

The Hilbert transform

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Abstract

The information about the Hilbert transform is often scattered in books about signal processing. Their authors frequently use mathematical formulas without explaining them thoroughly to the reader. The purpose of this report is to make a more stringent presentation of the Hilbert transform but still with the signal processing application in mind.

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1. Introduction

Signal processing is a fast growing area today and a desired effectiveness in utilization of bandwidth and energy makes the progress even faster. Special signal processors have been developed to make it possible to implement the theoretical knowledge in an efficient way. Signal processors are nowadays frequently used in equipment for radio, transportation, medicine and production etc.

In 1743 a famous Swiss mathematician named Leonard Euler (1707-1783) derived the formula

$$e^{jz} = \cos(z) + j \sin(z).$$

150 years later the physicist Arthur E. Kennelly and the scientist Charles P. Steinmetz used this formula to introduce the complex notation of harmonic wave forms in electrical engineering, that is

$$e^{j\omega t} = \cos(\omega t) + j \sin(\omega t).$$

Later on, in the beginning of the 20th century, the German scientist David Hilbert (1862-1943) finally showed that the function $\sin(\omega t)$ is the Hilbert transform of $\cos(\omega t)$. This gave us the $\pm\pi/2$ phase-shift operator which is a basic property of the Hilbert transform.

A real function $f(t)$ and its Hilbert transform $\hat{f}(t)$ are related to each other in such a way that they together create a so called strong analytic signal. The strong analytic signal can be written with an amplitude and a phase where the derivative of the phase can be identified as the instantaneous frequency. The Fourier transform of the strong analytic signal gives us a one-sided spectrum in the frequency domain. It is not hard to see that a function and its Hilbert transform also are orthogonal. This orthogonality is not always realized in applications because of truncations in numerical calculations. However, a function and its Hilbert transform has the same energy and therefore the energy can be used to measure the calculation accuracy of the approximated Hilbert transform.

The Hilbert transform defined in the time domain is a convolution between the Hilbert transformer $1/(\pi t)$ and a function $f(t)$ [7]. This is motivated later on.

Definition 1.1. *The Hilbert transform $\hat{f}(t)$ of a function $f(t)$ is defined for all t by*

$$\hat{f}(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau,$$

when the integral exists.

It is normally not possible to calculate the Hilbert transform as an ordinary improper integral because of the pole at $\tau = t$. However, the P in front of the

integral denotes the Cauchy principal value which expanding the class of functions for which the integral in Definition 1.1 exist. It can be defined by the following definition [5].

Definition 1.2. Let $[\alpha, \beta]$ be a real interval and let f be a complex-valued function defined on $[\alpha, \beta]$. If f is unbounded near an interior point ξ of $[\alpha, \beta]$, the integral of f over $[\alpha, \beta]$ does not always exist. However, the two limits

$$\lim_{\epsilon \rightarrow 0} \int_{\alpha}^{\xi-\epsilon} f(x)dx \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \int_{\xi+\epsilon}^{\beta} f(x)dx,$$

still may exist, and if they do their sum is called the improper integral of f over $[\alpha, \beta]$ and is denoted by the ordinary integration symbol

$$\int_{\alpha}^{\beta} f(x)dx.$$

Even if these two limits do not exist, it may happen that the "symmetric limit"

$$\lim_{\epsilon \rightarrow 0^+} \left(\int_{\alpha}^{\xi-\epsilon} f(x)dx + \int_{\xi+\epsilon}^{\beta} f(x)dx \right),$$

exists. If it does, it is called the principal value integral of f from α to β and is denoted by the symbol

$$P \int_{\alpha}^{\beta} f(x)dx.$$

Example 1.3. An ordinary real function $1/x$ that is integrated from $-a$ to a can be written as

$$\int_{-a}^a \frac{1}{x} dx = \lim_{\delta \rightarrow 0^+} \int_{-a}^{-\delta} \frac{1}{x} dx + \lim_{\epsilon \rightarrow 0^+} \int_{\epsilon}^a \frac{1}{x} dx,$$

and we see that it is not possible to calculate these integrals separately because of the pole in $x = 0$. However, if we apply the Cauchy principal value then δ and ϵ tend to zero at the same speed, that is

$$P \int_{-a}^a \frac{1}{x} dx = \lim_{\epsilon \rightarrow 0^+} \left(\int_{-a}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^a \frac{1}{x} dx \right) = 0,$$

and the integral converges.

2. Mathematical motivations for the Hilbert transform

In this chapter we motivate the Hilbert transform in three different ways. First we use the Cauchy integral in the complex plane and second we use the Fourier transform in the frequency domain and third we look at the $\pm\pi/2$ phase-shift which is a basic property of the Hilbert transform.

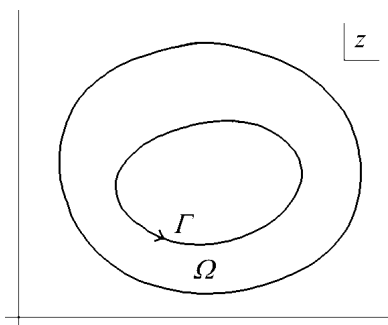


Figure 2.1: Γ is a piecewise smooth closed contour on an open domain -

2.1. The Cauchy integral

The Cauchy integral is a figurative way to motivate the Hilbert transform. The complex view helps us to relate the Hilbert transform to something more concrete and understandable.

