mathbf{P}$  can be recaptured from its values at the integers, which is achieved by the cardinal series representation of  $f$ , (YoR) p. 90.

The Paley-Wiener criterion is nothing more than the assertion that the mapping  $T: e_n \rightarrow f_n$  for  $n = 1, 2, \dots$  can be extended to an isomorphism on all of the separable Hilbert space  $H$  with its orthogonal basis  $\{e_n\}$  for which  $\|I - T\| < 1$ .

The trigonometric system is stable  $L_2(-\pi, \pi)$  under „sufficiently small“ perturbances of the integers. This means that if  $\lambda_n$  is a sequence of real or complex numbers for which  $\{\lambda_n - n\}$  is in some sense „small“, then the system  $\{e^{i\lambda_n t}\}$  will form a basis for  $L_2(-\pi, \pi)$ , in fact, a Riesz basis. Accordingly, every function  $f \in L_2(-\pi, \pi)$  will have an unique nonharmonic Fourier series expansion

$$f(t) = \sum_{-\infty}^{\infty} c_n e^{i\lambda_n t} \text{ (in the mean)}$$

with  $\sum_{-\infty}^{\infty} |c_n|^2 < \infty$ . The possibility of such nonharmonic expansions was discovered by Paley and Wiener. .... In the present setting that criterion takes the form

$$(*) \left\| \sum_{-\infty}^{\infty} c_n (e^{int} - e^{i\lambda_n t}) \right\| \leq \lambda < 1, \text{ whenever } \sum_{-\infty}^{\infty} |c_n|^2 \leq 1.$$

**Remark:** We note that the sequence  $\{|t|^\alpha e^{int}\}$  with  $0 < \alpha < 1/2$  is not a Riesz basis (YoR) p. 30.

**Remark:** From the Theorem of Levinson, (YoR), p. 100, it can be concluded that the system  $\{e^{i\lambda_n t}\}$  is complete in  $L_2(-\pi, \pi)$

**Remark:** A complete sequence of vectors in a separable Hilbert space is a Riesz basis if and only if its moment space is equal to  $l^2$ , (YoR) p. 141

When shall the sequence  $\{\lambda_n - n\}$  be considered „small“? Based on what has already been established, one might well suppose that the condition  $\lambda_n - n \rightarrow 0$  as  $n \rightarrow \pm\infty$  is, at the very least, necessary. Surprisingly, it is not, which is shown by, (YoR) p. 36,

**Kadec's  $\frac{1}{4}$ -Theorem:** If  $\lambda_n$  is a sequence of real numbers for which

$$|\lambda_n - n| \leq L < \frac{1}{4} \text{ for } n = 0, \pm 1, \pm 2, \pm \dots,$$

then  $\{e^{i\lambda_n t}\}$  satisfies the Paley-Wiener criterion and so forms a Riesz basis for  $L_2(-\pi, \pi)$ .

**Proof:** (YoR) p. 36; to prove the theorem it is to be shown that the Paley-Wiener criterion (\*) is fulfilled, whenever  $\sum_{-\infty}^{\infty} |c_n|^2 \leq 1$ , which is the case for  $\lambda = 1 - \cos(\pi L) + \sin(\pi L)$ . The trick to prove this is to expand the function  $1 - e^{i(\lambda_n - n)t}$  in a Fourier series relative to the complete orthogonal system  $\{1, \cos(nt), \sin((n - \frac{1}{2})t)\}_{n=1}^{\infty}$ .

**Remark** (YoR) p.37, p. 102: The result is sharp in the sense that the constant  $L$  cannot be improved. In fact, if  $L = \frac{1}{4}$  then the conclusion of Kadec's theorem no longer holds:

A counterpart is provided by the sequence  $\{\lambda_n^*\}$ , with

$$\lambda_n^* = \begin{cases} n - \frac{1}{4}, & n > 0 \\ 0, & n = 0 \\ n + \frac{1}{4}, & n < 0 \end{cases}$$

which is complete in  $L_2(-\pi, \pi)$ , and the entire function  $f(z) := a \int_{-\pi}^{\pi} (\cos \frac{1}{2}t)^{-1/2} e^{izt} dt$  (see also (Grl) 3.892, 8.338), where the constant  $a$  is chosen so that  $f(0) = 1$ . It holds  $f(\lambda_n) = 0$  for every  $\lambda_n$  and (see also (Grl) 8.325)

$$f(z) := \frac{\Gamma^2(\frac{3}{4})}{\Gamma(\frac{3}{4}+z)\Gamma(\frac{3}{4}-z)} = \prod_{k=0}^{\infty} \left[ \left(1 + \frac{z}{\frac{3}{4}+k}\right) \left(1 - \frac{z}{\frac{3}{4}+k}\right) \right].$$

The condition  $f(0) = 1$  is crucial to enable the application of Hadamard's factorization theorem to show that  $f(z)$  can vanish only at the  $\lambda_n$ 's. It further holds, when  $n$  is positive,

$$f'(\lambda_n) = (-1)^n \Gamma^2(\frac{3}{4}) \frac{\Gamma(n)}{\Gamma(n+1/2)} \quad \text{with} \quad \frac{\Gamma(n)}{\Gamma(n+1/2)} \sim \frac{1}{\sqrt{n}} \text{ as } n \rightarrow \infty.$$

Suppose to the contrary that the set  $\{e^{i\lambda_n t}\}$  were a Riesz basis for  $L_2(-\pi, \pi)$ . Then we could write  $1 = \sum_{-\infty}^{\infty} c_n e^{i\lambda_n t}$ . The sequence of coefficients  $\{c_n\}$  is unique and belongs to  $l^2$ . By means of the Fourier isometry, the entire discussion can be transferred into the Paley-Wiener space. The exponentials  $e^{i\lambda_n t}$  are transformed into the reproducing functions

$$K_n(z) = \frac{\sin \pi(z - \lambda_n)}{\pi(z - \lambda_n)}$$

and the expansion

$$\frac{\sin \pi z}{\pi z} = \sum_{-\infty}^{\infty} c_n K_n(z)$$

is valid in the topology of the Paley-Wiener space  $\mathbf{P}$ . Define a sequence  $\{f_n\}$  of functions of  $\mathbf{P}$  by setting

$$f_n(z) := \frac{f(z)}{f'(\lambda_n)(z - \lambda_n)}.$$

Then  $\{f_n\}$  and  $\{K_n\}$  are biorthogonal, since  $(f_n, K_m) = (f_n, \lambda_m) = \delta_{mn}$ . Therefore

$$c_n = \left( \frac{\sin \pi z}{\pi z}, f_n \right) = f_n(0) = \frac{-1}{f'(\lambda_n)\lambda_n}.$$

Using the asymptotic formula  $\frac{\Gamma(n)}{\Gamma(n+1/2)} \sim \frac{1}{\sqrt{n}}$  as  $n \rightarrow \infty$  we conclude that  $\sum_{-\infty}^{\infty} |c_n|^2$  diverges. This contradiction completes the proof.

**Remark:** A system  $\{e^{i\lambda_n t}\}$  of complex exponentials is a Riesz basis for  $L_2(-\pi, \pi)$  if and only if  $\{\lambda_n\}$  is a complete interpolating sequence for  $\mathbf{P}$ , (YoR) p. 143

**Remark:** (WaG) p. 503: for  $n > N$  the Bessel function  $J_0(x)$  has precisely one zero in each of the intervals  $(\pi(n - \frac{1}{4}), \pi(n + \frac{1}{4}))$

**Remark:** A striking generalization of "Kadec's 1/4-theorem" (YoR) p. 178, is

„Avdonin's Theorem 1/4 in the mean": Let  $\lambda_n = n + \delta_n$  ( $n = 0, \pm 1, \pm 2, \dots$ ) be a separated sequence of real or complex numbers. If there exists a positive integer  $N$  and a constant  $d$ ,  $0 \leq d < \frac{1}{4}$ , such that

$$\left| \sum_{k=mN+1}^{(m+1)N} \delta_k \right| \leq dN$$

for all integers  $m$ , then the system  $\{e^{i\lambda_n t}\}$  is a Riesz basis for  $L_2(-\pi, \pi)$ .

**Remark (LeN2):** For the sequence fulfilling  $|\lambda_n - n| \leq L < 1/4$ ,  $n = 0, \pm 1, \pm 2, \pm 3, \dots$  let

$$F(x) := \prod \left(1 - \frac{x}{\lambda_n}\right) \left(1 - \frac{x}{\lambda_{-n}}\right).$$

Then there exists an absolute constant  $A$  such that

$$\int_{-\infty}^{\infty} |F(x)|^2 dx < \frac{A}{(1-4L)^2}, \quad \int_c^{\infty} |F(x)|^2 dx < \frac{Ac^{-(1-4L)}}{(1-4L)^2}, \quad \text{for } c > 1, \quad \text{i.e. } F(x) \in L_2(-\infty, \infty).$$

The „generalized“ trigonometric moment problem in  $L_2(-\pi, \pi)$  is given by, (YoR) p.124,

$$(*) \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) e^{i\lambda_n t} dt = c_n.$$

As  $\{e^{i\lambda_n t}\}$  is a Riesz basis its moment space is  $l^2$  and (\*) admits a solution whenever  $c_n$  is square-summable, and for these sequences only. If  $\{g_n\}$  is the unique sequence in  $L_2(-\pi, \pi)$  biorthogonal to  $\{e^{i\lambda_n t}\}$ , then the unique solution to (\*) is given by the norm-convergent series

$$\varphi(t) = \sum_{-\infty}^{\infty} c_n g_n(t).$$

Putting

$$G(z) := z \prod_{k=0 \rightarrow \infty} \left(1 - \frac{z^2}{\lambda_n^2}\right)$$

then

$$G_n(z) := \frac{G(z)}{G'(\lambda_n)(z-\lambda_n)}$$

belongs to the Paley-Wiener space  $\mathbf{P}$  and  $g_n(t)$  is the inverse Fourier transform of  $G_n(z)$ , i.e. for almost all  $t \in [-\pi, \pi]$ ,

$$g_n(t) = \int_{-\infty}^{\infty} G_n(z) e^{ixt} dt.$$

By means of the Fourier isometry, we can transform the above into the Paley-Wiener space. The exponentials  $e^{i\lambda_n t}$  are transformed into the reproducing functions  $K_n(z) = \frac{\sin \pi(z-\lambda_n)}{\pi(z-\lambda_n)}$ ,  $g_n(t)$  is transformed into  $G_n(z)$ , while the moment problem itself becomes  $f(\lambda_n) = 0$ ,  $n = 0, \pm 1, \pm 2, \pm 3, \dots$ , since  $f(\lambda_n) = (f, K_n)$ . Here  $c_n \in l^2$  and  $f \in \mathbf{P}$  is to be found. The solution is given by

$$(**) \quad f(z) = \sum_{-\infty}^{\infty} c_n \frac{G_n(z)}{G'_n(\lambda_n)(z-\lambda_n)}.$$

Moreover, since the expansion  $f = \sum_{-\infty}^{\infty} (f, K_n) G_n$  is valid for every function  $f$  belonging to  $\mathbf{P}$  and  $\{G_n\}$  is a Riesz basis for  $\mathbf{P}$ ,  $\sum_{-\infty}^{\infty} |f(\lambda_n)|^2 < \infty$  for all  $f$ . Thus (\*\*) with  $c_n \in l^2$  represents the most general function in  $\mathbf{P}$ .

**Remark:** The Fourier function  $\log(n)$  is inappropriate for a harmonic Fourier series based analysis. However, the odd number based numbers  $h_n := \sum_{k=1}^n \frac{1}{2k-1}$  resp.  $c_n := \left(\frac{2}{n}\right)^2 \frac{h_n}{n}$ , are appropriate for a Hilbert space based analysis, whereby the link to the zeta function theory is given by, (EIL),

$$\sum \frac{h_n}{n} e^{-nx} = \frac{1}{4} \log^2 \left(\tanh \frac{x}{4}\right).$$

The link to the proposed nonharmonic Fourier series analysis framework is provided by the fact that  $c_n^* \in l^2$  with  $\sum_{n=1}^{\infty} c_n^{*2} = \frac{1}{2} < \infty$ , see also below. Regarding the Kadec condition we mention that  $\log\left(\frac{1}{4}\right) = -2\log 2$ .

**Remark:** The Fourier analysis for extended ( $n \rightarrow \lambda_n$ ) Cardinal functions (including coefficient terms like  $\log n$ , (WhJ1), the link to the Hilbert transform, (TiE1), and a „rationality result concerning  $\zeta(2n)$ , (TiE2)) is provided in (WhJ1-3), (TiE1-2). The formula (\*\*\*) above is a simple example of a „generalized“ Lagrange interpolation formula for an entire function with values  $c_n$  at the points  $\lambda_n$ . The formula is valid whenever  $\{\lambda_n\}$  is real, symmetric, and Kadec’s condition is fulfilled. If one choose  $\lambda_n = n$ , then (\*\*\*) reduces to the „cardinal series“ for  $f(z)$ , (YoR) p. 126.

**Theorem 8** (YoR) p. 139: A complete sequence of vectors belonging to a separable Hilbert space is a Riesz basis if and only if its moment space coincides with  $l^2$ .

Regarding interpolation in the Paley-Wiener space  $\mathcal{P}$  this leads to the

**Theorem 9** (YoR) p. 143: A system  $\{e^{i\lambda_n t}\}$  of complex exponentials is a Riesz basis for  $L_2(-\pi, \pi)$  if and only if  $\{\lambda_n\}$  is a complete interpolation sequence in  $\mathcal{P}$ .

A sub -class of the entire functions of exponential type  $\pi$  are sine-type-functions (regarding the considered Kummer function we refer to (SeA)):

**Theorem 10** (YoR) p. 144: if  $\{\lambda_n\}$  is the set of zeros of sine-type, then the system  $\{e^{i\lambda_n t}\}$  is a Riesz basis for  $L_2(-\pi, \pi)$ .

Regarding the stability of the class of Riesz bases  $\{e^{i\lambda_n t}\}$  in  $L_2(-\pi, \pi)$  Kadec’s theorem can be dramatically improved, first under „small“ displacements of the  $\lambda_n$ ’s and then under more general „vertical“ displacements.

The *modus operandi* is to combine the stability of interpolating sequences with the stability of frames, (YoR) pp. 160 ff.:

**Theorem 13.** If the system  $\{e^{i\lambda_n t}\}$  is a frame in  $L_2(-\pi, \pi)$ , then there is a positive constant  $L$  with the property that  $\{e^{i\mu_n t}\}$  is also a frame in  $L_2(-\pi, \pi)$  whenever  $|\lambda_n - \mu_n| \leq L$  for every  $n$ .

**Corollary.** If the system  $\{e^{i\lambda_n t}\}$  is a Riesz basis for  $L_2(-\pi, \pi)$  then there is a positive constant  $L$  with the property that  $\{e^{i\mu_n t}\}$  is also a Riesz basis for  $L_2(-\pi, \pi)$  whenever  $|\lambda_n - \mu_n| \leq L$  for every  $n$ .

**Theorem 14.** Let  $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$  be a sequence of points lying in a strip parallel to the real axis. If the system  $\{e^{iRe(\lambda_n)t}\}$  is a frame in  $L_2(-\pi, \pi)$ , then so is  $\{e^{i\lambda_n t}\}$ .

**Corollary 1.** Let  $\{\lambda_1, \lambda_2, \lambda_3, \dots\}$  be a sequence of points lying in a strip parallel to the real axis. If the system  $\{e^{iRe(\lambda_n)t}\}$  is a Riesz basis for  $L_2(-\pi, \pi)$ , then so is  $\{e^{i\lambda_n t}\}$ .

**Corollary 2.** If  $\{\lambda_n\}_{n=-\infty}^{n=\infty}$  is a sequence of scalars for which

$$|Re(\lambda_n) - n| < \frac{1}{4},$$

then the system  $\{e^{i\lambda_n t}\}_{n=-\infty}^{n=\infty}$  is a Riesz basis for  $L_2(-\pi, \pi)$ .

**Remark:** Assuming the series  $f(x) = \sum c_n e^{2\pi i k x}$  is real valued and the coefficients  $c_n \in R$ . Then  $f(x) = \sum c_n \cos(2\pi i k x)$ .

From the above we recall, (LeN2)

**Remark:** for  $F(x) := \prod(1 - \frac{x}{\lambda_n})(1 - \frac{x}{\lambda_{-n}})$  with  $\{\lambda_n\}_{n \in \mathbb{Z}}$  fulfilling  $|\lambda_n - n| \leq L < 1/4$ ,  $n = 0, \pm 1, \pm 2, \pm 3, \dots$  it holds  $F(x) \in L_2(-\infty, \infty)$ .

**Remark:** Let  $F(z)$  be a sine-type function, let the width of its indicator diagram be  $2\pi$ , and let  $\lambda_n$  be its zero set. Then the system of functions  $\left\{ \frac{F(z)}{F'(\lambda_n)(z - \lambda_n)} \right\}_{n=-\infty}^{n=\infty}$  is a Riesz basis in  $L^2_{\pi}$ , (LeB) p. 169.

Let  $\omega_n$  denote the imaginary parts of the zeros of the Kummer function  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi z\right)$ , whereby none of the zeros lie on the critical line (!), fulfilling the inequalities, (SeA),

$$2n - 1 < 2\omega_n < 2n < \omega_n + \omega_{n+1} < 2n + 1 \quad \text{resp.} \quad n - \frac{1}{4} < \lambda_n := \frac{3\omega_n + \omega_{n+1}}{4} < n + \frac{1}{4}, \quad n = 1, 2, \dots$$

Therefore, it holds the

**Theorem:** For any  $\omega_0 \in (-\frac{1}{2}, 0)$  (\*) and  $\lambda_{-n} := -\lambda_n$  the Kadec condition is fulfilled, i.e. it holds

$$|\lambda_n - n| \leq L < 1/4, \quad n = 0, \pm 1, \pm 2, \pm 3, \dots$$

In the following some cornerstones of a  $H_\alpha^\#(0,1)$  based Hilbert space Zeta function theory are collected, where the distributional Hilbert spaces  $H_\alpha^\#(0,1)$  are built on the separable Hilbert space  $L_2(0,1)$  with its underlying *harmonic* orthogonal basis  $\{e_n\}$ . With respect to a link back to the RH in the context of the considered Kummer function, the Paley-Wiener space  $\mathcal{P}$  and the related Paley-Wiener criterion we refer to (BaB), (BeA1).

**Remark:** The trigonometric moment problem in  $L_2(-\pi, \pi)$  is given by, (YoR) p.124,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(t) e^{int} dt = c_n.$$

**Remark:** The Mellin transforms of the  $\{\sin(nx), \cos(nx)\}$  system is given by, (Grl) 3.761

$$\int_0^\infty x^{s-1} \sin(ax) dx = \frac{\Gamma(s)}{a^s} \sin\left(\frac{\pi}{2}s\right), \quad \int_0^\infty x^{s-1} \cos(ax) dx = \frac{\Gamma(s)}{a^s} \cos\left(\frac{\pi}{2}s\right), \quad 0 < |Re(s)| < 1, a > 0.$$

resp.

$$\int_0^\infty x^{s-1} e^{iax} dx = \frac{\Gamma(s)}{a^s} e^{i\frac{\pi}{2}s}.$$

For the related zeta function representations we refer to (TiE) chapter II. The corresponding circular counterparts of the Mellin transforms above are given by, (Grl) 3.761

- i)  $\int_0^1 x^{s-1} \sin(ax) dx = -\frac{i}{2s} [{}_1F_1(s, s+1; ia) - {}_1F_1(s, s+1; -ia)], \quad Re(s) > -1, a > 0,$
- ii)  $\int_0^1 x^{s-1} \cos(ax) dx = \frac{1}{2s} [{}_1F_1(s, s+1; ia) + {}_1F_1(s, s+1; -ia)], \quad Re(s) > -1, a > 0.$

leading to

**Lemma 8** ((Gri) 3.761, (SeA)): For  $Re(s) > -1, a > 0$  it holds

$$\int_0^1 x^{s-1} e^{iax} dx = \frac{1}{s} {}_1F_1(s, s+1; ia) = \Gamma(s) (-ia)^{-s} \left[ 1 + O\left(\frac{1}{(ia)^2}\right) \right] + \frac{e^{ia}}{ia} \left[ 1 + \frac{1+s}{ia} + O\left(\frac{1}{(ia)^2}\right) \right].$$

**Remark:** The integral  $\int_0^\infty x^{s-1} e^{iax} dx$  links to the (generalized) Dirichlet series and integrals, (InA) V. The formula in lemma 8 is supposed to overcome the „domain gap“  $-1 < Re(s) < 0$  of the zeta function representation given by, (TiE) (2.1.7)

$$\zeta(s) = s \int_0^\infty \left( [x] - x + \frac{1}{2} \right) x^{-s-1} dx = s \int_0^\infty \left( \sum_{n=1}^\infty \frac{\sin(2\pi nx)}{\pi n} \right) x^{-s-1} dx.$$

**Remark:** From (BeA1) we recall for  $0 < Re(s), \rho(x) := x - [x] \in L_2^\#(0,1), 0 < \vartheta \leq 1$

$$\int_0^1 \rho\left(\frac{\vartheta}{x}\right) x^{s-1} dx = \frac{\vartheta}{s-1} - \frac{\vartheta^s \zeta(s)}{s}.$$

**Remark:** Riemann's density function  $J(x) = \frac{1}{2} \left[ \sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n} \right]$  is derived from the formula  $p^{-ns} = s \int_{p^n}^\infty x^{-s-1} dx$  (EdH) 1.11, i.e. lemma 6 above provides the basis for an alternative, circular density function with domain  $\Gamma = S^1(R^2)$  accompanied by properly chosen  $a > 0$ , where the winding number  $n$  is replaced by a non-harmonic sequence, enabling a corresponding  $H_\alpha^\#(0,1)$  based Fourier inversion, analogue to EdH) 1.12.