Consider an integral in the complex z -plane on the form

$$\oint_{\Gamma} \frac{f(z)}{z-a} dz,$$

which is known as a Cauchy integral. If f is analytic and Γ is a piecewise smooth closed contour in an open domain -, see Figure 2.1, then the Cauchy integral theorem is applicable as

$$\oint_{\Gamma} \frac{f(z)}{z-a} dz = \begin{cases} 2\pi i f(a) & \text{if } a \text{ is inside } \Gamma \\ 0 & \text{if } a \text{ is outside } \Gamma \end{cases}$$

To get a result when a lies on Γ we have to create a new contour Γ'_ε as in Figure 2.2 where

$$\oint_{\Gamma'_\varepsilon} \frac{f(z)}{z-a} dz = 2\pi i f(a). \quad (2.1)$$

If the radius ε of the semicircle γ_ε tends to zero, the contribution from the semicircle γ_ε to the integral along Γ'_ε approaches $\pi i f(a)$ according to Lemma 2.1 [7].

Lemma 2.1. *If g has a simple pole at $z = a$ and γ_r is the circular arc of Figure 2.3 defined by*

$$\gamma_r : z = a + r e^{i\theta} \quad (\Theta_1 \leq \theta \leq \Theta_2),$$

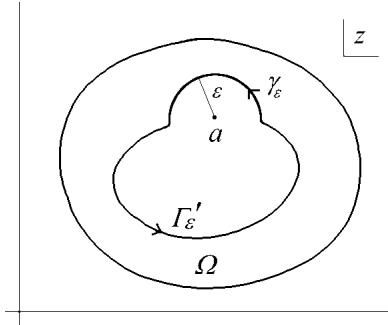


Figure 2.2: Γ'_ε is a piecewise smooth closed contour on an open domain Ω .

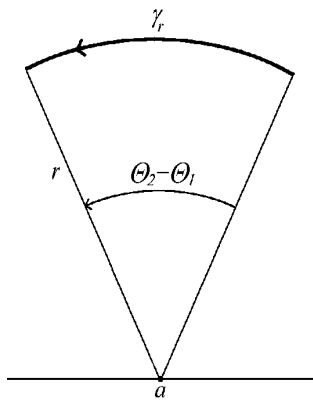


Figure 2.3: γ_r is a circular arc around a

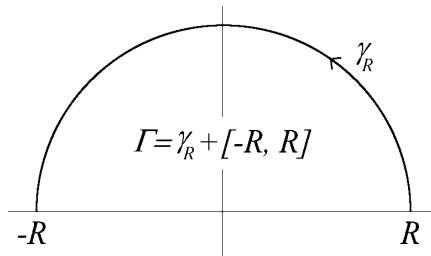


Figure 2.4: Γ is a piecewise smooth closed contour.

then

$$\lim_{r \rightarrow 0^+} \int_{\gamma_r} g(z) dz = i(\Theta_2 - \Theta_1) \operatorname{Res}_{z=a} g(z).$$

From Lemma 2.1 and from the definition of the Cauchy principal value, we see that

$$\lim_{\varepsilon \rightarrow 0} \oint_{\Gamma'_\varepsilon} \frac{f(z)}{z-a} dz = P \int_{\Gamma} \frac{f(z)}{z-a} dz + \lim_{\varepsilon \rightarrow 0} \int_{\gamma_\varepsilon} \frac{f(z)}{z-a} dz = 2\pi i f(a),$$

and that the integral of the Cauchy principal value is

$$P \int_{\Gamma} \frac{f(z)}{z-a} dz \equiv \lim_{\varepsilon \rightarrow 0} \int_{\Gamma_\varepsilon} \frac{f(z)}{z-a} dz = \pi i f(a). \quad (2.2)$$

where Γ_ε is a non closed contour without the indentation γ_ε . By (2.2) we have generalized the definition of the Cauchy principal value compared to Definition 1.2.

A particularly useful identity arises when we have a contour Γ as in Figure 2.4 that is closed by the semicircle γ_R and the real x -axis. If $f(z)$ is a function that is analytic in an open region that contains the upper half-plane and tends to zero at infinity in such a rate that the contribution from the semicircle γ_R vanishes as $R \rightarrow \infty$, then we have

$$P \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi-x} d\xi = \pi i f(x). \quad (2.3)$$

The integral over γ_R will disappear when $R \rightarrow \infty$ if

$$|f(z)| < \frac{C}{|z|},$$

for any positive constant C . The same yields if

$$|f(z)| < C |e^{imz}|,$$

for positive m according to Lemma 2.2 [7].

Lemma 2.2. (Jordan's lemma) *If $m > 0$ and P/Q is the quotient of two polynomials such that*

$$\text{degree } Q \geq 1 + \text{degree } P,$$

then

$$\lim_{\rho \rightarrow \infty} \int_{C_\rho^+} \frac{P(z)}{Q(z)} e^{imz} dz = 0,$$

where C_ρ^+ is the upper half-circle with radius ρ .

If we express $f(x)$ as

$$f(x) = g(x) - ih(x),$$

on both sides of (2.3) with arguments on the real x -axis and equating real and imaginary parts then we obtain for the real part

$$g(x) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{h(\xi)}{x - \xi} d\xi = -Hh(x),$$

and for the imaginary part

$$h(x) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(\xi)}{x - \xi} d\xi = Hg(x). \quad (2.4)$$

From Definition 1.1 we have that $h(x)$ in (2.4) is the Hilbert transform of $g(x)$ where H is the Hilbert transform operator. We also note that $g(x) = H^{-1}h(x)$ with H^{-1} as the inverse Hilbert transform operator. We see that $H \operatorname{Re} f(x) = \operatorname{Im} f(x)$ which motivates the following definition.

Definition 2.3. *A complex signal $f(x)$ that fulfils the conditions preceding (2.3) is called a strong analytic signal.*

Above we have just proved the following theorem.

Theorem 2.4. *For a strong analytic signal $f(x)$ we have that $H \operatorname{Re} f(x) = \operatorname{Im} f(x)$.*

2.2. The Fourier transform

The Fourier transform is important in the theory of signal processing. When a function $f(t)$ is real, we only have to look on the positive frequency axis because it contains the complete information about the waveform in the time domain. Therefore, we do not need the negative frequency axis and the Hilbert transform can be used to remove it. This is explained below.

Let us define the Fourier transform $F(\omega)$ of a signal $f(t)$ by

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt. \quad (2.5)$$

This definition makes sense if $f \in L^1(\mathfrak{R})$, that is if $\int_{-\infty}^{\infty} |f(t)| dt$ exists. It is important to be able to recover the signal from its Fourier transform. To do that we define the inverse Fourier transform as

$$\tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega.$$

If both f and F belong to $L^1(\mathfrak{R})$ then $f(t)$ is continuous and bounded for all real t and we have that $\tilde{f}(t) = f(t)$, that is [1]

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t} d\omega. \quad (2.6)$$

This result is a form of the Fourier inversion theorem.

Another variant of the inversion theorem [2] is that if f belongs to $L^1(\mathfrak{R})$, f is of bounded variation in a neighbourhood of t and f is continuous at t then

$$f(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \int_{-T}^T F(\omega)e^{i\omega t} d\omega.$$

This means that (2.6) is to be interpreted as a type of Cauchy Principal value. Further more general variants of the inversion theorem exist for $f \in L^1(\mathfrak{R})$ but are not mentioned here.

There is also a theory for the Fourier transform when $f \in L^2(\mathfrak{R})$ [2]. In this case we define the Fourier transform as

$$F(\omega) = \lim_{N \rightarrow \infty} \int_{-N}^N f(t)e^{-i\omega t} dt.$$

The mean limit

$$F(\omega) = \lim_{N \rightarrow \infty} F_N(\omega),$$

is to be interpreted as

$$\lim_{N \rightarrow \infty} \|F(\omega) - F_N(\omega)\|_2 = 0.$$

Now $F \in L^2(\mathfrak{R})$ and the inversion formula states that

$$f(t) = \lim_{N \rightarrow \infty} \frac{1}{2\pi} \int_{-N}^N F(\omega) e^{i\omega t} d\omega.$$

Often we need to consider the Fourier transform of a signal as it neither belongs to $L^1(\mathfrak{R})$ nor to $L^2(\mathfrak{R})$. The delta function is such an example. There is a theory of the Fourier transform, see e.g. [1], for there so called temperature distributions. This theory is, however, outside the scope of this report, but occasionally we use some results from this theory.

A useful relation is the Plancherel formula.

Theorem 2.5. *If f, g and G belong to $L^1(\mathfrak{R})$ or if f and g belong to $L^2(\mathfrak{R})$ then*

$$\int_{-\infty}^{\infty} f(t)g^*(t)dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)G^*(\omega)d\omega.$$

Proof. For a proof we refer to [1].

If $f(t)$ is a real function that can be represented by an inverse Fourier transform then we have the following relationship in the time domain

$$\begin{aligned} f(t) &= f^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(\omega) e^{-i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F^*(-\omega) e^{i\omega t} d\omega. \end{aligned}$$

This gives us the relation $F(\omega) = F^*(-\omega)$ or $F(-\omega) = F^*(\omega)$ in the frequency domain and we see that F for negative frequencies can be expressed by F^* for positive ones.