**Remark:** Riemann's Fourier inversion of  $\frac{\log \zeta(s)}{s}$  to derive the formula for  $J(x)$ , (EdH) 1.12, is built on the Fourier inverse formula  $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(\omega) e^{i(x-\omega)t} d\omega dt$ . Its periodical counterpart is given by

$$g(x) = \frac{1}{2\pi} \sum_{n=-\infty}^\infty \int_{-\pi}^\pi g(\omega) e^{i(x-\omega)n} d\omega = \sum_{n=-\infty}^\infty c_n e^{inx}, \quad c_n := \frac{1}{2\pi} \int_{-\pi}^\pi g(\omega) e^{i\omega n} d\omega.$$

**Remark:** From lemma 4 above we recall

$$F_n(s) := \frac{{}_1F_1\left(\frac{1}{2}; \frac{3}{2}; s\right)}{{}_1F_1\left(\frac{1}{2}; \frac{3}{2}; \alpha_n\right)(s - \alpha_n)} = -2e^{-\frac{2}{3}s} \prod\left(\frac{\alpha_k - s}{\alpha_n}\right) e^{\frac{s}{\alpha_n}} e^{-(\alpha_n - s)}.$$

**Remark:** For some specific properties of  ${}_1F_1(a, a + 1; z)$  we refer to the above and the Barnes-type contour integral representation, (AbM) 13.2.9,

$$a {}_1F_1(a, a + 1; z) = \frac{\Gamma(a+1)}{\Gamma(a)} {}_1F_1(a, a + 1; z) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(-s)\Gamma(a+s)}{\Gamma(a+1+s)} (-z)^s ds = -\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(1-s)}{(a+s)} (-z)^s \frac{ds}{s}.$$

**Remark:** With respect to the below mentioned replacement of the function  $\Psi(x)$  by  $\beta(x)$ , when changing to the „circular modelling framework“ we recall from (GrI) 4.532

- i)  $\int_0^1 t^x \arctan(t) \frac{dt}{t} = \frac{1}{2x} \left[ \frac{\pi}{2} + \beta\left(\frac{x+1}{2}\right) \right]$  for  $x > 0$
- ii)  $\int_0^\infty t^x \arctan(t) \frac{dt}{t} = \frac{1}{x} \frac{\pi/2}{\cos\left(\frac{\pi}{2}x\right)}$  for  $0 < x < 1$ .

**Remark:** The Hilbert kernel for the circular Hilbert transform can be obtained by making the Cauchy kernel  $\frac{1}{x}$  periodic, i.e.

$$\frac{1}{2} \cot \frac{x}{2} = \frac{1}{x} + \left[ \sum_{n=1}^{\infty} \frac{1}{x+2n\pi} + \frac{1}{x-2n\pi} \right], |x| \leq \pi,$$

resp.

$$\pi \cot \pi x = \frac{1}{x} + \left[ \sum_{n=-\infty}^{\infty} \frac{1}{n} + \frac{1}{x-n} \right] = \psi(1-x) - \psi(x), |x| \leq 1.$$

A corresponding Riesz basis based nonharmonic circular kernel is given by, (see also (GrI) (1.421), (8.365)),

$$\frac{1}{2} \cot^* \left(\frac{x}{2}\right) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda_n} + \frac{2\pi}{x-2\pi\lambda_n}, |x| \leq \pi, \text{ resp. } \pi \cot^*(\pi x) = \sum_{n=-\infty}^{\infty} \frac{1}{\lambda_n} + \frac{1}{x-\lambda_n} =: \psi^*(1-x) - \psi^*(x), |x| \leq 1.$$

**Remark:** We mention the formula  $\psi(n+1) + \gamma = H_n$ , (HaJ), and the approximate equation, (TiE) 4.14,

$$\zeta(s) - \sum_{n < x} n^{-s} = \sum_{n > x} n^{-s} = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} z^{-s} (\pi \cot(\pi z)) dz,$$

where without loss of generality  $x$  is half an odd integer; the function  $\pi \cot(\pi z)$  is holomorphic except the pole  $z = 1$ . We further note that for the zeros  $Z_n$  of the Kummer function  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right)$  it holds  $(\text{Re}(z_n)) > 1/2$ , while for the zeros of its derivative  ${}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; z\right)$  it holds  $(\text{Re}(z_n^*)) < 1/2$ , ((SeA), also with respect to the link to sine-type functions, playing an important role in nonharmonic analysis, which are not Fourier-Stieljes transforms.

**Remark:** The divergent Fourier series representation of  $\rho'_H(x)$ , (\*)

$$-\cot(\pi x) = -\sum_1^\infty \sin 2\pi v x = H[\rho'](x) = \rho'_H(x)$$

where, (ZyA) p. 5,  $\rho(x) = x - [x] = \frac{1}{2} - \sum_1^\infty \frac{\sin 2\pi v x}{\pi v} \in L_2^\#(0,1)$ , resp.

$$\rho_H(x) = \sum_1^\infty \frac{\cos 2\pi v x}{\pi v} = -\frac{1}{\pi} \log 2 |\sin(\pi x)| \in L_2^\#(0,1)$$

is convergent with respect to the weak  $H_{-1/2}^\#(0,1)$  topology (\*\*).

**Remark:** In the context of the Fourier character of conjugate series, lacunary series, and  $\alpha$  – resp. logarithmic capacities of sets and conjugate Fourier series like  $\sum_{n=1}^\infty \frac{\sin nx}{n}$ ,  $\sum_{n=1}^\infty \frac{\cos nx}{n}$  resp.  $\sum_{n=1}^\infty \frac{\sin nx}{n \log n}$ ,  $\sum_{n=1}^\infty \frac{\cos nx}{n \log n}$  we refer to (ZyA) I-2, VI-3, VII-2, XII-11.

(\*) we note the best bounds of harmonic sequence, (ChC),  $\frac{1}{2n+(1-\gamma)} \leq \sum_1^n \frac{1}{k} - \log n - \gamma < \frac{1}{2n+1}$ , where  $\frac{1}{1-\gamma} - 2 \sim 0,3652721..$

(\*\*) The series is Cesàro summable representing the  $\cot$  function if  $x$  is not an element of  $Z$ , and zero otherwise.

Regarding the distributional theory of asymptotic expansions we recall from (EsR) p. 139 the formula  $F.p. (P.v \int_0^\infty t^x \frac{dt}{t-1} = \begin{cases} 0 & x \in Z \\ \pi \cot(\pi x) & \text{otherwise} \end{cases}$ ,

where P.v. denotes „Principle value“ and F.p. denotes Hadarmard's „Finite part“. We mention the link between the  $\cot$  function and the Bernoulli numbers in the form, (BeB) 5, entry 13,

$$\frac{x}{2} \cot \frac{x}{2} = \sum_{n=0}^\infty (-1)^n \frac{B_{2n}}{(2n)!} x^{2n}, |x| < 2\pi$$

$$\Gamma(z) = \frac{(-1)^k}{k!} \left[ \frac{1}{z+k} + \Psi(1+k) + O(|z+k|) \right], \text{ (PeB) p. 114}$$



**Remark:** For a Riesz basis  $\{e^{i\lambda_n t}\}_{n \in \mathbb{Z}}$  and the related biorthogonal system  $\{\varphi_n\}_{n \in \mathbb{Z}}$  in  $L_2(-\pi, \pi)$  it holds

- i) any  $f \in L_2(-\pi, \pi)$  can be written as  $f = \sum_{n \in \mathbb{Z}} (f, e^{i\lambda_n(\cdot)}) \varphi_n = \sum_{n \in \mathbb{Z}} (f, \varphi_n) e^{i\lambda_n(\cdot)}$  (YoR) p. 22
- ii)  $2k - 1 < 2(\lambda_{n+k} - \lambda_n) > 2k + 1$ , i.e. the sequence of even integers  $\{2k\}_{k \in \mathbb{N}}$  could be replaced by  $\{2(\lambda_{n+k} - \lambda_n)\}_{k \in \mathbb{N}}$ , where e.g.  $n$  counts the  $n^{\text{th}}$  run on the unit circle.

**Remark:** In order to build an appropriate circular counterpart of  $\zeta(s)$  resp. to define a proper mapping anticipating the advantages of a Riesz basis framework we propose to consider the zeta function on the unit circle built on the following two sets of domains,

$$D_{\text{even}} := (-\infty, \frac{1}{2}] \text{ and } D_{\text{odd}} := (\frac{1}{2}, \frac{1}{2} + i\infty).$$

We note that none of the zeros  $\alpha_n$  of the considered Kummer function lie on the critical line and that cardinal series are interpolations at finite lattice points (WhJ3). The interval  $D_{\text{even}}$  contains all trivial zeros of  $\zeta(s)$ , while the interval  $D_{\text{odd}}$  contains all non-trivial zeros with positive imaginary part, *in case the RH is true*. The domain lines can be mapped by interpolating Möbius transforms to the unit circle governed by the winding number  $n$  applying the three point formulae to the lattice points governed by the two sequences  $\{\lambda_{-n}\}_{k \in \mathbb{N}_0}$  and  $\{\lambda_n\}_{k \in \mathbb{N}}$  in the following way

$$\left\{ \left(\frac{1}{2}, 0\right), \left(\frac{1}{2}, \frac{1}{2} + i \cdot (\text{Im}(\alpha_n))\right), \left(\frac{1}{2}, \infty\right) \right\}_{n \in \mathbb{N}} \leftrightarrow \{(1, 0), e^{2\pi i \lambda_n}, (-1, 0)\}_{n \in \mathbb{N}}$$

and

$$\{(-\infty, 0), (-2n, 0), (-1, 0)\}_{n \in \mathbb{N}} \leftrightarrow \{(-1, 0), e^{2\pi i \lambda_{-n}}, e^{2\pi i \lambda_0}\}_{n \in \mathbb{N}}.$$

We note that the sequence  $\{1, 2n\}_{n \in \mathbb{N}}$  has Snirelman density  $\frac{1}{2}$ . In the context of the Hilbert-Polya conjecture the „eigenstate“ or „eigenvibration value“  $\lambda_0$  might be interpreted as the „ground eigenstate“ of an (unbounded) normal self-adjoint operator, (BeM), in a corresponding distributional generalized Hilbert space  $H_\alpha^\#(0, 1)$ , based on the  $2\pi$ -periodic Hilbert space  $L_2^\#(\Gamma)$  with domain  $\Gamma := S^1(R^2)$ , which is the (*full*) boundary of the unit circle sphere, LaG) (\*).

**Remark:** We note that all zeros  $z_n$  of  $\Psi(x)$  lie on the negative real line, fulfilling the same properties as the imaginary parts of the zeros  $\alpha_n$  of the considered Kummer function. It holds, (Gil),

$$(*) \quad n - \frac{1}{2} < -z_n, \omega_n, \frac{\omega_n - z_n}{2} < n.$$

At the same time, all zeros of the function  $1/\Gamma(x)$  also lie on the negative real line, but at  $x = -n$ .

**Remark:** The inequalities (\*) enable the construction of appropriate sequences fulfilling the Kadec condition, allowing the construction of general (non-Z based) interpolation sequences in the context of (generalized) Paley-Wiener spaces and their related (generalized) Hilbert scales (\*\*), e.g. building interpolation function of  $\Psi(x)$ ,  $f(z) := \frac{1}{\Gamma(z)}$  (\*\*), or the modified von Mangoldt formula, (PrK) (4.37)

$$\psi_0^*(x) := x - \left[ \sum_{\text{Im}(\rho) > 0} \frac{x^\rho}{\rho} + \sum_{n \geq 1} \frac{x^{-2n}}{-2n} \right] - \frac{\zeta'(0)}{\zeta(0)}.$$

Their corresponding mapping onto the above (Riesz bases based) generalized Hilbert scale framework (with underlying unit circle domain), additionally provides the term  $e^{2\pi i \lambda_0}$  to govern numerical values like

$$\frac{\Gamma'(1)}{\Gamma(1)} = -\gamma = \lim_{\varepsilon \rightarrow 0} \left\{ \frac{1}{\varepsilon} - \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \right\}, \text{ (HeK) p.9), or } \frac{\zeta'(0)}{\zeta(0)} = \frac{-\log(2\pi)}{2} \cdot \frac{1}{-1/2} = \log(2\pi), \text{ (PrK) p.232.}$$

All zeros  $z_n$  of  $\Psi(x)$  lie on the negative real line. All zeros  $\alpha_n$  of the considered Kummer function lie on the right side of the „positive“ critical line, ( $\text{Re}(\alpha_n) > 1/2$ ). The two sets are proposed to be applied to the above interpolating Möbius transform mapping to the following functions

$$\Psi(z) := \frac{\Gamma'(z)}{\Gamma(z)}, \text{ (**) resp. } g(z) := 2\beta(x) := \Psi\left(\frac{z+1}{2}\right) - \Psi\left(\frac{z}{2}\right), \text{ (**).$$

**Remark:** We note that  $\Psi(z) = -\frac{f'(z)}{f(z)}$ ,  $-\frac{f'(z)}{\Gamma'(z)} = f^2(z)$ ,  $f(z)$ ,  $\frac{\Psi(z)}{\Gamma(z)} = f(z) \cdot \Psi(z) = -e^{2\gamma z} \prod \left\{ 1 - \frac{z}{x_k} \right\}^{x_k}$ , and  $\frac{\Psi'(z)}{\Gamma^2(z)} = \frac{(\Psi^2)'(z)}{(\Gamma^2)'(z)}$  are entire, transcendental functions where  $\Psi(1/2) = -\gamma - 2\log 2$ , i.e.  $\Psi(1) - \Psi\left(\frac{1}{2}\right) = 2\log 2$ .

(\*) The approach also puts the spot on Schoenberg's theory of totally positive functions, the Laguerre-Polya class, and to the Riemann zeta function, (GrK), (LeB). Regarding the condition  $O(|x|^{\frac{1}{2}+\varepsilon})$  we note that the embedding of  $H_{\frac{1}{2}+\varepsilon}$  into  $C^0$  is compact, i.e. for  $u \in H_{\frac{1}{2}+\varepsilon}$  it holds  $O(|x|^{\frac{1}{2}+\varepsilon})$ .

(\*\*) We note the properties (appendix, (NiN))

$$\begin{aligned} \gamma + \Psi\left(\frac{p}{q}\right) &= -\log q - \frac{\pi}{2} \cot\left(\pi \frac{p}{q}\right) - \sum_{k=1}^{\frac{q-p}{2}} (\cos 2k\pi \frac{p}{q}) (\log(2 - 2\cos(2k\pi \frac{p}{q}))) \\ \beta\left(\frac{p}{q}\right) &= \frac{\pi}{2 \sin(\pi \frac{p}{q})} - \sum_{k=0}^{\frac{q-p}{2}} (\cos(2k+1)\pi \frac{p}{q}) (\log(2 - 2\cos((2k+1)\pi \frac{p}{q}))) \\ \beta(nx) &= \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \beta\left(x + \frac{k}{n}\right) \text{ for } n \text{ odd}, \quad -\beta(nx) = \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \Psi\left(x + \frac{k}{n}\right) \text{ for } n \text{ even.} \end{aligned}$$

**Remark:** With respect to the theory of irrational and transcendental numbers it is proposed to revisit the theory of differences accompanied with the above proposed „2-Semi-Circle“ method:

(NöN): A „difference of 1<sup>st</sup> order“ (or difference) of a function  $f(x)$  is given by the averages

$$\Delta_{\omega}f(x) := \frac{f(x+\omega)-f(x)}{\omega}, \quad \nabla_{\omega}f(x) := \frac{f(x+\omega)+f(x)}{2}.$$

One of the most famous example of a solution of the difference equation is given by the Gamma function as solution of the difference equation  $f(x+1) - xf(x) = 0$ . The concept of „differences“ is accompanied by the concept of „fonctions interpolaires“ („divided differences“, „Steigungen“), relating „differences“ to the Newton and Lagrange interpolation formulae.

The most important and difficult problem of differences calculus is the solution of the difference equation  $f(x+1) - f(x) = \varphi(x)$  for given  $\varphi(x)$ . In case  $\varphi(x)$  is a polynomial of a certain degree the two differences definitions,  $\Delta_{\omega}f(x)$ ,  $\nabla_{\omega}f(x)$ , lead to the Bernoulli and Euler polynomials.

The most general solution of the difference equation is given by the sum of a specific polynomial solution and an arbitrary, periodical function  $\pi(x)$  with period 1, i.e.  $\pi(x+1) - \pi(x) = 0$ . In other words, every solution, which is not a polynomial solution is a transcendental function, i.e. a polynomial solution is a special solution characterized by simple function theoretical properties.

There are many trivial solutions of the difference equations. The main challenge of difference theory is to find the inverse operation of the difference operation, i.e. a function  $f(x)$  as the sum of  $\varphi(x)$  in that way, that  $\varphi(x)$  is the difference of  $f(x)$ , whereby  $f(x)$  is a so-called main solution, i.e. function theory relevant analytical function (like the Gamma function). The theory of differences is also concerned with properties of such main solutions, like analytical continuity, representation as faculty series, reciprocal differences and reciprocal derivatives, the Thiele's interpolation formula, and the solution of 2<sup>nd</sup> order differences and ordinary differential equations by Continued Fractions (CF).

**Remark:** : For CF-representations involving  $\Psi(z)$  we refer to (WaH) p. 372, e.g.

$$\Psi\left(\frac{1}{2} + z\right) - \frac{\Psi(z) + \Psi(z+1)}{2} = \frac{\left(\frac{1}{2}\right)^3}{z^2 + \frac{p_1}{1 + \frac{q_1}{z^2 + \frac{p_2}{1 + \frac{q_2}{z^2 + \dots}}}}}$$

with  $p_n = q_n = \frac{n(n+1)}{4}$ . From (NöN) S. 454, we recall the reciprocal differences and the related Continued Fraction (CF) representation of the function  $\Psi(x)$ , ( $\text{Re}(2x+y) > 1$ ),

$$\rho^{2n-1}(x-n+1, x-n+2, \dots, x+n) = n^2x, \quad \rho^{2n}(x-n, x-n+1, \dots, x+n) = \Psi(x) + 2\left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right)$$

$$\Psi(x+y) = \Psi(x) + \frac{y}{x + \frac{y-1}{2 + \frac{y+1}{1 + \frac{y-2}{3x + \frac{y+2}{2 + \frac{y-3}{5x + \frac{y+3}{3 + \dots}}}}}}} \quad H_n = \frac{n}{1 + \frac{n-1}{2 + \frac{n+1}{1 + \frac{n-2}{3 + \frac{n+2}{2 + \frac{n-3}{5 + \frac{n-3}{3 + \dots}}}}}}}$$

Taking the limit of  $\Psi(x+y) - \Psi(x)$  leads to, (NöN) S. 454, (WaH) p.373,

$$\Psi'(x) = \frac{1}{x - \frac{1}{2 + \frac{1^2}{2 + \frac{2^2}{3x - \frac{2^2}{2 + \frac{3^2}{5x - \frac{3^2}{2 + \dots}}}}}}} \quad g(x) = \frac{1}{x - \frac{1}{2 + \frac{1}{x - \frac{2}{2 + \frac{3}{x - \frac{3}{2 + \frac{3}{x - \dots}}}}}}}$$

**Remark:** For CF-representations of the considered Kummer functions we refer to ((PeO) §81 ff., (WaH) p. 347. For the CF representation of the „error term“  $J(z)$  in ( $\text{Re}(z) > 0$ )

$$\log \Gamma(z) = -z + \left(z - \frac{1}{2}\right) \log z + \log \sqrt{2\pi} + J(x)$$

we refer to (WaH) p. 364.

**Remark:** Beside the reciprocal differences of  $\Psi(z)$  another link to the harmonic series,  $H_n := \sum_{k=1}^n \frac{1}{k}$ ,  $H_0 := 0$ , is given by the curvature vector  $\rho$  at the zeros of  $\frac{1}{\Gamma(z)}$ . For the curvature vector  $\rho(-n)$  of the purely imaginary streamlines through the zeros  $z_n = -n$  of  $f(z)$  (and therefore also for the curvature vector  $\rho(-n)$  through the zeros of  $\Gamma(z)$ ) it holds, (Gil)

$$\rho(-n) = \frac{1}{2(\gamma - H_n)} \sim -\frac{1}{2\Psi(1+n)} \sim -\frac{1}{2\log(n+1/2)}.$$

With respect to the generalization of Lagrange's periodical CF criterion for algebraic numbers of 2<sup>nd</sup> order we refer to (MiH).