Theorem 2.6. *If $f(t)$ is a real function then*

$$f(t) = \frac{1}{2\pi} \int_0^{\infty} [F^*(\omega) e^{-i\omega t} + F(\omega) e^{i\omega t}] d\omega.$$

Proof. If we apply the Fourier transform on a real function $f(t)$ then

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^0 F(\omega) e^{i\omega t} d\omega + \frac{1}{2\pi} \int_0^{\infty} F(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_0^{\infty} F(-\omega) e^{-i\omega t} d\omega + \frac{1}{2\pi} \int_0^{\infty} F(\omega) e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_0^{\infty} (F^*(\omega) e^{-i\omega t} + F(\omega) e^{i\omega t}) d\omega. \end{aligned}$$

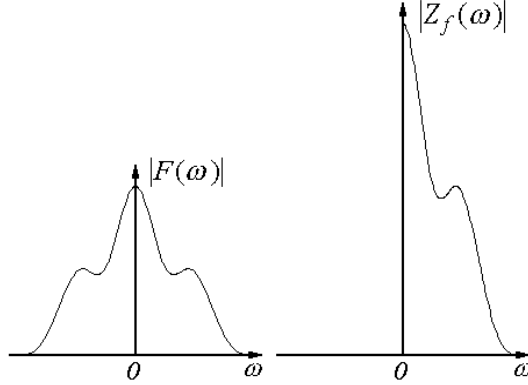


Figure 2.5: The amplitudes of $F(\omega)$ and $Z_f(\omega)$ as functions of ω .

This means that the positive frequency spectra is sufficient to represent a real signal. ■

Let us define a function $Z_f(\omega)$, that is zero for all negative frequencies and $2F(\omega)$ for all positive frequencies

$$Z_f(\omega) = F(\omega) + \text{sgn}(\omega)F(\omega), \quad (2.7)$$

where

$$\text{sgn}(\omega) = \begin{cases} 1 & \text{for } \omega > 0 \\ 0 & \text{for } \omega = 0 \\ -1 & \text{for } \omega < 0 \end{cases}$$

and $F(\omega)$ is the Fourier transform of the real function $f(t)$. In Figure 2.5 we see the relation (2.7) between $F(\omega)$ and $Z_f(\omega)$. The inverse transform of $Z_f(\omega)$ is therefore written as

$$z_f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} Z_f(\omega) e^{i\omega t} d\omega = \frac{1}{\pi} \int_0^{\infty} F(\omega) e^{i\omega t} d\omega,$$

where $z_f(t)$ is a complex function on the form

$$z_f(t) = f(t) + ig(t). \quad (2.8)$$

We will show below that $g(t)$ is real. From (2.7) and (2.8) we have that

$$f(t) + ig(t) \stackrel{F}{\Leftrightarrow} F(\omega) + \text{sgn}(\omega)F(\omega). \quad (2.9)$$

The definition of the Fourier transform tells us that $f(t) \stackrel{F}{\Leftrightarrow} F(\omega)$ and therefore we know from (2.9) that $F(\omega)\text{sgn}(\omega)$ is the Fourier transform of $ig(t)$, thus

$$g(t) \stackrel{F}{\Leftrightarrow} F(\omega) (-i\text{sgn}(\omega)).$$

It is a standard result that the inverse transform of $-i\text{sgn}(\omega)$ equals $1/(\pi t)$, that is

$$g(t) = f(t) * \frac{1}{\pi t} = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau = Hf(t) = \widehat{f}(t),$$

and we see that $g(t)$ can be written as $\widehat{f}(t)$ which is known as the Hilbert transform of $f(t)$. Further more $g(t)$ is real.

2.3. The $\pm\pi/2$ phaseshift

The $\pm\pi/2$ phaseshift is interpreted in the frequency domain as a multiplication with the imaginary value $\pm i$, thus

$$H(\omega) = \begin{cases} -i = e^{-i\frac{\pi}{2}} & \text{for } \omega > 0 \\ i = e^{i\frac{\pi}{2}} & \text{for } \omega < 0 \end{cases}$$

$H(\omega)$ is unfortunately not a property of Fourier transform but the problem can be solved by expressing $H(\omega)$ as a limit of a bounded function $G(\omega)$, that is

$$G(\omega) = \begin{cases} -ie^{-\sigma\omega} & \text{for } \omega > 0 \\ ie^{\sigma\omega} & \text{for } \omega < 0 \end{cases}$$

where

$$\lim_{\sigma \rightarrow 0} G(\omega) = H(\omega). \quad (2.10)$$

It is now possible to use the inverse Fourier transform on $G(\omega)$, thus

$$\begin{aligned} g(t) &= F^{-1}G(\omega) \\ &= \frac{1}{2\pi} \int_{-\infty}^0 ie^{\sigma\omega} e^{i\omega t} d\omega + \frac{1}{2\pi} \int_0^{\infty} -ie^{-\sigma\omega} e^{i\omega t} d\omega \\ &= \frac{i}{2\pi} \int_0^{\infty} (e^{-(\sigma+it)\omega} - e^{-(\sigma-it)\omega}) d\omega \\ &= \frac{i}{2\pi} \left[-\frac{1}{\sigma+it} e^{-(\sigma+it)\omega} + \frac{1}{\sigma-it} e^{-(\sigma-it)\omega} \right]_0^{\infty} \\ &= \frac{t}{\pi(\sigma^2 + t^2)}. \end{aligned}$$

where $g(t) \rightarrow h(t)$ when $\sigma \rightarrow 0$ and the inverse Fourier transform of the impulse response of $H(\omega)$ is

$$h(t) = \lim_{\sigma \rightarrow 0} g(t) = \lim_{\sigma \rightarrow 0} \frac{t}{\pi(\sigma^2 + t^2)} = \frac{1}{\pi t}.$$

A convolution between $f(t)$ and the impulse response $h(t)$ gives us

$$\hat{f}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau,$$

where $\hat{f}(t)$ is known as the Hilbert transform. Notice that this integral shall be considered as a principal-valued integral a limit that corresponds to the limit in (2.10). To make a rigorous presentation of this problem we should apply the distribution theory but that is beyond the scope of this report.

3. Properties of the Hilbert transform

In this chapter we look at some properties of the Hilbert transform. We assume that $F(\omega)$ does not contain any impulses for $\omega = 0$ and that $f(t)$ is a real valued function. Some of the formulas are to be interpreted in a distributional sense.

3.1. Linearity

The Hilbert transform that is a Cauchy principle-valued function, see Chapter 1, is expressed on the form

$$Hf(t) = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f(\tau)}{t - \tau} d\tau.$$

If we write the function $f(t)$ as $c_1 f_1(t) + c_2 f_2(t)$ where the Hilbert transform of $f_1(t)$ and $f_2(t)$ exists then

$$\begin{aligned} Hf(t) &= H(c_1 f_1(t) + c_2 f_2(t)) \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{c_1 f_1(\tau) + c_2 f_2(\tau)}{t - \tau} d\tau \\ &= c_1 \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f_1(\tau)}{t - \tau} d\tau + c_2 \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f_2(\tau)}{t - \tau} d\tau \\ &= c_1 Hf_1(t) + c_2 Hf_2(t). \end{aligned}$$

This is the linearity property of the Hilbert transform.

3.2. Multiple Hilbert transforms and their inverses

The Hilbert transform used twice on a real function gives us the same real function but with altered sign

$$H^2 = -I,$$

with I as the identity operator. The Hilbert transform used four times on the same real function gives us the original function back

$$H^2 H^2 = H^4 = I. \quad (3.1)$$

A more interesting property of multiple Hilbert transforms arises if we use the Hilbert transform 3 times, thus

$$H^3 H = I \Rightarrow H^{-1} = H^3.$$

This tells us that it is possible to use the multiple Hilbert transform to calculate the inverse Hilbert transform.

As we seen before the Hilbert transform can be applied in the time domain by using the definition of the Hilbert transform. In the frequency domain we simply multiply the Hilbert transform operator $-i\text{sgn}(\omega)$ to the function $F(\omega)$. By multiplying the Hilbert transform operator by itself we get an easy method to do multiple Hilbert transforms, that is

$$H^n f(t) \stackrel{F}{\Leftrightarrow} (-i\text{sgn}(\omega))^n F(\omega),$$

where n is the number of Hilbert transforms.

Example 3.1. We want to calculate the inverse Hilbert transform of the function $f(t)$ by using multiple Hilbert transforms in the frequency domain. First we have to Fourier transform the function $f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} dt,$$

and then use the Hilbert transform three times in the frequency domain, that is

$$H^3 = (-i\text{sgn}(\omega))^3.$$

Finally we use the inverse Fourier transform

$$H^{-1} f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H^3 F(\omega)e^{i\omega t} d\omega.$$

From above we see that we only have to calculate two infinite integrals in the frequency domain compared to three infinite integrals in the time domain. Another advantage in the frequency domain is that we formally can choose the number of times we want to use the Hilbert transform.

3.3. Derivatives of the Hilbert transform

Theorem 3.2. *The Hilbert transform of the derivative of a function is equivalent to the derivative of the Hilbert transform of a function, that is*

$$f'(t) \stackrel{H}{\Leftrightarrow} \frac{d}{dt} \widehat{f}(t). \quad (3.2)$$

Proof. From Definition 1.1 we have that

$$\widehat{f}(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(\tau)}{t - \tau} d\tau.$$

If we substitute τ with $t - s$

$$\widehat{f}(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f(t - s)}{s} ds,$$

and then apply the derivative of t on both sides we get

$$\frac{d}{dt} \widehat{f}(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f'(t - s)}{s} ds.$$

The substitution $s = t - \tau$ gives us that

$$\frac{d}{dt} \widehat{f}(t) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{f'(\tau)}{t - \tau} d\tau,$$

and the relation in (3.2) is valid. ■

From the proof above we conclude that the relation can be used repeatedly. Let us look at an example where we also make use of multiple Hilbert transforms, see Section 3.2.

Example 3.3. *By (3.2) we may calculate the Hilbert transform of the delta function $\delta(t)$ and its derivatives. At the same time we get the Hilbert transform representation of the delta function. Consider the Hilbert transform of the delta function*

$$H\delta(t) = \frac{1}{\pi t}.$$

The derivative of the delta function is calculated to

$$H\delta'(t) = -\frac{1}{\pi t^2}, \quad (3.3)$$

and if we apply the Hilbert transform on both sides then we get

$$\delta'(t) = H\left(\frac{1}{\pi t^2}\right).$$

The derivative of (3.3) is

$$H\delta''(t) = \frac{2}{\pi t^3},$$

and when we apply the Hilbert transform on both sides we get

$$\delta''(t) = H\left(-\frac{2}{\pi t^3}\right).$$

This procedure can be continued.