With respect to the measure theory of continued fractions we recall from (KhA) extracts from

### Chapter III, 11, The Measure Theory of Continued Fractions

Beside the basic divisions of the real numbers into rational and irrational or algebraic and transcendental numbers, there are several considerably finer subdivisions of these numbers based on a whole series of criteria characterizing their arithmetic nature (most importantly, criteria involving the approximation by rational fractions that these numbers admit). Thus, we know that numbers exist admitting approximation by rational fractions of the form  $p/q$  with order of accuracy not exceeding  $1/q^2$  (for example, all quadratic irrational numbers); but we also know that there exists numbers admitting approximation of much higher order (Theor. 22, Chap. II). The following question naturally arises: which of these opposite properties should we consider the more „general“, that is, which of these two types of real numbers do we „encounter more often“? ... The question is ... clearly reduces to a comparative study of the two sets, (1) the set of numbers possessing a property, (2) the set of numbers not possessing it, with the purpose of determining which of them contains more numbers. ... As regards both methods and results, the study of the measure of sets of numbers defined by a given property of their elements has proven the most interesting. This study ... we shall call *the measure arithmetic of the continuum* ... As with every study of the arithmetic nature of irrational numbers, the apparatus of continued fractions is the most natural and the best investigating instrument. ... We must, in other words, learn to determine the measure of numbers whose expansions in CF possess some previously stated property. Questions of this kind can be quite varied; ... The methods used in solving problems such as these constitute *the measure theory of continued fractions*.

### Chapter III, 15, Gauß's problem and Kusmin's theorem

Setting  $\alpha = [0; \alpha_1, \alpha_2, \dots, \alpha_n, \dots]$ ,  $r_n = r_n(\alpha) = [\alpha_n; \alpha_{n+1}, \alpha_{n+2}, \dots]$  we denote by  $z_n = z_n(\alpha)$  the value of the continued fraction  $[0; \alpha_{n+1}, \alpha_{n+2}, \dots]$ . Obviously we

$$0 \leq z_n = r_n - a < 1.$$

We denote by  $m_n(x)$  the measure of the set of numbers  $\alpha$  in the interval  $(0,1)$  for which  $z_n(\alpha) < x$ , satisfying the functional equation

$$m'_{n+1}(x) = \sum_{k=1}^{\infty} \left\{ m_n\left(\frac{1}{k}\right) - m_n\left(\frac{1}{k+x}\right) \right\}, \quad 0 \leq x \leq 1, \quad n \geq 0.$$

In 1928, Kusmin published his proof of the Gauß problem and gave a good approximation estimate, i.e. it holds

$$\lim_{n \rightarrow \infty} m_n(x) = \frac{\log(1+x)}{\log 2}, \quad 0 \leq x \leq 1$$

and

$$\left| m_n(x) - \frac{\log(1+x)}{\log 2} \right| < A_1 e^{-A_2 \sqrt{n}}, \quad A_1, A_2 \text{ absolute positive constants.}$$

**Remark:** Kusmin's approximation estimate can be applied to obtain an approximation of the measure of the set of points for which  $a_n = k$  for sufficiently large values of  $n$ , given by

$$\left| ME\left(\frac{n}{k}\right) - \frac{\log\left(1 + \frac{1}{k(k+2)}\right)}{\log 2} \right| < \frac{A_1}{k(k+1)} e^{-A_2 \sqrt{n-1}}.$$

**Remark:** The central element of the proof of the Gauß conjecture is *Theorem 33* ((KhA) p. 74). It also enables a more general result regarding the measure  $M_n(x)$  of the set of numbers belonging to some fixed interval of rank  $k$  and satisfying the condition  $z_{k+n} < x$ .

### Chapter III, 16, Average values

The solution of the Gauß problem enable a proof of the following general proposition

**Theorem 35:** Suppose that  $f(r)$  is a non-negative function of a natural argument  $r$  and suppose that there exist positive constants  $C$  and  $\delta$  such that

$$f(r) < Cr^{\frac{1}{2}-\delta}, \quad r = 1, 2, \dots$$

Then, for all numbers in the interval  $(0,1)$  with the exception of a set of measure zero,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n f(a_k) = \sum_{k=1}^n f(r) \frac{\log\left(1 + \frac{1}{k(k+2)}\right)}{\log 2}.$$

**Remark:** Theorem 33 enables the establishment of quite a number of properties of continued fractions that are satisfied for almost all irrational numbers. For example, let us set

$$f(r) := 1, \text{ for } r = k, \text{ and } f(r) := 0, \text{ for } r \neq k,$$

where  $k$  is some (arbitrary) natural number. In this case, the sum

$$\psi_n(k) := \sum_{i=1}^n f(a_i)$$

obviously represents the number of times the integer  $k$  occurs among the first  $n$  elements of a given continued fraction. On the other hand, the ratio

$$\frac{\psi_n(k)}{n} = \frac{1}{n} \sum_{i=1}^n f(a_i)$$

gives us the density of the number  $k$  among the first  $n$  elements of the given continued fraction. Finally, the limit

$$\lim_{n \rightarrow \infty} \frac{\psi_n(k)}{n} = d(k),$$

if it exists, is naturally interpreted as the density of the number  $k$  in the entire sequence of elements of a given continued fraction.

Since the function  $f(r)$  that we have defined clearly satisfies all of the requirements of *Theorem 35*, we conclude, on the basis of that theorem, that, *for arbitrary  $k$ , this density exists almost everywhere and that it has the same value almost everywhere*. Furthermore, the same theorem makes it possible for us to calculate the value of that density. Obviously we have almost everywhere

$$d(1) = \frac{\log 4 - \log 3}{\log 2}, \quad d(2) = \frac{\log 9 - \log 8}{\log 2}, \quad d(3) = \frac{\log 16 - \log 15}{\log 2},$$

and so on. Thus, an arbitrary natural number occurs as an element in the expansion of almost all numbers with equal average frequency.

We obtain another interesting result by setting

$$f(r) := \log r \text{ for } r = 1, 2, 3, \dots$$

All the conditions of *Theorem 35* are then satisfied. Therefore, we see that almost everywhere

$$\frac{1}{n} \sum_{i=1}^n \log(a_i) \rightarrow \sum_{k=1}^{\infty} \log(r) \frac{\log(1 + \frac{1}{k(k+2)})}{\log 2}, \quad n \rightarrow \infty,$$

or, equivalently,

$$g(\sqrt[n]{a_1 a_2 \dots a_n}) \rightarrow \prod \left\{ 1 + \frac{1}{k(k+2)} \right\}^{\frac{\log r}{\log 2}}.$$

Thus, the geometric mean of the first elements approaches the absolute constant

$$\prod \left\{ 1 + \frac{1}{k(k+2)} \right\}^{\frac{\log r}{\log 2}} = 2, 6, \dots$$

almost everywhere as  $n \rightarrow \infty$ .

Obviously *Theorem 35* makes it possible to establish analogous results for a series of other types of average values. However, investigating of arithmetic mean

$$(*) \quad \frac{1}{n} \sum_{i=1}^n a_i$$

by this method is impossible, because the corresponding function  $f(r) := \log r$  does not satisfy the conditions of *Theorem 35*. However, it is easy to see from more elementary considerations that, almost everywhere, the expression (\*) cannot have any kind of finite limit. The *Theorem 30* (sec 13) tells us that almost everywhere  $a_n > n \log n$  for an infinite number of values of  $n$ , and hence, a fortiori,

$$\sum_{i=1}^n a_i > n \log n, \quad \text{and hence, } \frac{1}{n} \sum_{i=1}^n a_i > \log n.$$

Thus, the quantity (\*) is almost everywhere unbounded and therefore, as we stated, cannot have finite limit.

With respect to orthogonal polynomials on the unit circle we recall from (SzG) the section

### 11.1 Definition; preliminaries

Let  $f(\theta)$  be a non-negative function of period  $2\pi$ , integrable on  $[-\pi, \pi]$  in Lebesgue's sense, and assume  $\int_{-\pi}^{\pi} f(\theta) d\theta > 0$ . We introduce the Fourier constants

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cdot e^{-in\theta} d\theta \quad n = 0, \pm 1, \pm 2, \dots$$

Obviously,  $c_{-n} = \bar{c}_n$  so that the matrix of "Toeplitz type"  $T_n = (c_{\nu-\mu})$ ,  $\nu, \mu = 0, 1, 2, \dots, n$  is Hermitian. The corresponding Hermitian form

$$H_n = \sum_{\nu=0}^n \sum_{\mu=0}^n c_{\nu-\mu} u_{\nu} \bar{u}_{\mu} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cdot |u_0 + u_1 z + u_2 z^2 + \dots + u_n z^n|^2 d\theta$$

where  $z = e^{i\theta}$ , is positive definite and has the positive determinant  $D_n = [c_{\nu-\mu}]$ ,  $\nu, \mu = 0, 1, 2, \dots, n$ .

*Definition:* If we orthogonalize the system

$$\{z^n \cdot \sqrt{f(\theta)}\}, \quad z = e^{i\theta} \quad n = 0, 1, 2, \dots, n$$

we obtain a system of polynomials  $\varphi_0(z), \varphi_1(z), \varphi_2(z), \dots, \varphi_n(z), \dots$  with the following properties:

- i)  $\varphi_n(z)$  is a polynomial of precise degree  $n$  in which the coefficients of  $z^n$  is real and positive;
- ii) the system  $\{\varphi_n(z)\}$  is orthogonal; that is,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \varphi_n(z) \bar{\varphi}_m(z) d\theta = \delta_{nm}, \quad z = e^{i\theta}, \quad n, m = 0, 1, 2, \dots$$

Moreover, the system  $\{\varphi_n(z)\}$  is uniquely determined by the conditions i) and ii). If  $f(\theta)$  is an even function the coefficients of  $\varphi_n(z)$  are real. The coefficient of  $z^n$  in  $\varphi_n(z)$  is  $\sigma_n = \sqrt{D_{n-1} D_n^{-1}}$ .

*Theorem 11.1.1:* Let  $F(e^{-i\theta})$  be a given measurable function for which  $\int_{-\pi}^{\pi} f(\theta) |F(e^{-i\theta})|^2 d\theta$  exists. The weighted quadratic deviation

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) |F(z) - \rho(z)|^2 d\theta$$

where  $\rho(z)$  ranges over the set of all (polynomials of degree  $n$ )  $\pi_n$ , is a minimum, if  $\rho(z)$  is the  $n^{\text{th}}$  partial sum of the Fourier expansion

$$F(z) \sim F_0 \varphi_0(z) + F_1 \varphi_1(z) + \dots + F_n \varphi_n(z) + \dots$$

$$F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) F(z) \overline{\varphi_n(z)} d\theta, \quad z = e^{i\theta}, \quad n = 0, 1, 2, \dots$$

As a ready consequence, there follows Bessel's inequality

$$|F_0|^2 + |F_1|^2 + \dots + |F_n|^2 + \dots \leq \int_{-\pi}^{\pi} f(\theta) |F(e^{-i\theta})|^2 d\theta.$$

In addition, the equality sign holds (i.e. the Parseval's formula) if one of the following sets of conditions is satisfied:

- i)  $F(z)$  is regular and bounded for  $|z| < 1$
- ii)  $f(\theta)$  is bounded and  $F(z)$  is of the class  $H_2$  (see §10.1; Fatou's theorem)

A consequence of theorem 11.1.1 is the following

*Theorem 11.1.2:* The polynomial  $\sigma_n^{-1} \varphi_n(z)$  minimizes the integral

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \cdot |z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n|^2 d\theta, \quad z = e^{i\theta}$$

if  $z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$  ranges over the set of all  $\pi_n$  with the highest term  $z^n$ . The minimum is  $\sigma_n^{-2}$ .

#### 6.4 Theorem of Pólya-Szegő on trigonometric polynomials with monotonic coefficients

**Theorem 6.4:** Let  $a_0 > a_1 > \dots > a_n > 0$ . Then the functions

$$f(t) = a_0 \cos(mt) + a_1 \cos((m-1)t) + \dots + a_{n-1} \cos t + a_n$$

$$g(t) = a_0 \cos\left(m + \frac{1}{2}\right)t + a_1 \cos\left(m - \frac{1}{2}\right)t + \dots + a_{n-1} \cos \frac{3}{2}t + a_n \cos\left(\frac{1}{2}t\right)$$

have only real and simple zeros; there is, respectively, exactly one zero in each of the intervals

$$\frac{\mu - \frac{1}{2}}{m+1/2} \pi < t < \frac{\mu + \frac{1}{2}}{m+1/2} \pi \quad \text{and} \quad \frac{\mu - \frac{1}{2}}{m+1} \pi < t < \frac{\mu + \frac{1}{2}}{m+1} \pi$$

where  $\mu = 1, 2, \dots, 2m$ , and  $\mu = 1, 2, \dots, 2m + 1$ , respectively.

**Remark:** Bagchi's "Hilbert space based reformulation of the Nyman-Beurling RH criterion" (BaB), is about the Hilbert space of all sequences  $a = \{a_n | n \in \mathbb{N}\}$  of complex numbers such that

$$\sum_{n=1}^{\infty} \sigma_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \leq \sigma_n \leq \frac{c_2}{n^2}$$

which is isomorph to the Hilbert space  $H_{-1} \cong l_2^{-1}$ . For  $\gamma = \{1, 1, 1, 1, \dots\}$  it holds  $\|\gamma\|_{-1}^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ , i.e.  $\gamma \in l_2^{-1}$ .

**Remark:** In order to link Bagchi's conclusion text (\*) in (BaB) to the above Riesz basis we mention one of the characteristic properties of Riesz bases (YoR) p. 27:

*The sequence  $\{\lambda_n\}$  is complete in  $H$ , and its Gram matrix  $((\lambda_i, \lambda_j))_{i,j=1}^{\infty}$  generates a bounded invertable operator on  $\{l^2\}$ .*

**Remark:** In order to link the considered vector  $\gamma = \{1, 1, 1, 1, \dots\}$  in (BaB) we note that the Haar wavelet  $\text{sinc}(x)$ -function with its infinite product representation in the form

$$\text{sinc}(\pi x) = e^{-i\pi x} \prod (F_{0,m} \left(\frac{x}{m^n}\right)) = \prod \left(1 - \left(\frac{x^2}{n^2}\right)\right) = \frac{1}{\Gamma(1+x)\Gamma(1-x)}, \quad \text{where} \quad F_{0,m}(x) := \frac{1}{m} \sum_{k=0}^{m-1} e^{2\pi i k x}$$

is the Fourier transform of the scaling vector  $\gamma$ , (ReH) p.145.

(\*) „So where does the undoubtedly elegant reformulation of RH in Theorem 1 leave us? One possible approach is as follows. For positive integers  $L$ , let  $D(L)$  denote the distance of the vector  $\gamma \in H$  from the  $(L-1)$ -dimensional subspace of  $H$  spanned by  $\gamma_1, \gamma_2, \gamma_3, \dots, \gamma_L$ . In view of Theorem 1, RH is equivalent to the statement  $D(L) \rightarrow 0$  as  $L \rightarrow \infty$ . So one might try to estimate  $D(L)$ . Indeed, as a discrete analogue of a conjecture of Baez-Duarte et. al. in [3], one might expect that  $D^2(L)$  is asymptotically equal to  $\frac{A}{\log L}$  for  $A = 2 + C - \log(4\pi)$ , where  $C$  is Euler's constant. (But, of course, this is far stronger than RH itself.) A standard formula gives  $D^2(L)$  as a ratio of two Gram determinants, i.e., determinants with the inner products  $(\gamma_i, \gamma_m)$  as entries. It is easy to write down these inner products as finite sums involving the logarithmic derivative of the Gamma function. But such formulae are hardly suitable for calculation/estimation of determinants.“

In the late 1930s A. Beurling observed that the entire function of exponential type  $2\pi$

$$B(z) = \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=0}^{\infty} (z-n)^{-2} - \sum_{m=-\infty}^{-1} (z-m)^{-2} - 2z^{-1} \right\}$$

satisfies simple and useful extremal property (\*). It holds  $B(z) \geq \operatorname{sgn}(x)$  for all real  $x$ . Beurling showed that if  $F(z)$  is any entire function of exponential type  $2\pi$  satisfying  $F(x) \geq \operatorname{sgn}(x)$  for all real  $x$ , then

$$\int_{-\infty}^{\infty} F(x) - \operatorname{sgn}(x) dx \geq 1$$

and equality occurs if and only if  $F(z) = B(z)$ .

In 1974 A. Selberg used the function  $B(z)$  to obtain a sharp form of the large sieve inequality, see e.g. (CaJ1). Selberg noted that if  $\sigma_I$  is the characteristic function of the interval  $I := ]a, b]$  and

$$C_I(z) := \frac{1}{2} \{ B(\beta - z) + B(z - \alpha) \}$$

then

$$C_I(z) \geq \sigma_I$$

for all real  $x$ . Moreover, the Fourier transform

$$\hat{C}_I(z) = \int_{-\infty}^{\infty} C_I(x) e(-tx) dx$$

vanishes outside the interval  $[-1, 1]$  (\*).

Beurling found the following interesting inequality for almost periodic functions:

**Theorem 15** (VaJ): Let  $\lambda_1, \lambda_2, \dots, \lambda_N$  be real numbers and

$$f(x) := \sum_{n=1}^N a(n) e(\lambda_n x) = \sum_{n=1}^N a(n) e^{2\pi i \lambda_n x}.$$

If  $|\lambda_n| \geq \delta > 0$  for  $n = 1, 2, \dots, N$ , then

$$\sup_x |f(x)| \leq (4\delta)^{-1} \sup_{\omega} |f'(\omega)|.$$

If, additionally,  $f(x)$  is real valued, then

$$\sup_x |f(x)| \leq (2\delta)^{-1} \sup_{\omega} |f'(\omega)|.$$

Moreover, the constants  $(4\delta)^{-1}$  and  $(2\delta)^{-1}$  are asymptotically best possible as  $N \rightarrow \infty$ .

In (VaJ) the Beurling function is applied for a simple proof of a general form of Hilbert's inequality, which was first obtained by Montgomery and Vaughan, (see also (SeA1):

**Theorem 16** (VaJ): Let  $\lambda_1, \lambda_2, \dots, \lambda_N$  be real numbers satisfying  $|\lambda_m - \lambda_n| \geq \delta > 0$  whenever  $m \neq n$ , and let  $a(1), \dots, a(n)$  be arbitrary complex numbers. Then

$$\left| \sum_{\substack{m=1 \\ m \neq n}}^N \sum_{\substack{n=1 \\ n \neq m}}^N \frac{a(m) \overline{a(n)}}{\lambda_m - \lambda_n} \right| \leq \frac{\pi}{\delta} \sum_{n=1}^N |a(n)|^2.$$

(\*) (GrS) appendix, see also (MoH) chapter 2: „van der Corput's sets“

We briefly sketch the main formulae and their relationships, (GrS), (VaJ):

Putting

$$H(z) := \left(\frac{\sin \pi z}{\pi}\right)^2 \left\{ \sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2} + \frac{2}{z} \right\},$$

$$J(x) := \frac{1}{2} H'(z), \quad zK(z) := \left(\frac{\sin \pi z}{\pi}\right)^2$$

and

$$E(x) = H(x) - \operatorname{sgn}(x) \text{ for real } x,$$

it holds:

**Lemma A.1:** If  $x$  is real then  $|H(x)| \leq 1$  and  $|E(x)| \leq K(x)$ .

**Lemma A.2:** The Fourier transform  $\hat{f}(t)$  satisfies

$$\hat{f}(t) = \begin{cases} 1 & \text{if } t = 0 \\ \pi t(1 - |t|) \cot(\pi t) + |t|, & \text{if } 0 < |t| < 1 \\ 0 & \text{if } |t| \geq 1 \end{cases}$$

**Corollary A.3:** The Fourier transform of  $E(x) = H(x) - \operatorname{sgn}(x)$  is

$$\hat{E}(t) = \begin{cases} 0 & \text{if } t = 0 \\ \frac{\hat{f}(t) - 1}{\pi i t} & \text{if } t \neq 0 \end{cases}$$

**Theorem A.4:** Define  $B(z) := H(z) + K(z)$  and  $b(z) := H(z) - K(z)$ . Let  $I$  be the interval  $[\alpha, \beta]$  and let  $\sigma_I$  be the characteristic function of  $I$ . Finally, define

$$C_I(z) := \frac{1}{2} \{B(\beta - z) + B(z - \alpha)\}, \quad c_I(z) := \frac{1}{2} \{b(\beta - z) + b(z - \alpha)\}.$$

- i) If  $x$  is real, then  $b(x) \leq \operatorname{sgn}(x) \leq B(x)$
- ii) If  $x$  is real, then  $c_I(x) \leq \sigma_I(x) \leq C_I(x)$
- iii)  $\hat{c}_I(0) = \beta - \alpha - 1$  and  $\hat{C}_I(0) = \beta - \alpha + 1$
- iv) If  $|t| \geq 1$  then  $\hat{c}_I(t) = 0$  and  $\hat{C}_I(t) = 0$ .