3.4. Orthogonality properties

A symmetry found in Section 2.2 about the Fourier transform $F(\omega)$ of a real function $f(t)$ leads us to do the following definition [6].

Definition 3.4. A complex function is called Hermitian if its real part is even and its imaginary part is odd.

From this we have that the Fourier transform $F(\omega)$ of a real function $f(t)$ is Hermitian.

Theorem 3.5. A real function $f(t)$ and its Hilbert transform $\hat{f}(t)$ are orthogonal if f, \hat{f} and F belong to $L^1(\mathfrak{R})$ or if f and \hat{f} belong to $L^2(\mathfrak{R})$.

Proof. From Theorem 2.5 we have that

$$\begin{aligned} \int_{-\infty}^{\infty} f(t)\hat{f}(t)dt &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) (-i\text{sgn}(\omega)F(\omega))^* d\omega \\ &= \frac{i}{2\pi} \int_{-\infty}^{\infty} \text{sgn}(\omega)F(\omega)F^*(\omega)d\omega \\ &= \frac{i}{2\pi} \int_{-\infty}^{\infty} \text{sgn}(\omega) |F(\omega)|^2 d\omega, \end{aligned}$$

where $\text{sgn}(\omega)$ is an odd function and the fact that $F(\omega)$ is Hermitian gives us that $|F(\omega)|^2$ is an even function. We conclude that

$$\int_{-\infty}^{\infty} f(t)\hat{f}(t)dt = 0,$$

and a real function and its Hilbert transform are orthogonal. ■

3.5. Energy aspects of the Hilbert transform

The energy of a function $f(t)$ is closely related to the energy of its Fourier transform $F(\omega)$. Theorem 2.5 with $f(t) = g(t)$ is called the Rayleigh theorem and it helps us to define the energy of $f(t)$ and $F(\omega)$ as

$$E_f = \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega. \quad (3.4)$$

Here it is natural to assume that $f \in L^2(\mathfrak{R})$ which means that E_f is finite. The same theorem is used to define the energy of the Hilbert transform of $f(t)$ and $F(\omega)$, that is

$$E_{\hat{f}} = \int_{-\infty}^{\infty} |\hat{f}(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |-i\text{sgn}(\omega)F(\omega)|^2 d\omega, \quad (3.5)$$

where $|-i\text{sgn}(\omega)|^2 = 1$ except for $\omega = 0$. But, since $F(\omega)$ does not contain any impulses at the origin we get $E_{\hat{f}} = E_f$.

A consequence of (3.5) is that $f \in L^2(\mathfrak{R})$ induces that $\hat{f} \in L^2(\mathfrak{R})$. The accuracy of the approximated Hilbert transform operator can be measured by comparing the energy in (3.4) and (3.5). However, a minor difference in energy always exists in real applications due to unavoidable truncation errors.

3.6. The Hilbert transform of strong analytic signals

From Section 3.2 we have that the Hilbert transform of a strong analytic signal $z(t)$ is

$$Hz(t) = H(f(t) + i\hat{f}(t)) = \hat{f}(t) - if(t) = -iz(t). \quad (3.6)$$

From this follows the result of the Hilbert transform of two multiplied strong analytic signals.

Theorem 3.6. *The product of $H(z_1(t))z_2(t)$ is identical with the product of $z_1(t)H(z_2(t))$ if $z_1(t)$ and $z_2(t)$ are strong analytic signals.*

Proof. Since $z_1(t)$ and $z_2(t)$ are strong analytic signals then

$$H(z_1(t))z_2(t) = (\hat{f}_1(t) - if_1(t))(f_2(t) + i\hat{f}_2(t)) \quad (3.7)$$

$$= -i(f_1(t) + i\hat{f}_1(t))(f_2(t) + i\hat{f}_2(t))$$

$$= -iz_1(t)z_2(t)$$

$$= (f_1(t) + i\hat{f}_1(t))(\hat{f}_2(t) - if_2(t)) \quad (3.8)$$

$$= z_1(t)H(z_2(t)), \quad (3.9)$$

where we make use of (3.6) in (3.7) and (3.8). ■

Theorem 3.7. *The product of $z_1(t)z_2(t)$ is identical with the product $iH(z_1(t))z_2(t) = iz_1(t)H(z_2(t))$ if $z_1(t)$ and $z_2(t)$ are strong analytic signals.*

Proof. Since $z_1(t)$ and $z_2(t)$ are strong analytic signals then

$$\begin{aligned} z_1(t)z_2(t) &= (f_1(t) + i\hat{f}_1(t))(f_2(t) + i\hat{f}_2(t)) \\ &= i(\hat{f}_1(t) - if_1(t))(f_2(t) + i\hat{f}_2(t)) \\ &= iH(z_1(t))z_2(t) = iz_1(t)H(z_2(t)), \end{aligned}$$

and the theorem follows. ■

The Hilbert transform of the product of two strong analytic signals gives us the same result as in (3.9). To prove this we first need to show that the product of two strong analytic signals is strong analytic.