**Theorem A.5:** Let  $N$  and  $T$  be positive real numbers, and let  $a_n$  be a sequence of complex numbers with  $a_n = 0$  if  $n \leq N$  or  $n > 2N$ . Suppose  $g$  is a real-valued function with  $|g(m) - g(n)| > \delta$  whenever  $N < m, n \leq 2N$  and  $m \neq n$ . Then

$$(T - \delta^{-1}) \sum_n |a_n|^2 \leq \int_T^{2T} |\sum_n a_n e(itg(n))|^2 dt \leq (T + \delta^{-1}) \sum_n |a_n|^2.$$

In 1985, J. Vaaler showed how Beurling's function could be used to construct trigonometric polynomial approximation to  $\psi(x) := \rho(x) - \frac{1}{2} = x - [x] - \frac{1}{2} = -\sum_1^\infty \frac{\sin 2\pi vx}{\pi v}$ , (GrS) appendix:

**Theorem A.6:** The trigonometric polynomial

$$\psi^*(x) = -\sum_{1 \leq |n| \leq N} \frac{1}{2\pi i n} \hat{f}_{N+1}(n) e(nx)$$

satisfies

$$|\psi(x) - \psi^*(x)| \leq \frac{1}{N+1} \sum_{1 \leq |n| \leq N} \left(1 - \frac{|n|}{N+1}\right) e(nx).$$



Turán's first main theorem, (see (MoH) chapter 5), is concerned with lower bounds of exponential sums, resulting into the representative application of

**Fabry's Gap Theorem:** Suppose that  $T(x)$  is an exponential polynomial of  $N$  terms and period 1, say

$$T(x) = \sum_{n=1}^N b_n e(\lambda_n x)$$

where the  $\lambda_n$  are integers. Let  $I$  be a closed arc on the circle group  $T$ , and let  $L$  denote the length of  $I$ . Then

$$\max_{x \in I} |T(x)| \geq \left(\frac{L}{2e}\right)^{N-1} \max_{x \in T} |T(x)|.$$

**Remark:** The striking feature of this bound is that it does not depend on the size of the  $\lambda_n$ .

## Appendix

**Remark:** The  $\zeta(s)$  function representation, (TiE) (2.1.5),

$$\zeta(s) = s \int_0^{\infty} \frac{[x]-x}{x^s} \frac{dx}{x} = -s \int_0^{\infty} x^{-s} \rho(x) \frac{dx}{x} = \int_0^{\infty} x^{1-s} (-\rho'(x)) \frac{dx}{x} \quad 0 < \operatorname{Re}(s) < 1$$

in combination with the formulas, (Grl) 3.761,

$$\int_0^{\infty} x^s \sin(ax) \frac{dx}{x} = \frac{\Gamma(s)}{a^s} \sin\left(\frac{\pi}{2}s\right) \text{ for } 0 < |\operatorname{Re}(s)| < 1 \text{ and } \int_0^{\infty} x^s \cos(x) \frac{dx}{x} = \frac{\Gamma(s)}{a^s} \cos\left(\frac{\pi}{2}s\right) \text{ for } 0 < \operatorname{Re}(s) < 1$$

leads to (see also (PeB))

$$\int_0^{\infty} x^{1-s} (-\rho'_H(x)) \frac{dx}{x} = \int_0^{\infty} x^{1-s} \cot(\pi x) \frac{dx}{x} = \zeta(1-s)(2\pi)^{s-1} \Gamma(1-s) \sin\left(\frac{\pi}{2}(1-s)\right), \text{ for } 0 < |\operatorname{Re}(s)| < 1$$

resp.

$$(*) \quad \int_0^{\infty} x^s (-\rho'_H(x)) \frac{dx}{x} = \int_0^{\infty} x^s \cot(\pi x) \frac{dx}{x} = \zeta(s)(2\pi)^{-s} \Gamma(s) \sin\left(\frac{\pi}{2}s\right) = \zeta(s) \tan\left(\frac{\pi}{2}s\right) (2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi}{2}s\right).$$

Applying the duality equations in the form

$$\zeta(s) = \chi(s)\zeta(1-s) \text{ resp. } \zeta(1-s) = \chi(1-s)\zeta(s)$$

with

$$\chi(s) = \frac{1}{\pi} (2\pi)^s \Gamma(1-s) \sin\left(\frac{\pi}{2}s\right) \text{ resp. } \chi(1-s) = 2(2\pi)^{-s} \Gamma(s) \cos\left(\frac{\pi}{2}s\right)$$

results into the following Mellin transform representations

$$\int_0^{\infty} x^{1-s} \pi \cot(\pi x) \frac{dx}{x} = \zeta(s) \frac{\pi}{2} \cot\left(\frac{\pi}{2}s\right) = \frac{\zeta(s)}{s} \left[ \left(\frac{\pi}{2}s\right) \cot\left(\frac{\pi}{2}s\right) \right] \text{ resp. } \int_0^{\infty} x^s \pi \cot(\pi x) \frac{dx}{x} = \zeta(1-s) \frac{\pi}{2} \tan\left(\frac{\pi}{2}s\right).$$

We note the formulas

$$\pi \cot(\pi x) = \frac{\pi}{2} \cot\left(\frac{\pi}{2}x\right) - \frac{\pi}{2} \cot\left(\frac{\pi}{2}(1-x)\right) \text{ and } \left(\frac{\pi}{2}x\right) \cot\left(\frac{\pi}{2}x\right) = 1 - \sum_{k=1}^{\infty} \frac{|B_{2k}|}{(2k)!} (\pi x)^{2k}, \quad x^2 < 4.$$

From the latter one the following approximation formula for  $\zeta(s)$  can be derived:

$$\zeta(s) \approx s \int_0^{\infty} x^{1-s} \pi \cot(\pi x) \frac{dx}{x} = M \left[ \frac{\pi x}{\sin^2(\pi x)} \right] (1-s) = \int_0^{\infty} x^{1-s} \frac{\pi x}{\sin^2(\pi x)} \frac{dx}{x}.$$

The duality equation

$$\zeta(s) = \frac{1}{\pi} (2\pi)^s \Gamma(1-s) \sin\left(\frac{\pi}{2}s\right) \zeta(1-s) = \frac{\frac{s}{\pi^2} \Gamma\left(\frac{1-s}{2}\right)}{\frac{1-s}{\pi^2} \Gamma\left(\frac{s}{2}\right)} \zeta(1-s)$$

In combination with the formulas

$$\begin{aligned} \sin\left(\frac{\pi}{2}s\right) \Gamma(s) &= \sqrt{\pi} 2^{s-1} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \tan\left(\frac{\pi}{2}s\right) \\ \cos\left(\frac{\pi}{2}s\right) \Gamma(1-s) &= \sqrt{\pi} 2^{-s} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \cot\left(\frac{\pi}{2}s\right) \end{aligned}$$

results into the following representations

$$\begin{aligned} \int_0^{\infty} x^{1-s} \cot(\pi x) \frac{dx}{x} &= \frac{1}{2} \cot\left(\frac{\pi}{2}s\right) \zeta(s) = \frac{(2\pi)^s}{2\pi} \cos\left(\frac{\pi}{2}s\right) \Gamma(1-s) \zeta(1-s) = \frac{1}{2} \frac{\frac{s}{\pi^2} \Gamma\left(\frac{1-s}{2}\right)}{\frac{1-s}{\pi^2} \Gamma\left(\frac{s}{2}\right)} \cot\left(\frac{\pi}{2}s\right) \zeta(1-s) \\ \int_0^{\infty} x^s \cot(\pi x) \frac{dx}{x} &= \frac{1}{2} \tan\left(\frac{\pi}{2}s\right) \zeta(1-s) = \frac{(2\pi)^{1-s}}{2\pi} \sin\left(\frac{\pi}{2}s\right) \Gamma(s) \zeta(s) = \frac{1}{2} \frac{\frac{1-s}{\pi^2} \Gamma\left(\frac{s}{2}\right)}{\frac{s}{\pi^2} \Gamma\left(\frac{1-s}{2}\right)} \tan\left(\frac{\pi}{2}s\right) \zeta(s) \end{aligned}$$

which we summarize in the

**Lemma:** In the critical stripe the distribution valued Mellin transform of the (in a weak  $H_{-1/2}^{\#}(0,1)$  sense convergent (\*)) Fourier series representation of  $\cot(\pi x) = \sum_1^{\infty} \sin 2\pi vx$  is given by

- i)  $\frac{1}{\sqrt{\pi}} \int_0^{\infty} x^{1-s} \cot(x) \frac{dx}{x} = \frac{\pi^{1/1-s}}{2} \cot\left(\frac{\pi}{2}s\right) \zeta(s) = \frac{1}{2} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \cot\left(\frac{\pi}{2}s\right) \zeta(1-s)$
- ii)  $\frac{1}{\sqrt{\pi}} \int_0^{\infty} x^s \cot(x) \frac{dx}{x} = \frac{\pi^{s-1/2}}{2} \tan\left(\frac{\pi}{2}s\right) \zeta(1-s) = \frac{1}{2} \frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \tan\left(\frac{\pi}{2}s\right) \zeta(s).$

**Corollary:** In the critical stripe it holds

$$\Gamma\left(\frac{s}{2}\right) \cot\left(\frac{\pi}{2}(1-s)\right) \int_0^{\infty} x^{1-s} \cot(\sqrt{\pi}x) \frac{dx}{x} = \Gamma\left(\frac{1-s}{2}\right) \cot\left(\frac{\pi}{2}s\right) \int_0^{\infty} x^s \cot(\sqrt{\pi}x) \frac{dx}{x}.$$

**Remark (NiN):** For the tool set functions

$$\log \Gamma(x) = \int_1^x \Psi(t) dt, \nu(x) := \log x - \Psi(x),$$

$$\beta(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{x+k} = \frac{1}{2} \left( \Psi\left(\frac{x+1}{2}\right) - \Psi\left(\frac{x}{2}\right) \right), \vartheta(x) := \frac{1}{2} - x + [x] = \sum_1^{\infty} \frac{\sin 2\pi \nu x}{\pi \nu}$$

the following properties are valid

- i)  $\Psi(x) = -\gamma + \sum_{k=0}^n \left( \frac{1}{1+k} - \frac{1}{x+k} \right), \Psi\left(\frac{1}{2}\right) = -\gamma - 2 \log 2, \Psi(1) = -\gamma, \Psi(n+1) = -\gamma + \sum_{k=1}^n \frac{1}{k}$
- ii)  $\Psi(1-x) - \Psi(x) = \pi \cot(\pi x), \Psi(x+1) - \Psi(x) = \frac{1}{x}, \Psi(x) = \lim_{n \rightarrow \infty} \left( \log n - \sum_{k=0}^{n-1} \frac{1}{x+k} \right)$
- iii)  $\Psi(s) = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right), \Psi'(s) = \frac{1}{s} + O\left(\frac{1}{|s|^2}\right), \frac{1}{2s} |\arg(s)| \leq \pi - \delta, \delta > 0, (\text{PrK}) \S 7$
- iv)  $\Psi^{(0)}(1) = -\gamma, \text{ but } \Psi^{(n)}(1) = (-1)^{n+1} n! \zeta(n+1), \text{ resp. } \Psi(1+z) = -\gamma + \sum_{n=1}^{\infty} (-1)^{n+1} \zeta(1+n) z^n, |z| < 1$
- v)  $\frac{1}{n} \sum_{k=0}^{n-1} \Psi\left(x + \frac{k}{n}\right) = \Psi(nx) - \log n, (\text{NöN}) \text{ chapter V, } \S 1, \text{ multiplication theorem}$
- vi)  $\beta(x) = \int_0^1 \frac{t^{x-1}}{1+t} dt = \int_0^{\infty} \frac{e^{-tx}}{1+e^{-t}} dt, \beta(x) + \beta(1-x) = \frac{\pi}{\sin \pi x}, \beta(1) = \log 2, \beta\left(\frac{1}{2}\right) = \frac{\pi}{2}, \lim_{\text{Re}(x) \rightarrow \infty} \beta(x) = 0$
- vii)  $\beta(x) = \frac{1}{2n+1} \sum_{k=1}^{2n} (-1)^k \beta\left(\frac{x+k}{2n+1}\right), \beta(x) = \sum_{k=1}^{\infty} \frac{k!}{x(x+1)\dots(x+k)} \frac{1}{2^{k+1}}, \beta(x+1) + \beta(x) = \frac{1}{x}$
- viii)  $\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \beta\left(x + \frac{k}{n}\right) = \beta(nx)$  for  $n$  odd,  $\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k \Psi\left(x + \frac{k}{n}\right) = -\beta(nx)$  for  $n$  even, (NöN) chapter V, §1
- ix)  $\Psi(2x) = \frac{1}{2} \left[ \Psi\left(x + \frac{1}{2}\right) + \Psi(x) \right] + \log 2, \beta(2x) = \frac{1}{2} \left[ \Psi\left(x + \frac{1}{2}\right) - \Psi(x) \right], (\text{NöN}) \text{ chapter V, } \S 1$
- x)  $\nu(x) = \int_0^{\infty} \left[ \frac{1}{e^t-1} - \frac{1}{t} + 1 \right] e^{-tx} dt = \sum_{k=0}^{\infty} \left( \frac{1}{x+k} - \log\left(1 + \frac{1}{x+k}\right) \right), \nu(x) = \frac{1}{2x} - \int_0^{\infty} \frac{\vartheta(t)}{(t+x)^2} dt$
- xi)  $\Psi\left(\frac{p}{q}\right) + \gamma = \log q - \frac{\pi}{2} \cot\left(\pi \frac{p}{q}\right) + \sum_{k=1}^{\frac{q}{2}} \cos\left(\frac{2k\pi}{q} p\right) \left( \log\left(2 - 2 \cos\left(\frac{2k\pi}{q}\right)\right) \right) \quad p, q \in \mathbb{N}, p < q$
- xii)  $\beta\left(\frac{p}{q}\right) = \frac{\pi}{2 \sin\left(\frac{\pi p}{q}\right)} - \sum_{k=0}^{\frac{q-1}{2}} \left( \cos\left(\frac{2k+1}{q} \pi\right) p \right) \left( \log\left(2 - 2 \cos\left(\frac{(2k+1)\pi}{q}\right)\right) \right), \text{ because of } 2\beta\left(\frac{p}{q}\right) = \Psi\left(\frac{p+q}{2q}\right) - \Psi\left(\frac{p}{2q}\right)$
- xiii)  $\Psi(x) = \log x - \frac{1}{2x} + 2 \int_0^{\infty} \frac{1}{x^2+t^2} \frac{t dt}{(1-e^{-2\pi t})}, \Psi\left(x + \frac{1}{2}\right) = \log x + 2x \int_0^{\infty} \frac{1}{x^2+t^2} \frac{t dt}{(1+e^{-2\pi t})} \quad (\text{NöN}) \text{ chapter V, } \S 1$
- xiv)  $\beta\left(x + \frac{1}{2}\right) = x \int_0^{\infty} \frac{1}{x^2+t^2} \frac{dt}{\cosh(\pi t)} \quad (\text{NöN}) \text{ chapter V, } \S 1$
- xv)  $\frac{1}{\Gamma(z)} = z \prod \left(1 + \frac{z}{k}\right) \left(1 + \frac{1}{k}\right)^{-z} = z e^{\gamma z} \prod \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}, \text{ Re}(z) > 0, (\text{Grl}) (8.322)$
- xvi)  $e^{\Psi(x)} = x \prod \left(1 + \frac{1}{x+k}\right) e^{-\frac{1}{x+k}}, (\text{Grl})(8.363), (\text{NiN}) \text{ p. 65.}$

**Remark:** The circular counterparts of the  $\log x$  – function for the full circle  $|x| \leq \pi$ , resp. for the semi-circle  $|x| \leq \pi/2$  are given by, (Grl) (1.518),

$$\begin{aligned} \text{i)} \quad \log(\sin x) &= \log(\bar{x}) + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-1)^k}{k} B_{2k} \frac{(2x)^{2k}}{(2k)!} . \\ \text{ii)} \quad \log(\tan x) &= \log(x) + \sum_{n=1}^{\infty} \frac{(-1)^k}{k} (1 - 2^{2k-1}) B_{2k} \frac{(2x)^{2k}}{(2k)!} . \end{aligned}$$

**Remark:** From Euler's formula, (NiN) §4, (6), it follows for  $0 < \operatorname{Re}(x) < 1$ , (NiN) §53, (17),

$$\log\left(\tan \frac{\pi}{2} x\right) = \int_0^1 t^{-1/2} \frac{t^{x-1/2} - t^{-x+1/2}}{(1+t)} \frac{dt}{\log t} = \int_0^{\infty} \frac{\sinh\left[\left(\frac{1}{2}-x\right)t\right]}{(1+e^t)} \frac{dt}{t}.$$

From (Grl) we recall the following formulae

**Lemma 9:**

$$\begin{aligned} \text{i)} \quad \int_0^{\pi/2} \log(\sin x) dx &= -\frac{\pi}{2} \log 2 \quad , & \text{(Grl) (4.224)} \\ \text{ii)} \quad \int_0^{\pi/2} \log^2(\sin x) dx &= \frac{\pi}{2} ((\log 2)^2 + \frac{\pi^2}{12}), & \text{(Grl) (4.224)} \\ \text{iii)} \quad \int_0^{\pi/4} \log(\tan x) dx &= -G \text{ with the Catalan constant } G \sim 0,915965594 \dots & \text{(Grl) (4.224)} \\ \text{iv)} \quad \int_0^{\pi/4} \log^2(\tan x) dx &= \frac{\pi^3}{16}, & \text{(Grl) (4.227)} \\ \text{v)} \quad \int_0^{\pi/2} \log(\tan x) \sin x dx &= \log 2 \quad , \quad \int_0^{\pi/2} \log(\tan x) \cos x dx = -\log 2 & \text{(Grl) 4.393} \\ \text{vi)} \quad \int_0^{\pi/2} \log\left(\cot \frac{x}{2}\right) \sin x dx &= \log 2 & \text{(Grl) (4.393).} \end{aligned}$$

**Corollary:**

$$\begin{aligned} \text{i)} \quad \int_{-\pi}^{\pi} \log\left(\frac{1}{2} \left|\sin\left(\frac{x}{2}\right)\right|\right) dx &= -4\pi \log 2 \\ \text{ii)} \quad \int_{-\pi}^{\pi} \log^2\left(\frac{1}{2} \left|\sin\left(\frac{x}{2}\right)\right|\right) dx &= \frac{\pi}{2} ((4 \cdot \log 2)^2 + \frac{\pi^2}{3}) \\ \text{iii)} \quad \int_{-\pi/2}^{\pi/2} \log\left|\tan\left(\frac{x}{2}\right)\right| dx &= -4G \\ \text{iv)} \quad \int_{-\pi/2}^{\pi/2} \log^2\left(\frac{1}{2} \left|\tan\left(\frac{x}{2}\right)\right|\right) dx &= 8G \cdot \log 2 + \pi((\log 2)^2 + \frac{\pi^2}{4}) \\ \text{i.e. } \log\frac{1}{2} \left|\sin\frac{x}{2}\right| &\in L_2(-\pi, \pi) \text{ and } \log\frac{1}{4} \left|\tan\frac{x}{4}\right| \in L_2(-\pi, \pi). \end{aligned}$$

Putting  $T(r) := \int_0^{r\pi/2} \log(\tan x) dx$ ,  $0 \leq r \leq 1$ , we recall from (BrD) the following properties

**Lemma 10:**

$$\begin{aligned} \text{i)} \quad \log(\tan x) &= -2 \sum_{n=0}^{\infty} \frac{\cos(2(2n+1)x)}{2n+1} \quad , \quad -\int_0^x \log(\tan y) dy = \sum_{n=0}^{\infty} \frac{\sin(2(2n+1)x)}{(2n+1)^2}, \text{ for } x \in (0, \frac{\pi}{2}) \\ \text{ii)} \quad T(r) &= T\left(\frac{1}{2} - r\right) \text{ for } 0 \leq r \leq 1 \quad \text{whereby } T(0) = T(1) = 0 \\ \text{iii)} \quad \frac{T((2n+1)r)}{(2n+1)} &= \sum_{j=0}^n T\left(\frac{j}{2n+1} + r\right) - \sum_{j=1}^n T\left(\frac{j}{2n+1} - r\right) \text{ for } 0 \leq r \leq \frac{1}{2(2n+1)}. \end{aligned}$$

**Remark (BoJ):** Let  $E_n(x)$  denote the Euler polynomials and  $E_n = 2^n E_n(\frac{1}{2})$ ,  $E_{2n+1} = 0$ , the corresponding Euler numbers. Putting

$$T_n := (-1)^n 2^n E_n(1), \quad T_0 := 1$$

the series representations of the  $\tan$  – and  $\sec$  – functions are given by ( $E_{2n+1} = T_{2n} = 0$ )

$$\tan(z) = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{T_{2n+1} z^{2n+1}}{(2n+1)!} \quad , \quad \sec(z) = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n} z^{2n}}{(2n)!}, \quad |z|^2 < \frac{\pi^2}{4},$$

whereby for  $n \geq 1$ ,  $\sum_{k=0}^n \binom{n}{k} 2^k T_{n-k} + T_n = 0$ ,  $\sum_{k=0}^n \binom{2n}{2k} E_{2k} = 0$ , (Grl) 1.411.