Theorem 3.8. *The product of two strong analytic signals is strong analytic.*

Proof. Let $z_1(t'')$ and $z_2(t'')$ be analytic signals of the complex variable $t'' = t + it$ in the open upper half-plane. Then $z_1(t'')z_2(t'')$ is also an analytic signal in the same region. Assume that $z_1(t'')$ and $z_2(t'')$ are decreasing in such a rate at infinity that the discussion in Section 2.1 is true then $z_1(t)$ and $z_2(t)$ are strong analytic signals. If $z_1(t'')$ and $z_2(t'')$ are decreasing sufficiently rapid at infinity then $z_1(t'')z_2(t'')$ have to decrease faster than the one of $z_1(t'')$ and $z_2(t'')$ that is decreasing with the least rate. From this we have that $H(\operatorname{Re}(z_1(t)z_2(t))) = \operatorname{Im}(z_1(t)z_2(t))$ and that $z_1(t)z_2(t)$ is a strong analytic signal. ■

Theorem 3.9. *$H(z_1(t)z_2(t)) = -iz_1(t)z_2(t)$ if $z_1(t)$ and $z_2(t)$ are strong analytic signals.*

Proof. Since $z_1(t)$ and $z_2(t)$ are strong analytic signals then

$$\begin{aligned} H(z_1(t)z_2(t)) &= H(f_1(t) + i\hat{f}_1(t))(f_2(t) + i\hat{f}_2(t)) \\ &= H(f_1(t)f_2(t) - \hat{f}_1(t)\hat{f}_2(t) \\ &\quad + i(f_1(t)\hat{f}_2(t) + \hat{f}_1(t)f_2(t))) \\ &= f_1(t)\hat{f}_2(t) + \hat{f}_1(t)f_2(t) \\ &\quad - i(f_1(t)f_2(t) - \hat{f}_1(t)\hat{f}_2(t)) \tag{3.10} \\ &= -i(f_1(t)f_2(t) + i\hat{f}_1(t)\hat{f}_2(t) \\ &\quad + i\hat{f}_1(t)f_2(t) - i\hat{f}_1(t)\hat{f}_2(t)) \\ &= -i(f_1(t) + i\hat{f}_1(t))(f_2(t) + i\hat{f}_2(t)) \\ &= -iz_1(t)z_2(t), \end{aligned}$$

where we make use of Theorem 3.8 in (3.10). ■

Consequently it is possible to apply the Hilbert transform on the product of two strong analytic signals in several different ways, thus

$$H(z_1(t)z_2(t)) = H(z_1(t))z_2(t) = z_1(t)H(z_2(t)) = -iz_1(t)z_2(t).$$

It does not matter on which strong analytic signal we apply the Hilbert transform. We conclude that the Hilbert transform of the product of n strong analytic signals form the equation

$$Hz^n(t) = H(z(t))z^{n-1}(t) = -iz(t)z^{n-1}(t) = -iz^n(t).$$

3.7. Analytic signals in the time domain

The Hilbert transform can be used to create an analytic signal from a real signal. Instead of studying the signal in the frequency domain it is possible to look at a rotating vector with an instantaneous phase $\varphi(t)$ and an instantaneous amplitude $A(t)$ in the time domain, that is

$$z(t) = f(t) + i\hat{f}(t) = A(t)e^{i\varphi(t)}.$$

This notation is usually called the polar notation where

$$A(t) = \sqrt{f^2(t) + \hat{f}^2(t)},$$

and

$$\varphi(t) = \arctan\left(\frac{\hat{f}(t)}{f(t)}\right).$$

If we express the phase with a Taylor series then

$$\varphi(t) = \varphi(t_0) + (t - t_0)\varphi'(t_0) + R,$$

where R is small when t is close to t_0 . The analytic signal becomes

$$z(t) = A(t)e^{i\varphi(t)} = A(t)e^{i(\varphi(t_0) - t_0\varphi'(t_0))} e^{it\varphi'(t_0)} e^{iR},$$

and we see that $\varphi'(t_0)$ has the role of frequency if R is neglected. This makes it natural to introduce the notion of instantaneous (angular) frequency, that is

$$\omega(t) = \frac{d\varphi(t)}{dt}.$$

Example 3.10. We have a real signal and its Hilbert transform

$$\begin{aligned}f(t) &= \cos(\omega_0 t), \\ \hat{f}(t) &= \sin(\omega_0 t).\end{aligned}$$

Together they form an analytic signal where the instantaneous amplitude is

$$A(t) = \sqrt{\cos^2(\omega_0 t) + \sin^2(\omega_0 t)} = 1.$$

The instantaneous frequency is easy to calculate from the phase $\varphi(t) = \omega_0 t$, that is

$$\omega(t) = \omega_0.$$

We see that in this particular case the instantaneous frequency is the same as the real frequency.

4. Numerical calculations of the Hilbert transform

The purpose of this chapter is to study different types of numerical calculation methods for the Hilbert transform.

4.1. Continuous time

4.1.1. Numerical integration.

Numerical integration works fine on smooth functions that decrease rapidly at infinity. When we want to calculate the Hilbert transform by Definition 1.1 we are facing some problems. In numerical integration we use finite intervals and it is therefore important to consider the integration region to control the calculation error. This is the reason why a rapid decrease at infinity is an advantage. Another problem is that the integrand in Definition 1.1 is infinite when the nominator vanishes. However, by using more integration grid points in the numerical integration close to this value we get a better approximation.