We note that the Euler numbers are integral and the tangent numbers are integers.

## The three Millennium problem solutions, RH, NSE, YME, and a Hilbert scale based quantum geometrodynamics

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„looking back, part (A)“

on a 10 year journey to ...

### Riemann Hypothesis solutions

A modified Zeta function theory is proposed to overcome current challenges

- (a) to verify several **Riemann Hypothesis** (RH) criteria
- (b) to prove the binary **Goldbach conjecture**

The current two baseline functions to define the Zeta functions, the Gaussian function and the (periodical) fractional part function (resp. their corresponding Mellin transforms) are replaced by their corresponding Hilbert transforms, which are the Dawson function (which is a specific Kummer function) and the Fourier series representation of the  $\log(\sin x)$ -function. The convergence analysis is based on corresponding Hilbert space frameworks, supporting especially Cardon's „convolution operator representation“ and Bagchi's "Nyman-Beurling" RH criteria applied to the modified entire Zeta function. Thereby, the convolution operator representation goes along with convergent (Mellin transform) integrals, overcoming the corresponding challenge of an only formally valid self-adjoint invariant operator representation of the standard entire Zeta function ((EdH) 10.3).

Basically, the Bagchi RH criterion and the Cardon RH criterion are two sides of the same coin, which is about the construction of appropriately defined operators, i.e. a defined mapping rule in combination with a defined domain.

The Bagchi-"Nyman-Beurling" RH criteria comes along with the  $\log(\sin x) L_2^\#(0,1)$ -function (the Hilbert transform of the fractional part function  $\rho(x)$ ) in the form

$$\rho_H(x) = \sum_1^{\infty} \frac{\cos 2\pi vx}{\pi v} = -\frac{1}{\pi} \log 2 \sin(\pi x) \in L_2^\#(0,1)$$

defined in the (periodical) Hilbert scale framework  $H_0^\#(0,1)$ . It enables the definition of new arithmetical functions going along with a Hilbert space based "circle method" defined on the „boundary of the unit circle“, alternatively to the "open unit disk" domain of the Hardy-Littlewood circle method. The counterpart of the zeros of the  $e^{ix}$ -function are the zeros of the Kummer function  ${}_1F_1\left(\frac{1}{2}; \frac{3}{2}; -x\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^{n+1} n!}$  enabling the definition of a pair of two different arithmetical functions to analyse  $(p, q)$ -binary number theoretical problems. The concept also supports the verification of the Snirelmann density criterion to prove the Goldbach conjecture.

The proposed modified Zeta function theory supports the proof of several RH criteria, which might be grouped into the following three classes (A1)-(A3), basically defined by the applied underlying function space frameworks:

(A1) this class is about RH criteria which can be re-formulated in terms of distributional Hilbert scale functions  $H_\alpha(-\infty, \infty)$

(A2) this class is about RH criteria which can be re-formulated in terms of periodical distributional Hilbert scale functions  $H_\alpha^\#(0,1)$

(A3) this class is about RH criteria formulated by distributional arithmetical functions going beyond the distributional arithmetical functions applied for "a distributional way to prove the Prime Number Theorem" (ViJ).

The common key challenge is in the context of the "Polya theorem", where the not-vanishing constant Fourier term provides the key handicap. From the related "Polya theorem" pdf (see pdf in the "literature" page) we quote:

"Attempts have been made to apply Polya's general theorem about the zeros of the Fourier transform of a real function to the Zeta function (PoG). After a change of variable  $t \rightarrow \log x$  and approximating the integral over the half-line (positive  $x$ -axis) by integrals over finite intervals (which are the essential restriction/handicap of Polya's theorem to be applied to the Zeta function), one can obtain a theorem about zeros of Mellin transforms. For the Zeta function, the Müntz formula has been used, but the Polya theorem only applies in the interval  $(0,1)$ , and no information is obtained about the zeros of the Zeta function in the critical open stripe of validity of Müntz's formula".

(OIJ): „The theory of Dirichlet series offers a bridge between number theory and analysis. Perhaps the most appealing example of the power of this connection is given by the tauberian approach to the classical prime number theorem.“ ... The asymptotic behavior in terms of the related Chebyshev-type inequality can be connected to local function theoretic properties of (distributional) (Bagchi-type) Hilbert spaces  $H_\omega$ .

The Hilbert-Polya conjecture (which is about the existence of a proper self-adjoint integral operator going along with the concept of convolution operators, (CaD)) needs to overcome the mathematical problem of the not vanishing constant Fourier term of the Jacobian theta function.

Every Hilbert transformed  $L_2$  function has a vanishing constant Fourier term. The Paley-Wiener functions form a Hilbert subspace of  $PW$  of  $L_2(\mathbb{R})$ , which is a reproducing kernel Hilbert space (HiJ). Thus, the inner product of any  $f \in PW$  with  $w := w(t, x) = \frac{\sin(\pi(t-x))}{\pi(t-x)}$ , i.e.  $f(t) = (f, w)_0 = \int_{-\infty}^{\infty} f(x)w(t-x)dt$ .  $PW$  and  $PW^\perp$  are invariant subspaces for the Hilbert transforms on  $L_2(\mathbb{R})$ . Calderón's reproducing formula provided the baseline for the continuous wavelet transformation theory coming along with the concept of „windowed Fourier transforms“ (LoA). The  $PW_\pi$  reproducing kernel provided the baseline for the cardinal series theory, addressing e.g. the question „how power series coefficients  $a_n$  of a function  $f(z) = \sum a_n z^n$  determine its singularities“, (HiJ). Corresponding  $\pi$ -cardinal series are convergent under the condition  $\sum \frac{|a_n|}{n} < \infty$ . Cardinal series can be obtained formally by considering the Lagrange interpolation formula, which is proposed alternatively to the Euler-Mclaurin /Newton-Gauss summation formulas.

The canonical resolution of the „ $E_\mu$  spectrum of the dynamical Hamiltonian mechanics system“ (consisting of a discontinuous and a continuous part) is concerned with certain self-adjoint and unitary operators in a Hilbert space. The conception that such a spectrum reveals the mechanical properties of the system in its own structure lead to the concepts of an inner product  $(E_\mu f, g)$  on the Hilbert space,  $PW$ -like formulas (KoB1), which is the well known spectral theory in Hilbert spaces for unitary operators (e.g. (HiF)). The physical motivations for related ergodic theorems are based on the Maxwell-Boltzmann gas theory and the Gibbs statistical mechanics (HoE).

With respect to part B, the common mathematical tools to analyze the proposed complementary kinematical energy space  $H_1^\perp$  and the non-linear Landau damping phenomenon are

- the wavelets, where a vanishing constant Fourier term of a  $L_2$  function is a sufficient condition to be a wavelet function (HoM), (MeY); the underlying „theory of frames“, especially the Nonharmonic, but exact Fourier frames are closely related to the Riesz basis of a separable Hilbert space, (YoR) p. 157, (NaA); the latter one plays a key role in the context of the Kadec 1/4-theorem (related to the Paley-Wiener space  $PW$  (DuR), resp. indefinite metrics, (AzT), (BoJ), as applied in (A2), (A3) below; general references to frames are provided in (YoR) p. 190.
- ergodic theory on a Hilbert space (HaP), which is about measure-preserving transformation, e.g. in relationship to turbulence problems in hydrodynamics (HoE), (LiP), in combination with non-harmonic Fourier series theory (YoR), based on complex exponentials forming a Riesz basis for  $L_2^\#$  and sequences of real or complex numbers with uniform density one (DuR).

The parts (A1-A3) below are concerned with Hilbert scales in the form  $H_\alpha, l_2^\alpha, \alpha \in R$ :

The Dirichlet series theory is an extension of the concept of power series replacing  $\sum_1^\infty a_n e^{-xn} \rightarrow \sum_1^\infty a_n e^{-x \log n}$ . The relationship between the Dirichlet series

$$f(s) := \sum_1^\infty a_n e^{-s \log n} \quad g(s) := \sum_1^\infty b_n e^{-s \log n}$$

and the Hilbert space  $H_{-1/2}^\# \cong l_2^{-1/2}$  on the critical line is given by ([LaE] §227, Satz 40):

$$((f, g))_{-1/2} := \lim_{\omega \rightarrow \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f\left(\frac{1}{2} + it\right) g(1/2 - it) dt = \sum_1^\infty \frac{1}{n} a_n b_n.$$

The cardinal series theory is an extension of the Dirichlet series theory. In our case this is about absolute convergent series  $C(x) = \frac{\sin(\pi(x-\lambda))}{\pi} \sum_{n=1}^\infty (-1)^n \frac{c(n+\lambda)}{x-\lambda-n}$  with cardinal series  $C(x) =$

$\frac{w}{\pi} \sum_{-\infty}^\infty a_n \frac{\sin \frac{w}{\pi}(x-a-nw)}{x-a-nw}$  for the points  $(a+nw, a_n)$ , whereby  $\sum_2^\infty (|a_n| + |a_{-n}|) \frac{\log n}{n} < \infty$  to ensure that the cardinal series  $\frac{\sin \pi x}{\pi} \left[ \frac{a_0}{x} + \sum_{n=1}^\infty (-1)^n \left( \frac{a_n}{x-n} + \frac{a_{-n}}{x+n} \right) \right]$  is absolutely convergent (WhJ1).

Remark: The inverse mapping of  $\sigma(x) := \tilde{T}(\log x) := \sum_{n \leq x} \frac{1}{n} \log \left( \frac{x}{n} \right), x \geq 1$ , is given by ((ScW), lemma 3.3, p.36)  $\sigma^{-1}(x) = \tilde{T}^{-1}(\log x) = \sum_{n \leq x} \frac{\mu(n)}{n} \log \left( \frac{x}{n} \right)$ . The link to the Riemann, von Mangoldt and Landau density functions  $J(x) := \frac{1}{2} \left[ \sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n} \right], \psi(x) := \sum_{n \leq x} \Lambda(n), \vartheta(x) := \sum_{n=1}^\infty \Lambda(n) \log \left( \frac{x}{n} \right) = \psi(x) \log x$  (fulfilling  $d[x\vartheta'] = d\psi = \log x dj$ ) is given by  $x\sigma'(x) = \sum_{n \leq x} \frac{1}{n}$ . The considered sequences below enable modified Riemann density functions  $J^*(x), \psi^*(x), \vartheta^*(x)$  with  $dJ = dJ^*, d\psi^* = d\psi, d\vartheta = d\vartheta^*$  (see also lemma 5 v), vii), viii) below).

From (ApT) p. 66 we recall that the PNT can be derived from the formula  $\sum_{n=1}^\infty \frac{\mu(n)}{n} = 0$ . What cannot be derived from the PNT is the convergence of the series, (LaE) §160,  $\sum_{n=1}^\infty \frac{\mu(n)}{n} \log \left( \frac{1}{n} \right) = -\sum_{n=2}^\infty \mu(n) \frac{\log n}{n} = 1$ .

Putting  $a_n := \sqrt{n} \mu(n) \log \left( \frac{1}{n} \right)$  ( $a_n^* := \mu(n) \log \left( \frac{1}{n} \right)$ ) and  $b_n := \frac{1}{\sqrt{n}}$  ( $b_n^* := 1$ ) one gets

$$((f, g))_{-1/2} = ((f^*, g^*))_{-1/2} = 1 \quad \text{and} \quad ((g, g))_{-1/2} = ((g^*, g^*))_{-1} = \sum_1^\infty \frac{1}{n} b_n^2 = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

i.e.  $f\left(\frac{1}{2} + it\right), g\left(\frac{1}{2} + it\right) \in H_{-1/2}$  resp.  $g^*\left(\frac{1}{2} + it\right) \in H_{-1}$  and therefore  $f^*\left(\frac{1}{2} + it\right) \in H_0$ .

We note that the RH is equivalent to an order of magnitude  $O(x^{\frac{1}{2}+\varepsilon})$  ( $\varepsilon > 0$ ) of  $M(x) := \sum_{n \leq x} \mu(n)$ , ((ApT) p. 301), i.e., that  $M \in H_{\frac{1}{2}+\varepsilon}$ . From (LaE6), (MiM), (MiM1), we recall the equivalent Farey series based criterion of  $M(x) := \sum_{n \leq x} \mu(n) = \sum_{k=1}^{A(x)} \cos(2\pi r_k)$  with  $A(x) := \sum_{n \leq x} \varphi(n)$ . From the Sobolev embedding theorem we recall, that  $H_{\frac{1}{2}+\varepsilon}$  is a sub-space of  $C^0$  (more precisely the space of Hölder continuous functions with exponent  $1/2$ ) and that the Dirac function  $\delta$  is an element of the dual space  $\delta \in H_{-\frac{1}{2}-\varepsilon}$ . The link to „a quick distributional way to the Prime Number Theorem (PNT)“, (ViJ) is given by the distributions  $\delta, H[\delta] \in H_{-1/2-\varepsilon}$  where  $\pi H[\delta(x)] = \frac{1}{x} = \log' \left( \frac{x}{n} \right)$  and  $\psi'(x) = \sum_{n \leq x} \Lambda(n) \delta(x-n)$ . We further note, that  $H_{\alpha+\varepsilon}$  is compactly embedded into  $H_\alpha$ .

We mention the Riemann error function

$$\int_x^\infty \frac{dt}{t(t^2-1) \log t} = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log \Gamma(1+\frac{s}{2})}{s} \right] x^s ds \cong -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \log \Gamma(1+\frac{s}{2}) x^s \frac{ds}{s},$$

derived from the  $\Gamma(1+\frac{s}{2})$  term of the entire Zeta function ((EdH) 1.16).

The parts (A1-A3) below are also concerned with the Kummer functions are  ${}_1F_1\left(1, \frac{3}{2}; -x\right)$  and  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right)$ , which are connected by the formula  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) = e^x {}_1F_1\left(1, \frac{3}{2}; -x\right)$ , (AbM) (7.1.21), (GrI) 9.212.

The challenging part to verify the RH criterion  $\pi(x) - li(x) = O(\sqrt{x} \log x) = O(x^{\frac{1}{2}+\epsilon})$ , (which could be even  $O(x^{\frac{1}{2}})$  in a variational representation avoiding the  $L_\infty$ -norm) is the asymptotical behavior of the exponential (integral) function ((EdH) 1.14 ff., (BeB) IV)

$$Ei(x) := - \int_x^\infty e^{-y} d \log y = - \int_x^\infty e^{-y} \frac{dy}{y} = \int_{-\infty}^x \frac{e^t}{t} dt \approx \frac{e^{-x^2}}{x^2} \sum_{k=0}^\infty (-1)^{k+1} \frac{k!}{x^{2k}}.$$

In the following we summarize some properties of the concerned Kummer functions and their relationships to the baseline functions of the Zeta function theory:

The Kummer function based representations of the  $li(x)$  function is given by

$$li(x) = \int_{-\infty}^x \frac{dt}{\log t} dt = -x {}_1F_1(1; 1, -\log x) = Ei(\log x) = - \int_{-\log x}^\infty \frac{e^{-t}}{t} dt = \int_{-\infty}^{\log x} \frac{e^t}{t} dt \sim \frac{1}{\log x}$$

with

$$Ei(x) = \log x + \gamma + \sum_{k=1}^\infty \frac{1}{k} \frac{x^k}{k!}.$$

The series representations and asymptotics of the error function

$$Erf(z) := \int_0^z e^{-t^2} dt \text{ resp. } Erfc(z) := \int_z^\infty e^{-t^2} dt$$

are given by (AbM) 7.1.1, 7.1.5, 7.1.23, (OIF) 3 §1,

$$Erf(z) := z \sum_{k=0}^\infty \frac{(-1)^k z^{2k}}{2^{k+1} k!}, \quad Erfc(z) \approx \frac{1}{z} e^{-z^2} \left[ 1 + \sum_{k=1}^\infty (-1)^k \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{x^{2k}} \right].$$

The relationships to the concerned Kummer functions and the related functions  $\Psi(a, c, z^2)$  and  $\frac{e^x-1}{x}$  are given by (LeN) 9.13,

$$li(z) = -z {}_1F_1(1, 1, -\log z), \quad Ei(z) = -e^z \Psi(1, 1, -z), \quad \frac{e^x-1}{x} = {}_1F_1(1, 2; x)$$

$$Erf(z) = z {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -z^2\right) = ze^{-z^2} {}_1F_1\left(1, \frac{3}{2}, z^2\right)$$

$$Erfc(z) = \frac{1}{2} e^{-z^2} \Psi\left(\frac{1}{2}, \frac{1}{2}, z^2\right), \quad Erfc(z) = \frac{1}{2} ze^{-z^2} \Psi\left(1, \frac{3}{2}, z^2\right).$$

The asymptotics of the Kummer functions are given by, (LeN) 9,

$${}_1F_1(a, c; z) = \frac{\Gamma(c)}{\Gamma(c-a)} e^{\pm i\pi a} z^{-a} \left[ \sum_{k=0}^n \frac{(-1)^k}{k!} (a)_k (1+a-c)_k z^{-k} + O(|z|^{-n-1}) \right] + \frac{\Gamma(c)}{\Gamma(a)} e^{-(c-a)} z^{-(c-a)} \left[ \sum_{k=0}^n \frac{(-1)^k}{k!} (1-a)_k (c-a)_k z^{-k} + O(|z|^{-n-1}) \right],$$

It especially holds  $z {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, z\right) = \frac{1}{2e} \left[ \sum_{k=0}^n (-1)^k \left(\frac{1}{2}\right)_k z^{-k} + O(|z|^{-n-1}) \right]$ , i.e.

$$Ei(x) \approx -2e {}_1F_1\left(\frac{1}{2}, \frac{3}{2}, -x\right) \approx \frac{1}{x} \left[ \sum_{k=0}^\infty \left(\frac{1}{2}\right)_k \left(\frac{1}{x}\right)^k \right].$$

We further note that the function represented by the series  $\sum_{k=0}^\infty \frac{(-1)^k}{(2k+1) \cdot x^{2k+1}}$ ,  $|x| > 1$ , has the value  $\pi/2$  as  $x \rightarrow 0$ ,  $x > 0$ , (BeB) IV, (10.2).

The Hilbert transform of the Gaussian function is the Dawson function given by

$$F(x) := e^{-x^2} \int_0^x e^{t^2} dt = \int_0^\infty e^{-t^2} \sin(2xt) dt = x {}_1F_1\left(1, \frac{3}{2}; -x^2\right) = xe^{-x^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right).$$

and it holds (GaW)

$$H[e^{-x^2}] = 2\sqrt{\pi}F(x)$$

resp.

$$H^+ \left[ \frac{e^{-x}}{\sqrt{\pi x}} \right] = 2 \frac{F(\sqrt{x})}{\sqrt{x}} \approx \frac{1}{x} \left[ 1 + \sum_{k=1}^\infty \frac{1 \cdot 3 \cdot \dots \cdot (2k-1)}{(2x)^k} \right].$$



The link of the Hermite polynomials is given by (AbM), 7.1.15, resp. (RyG)

$$\frac{1}{\sqrt{\pi}} H[e^{-x^2}] = F(x) = \frac{1}{\sqrt{\pi}} \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{H_k^{(n)}}{x - x_k^{(n)}} = \frac{1}{\sqrt{\pi}} \lim_{h \rightarrow 0} \sum_{n \text{ odd}} \frac{e^{-(x-nh)^2}}{n} \approx \frac{1}{x}.$$

where  $x_k^{(n)}$  and  $H_k^{(n)}$  are the zeros and weight factors of the Hermite polynomials.

Let  $f(x) := \int_0^\infty \frac{e^{-t^2}}{t+x} dt$ , then,  $f(x) = \sqrt{\pi} F(x) - \frac{e^{-x^2}}{2} Ei(x^2) = -\log x - \frac{\gamma}{2} + o(1)$ ,  $x \rightarrow 0$ , (OIF) 2 §4.

Remark: The functions  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -z\right)$  and  ${}_1F_1\left(1, \frac{3}{2}; z\right)$  possesses the same zeros (SeA), as the related Whittaker function  $z^{-3/4} M_{\frac{1}{4}, \frac{1}{4}}(z)$ , which is the solution of the corresponding self-adjoint (ODE) Whittaker operator, (BuH), §17.1.

Lemma:

- i)  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) = \sum_{k=0}^\infty \frac{1}{k+\frac{1}{2}} \frac{x^k}{k!} = \sum_{k=0}^\infty \frac{2}{2k+1} \frac{x^k}{k!} = \frac{1}{2} \int_0^1 \frac{e^{xt}}{\sqrt{t}} dt$ ,  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 0\right) = 1$  (LeN) (9.11.1)
- ii)  $x e^{-x^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) = x {}_1F_1\left(1, \frac{3}{2}; -x^2\right) = \int_0^\infty e^{-t^2} \sin(2xt) dt$
- iii)  ${}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; x\right) = \frac{1}{3} {}_1F_1\left(\frac{3}{2}, \frac{5}{2}; x\right)$
- iv)  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) + x {}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; x\right) = e^x$
- v)  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right) = \frac{2}{x} \int_0^x e^{t^2} dt$  resp.  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right) = \frac{1}{\sqrt{x}} \int_0^x e^{t/\sqrt{y}} dy$  (AbM) 7.1.5, (GaW)
- vi)  $\int_0^\infty x^{s/2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x} = \frac{\Gamma(\frac{s}{2})}{1-s}$  resp.  $\int_0^\infty x^{s/2} \left[-x {}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; -x\right)\right] \frac{dx}{x} = \frac{s}{1-s} \Gamma(\frac{s}{2})$  (GrI) 7.612  
resp.  $\Gamma\left(1 + \frac{s}{2}\right) = s(1-s) \int_0^\infty x^s {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x^2\right) \frac{dx}{x}$  in the critical stripe

Putting

$$g(x) := {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) + x {}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; -x\right) = \sum_{k=0}^\infty \frac{k+1}{k+1/2} \frac{(-x)^k}{k!}$$

one gets

$$\begin{aligned} -xg(x) &= \sum_{k=0}^\infty \frac{k+1}{k+\frac{1}{2}} \frac{(-x)^{k+1}}{k!} = -x {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) - x^2 {}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; -x\right) \approx \left[ \sum_{k=0}^\infty \binom{1}{2}_k \left(\frac{1}{x}\right)^k \right] + \left[ \sum_{k=0}^\infty \binom{1}{2}_k (k+1) \left(\frac{1}{x}\right)^k \right] \\ &= -\int_0^\infty x^{1-s} g(x) \frac{dx}{x} = -\int_0^\infty x^{1-s} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -x\right) \frac{dx}{x} + \int_0^\infty x^{1-s} \left[-x {}_1F_1'\left(\frac{1}{2}, \frac{3}{2}; -x\right)\right] \frac{dx}{x} = \frac{s\Gamma(1-s)}{1-2s}. \end{aligned}$$

Putting  $T(x) := \int_0^x g(t) dt$  resp.  $\tilde{T}(x) := -\int_0^x t g(t) dt$  it follows

- i)  $T(x) = \int_0^x \left[ \sum_{k=0}^\infty (-1)^k \frac{k+1}{k+\frac{1}{2}} \frac{t^k}{k!} \right] dt = x \cdot \left[ \sum_{k=0}^\infty (-1)^k \frac{1}{k+\frac{1}{2}} \frac{x^k}{k!} \right]$  with  $dT(x) = g(x) dx$
- ii)  $\lim_{x \rightarrow \infty} \frac{1}{x} \int_0^x g(t) dt = \lim_{x \rightarrow \infty} \frac{T(x)}{x} = 1$  i.e.  $T(x) \sim x$ , as (\*)  $sf(s) = s \int_0^\infty x^{-s} g(x) \frac{dx}{x} = -\frac{s+1}{2s+1} \Gamma(1-s)$
- iii)  $g(x) = \sum_{k=0}^\infty \frac{k+1}{k+1/2} \frac{(-x)^k}{k!} = \sum_{k=0}^\infty \frac{1}{k+1/2} \frac{(-x)^k}{k!} + \sum_{k=0}^\infty \frac{k}{k+1/2} \frac{(-x)^k}{k!} = \frac{T(x)}{x} + \sum_{k=0}^\infty \frac{k}{k+1/2} \frac{(-x)^k}{k!}$
- iv)  $\int_0^\infty x^{-s} \frac{d\tilde{T}(x)}{x} = -\int_0^\infty x^{1-s} g(x) \frac{dx}{x} = \frac{s\Gamma(1-s)}{1-2s}$  i.e.  $\lim_{x \rightarrow \infty} \frac{\tilde{T}(x)}{x} = 0$ .

(\*) Lemma: (OsH) Vol1, 89): Let  $T(x)$  monotone increasing, and

$$f(s) = \int_0^\infty e^{-sx} dT(x) = \int_0^\infty x^{-s} dT(\log x) = \int_0^\infty x^{-s} \frac{dT(x)}{x}, \quad s = \sigma + it, \quad \sigma > 0$$

convergent. Then, if  $\lim_{s \rightarrow 0^+} sf(s)$  exists, this holds also for  $\lim_{x \rightarrow \infty} \frac{T(x)}{x}$ , and both limits are identical, i.e.  $\lim_{s \rightarrow 0^+} sf(s) = \lim_{x \rightarrow \infty} \frac{T(x)}{x}$ .

In the context with „the connection between  $\zeta(s)$  and primes“ we refer to (EdH) 1.11.

In the context with Polya’s theorem (PoG) we recall from (EdH) 12.5, „Transforms with zeros on the line“:

„Polya’s theorem is that a real self-adjoint<sup>(\*)</sup> operator of the form  $f(x) \rightarrow \int_{1/a}^a f(ux)F(u)du$  (where  $F(u)$  is real and satisfies  $u^{-1}F(u^{-1}) = F(u)$ ) which has the property that  $\frac{1}{u^2}F(u)$ <sup>(\*\*)</sup> is nondecreasing on the interval  $[1, a]$  has the property that the zeros of its transform all lie on the line  $Re(s) = 1/2$ .

<sup>(\*)</sup> The notion „self-adjoint“ is different from the definition in functional analysis, but very much close to it, when considering the operator equation in a Hilbert space framework for functions with domain on the critical line. In this case it means that the Polya condition has the same effect than replacing  $s$  by  $(1-s)$ .

<sup>(\*\*)</sup> The unnatural-seeming factor  $u^{-1}$  can be eliminated by renormalizing so that  $\int_0^\infty x^{-s}F(x)dx$  is written  $\int_0^\infty x^{\frac{1}{2}-s} [x^{\frac{1}{2}}F(x)] d\log(x) = \int_0^\infty x^{-s}\tilde{F}(x)d\log(x)$ . Then the self-adjoint condition is simply  $\tilde{F}(u^{-1}) = \tilde{F}(u)$ . Polya’s condition is that  $\tilde{F}$  be non-decreasing on  $[1, a]$ , and the conclusion of the theorem is that the zeros lie on  $Im(s) = 0$ .

In the context with „the connection between  $\zeta(s)$  and Tauberian theorems resp. Abel or Cesaro average“ we refer (EdH) 12.7.

In the context with some relevance of the considered Kummer functions to plasma physics we refer to (KoV), (PaY) regarding

- the linear response of magnetized Bose plasmas at  $T=0$  for large and small values of its parameter; the large parameter expansion plays a determining role in the behaviour of these Bose systems in the limit that the external magnetic field  $B$  approaches zero. This particular expansion is generalized for the Hurwitz zeta function, (KoV)
- 
- the linearized collision operator in the Boltzmann equation with repulsive intermolecular (inverse-power) potentials  $V(r) = a \cdot r^{-\alpha}$  for  $\alpha > 2$ ; the collision operator has a purely discrete spectrum and its eigenfunctions are infinitely differentiable  $L_2$ -functions which are complete in  $L_2$ . The proof relies on the formalism of pseudo-differential operators; the special case  $\alpha = 2$  is about the Maxwell’s molecules, (PaY).

In the context with the building of distributional Hilbert scales based on a linear operator with discrete spectrum and eigenfunctions, which are complete in  $L_2$ , its underlying approximation theory, and an „exponential decay“ inner product resp. norm with parameter  $t > 0$ , given by

$$(x, y)_{\alpha(t)} = \sum_k \sigma_k^\alpha e^{-\sqrt{\sigma_k}t} (x, \varphi_k)(y, \varphi_k) \quad , \quad \|x\|_{\alpha(t)}^2 := (x, x)_{\alpha(t)}$$

governing all „polynomial decay“ Hilbert scale norms we refer to (NiJ), (NiJ1).

In the context with some relevance of the considered Kummer functions to the Navier-Stokes equation we refer to (PR1) regarding an integral representation of the Navier-Stokes equations for an incompressible viscous fluid. „Making use of standard integral transform methods and considering the longitudinal components of the velocity field, thereby eliminating the pressure field, the Navier-Stokes equations are cast in integral form. The intrinsically non linear character of the equations has proved to be an unsurmountable difficulty that has severely restricted their practical use. The limited understanding of the turbulent motion of fluids and the lack of a comprehensive theory of turbulence is a consequence of this mathematical complication. ... The final result is a non linear integral equation for the velocity field alone, involving a single convolution over the space and time variables.“

The convolution kernel of the integral representation of the Navier-Stokes equations is build on the functions

$$I_0(\vec{r}, t) := \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{\vec{r}^2}{4\nu t}} \quad , \quad I_1(\vec{r}, t) := \left(\frac{\nu t}{\vec{r}^2}\right) \frac{1}{(4\pi\nu t)^{3/2}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -\frac{\vec{r}^2}{4\nu t}\right) .$$

The following part (A1) is related to RH criteria which can be re-formulated in terms of distributional Hilbert scale functions  $H_\alpha(-\infty, \infty)$ .

Part (A2) is related to RH criteria which can be re-formulated in terms of periodical distributional Hilbert scale functions  $H_\alpha^\#(0,1)$ .

Part (A3) is about RH criteria formulated by distributional arithmetical functions going beyond the distributional arithmetical functions applied for "*a distributional way to prove the Prime Number Theorem*" (ViJ).

The parts (A2) and (A3) are concerned with appropriate sequences of real numbers and sequences of vectors; the latter ones are called Riesz sequences, if the sequence is a Riesz basis in the closure of the space spanned by those vectors.

With respect to the Bagchi criterion below we emphasize that there is only a positive answer to the following question in a certain „weak“ sense (SeK): „*can every frame of complex exponentials  $\{e^{i\lambda_n x}\}$  in  $L_2(-\pi, \pi)$  be made into a Riesz basis by removing from  $\{e^{i\lambda_n x}\}$  a suitable collection of the functions  $e^{i\lambda_n x}$  resp. can every Riesz sequence  $\{e^{i\lambda_n x}\}$  in  $L_2(-\pi, \pi)$  be made into a Riesz basis by adjoining to  $\{e^{i\lambda_n x}\}$  a suitable collection of exponentials  $e^{i\lambda x}$  not elements of  $\{e^{i\lambda_n x}\}$ ?*“ For the critical sequence in our cases (part (A2) and (A3)) this is about  $\lambda := \beta_1 := \frac{\omega_1 + \omega_2}{2} - \frac{1}{4}$  to ensure a Snirelmann density of  $1/2$  of the concerned sequence  $\{\beta_n\}_{n \in \mathbb{N}}$  (see also (ReR)).

Following part (A3), there is a section about a new tool set to prove the binary Goldbach conjecture, taking advantage of (A2-A3), e.g. the sequences based on the imaginary parts of the zeros of the considered Kummer functions with underlying domains with Snirelmann density of  $1/2$ . The final tool is about a „*2-semi-(truly)-circle (even/odd integer) method*“ with an underlying „*major/minor arcs I/II*“ concept alternatively to the Hardy-Littlewood circle method.

(A1) This class is about RH criteria which can be re-formulated in terms of distributional Hilbert scale functions  $H_\alpha(-\infty, \infty)$ .

(A1) The Euler product formula connects the Riemann function  $\zeta(s)$  (the analytic continuation of the power function representation beyond the halfplane  $Re(s) > 1$ ) with primes. Taking the log of both sides of the Euler product formula results into a Stieltjes integral representation of the  $\log\zeta(s)$  function with a (prime) density  $dJ(x)$  in the form ((EdH) 1.11)

$$\log \zeta(s) = s \int_0^{\infty} x^{-s-1} J(x) dx = \int_0^{\infty} x^{-s} dJ(x)$$

The entire Zeta function

$$\xi(s) := \frac{s}{2} \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s) = \xi(1-s)$$

is basically a product of the three functions  $\zeta(s)$ ,  $(s-1)$  and  $\Gamma(1+\frac{s}{2})$ , whereby  $\Gamma(s)$  denotes the Gamma function ((EdH) 1.13).

The method for deriving the formula for  $J(x)$  is basically about the calculation of the Fourier inverses of  $\log\xi(s)$ ,  $-\log(s-1)$  and  $-\log\Gamma(1+\frac{s}{2})$ . The Fourier inverse of the principle term  $-\log(s-1)$  becomes the logarithmic integral representation of the  $li(x)$  function. The Fourier inverse of the term  $-\log\Gamma(1+\frac{s}{2})$  leads to the famous Riemann approximation error function between the prime density function  $J(x)$  and the  $li(x)$  function, ((EdH) 1.16),

$$\int_x^{\infty} \frac{dt}{t(t^2-1)\log t} = \int_x^{\infty} \frac{1}{t\log t} (\sum t^{-2n}) dt = \sum_{n=1}^{\infty} \int_x^{\infty} t^{-2n} \frac{dt}{t\log t} = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[ \frac{\log\Gamma(1+\frac{s}{2})}{s} \right] x^s ds$$

An appropriate convergence behavior of the Riemann error function is one of several RH criteria. Let  $f(x)$  denote the Gaussian function. Then the entire Riemann Zeta function is given by

$$\xi(s) = \frac{s}{2} \Gamma\left(\frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s) = (1-s) \cdot \zeta(s) M[-x f'(x)](s) = \xi(1-s).$$

The Hilbert transform of the Gaussian function is the Dawson function given by

$$F(x) := e^{-x^2} \int_0^x e^{t^2} dt = \int_0^{\infty} e^{-t^2} \sin(2xt) dt = x {}_1F_1\left(1, \frac{3}{2}; -x^2\right) = x e^{-x^2} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x^2\right).$$

Replacing the Gaussian function by its Hilbert transform leads to an alternative entire Zeta function definition  $\xi^*(s)$  with same zeros in the critical stripe as  $\xi(s)$  resulting into a modified Riemann error function with improved approximation property. It is given by

$$\xi^*(s) := \frac{1}{2} (s-1) \pi^{\frac{1-s}{2}} \Gamma\left(\frac{s}{2}\right) \tan\left(\frac{\pi}{2}s\right) \cdot \zeta(s) = \zeta(s) \cdot M\left[\frac{d}{dx} [-x \cdot f_H(x)]\right](s)$$

resp.

$$\zeta^*(s) := \frac{\cot(\frac{\pi}{2}s)}{1-s} \cdot \zeta(s) = \frac{\tan(\frac{\pi}{2}(1-s))}{1-s} \cdot \zeta(s)$$

with same zeros as  $\xi(s)$ , as it holds  $s(1-s)\xi^*(s)\xi^*(1-s) = \pi\xi(s)\xi(1-s)$ . It is basically about a replacement of the term  $\pi^{\frac{s}{2}}$  of the term  $\pi\Gamma\left(1+\frac{s}{2}\right) = \frac{\pi}{2}s\Gamma\left(\frac{s}{2}\right)$  by the term  $\tan(\frac{\pi}{2}s) = \cot(\frac{\pi}{2}(1-s))$ . The method for deriving the formula for  $J(x)$  is then about the calculation of the Fourier inverses of  $\log\xi(s)$ ,  $-\log(s-1)$ ,  $-\log\Gamma(\frac{s}{2})$  and  $-\log(\tan(\frac{\pi}{2}s)) = -\log(\cot(\frac{\pi}{2}(1-s)))$ .

With the series expansion of  $\tan(\frac{\pi}{2}s)$  resp.  $\frac{\pi}{2}s \cdot \cot(\frac{\pi}{2}s)$  ((GrI) 1.421) it follows

$$\xi^*(s) = \frac{\xi(s)}{\sqrt{\pi}} \sum_{k=1}^{\infty} \frac{1}{(k-\frac{1}{2})^2 - (\frac{s}{2})^2} \quad \text{resp.} \quad \xi(s) = \frac{\xi^*(s)}{\sqrt{\pi}} \cdot \frac{2}{\pi} \left[ 1 + \sum_{k=1}^{\infty} \frac{s^2}{s^2 - (2k)^2} \right].$$

(A2) This class is about RH criteria which can be re-formulated in terms of periodical distributional Hilbert scale functions  $H_{\alpha}^{\#}(0,1)$

Bagchi's "Hilbert space based reformulation of the Nyman-Beurling RH criterion" (BaB) is based on the (periodical) fractional part  $L_2^{\#}$ -function

$$\rho(x) = x - [x] = \frac{1}{2} - \sum_1^{\infty} \frac{\sin 2\pi vx}{\pi v}.$$

Its convergent Mellin transform in the critical stripe defines the zeta function  $\zeta(s)$  given by ((TiE) II 2.1),

$$\zeta(s) = s \int_0^{\infty} \frac{[x]-x}{x^s} \frac{dx}{x} = -s \int_0^{\infty} x^{-s} \rho(x) \frac{dx}{x} \quad 0 < \text{Re}(s) < 1.$$

The conceptual challenge becomes obvious condering the general Mellin transform property  $M[xh'](s) = -sM[h](s)$  and the related (insufficient) Mellin transform zeta function  $\zeta(s)$  representation in the form

$$\zeta(s) = s \int_0^{\infty} \frac{[x]-x+\frac{1}{2}}{x^s} \frac{dx}{x} = -s \int_0^{\infty} x^{-s} \left[ \rho(x) - \frac{1}{2} \right] \frac{dx}{x} \quad -1 < \text{Re}(s) < 0.$$

The considered Hilbert space in [BaB] is about of all sequences  $a = \{a_n | n \in \mathbb{N}\}$  of complex numbers such that

$$\sum_{n=1}^{\infty} \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \leq \omega_n \leq \frac{c_2}{n^2}$$

which is isomorph to the Hilbert space  $H_{-1} \cong l_2^{-1}$ . For  $\gamma := \{1,1,1,1,\dots\}$  it holds

$$\|\gamma\|_{-1}^2 = \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e.  $\gamma \in l_2^1$  resp.  $\gamma \in (l_2^{-1/2})^{\perp}$ , with  $l_2^{-1} = l_2 \otimes l_2^1$ ,  $l_2^{-1} = l_2^{-1/2} \otimes (l_2^{-1/2})^{\perp}$ . For the Zeta function on the critical line it also holds  $\mathcal{E} \in H_{-1}(-\infty, \infty) = H_{-1} = H_0 \otimes H_0^{\perp}$ , and more specifically,  $\mathcal{E}, \gamma \in H_{-1/2}^{\perp}$ , i.e. the orthogonal projection of  $\mathcal{E}, \gamma$  onto  $H_{-1/2}$  is 0.

Theorem (Bagchi-Nyman criterion, (BaB)): Let

$$\gamma_k := \left\{ \rho\left(\frac{m}{k}\right) \mid m = 1, 2, 3, \dots \right\} \quad \text{for } k = 1, 2, 3, \dots$$

and  $\Gamma_k$  be the closed linear span of  $\gamma_k$ . Then the Nyman criterion states that the following statements are equivalent:

- i) The Riemann Hypothesis is true
- ii)  $\gamma \in \bar{\Gamma}_k$ .

The verification of the Bagchi criterion is enabled by ergodic theory on a Hilbert space, specific properties of the Mellin and Hilbert transforms regarding the basis functions  $e_v$  and the fractional function in combination with non-harmonic Fourier series theory (based on complex exponentials forming a Riesz basis for  $L_2^{\#}$ , ((DuR), (YoR)) and related approximation theory in Hilbert scales.

The concerned  $\gamma := \{1,1,1,1,\dots\} \in l_2^{-1}$  can be represented in the form  $\gamma = \left\{ \frac{1}{n} \gamma_1 \right\}_{n \in \mathbb{N}} = e_n^{odd} + e_n^{even}$  with  $e_{odd} := (1,0,1,0,1,0,1 \dots)$  and  $e_{even} := (1,0,1,0,1,0,1 \dots) + (0,1,0,1,0, \dots)$ . The analogue split for  $\gamma_k$  with  $k \geq 2$  is given by

$$\begin{aligned} \tilde{\gamma}_{2k} &:= \left\{ \frac{1}{n} \gamma_{2k} \right\}_{n \in \mathbb{N}} = \left\{ \frac{e_n^{odd}}{n} \gamma_{2k} \right\}_{n \in \mathbb{N}} + \left\{ \frac{e_n^{even}}{n} \gamma_{2k} \right\}_{n \in \mathbb{N}} \\ \tilde{\gamma}_{2k+1} &:= \left\{ \frac{1}{n} \gamma_{2k+1} \right\}_{n \in \mathbb{N}} = \left\{ \frac{e_n^{odd}}{n} \gamma_{2k+1} \right\}_{n \in \mathbb{N}} + \left\{ \frac{e_n^{even}}{n} \gamma_{2k+1} \right\}_{n \in \mathbb{N}}. \end{aligned}$$

Let  $S_n^{odd} := \text{span}\{\tilde{\gamma}_{2k}\}_{k=1,\dots,n+1}$  and  $S_n^{even} := \text{span}\{\tilde{\gamma}_{2k+1}\}_{k=1,\dots,n+1}$  denote the n-dimensional subspaces of  $H_{-1/2} \cong l_2^{-1/2}$  and  $\gamma_{odd}^{(n)} := P_n(e_n^{odd})$  and  $\gamma_{even}^{(n)} := P_n(e_n^{even})$  be the orthogonal projections  $P_n : H_{-1} \rightarrow S_n^{odd}, S_n^{even}$ , of  $\gamma$  onto  $S_n^{odd}, S_n^{even}$ . Then the Bagchi criterion is equivalent to

$$(\gamma - \gamma_{odd}^{(n)}, v)_{-1/2} = (\gamma - \gamma_{even}^{(n)}, v)_{-1/2} = 0, \quad \forall v \in l_2^{-1/2}.$$

Lemma ([GrI] 3.761): For  $a > 0$  it holds

$$\begin{aligned} M[\sin(a \cdot)](s) &= \int_0^\infty x^s \sin(ax) \frac{dx}{x} = \frac{\Gamma(s)}{a^s} \sin\left(\frac{\pi}{2}s\right) & 0 < \operatorname{Re}(s) < 1 \\ M[\cos(a \cdot)](s) &= \int_0^\infty x^s \cos(ax) \frac{dx}{x} = \frac{\Gamma(s)}{a^s} \cos\left(\frac{\pi}{2}s\right) & 0 < \operatorname{Re}(s) < 1 \end{aligned}$$

and  $H[\sin](\omega x) = -\cos(\omega x)$  resp.  $H[\cos](\omega x) = \sin(\omega x)$ .

The ergodic theorem on a Hilbert space is about the convergent sequence  $x_k := \frac{1}{k}(T^1 x + T^2 x + \dots + T^k x)$  with respect to the norm topology. The underlying operator

$$T^* := \lim_{k \rightarrow \infty} \frac{1}{k}(T^1 + T^2 + \dots + T^k)$$

is bounded and fulfills  $T^k \circ T^* = T^*$ ,  $T^* \circ T^k = T^*$ ,  $T^* \circ T^* = T^*$ . „The mean ergodic theorem ((HaP) p. 16) is an amusing and simple piece of classical analysis in case the underlying Hilbert space is one-dimensional“ (HaP) p. 14.

Lemma ((HiF)): Let  $E$  denote the eigen-space of the operator  $T^*$  with respect to the eigenvalue 1, i.e.  $E := \{x \in H | T^* x = x\}$  then it holds

- i)  $E = (T^* - \operatorname{Id})^{-1}(0)$  is closed
- ii) the operator  $T^*$  is a projector onto the eigen-space  $E$  (because of  $T^*|_E = \operatorname{Id}|_E$ ).

The chosen operator  $T^1 := H$  is the Hilbert operator (e.g. (LiI1) 11.1)

$$\underline{H}[u](t) := \frac{1}{i} \int_0^1 \cot(\pi(s-t)) u(s) ds, \quad u \in \bigcap_{\alpha \in \mathbb{R}} H_\alpha.$$

For each  $\alpha \in \mathbb{R}$  the extended operators  $\underline{H}^{(\alpha)}: H_\alpha \rightarrow H_\alpha$  with domain  $H_\alpha$  are bounded Fredholm operators of index zero with  $\|\underline{H}^{(\alpha)} u\|_\alpha = \|u\|_\alpha$ , with kernel  $N(\underline{H}^{(\alpha)}) = \operatorname{span}(1)$  (the set of all constants), and their range is  $R(\underline{H}^{(\alpha)}) = \{v \in H_\alpha | (v, 1) = 0\}$ , ((LiI1) p.316); and for  $e_\nu(x) := e^{2\pi i \nu x}$ ,  $\nu \in \mathbb{Z}$  it holds  $\underline{H} e_\nu = \operatorname{sign}(\nu) \cdot e_\nu$  (LiI1).

Putting  $T^1 := \underline{H}^{(\alpha)}$  one gets with  $T^k e_\nu = \begin{cases} \operatorname{sign}(\nu) e_\nu & \text{for } k \text{ odd} \\ e_\nu & \text{for } k \text{ even} \end{cases}$  for  $\nu \neq 0$

$$\sum_{i=1}^{2k-1} T^i e_\nu = \begin{cases} 2k-1 & \text{for } \nu > 0 \\ -1 & \text{for } \nu < 0 \end{cases}, \quad \sum_{i=1}^{2k} T^i e_\nu = \begin{cases} 2k & \text{for } \nu > 0 \\ 0 & \text{for } \nu < 0 \end{cases}.$$

Applying the Hilbert operator to the (periodical) fractional part  $L_2^\#$ -function ( $\alpha = 0$ )

$$\rho(x) = x - [x] = \frac{1}{2} - \sum_1^\infty \frac{\sin 2\pi \nu x}{\pi \nu}$$

leads to (\*)

$$\sum_{i=1}^{2k-1} T^i [\rho](x) = \begin{cases} \sum_1^\infty \frac{\cos 2\pi \nu x}{\pi \nu} & \text{for } 2k-1 = 1,5,9,13, \dots \\ \sum_1^\infty \frac{\sin 2\pi \nu x}{\pi \nu} & \text{for } 2k-1 = 3,7,11, \dots \end{cases}$$

and

$$\sum_{i=1}^{2k} T^i [\rho](x) = \begin{cases} \sum_1^\infty \frac{\sin 2\pi \nu x + \cos 2\pi \nu x}{\pi \nu} & \text{for } 2k = 2,6,10 \dots \\ 0 & \text{for } 2k = 4,8,12, \dots \end{cases}$$

whereby  $\sum_1^\infty \frac{\cos 2\pi \nu x}{\pi \nu} = -\frac{1}{\pi} \log |2 \sin(\pi x)| =: \rho_H(x) \in L_2^\#(0,1)$  and  $\sum_1^\infty \frac{\sin 2\pi \nu x}{\pi \nu} = \frac{1}{2} - \rho(x)$  (\*\*).

We note that the first derivative of the Hilbert transform of  $\rho(x)$  is given by the divergent (Cesàro summable) Fourier series representation of the  $\cot(\pi x)$  function,  $\cot(\pi x) = 2 \sum_1^\infty \sin(2\pi \nu x) = -\rho_H'(x)$  (see also (BeB)); it is an element of the Hilbert space  $H_{-1}^\#(0,1)$ , i.e. the formulas above are valid in similar form for  $\alpha = -1$  applied to the (in  $H_{-1}^\#(0,1)$  convergent) Fourier series representation  $\frac{1}{2} \cot(\pi x) = \frac{a_0}{2} + \sum_1^\infty a_\nu \cos(2\pi \nu x) + b_\nu \sin(2\pi \nu x)$  with  $a_\nu = 0$  for  $\nu \geq 0$  and  $b_\nu = 1$  for  $\nu > 0$  (i.e.  $\gamma := \{b_1, b_2, b_3, \dots\}$ ).

(\*) see also (CoG):

$$\begin{aligned} T^1[\rho](x) &= \sum_1^\infty \frac{\cos 2\pi \nu x}{\pi \nu}, \quad T^2[\rho](x) = \sum_1^\infty \frac{\sin 2\pi \nu x}{\pi \nu}, \quad T^3[\rho](x) = -\sum_1^\infty \frac{\cos 2\pi \nu x}{\pi \nu} = -T^1[\rho](x), \quad T^4[\rho](x) = -\sum_1^\infty \frac{\sin 2\pi \nu x}{\pi \nu} = -T^2[\rho](x), \quad T^5[\rho](x) = \\ \sum_1^\infty \frac{\cos 2\pi \nu x}{\pi \nu} &= T^1[\rho](x), \quad T^6[\rho](x) = \sum_1^\infty \frac{\sin 2\pi \nu x}{\pi \nu} = T^2[\rho](x), \quad T^7[\rho](x) = -\sum_1^\infty \frac{\cos 2\pi \nu x}{\pi \nu} = -T^1[\rho](x), \quad T^8[\rho](x) = -\sum_1^\infty \frac{\sin 2\pi \nu x}{\pi \nu} = -T^2[\rho](x), \dots \\ (*[x] + [x + \frac{1}{2}] &= [2x] \text{ resp. more general } \rho(nx) = \sum_{k=0}^{n-1} \rho\left(x + \frac{k}{n}\right), \text{ (ApT) p. 72} \end{aligned}$$

The sequence  $\omega_n$  (the imaginary parts of the Kummer function  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi iz\right)$ ) enables the definition of two sequences  $\lambda_n^{(1)} = \omega_n$  and  $\lambda_n^{(2)} = \frac{\omega_n + \omega_{n+1} - 1}{2}$ , leading to „nearly“ harmonic even and odd winding numbers of the corresponding exponentials  $\{e^{2\pi i \lambda_n x}\}_{n=0, \pm 1, \pm 2, \dots}$  occurring on the left and right part of the unit circle. From (YoR) p. 36, we recall

Kadec's  $\frac{1}{4}$ -Theorem (14): If  $\sigma_v$  is a sequence of real numbers for which

$$|\sigma_v - v| \leq L < \frac{1}{4}, \quad v = 0, \pm 1, \pm 2, \dots$$

Then  $\{e^{i\sigma_v x}\}_{v=0, \pm 1, \pm 2, \dots}$  satisfies the Paley-Wiener criterion and so forms a Riesz basis for  $L_2^\#(-\pi, \pi) = H_0^\#(-\pi, \pi)$ .

For the extension of Kadec's theorem to frames (being built by two real sequences  $\{\theta_k^{(1)}\}_{k \in \mathbb{Z}}$   $\{\theta_k^{(2)}\}_{k \in \mathbb{Z}}$ ) we refer to (ChO) 9.8.

If the sequence  $\sigma_v$  is symmetric, then the product  $\prod\left(1 - \frac{z^2}{\sigma_n^2}\right)$  converges to an entire function  $f(z)$  which belongs to the Paley-Wiener space, (YoR) p. 124.

The exponentials  $e^{i\sigma_v x}$  can be transformed into the reproducing functions

$$K_n(z) = \frac{\sin(\pi(z - \sigma_n))}{\pi(z - \sigma_n)}.$$

For  $G(z) := z \prod\left(1 - \frac{z^2}{\sigma_n^2}\right)$  the related function  $G_n(z) := \frac{G(z)}{G'(\sigma_n)(z - \sigma_n)}$  can be transformed into  $g_n(t) := \int_{-\infty}^{\infty} G_n(x) e^{ixt} dt$ . The solution of the moment problem  $f(\sigma_v) = c_v$   $v = 0, \pm 1, \pm 2, \dots$ , is given by (YoR) p.126,

$$(*) \quad f(z) = \sum_{v=-\infty}^{\infty} c_v \frac{G(z)}{G'(\sigma_v)(z - \sigma_v)}.$$

Remark (YoR) p.126: The formula (\*) is a simple example of a „generalized“ Lagrange interpolation formula for an entire function assuming the values  $c_v$  at the points  $\sigma_v$ .

Remark: The product representation of the considered Kummer function is given by ((BuH) p.184)

$${}_1F_1(a, c, z) = \frac{1}{\Gamma(c)} e^{\frac{a}{c}z} \prod\left(1 - \frac{z}{\alpha_n}\right) e^{z/\alpha_n}.$$

The considered imaginary parts of its zeros fulfill the Kadec requirements with

$$\sigma_n = \alpha_n := \omega_n + \frac{1}{4}, \quad \sigma_n = \beta_n := \frac{\omega_n + \omega_{n+1}}{2} - \frac{1}{4}.$$

We mention that the approximation solution to  $\cot(\pi x)$ , being testing against  $H_0^\#(0,1)$  test functions, is an element of  $H_{-1/2}^\#(0,1)$ .

With respect to (A1) and the newly proposed entire Zeta function  $\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right) \cdot \zeta(s)$  we note the related (classical) series representations for the  $\log\left(\tan\left(\frac{\pi}{2}x\right)\right)$  (whereby all summands of the first series are positive), and the (A2)-related series representations for the  $\log \sin(\pi x)$  functions, given by (GrI) 1.518,

$$\log\left(\tan\left(\frac{\pi}{2}x\right)\right) = \log\left(\frac{\pi}{2}x\right) + \sum_{1}^{\infty} (-1)^{k+1} \frac{1}{k} \frac{(2^{2k-1}-1)}{(2k)!} B_{2k}(\pi x)^{2k}, \quad x^2 < 1$$

$$\log \sin(\pi x) = \log(x\pi) + \sum_{1}^{\infty} (-1)^k \frac{1}{k} \frac{2^{2k-1}}{(2k)!} B_{2k}(\pi x)^{2k}, \quad x^2 < 1.$$

The term  $-\log\left(\tan\left(\frac{\pi}{2}s\right)\right)$  is  $L_2^\#$ -integrable and can be represented as a convergent series (EIL). It results into a correspondingly modified Riemann error function or a correspondingly modified definition of the  $li(x)$  function. In both cases the approximation behavior between the  $li(x)$  function and the prime density function gets improved.

(A3) This class is about RH criteria formulated by distributional arithmetical functions going beyond the distributional arithmetical functions applied for "*a distributional way to prove the Prime Number Theorem*" (ViJ).

In 1927 Polya published a very different sort of theorem as in (PoG) on the same general subject ("Über trigonometrische Integrale mit nur reellen Nullstellen") with quite weak conditions to the baseline function  $F(u)$  (see also (EdH) 12.5). All attempts failed so far to build a proper (classical) baseline function  $F(u)$  to prove the RH. The proposed distributional Hilbert space framework of this homepage provides an additional opportunity to build such a baseline function.

According to E. Landau the classical proofs of the Prime Number Theorem (PNT) are not being scalable for "*deeper going number theory problems*". Ikehara's theorem ((EdH) 12.7) is about a deduction from Wiener's general Tauberian theorem. In (ViJ) "*a distributional way to prove the PNT*" is based on the Mellin transform of e.g. the Dirac delta "function". The part B section is about a (distributional) quantum element/energy Hilbert space framework enabling a quantum gravity model. From a physical modelling perspective it goes along with a replacement of Dirac's model of the "density" of an "idealized point mass" resp. an "idealized point charge" (modelled by the Dirac or Delta "function") by Plemelj's concept of a "mass element  $dm$ " (PIJ) of the proposed "bosons" quantum state Hilbert space  $H_{-1/2}$ . "Tauberian Theorems for Generalized Functions" come along with the concept of an "automodel function". It is claimed to be an appropriate framework to apply the Polya theorem to prove the RH, as the crucial condition of Polya's theorem is equivalent to one of the characterization criteria of an "automodel function", ((VIV) p. 57).

The proposed periodical, distributional Hilbert scale framework  $H_\alpha^\#$  also enables also a truly unit circle method to tackle the binary Goldbach problem. It overcomes current handicaps of the famous Hardy-Littlewood circle method, which is based on the open unit circle disk.

Vinogradov applied the Hardy-Littlewood circle method to derive his famous (currently best known, but not sufficient) estimate regarding the tertiary Goldbach problem. Vinogradov's theorem states that any sufficiently large odd integer can be written as a sum of three prime numbers. The "any sufficiently large odd integer" condition is due to a not sufficiently "good" estimate: it is derived from two components based on a decomposition of the (Hardy-Littlewood) circle into two parts, the "major arcs" (also called "basic intervals") and the "minor arcs" (also called "supplementary intervals"). The sufficiently good estimate is based on "major arcs" estimate using also Goldbach problem relevant data; the not sufficiently good "minor arcs" estimate are purely Weyl sums estimates taking not any Goldbach problem relevant data into account. However, this estimate is optimal with respect to Weyl sums properties. In other words, the major/minor arcs decomposition is inappropriate to solve both Goldbach problems. The proposed periodical, distributional Hilbert scale framework  $H_\alpha^\#$  with its underlying domain (the boundary of the unit disk, the unit circle) is claimed to enable a truly unit circle based method to prove the binary Goldbach problem.

The proposed change also enables the definition of *two* appropriately different arithmetical functions to analyze binary number theoretical problems for any prime number pair  $(p, q)$ . We note that the winding numbers (i.e. the set of integers) of the unit circle are related to the zeros of the Weyl sum components, which are the basis functions  $e^{2\pi i n x}$ . The sequence  $\omega_n$  (the imaginary parts of the Kummer function  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi i z\right)$ ) enables the definition of two sequences  $\lambda_n^{(1)} = \omega_n$  and  $\lambda_n^{(2)} = \frac{1}{2}(\omega_n + \omega_{n+1})$ , leading to "nearly" harmonic even and odd winding numbers of the corresponding exponentials  $\{e^{2\pi i \lambda_n x}\}_{n=0, \pm 1, \pm 2, \dots}$  occurring on the left and right semicircle of the unit circle.

Referring back to (A2) and the considered  $\gamma = \{1, 1, 1, 1, \dots\}$  with  $\gamma \in (L_2^{-1/2})^\perp$  we mention that  $\gamma$  can be interpreted as a "winding number list", while running periodically through the unit circle

The links of non-harmonic Fourier series theory to the cardinal series based interpolation theory in the Paley-Wiener space  $PW$  are given by the following theorems (YoR) p. 170, p. 157):

for a complex exponentials system  $\{e^{2\pi i \lambda_n x}\}_{n=0, \pm 1, \pm 2, \dots}$  to form a Riesz basis for  $L_2^\#$  it is necessary and sufficient to the interpolation problem  $f(\lambda_n) = c_n$ , where  $f \in PW$

a sequence of vectors  $\varphi_n$  belonging to a separable Hilbert space  $H$  is a Riesz basis if and only if it is an exact frame, i.e.  $A\|f\|^2 \leq \sum_{n=1}^{\infty} |(f, \varphi_n)|^2 \leq B\|f\|^2$  (if  $\varphi_n$  is a complete orthogonal sequence in  $H$ , then it holds  $A = B = 1$ ), or if and only if  $\varphi_n$  is both, a frame and a Riesz sequence (SeK)

The link to a specific class of non-harmonic Fourier series is given by the considered sequences of real or complex numbers with uniform density one (DuR).



## A tool set to prove the binary Goldbach conjecture

On the occasion of the 59th birthday  
of my wife, Vibhuta  
August 25, 2020

In the following we summaries the proposed tool set to tackle the binary Goldbach conjecture.

The following two lemmata are about the building of arithmetical functions based on a certain integrals:

Lemma 1 (Landau, (PoG1)): Let  $q_n$  denote a divergent sequence of positive numbers  $0 < q_1 \leq q_2 \leq q_3 \leq \dots$   $\lim_{n \rightarrow \infty} q_n = \infty$ ,  $\tau(x)$  the corresponding counting function of the numbers of  $q_n$  less than  $\leq x$  and  $w(x)$  a positive, non-decreasing function with  $\lim_{x \rightarrow \infty} \frac{w(2x)}{w(x)} = \lim_{x \rightarrow \infty} \frac{\tau(x)w(x)}{x} = 1$ .

Then

$$\lim_{x \rightarrow \infty} \frac{1}{\tau(x)} \sum_{q \leq x} \rho\left(\frac{x}{q}\right) = 1 - \gamma,$$

where  $\rho(x)$  denotes the fractional part function.

In (PoG1) lemma 1 is generalized by

Lemma 2: Let  $w(x)$  a positive, non-decreasing function with  $\lim_{x \rightarrow \infty} \frac{w(\beta x)}{w(\alpha x)} = 1$  with  $\alpha, \beta$  positive numbers. Then

$$\lim_{x \rightarrow \infty} \frac{w(x)}{x} \sum_{n \leq x} f\left(\frac{x}{n}\right) = \int_0^1 f(t) dt.$$

Remark 1: Lemma 2 is valid for two kind of conditions about  $f(t)$ :

- i)  $f(t)$  is Riemann integrable in  $[0,1]$
- ii)  $f(t)$  is integrable in every closed sub-interval of  $[0,1]$ , which does not contain 0, and it exists an  $\alpha \in (0,1)$  with  $\lim_{t \rightarrow 0} t^{1-\alpha} f(t) = 0$ .

Remark 2: The condition  $\lim_{x \rightarrow \infty} \frac{w(\beta x)}{w(\alpha x)} = 1$  is related to the theory of quasi-asymptotics of generalized functions with its underlying concept of (slow) regular varying (automodel) functions  $\vartheta(x)$ , where  $\lim_{x \rightarrow \infty} \frac{\vartheta(\beta x)}{\vartheta(x)}$  (resp.  $\lim_{x \rightarrow \infty} \frac{x\vartheta'(x)}{\vartheta(x)}$ ) exists, (VIV) p. 56/57; (EsR). The latter condition,  $\lim_{x \rightarrow \infty} \frac{x\vartheta'(x)}{\vartheta(x)} = c$ , links back to the famous Polya criterion in (PoG) given by  $\alpha \leq -\frac{xf'(x)}{f(x)} \leq \beta$  being valid for only (!) closed interval domains. We note that  $-\log x$  is slowly varying at  $x = 0^+$ , (SeE) p.47. We further note with respect to lemma 5 v) below, that for  $\tilde{h}(x) := {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right)$  it holds  $-\frac{x\tilde{h}'(x)}{\tilde{h}(x)} = 1 - \frac{e^x}{h\tilde{h}x}$ .

In (LaE) §56, the following following lemma is provided. It has been applied in (LaE1) to derive the asymptotics integral formula for the Goldbach number counting function

$$H(2n) := \sum_{k=1}^{2n} G_{2k} = \sum_{p \leq 2n} \pi(2n-p), \quad p+q \leq 2n,$$

in the form

$$H(2n) \sim 2n \int_2^n \frac{1}{\log(2n-u) \log u} du \sim 2n \int_2^n \frac{1}{\log(2n-\frac{u}{2}) \log u} du \sim \frac{1}{2} \left(\frac{2n}{\log(2n)}\right)^2, \quad 0 \leq \vartheta \leq 1$$

by putting  $F(u, x) := \pi(2n-p)$ .

We note that  $\sum_{k \text{ odd}}^{2x} G_k \sim \frac{2x}{\log(2x)}$ , while  $\sum_{k=1}^x G_{2k} \sim \frac{1}{2} \left(\frac{2x}{\log(2x)}\right)^2$ , because the odd  $k$  can be neglected as every odd number can be represented as sum of two primes, if  $k-2$  is prime, otherwise not (LaE1).

We further note that  $d\left[\sum_{k=1}^x \frac{1}{\varphi(k)}\right] \sim d[\log x] \sim \frac{dx}{x}$  (LaE1), and  $d\vartheta = \left[\sum_{n \leq x} \frac{\mu(n)}{n} \log \left(\frac{x}{n}\right)\right] = \frac{1}{x} \left[\sum_{n \leq x} \frac{\mu(n)}{n}\right] dx \sim \frac{dx}{x}$  (ApT) p.97.

Lemma 3: Let  $F(u, x)$  be a function with real arguments with  $2 \leq u \leq x$  fulfilling the following conditions

- i)  $F(u, x) \geq 0$
- ii) For fixed  $x > 2$  the function  $\frac{F(u, x)}{\log u}$  is never increasing for  $u \in (2, x)$
- iii)  $F(2, x) = o\left(\int_2^x \frac{F(u, x)}{\log u} du\right)$ ,

then it holds

$$\sum_{p \leq x} F(p, x) \sim \int_2^x \frac{F(u, x)}{\log u} du .$$

The considered Stäckel approximation formula in (LaE1) is given by

$$\tilde{H}(2n) := \sum_{k=1}^n \tilde{G}_{2k} = c_\lambda \cdot \sum_{\substack{k=2 \\ k \text{ even}}}^{2n} \frac{1}{\varphi(k)} \left(\frac{k}{\log(k)}\right)^2 \quad \text{with } c_\lambda := \frac{\pi^4}{105 \cdot \zeta(3)} \sim 0,772 \dots$$

The factor  $c_\lambda$  was suggested to ensure that  $\tilde{H}(2n) < H(2n)$ .

Remark 3: We note that the above (appreciated) identical asymptotical behavior of both summation formulas does not allow a corresponding conclusion to the approximation behavior of the underlying Goldbach numbers  $G_{2k}$  resp.  $\tilde{G}_{2k}$ , as in the sum formula  $\sum_{k=1}^n G_{2k} - \tilde{G}_{2k}$  positive and negative terms cancel each other out.

Remark 4: With respect to (A2) above and the the term  $\frac{1}{\varphi(n)}$  in the Stäckel approximation formula we note the inequality, (ApM) p. 71,  $\frac{\sigma(k)}{k^2} \leq \frac{1}{\varphi(k)} \leq \frac{\pi^2}{6} \frac{\sigma(k)}{k^2} = \zeta(2) \frac{\sigma(k)}{k^2}$ ,  $k \geq 2$ , where  $\sigma(n) = \sigma_1(n)$  denotes the sum of the divisors of  $n$  ((ApM) p. 38), and  $\varphi(n)$  denotes the Euler totient function. It can be represented as discrete Fourier transform of the greatest common divisor in the form  $\varphi(n) = \sum_{k=1}^n \gcd(k, n) \cdot \cos(2\pi \frac{k}{n})$ . We further mention that for even integers it holds  $\varphi(2k) = 2\varphi(k) > \sqrt{k}$ . At the same point in time, it holds  $\frac{k}{\log^2 k} > \sqrt{k}$  for  $k \geq 2$ .

Remark 5: It holds  $\sum_{k \leq x} \frac{1}{\varphi(k)} = O(\log x)$ ,  $\frac{1}{x} \sum_{k \leq x} d(n) = \log x + (2\gamma - 1) + O(\sqrt{x})$ , and  $\sum_{k \leq x} \frac{k}{\varphi(k)} = O(x)$ ,  $\sum_{k \leq x} \frac{1}{\varphi(k)} = O(\log x)$  (ApM) p. 71. The average order of  $d(n)$  is  $\log n$  i.e.  $\frac{1}{x} \sum_{k \leq x} d(k) = \log x + 2\gamma - 1 + O(\sqrt{x}) \sim \log x$ , (ApM) p. 52. For  $\vartheta(x) := \sum_{n \leq x} \frac{\mu(n)}{n} \log \left(\frac{x}{n}\right)$  we recall from (ApT) p. 97, that the PNT theorem is equivalent to  $d\vartheta(x) = \sum_{n \leq x} \frac{\mu(n)}{n} \frac{dx}{x} \sim \frac{dx}{x}$ .

Remark 6: ((LaE) §56 p. 214): the  $n$ -th prime number is asymptotically equal to  $n \cdot \log n$ .

Building an arithmetical function for the term  $F(u, x) := \pi(2n - p)$  is challenging because of

Lemma 4, ((LaE) §56, p. 215):

from a certain number  $x$  on, there are more primes in the interval  $(1, x)$  than in the interval  $(x, 2x)$ .

Remark 7: Lemma 4 cannot be proven with the PNT; the proof requires the asymptotics

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + o\left(\frac{x}{\log^2 x}\right)$$

Remark 8: The functions  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -z\right)$  and  ${}_1F_1\left(1, \frac{3}{2}; z\right)$  possesses the same zeros as the related Whittaker function  $z^{-3/4} M_{\frac{1}{4}, \frac{1}{4}}(z)$ , which is the solution of the corresponding self-adjoint (ODE) Whittaker operator ((BuH), §17.1). The latter one also plays key role for the radial Schrödinger operator with a Coulomb potential or with a morse potential on the half line (DeJ), (LaJ1).

Corresponding integrals of oscillatory type for  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; x\right)$  (when the variables are real) are the Fresnel integrals  $C(x), S(x)$  with  $\lim_{x \rightarrow \infty} C(x) = \lim_{x \rightarrow \infty} S(x) = \frac{1}{2}$  with the exact error formula (OIF) p. 67,

$$C\left(\sqrt{\frac{2x}{\pi}}\right) + i \cdot C\left(\sqrt{\frac{2x}{\pi}}\right) = \sqrt{\frac{2x}{\pi}} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; ix\right) = \int_0^x \frac{\cos t dt}{\sqrt{2\pi t}} + i \cdot \int_0^x \frac{\sin t dt}{\sqrt{2\pi t}} = \left(\frac{1}{2} + i\frac{1}{2}\right) - \frac{ie^{ix}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{(2ix)^n}.$$

The formula allows an alternative „2-semicircle (even/odd integer) method“ with an underlying „major/minor arcs I/II“ concept based on two appropriately defined Fresnel integral distribution functions ( $\cos(t) > 0, \sin(t) > 0$ ), enabled by the real part values  $\omega_n$  of  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi iz\right)$ . With respect to the link to exponentials Riesz bases we refer to (KoG), (NaA) and the references cited there.

Lemma 5:

For the real part values  $\omega_n$  of  ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; 2\pi iz\right)$  it holds (SeA)

- i)  $2n - 1 < 2\omega_n, \omega_n + \omega_{n+1} - 1 < 2n < 2\omega_n + 1, \omega_n + \omega_{n+1} < 2n + 1$
- ii)  $n - \frac{3}{4} < \lambda_n^{(1)} - \frac{1}{4} := \omega_n - \frac{1}{4}, \lambda_n^{(2)} - \frac{1}{4} := \frac{\omega_n + \omega_{n+1} - 1}{2} - \frac{1}{4} < n - \frac{1}{4} < \omega_n + \frac{1}{4}, \frac{\omega_n + \omega_{n+1} - 1}{2} + \frac{1}{4} < n + \frac{1}{4}$
- iii)  $n - \frac{1}{2} < \omega_n < n, \frac{1}{2} < \omega_1, \frac{\omega_1 + \omega_2 - 1}{2} < 1, s_n := \frac{\omega_n}{n} \rightarrow 1 \quad n \in \mathbb{N}$
- iv) the non-integer sequences  $2\omega_n$  and  $\omega_n + \omega_{n+1}$  fulfill a kind of Hadamard gap condition
 
$$\frac{\omega_{n+1}}{\omega_n} > \frac{n + \frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1 \quad \text{resp.} \quad \frac{\omega_{n+1} + \omega_{n+2}}{\omega_n + \omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1$$
- v)  $\theta := \frac{1}{4} < \omega_{n+1} - \omega_n < 1 - \frac{1}{4} = 1 - \theta$
- vi) putting  $a_n := \frac{\omega_n}{n} - \frac{1}{2}, b_n := \frac{\omega_1 + \omega_2 - 1}{2n} - \frac{1}{2}$  it follows
 
$$0 < a_1 = \omega_1 - \frac{1}{2} \leq a_n \rightarrow \frac{1}{2}, \quad \frac{1}{2} \leftarrow b_n < b_1 = \frac{\omega_1 + \omega_2 - 1}{2} < 1, \quad a_n, b_n - a_n \in \left(0, \frac{1}{2}\right), \quad b_n, 1 - a_n \in \left(\frac{1}{2}, 1\right).$$

Lemma 6, (BaR), (KaD):

The following inequalities are valid

$$c_1 \cdot \left[ {}_1F_1\left(\frac{1}{2}, c; x\right) \right]^2 \leq {}_1F_1(a_n, c; x) \cdot {}_1F_1(1 - a_n, c; x) \leq \left[ {}_1F_1\left(\frac{1}{2}, c; x\right) \right]^2.$$

Remark 9: For the sequences  $a_n := \frac{\omega_n}{n} - \frac{1}{2}, b_n := \frac{\omega_n + \omega_{n+1} - 1}{2n} - \frac{1}{2}$  it holds  $a_n \in \left(0, \frac{1}{2}\right)$  resp.  $b_n \in \left(\frac{1}{2}, 1\right)$  with  $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = 1/2$ , and the domain of each sequence has Snirelmann density  $\frac{1}{2}$ . Therefore, it holds

$$-x {}_1F_1(a_n; a_n + 1, -\log x), -x {}_1F_1(b_n; b_n + 1, -\log x) \rightarrow li(x) = -x {}_1F_1(1; 1, -\log x).$$

Remark 10: In the context of non-harmonic Fourier series theory the counterpart of  $\gamma := \{1, 1, 1, \dots\}$  can be defined by  $(a, b)_n := \{a_1, b_1, a_2, b_2, \dots\}$ . It enables „two check points“, while running once through the unit circle, one „check point“ for each prime number  $p, q$ .

Remark 11: The sequences  $s_n := \frac{\omega_n}{n}$  and  $t_n := \frac{\omega_n + \omega_{n+1} - 1}{2n}$  fulfill the Hardy-Littlewood condition  $|s_{n+1} - s_n| < \frac{K}{n}$ , e.g.  $s_n$  has a defined Abel average ((EdH) 12.7)  $\lim_{r \uparrow 1} \frac{s_1 r + s_1 r^2 + s_3 r^3 + \dots}{r + r^2 + r^3 + \dots} = L$ .

Remark 12 ((KaM), (KoA), (ZyA)): Let  $\{n_k\}$  be a sequence of integers satisfying the Hadamard gap condition, i.e.  $\frac{n_{k+1}}{n_k} > q > 1$ , then trigonometric gap series  $\sum_{k=1}^{\infty} c_k \sin(2\pi n_k x)$  converges almost everywhere iff  $\sum_{k=1}^{\infty} c_k^2 < \infty$ .

Remark 13: We note that for  $\tilde{T}(\log x) := \sum_{n \leq x} \frac{1}{n} \log\left(\frac{x}{n}\right), x \geq 1$ , the inverse mapping is given by ([ScW] lemma 3.3)

$$\sigma(x) := \tilde{T}^{-1}(\log x) = \sum_{n \leq x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right).$$

Remark 14: With respect to Remark 7 we note that what also cannot be derived from the PNT is the convergence of the series

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} \log\left(\frac{1}{n}\right) = 1 .$$

„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived“ ((LaE) §160). From (ApT) p. 66 we recall that the PNT can be derived from  $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$ .

The related point measure is given by  $\vartheta(x) = \sum_{n \leq x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right)$  with  $d\vartheta = \frac{1}{x} \left[ \sum_{n \leq x} \frac{\mu(n)}{n} \right] dx$ . Because of  $\frac{d}{dx} \log\left(\frac{x}{a}\right) = \frac{1}{x}$  it can be replaced by

$$\vartheta^*(x) = \sum_{\substack{n \leq x \\ n \text{ odd}}} \frac{\mu(n)}{n} \log\left(\frac{2x}{2\omega_n - \frac{1}{2}}\right) + \sum_{\substack{n \leq x \\ n \text{ even} \\ n=1}} \frac{\mu(n)}{n} \log\left(\frac{2x}{\omega_n + \omega_{n+1} - \frac{1}{2}}\right)$$

with  $d\vartheta = d\vartheta^* \sim \frac{dx}{x} = d(\log x)$  resp.

$$\sum_{\text{odd}} \frac{\mu(n)}{n} \log\left(\frac{2}{2\omega_n - \frac{1}{2}}\right) + \sum_{\text{even}} \frac{\mu(n)}{n} \log\left(\frac{2}{\omega_n + \omega_{n+1} - \frac{1}{2}}\right) \leq 1$$

The additional term  $\omega_n + \omega_{n+1} - \frac{1}{2}$  for the „ $n$  even“ series (i.e. not starting the series  $\omega_n + \omega_{n+1} + \frac{1}{2}$ ) is to get „ $n = 1$ “ as an element of the underlying domain, i.e. both domains get a Snirelmann density of  $\frac{1}{2}$ . Additionally both related sequences,  $\left(\frac{2}{2\omega_n - \frac{1}{2}}\right)^{-1} = \omega_n - \frac{1}{4}$  and  $\left(\frac{2}{\omega_n + \omega_{n+1} - \frac{1}{2}}\right)^{-1} = \frac{\omega_n + \omega_{n+1}}{2} - \frac{1}{4}$  fulfill the prerequisite of the ((YoR) p. 36)

Kadec  $\frac{1}{4}$ -Theorem: If  $\sigma_\nu$  is a sequence of real numbers for which

$$|\sigma_\nu - \nu| \leq L < \frac{1}{4}, \quad \nu = 0, \pm 1, \pm 2, \dots,$$

then  $\{e^{i\sigma_\nu x}\}_{\nu=0, \pm 1, \pm 2, \dots}$  satisfies the Paley-Wiener criterion and so forms a Riesz basis for  $L^2(-\pi, \pi) = H_0^\#(-\pi, \pi)$ .

The below strongly increasing sequences  $(A_n^{(1)}, \bar{A}_n^{(1)})$  and  $(A_n^{(2)}, \bar{A}_n^{(2)})$  (lemma 9) enable the definition of a twofold finer sieve method (HaH): there is the split above between „odd“ and „even“ series, both with underlying Snirelmann density  $\frac{1}{2}$ , governing two different arithmetical functions on the first level; on a second level per each series there can be built specific sieve methods per underlying considered „prime interval“ domains  $[1, p]$  and  $[p, 2p]$ , to overcome the mentioned challenge in lemma 4 to build an arithmetical function for  $F(u, x) := \pi(2n - p)$  based on lemma 2.

From (LaE1) we note that  $\sum_{k \text{ odd}}^{2x} G_k \sim \frac{2x}{\log(2x)}$ , while  $\sum_{k=1}^x G_{2k} \sim \frac{1}{2} \left(\frac{2x}{\log(2x)}\right)^2$ , because the odd  $k$  can be neglected as every odd number can be represented as sum of two primes, if  $k - 2$  is prime, otherwise not.

Lemma 7:

For the sequences pair  $(\lambda_k^{(1)}, \lambda_k^{(2)})$  it holds  $\lambda_k^{(1)} \in \left(k - \frac{1}{2}, k\right)$ ,  $\lambda_k^{(2)} \in \left(k, k + \frac{1}{2}\right)$ , and therefore,

$$\frac{2k-1}{2k+1} < \frac{\lambda_k^{(1)}}{k} \cdot \frac{\lambda_k^{(2)}}{k + \frac{1}{2}} < 1 \quad \text{resp.} \quad \frac{1}{2k} \cdot \frac{1}{2(k+1)} < \frac{1}{2k} \cdot \frac{2k-1}{2k+1} < \frac{1}{2k} \cdot \frac{\lambda_k^{(1)}}{k} \cdot \frac{\lambda_k^{(2)}}{k + \frac{1}{2}} < \frac{1}{2k}$$

resp.

$$\frac{n}{2} n < \sum_{k=1}^n \lambda_k^{(1)} < \frac{n}{2} (n+1) < \sum_{k=1}^n \lambda_k^{(2)} < \frac{n}{2} (n+2).$$

Lemma 8:

Putting  $a_k^{(1)} := \lambda_k^{(1)} - \frac{1}{2}$ ,  $a_k^{(2)} := \lambda_k^{(2)} - \frac{1}{2}$  one gets

$$(2n-1) < A_n^{(1)} := \frac{1}{n} \sum_{k=1}^{2n} a_k^{(1)}, \quad \bar{A}_n^{(1)} := \frac{1}{n} \sum_{k=1}^{2n} \left[2n - a_k^{(2)}\right] < 2n < \\ < A_n^{(2)} := \frac{1}{n} \sum_{k=1}^{2n} a_k^{(2)}, \quad \bar{A}_n^{(2)} := \frac{1}{n} \sum_{k=1}^{2n} \left[2n - a_k^{(1)}\right] < (2n+1)$$

and each of the two domains of the two merged, strongly increasing sequences  $(A_n^{(1)}, \bar{A}_n^{(1)})$  and  $(A_n^{(2)}, \bar{A}_n^{(2)})$  both have Snirelmann density  $\frac{1}{2}$ .

The link to the proposed Kummer function based Zeta function theory is given by the theory of quasi-asymptotics in distributional Hilbert scales, defined via the eigenvalues of the specific Whittaker operator with the fundamental solutions, (AbM) 13.1.31, (GrI) 9.212,

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right) = z^{-\frac{3}{4}} e^{\frac{z}{4}} M_{\frac{1}{4}}(z) \quad \text{and} \quad {}_1F_1\left(1, \frac{3}{2}; -z\right) = z^{-\frac{3}{4}} e^{-\frac{z}{4}} M_{\frac{1}{4}}(z).$$

From the (A1) lemma we recall the asymptotics

$$d\left[{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log x\right)\right]^2 \sim \left(\frac{x}{\log x}\right)^2 dx.$$

For the construction of a Kummer function related arithmetical function one gets from lemma 5 vii) with  $s = 1/2$

$$\int_0^\infty t^{1/4} {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -t\right) \frac{dt}{t} = 2\Gamma\left(\frac{1}{4}\right),$$

i.e. ( $t \rightarrow -\log x = \log\left(\frac{1}{x}\right)$ ) it holds

$$\int_0^1 h(x) dx = 1 \quad \text{for} \quad h(x) := \frac{1}{x} \frac{\log^{\frac{1}{4}}\left(\frac{1}{x}\right)}{2\Gamma\left(\frac{1}{4}\right)} \frac{{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log x\right)}{\log x} = \frac{1}{2\Gamma\left(\frac{1}{4}\right)} \frac{1}{\sqrt{x \log x}} \frac{M_{\frac{1}{4}}(\log x)}{\log x}.$$

Remark: Recalling  ${}_1F_1(a, a+1; x) \sim \frac{1}{\Gamma(a)} \frac{e^x}{x}$  and  $\frac{1}{2} \frac{d}{dx} F_a^2(x) = \frac{a}{a+1} F_a(x) \cdot F_{a+1}(x)$  two (strongly increasing resp. strongly decreasing) sequences  $\vartheta_n^{(i)}$  with  $0 < \theta_n < 1$  and  $\lim_{n \rightarrow \infty} \vartheta_n^{(i)} = \frac{1}{2}$  ( $i = 1, 2$ ) allow to approximate the  $d\left[{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log t\right)\right]^2$  integral density in the following form

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[ {}_1F_1\left(\vartheta_n^{(1)}, \vartheta_n^{(1)} + 1; \log x\right) \cdot {}_1F_1\left(\vartheta_n^{(2)} + 1, \vartheta_n^{(2)} + 2; \log x\right) \right] &= \frac{1}{2} \frac{d}{dx} \left[ {}_1F_1\left(\frac{1}{2}, \frac{3}{2}; \log x\right) \right]^2 \\ \frac{\pi}{2} {}_1F_1\left(\vartheta_n^{(1)}, \vartheta_n^{(1)} + 1; \log x\right) \cdot {}_1F_1\left(\vartheta_n^{(2)} + 1, \vartheta_n^{(2)} + 2; \log x\right) &\sim \frac{\pi}{2} \frac{1}{\Gamma(\vartheta_n^{(1)})} \frac{1}{\Gamma(\vartheta_n^{(2)})} \left[ \frac{x}{\log x} \right]^2 \sim \frac{1}{2} \left[ \frac{x}{\log x} \right]^2. \end{aligned}$$

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