4.1.2. Hermite polynomials

The numerical integration is inefficient when a function decreases in a slow rate at infinity. It is sometimes better to use a series of orthogonal polynomials where the function does not have to decrease rapid at infinity. In this section we use the Hermite polynomials to calculate the Hilbert transform. First we need to take a look at the definition of the Hermite polynomials.

The successive differentiation of the Gaussian pulse e^{-t^2} generates the n th order Hermite polynomial which is defined by Rodrigues' formula as [4]

$$H_n(t) = (-1)^n e^{t^2} \frac{d^n}{dt^n} e^{-t^2}.$$

It is also possible to calculate the Hermite polynomials by the recursion formula [3]

$$H_n(t) = 2tH_{n-1}(t) - 2(n-1)H_{n-2}(t), \quad (4.1)$$

with $n = 1, 2, 3, \dots$ and the start condition

$$H_0(t) = 1.$$

Let us also define the weighted Hermite polynomials that is weighted by the generating function e^{-t^2} on the form [3]

$$g_n(t) = H_n(t)e^{-t^2} = (-1)^n \frac{d^n}{dt^n} e^{-t^2}.$$

The weighted Hermite polynomials $g_n(t)$ do not represent an orthogonal set in L_2 since the scalar product

$$\int_{-\infty}^{\infty} g_n(t)g_m(t)dt = \int_{-\infty}^{\infty} H_n(t)H_m(t)e^{-2t^2} dt,$$

is in general different from zero when $n \neq m$. The solution is to replace the weighting function e^{-t^2} with $e^{-\frac{t^2}{2}}$, that is

$$\int_{-\infty}^{\infty} H_n(t)H_m(t)e^{-t^2} dt = \begin{cases} 0 & \text{for } n \neq m \\ 2^n n! \sqrt{\pi} & \text{for } n = m \end{cases}$$

By that the weighted Hermite polynomials fulfil the condition of orthogonality. The weighted polynomial $e^{-\frac{t^2}{2}} H_n(t)$ divided by their norm $\sqrt{2^n n! \sqrt{\pi}}$ yields a set of orthonormal functions in L_2 and is called the Hermite function

$$\varphi_n(t) = \frac{e^{-\frac{t^2}{2}} H_n(t)}{\sqrt{2^n n! \sqrt{\pi}}}. \quad (4.2)$$

If we combine (4.1) and (4.2) we get the recursive algorithm

$$\varphi_n(t) = \sqrt{\frac{2(n-1)!}{n!}} t \varphi_{n-1}(t) - (n-1) \sqrt{\frac{(n-2)!}{n!}} \varphi_{n-2}(t), \quad (4.3)$$

which can be used to derive the Hilbert transform of the Hermite functions by applying the "multiplication by t " theorem [3].

Theorem 4.1. If we assume that $f(t)$ and $\widehat{f}(t)$ belong to L_1 then the Hilbert transform of $tf(t)$ is given by the equation

$$H(tf(t)) = t\widehat{f}(t) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(\tau)d\tau.$$

The integral is a constant defined by the function $f(t)$. For odd functions this constant equals zero.

Proof. Consider the Hilbert transform of $tf(t)$

$$H(tf(t)) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\tau f(\tau)}{t - \tau} d\tau.$$

The insertion of a new variable $s = t - \tau$ yields

$$\begin{aligned} H(tf(t)) &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{(t-s)f(t-s)}{s} ds \\ &= \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{tf(t-s)}{s} ds - \frac{1}{\pi} \int_{-\infty}^{\infty} f(t-s) ds \\ &= tH(f(t)) - \frac{1}{\pi} \int_{-\infty}^{\infty} f(\tau) d\tau, \end{aligned}$$

and the theorem follows. ■

From Theorem 4.1 and from (4.3) we have that

$$\begin{aligned} H(\varphi_n(t)) &= \widehat{\varphi}_n(t) = \sqrt{\frac{2(n-1)!}{n!}} \left(t\widehat{\varphi}_{n-1}(t) - \frac{1}{\pi} \int_{-\infty}^{\infty} \varphi_{n-1}(\eta) d\eta \right) \\ &\quad - (n-1) \sqrt{\frac{(n-2)!}{n!}} \widehat{\varphi}_{n-2}(t), \end{aligned} \quad (4.4)$$

where $n = 1, 2, 3, \dots$. The first term $\varphi_0(t)$ can be calculated by using the Fourier transform on equation (4.2), that is

$$\varphi_0(t) = \pi^{-\frac{1}{4}} e^{-\frac{t^2}{2}} \stackrel{F}{\Leftrightarrow} \sqrt{2} \pi^{\frac{1}{4}} e^{-\frac{\omega^2}{2}}.$$

In the frequency domain we multiply the Hermite function $\varphi_0(t)$ by the Hilbert transformer $-i \operatorname{sgn}(\omega)$ and finally we use the inverse Fourier transform to get the Hilbert transform of the Hermite function, that is

$$H(\varphi_0(t)) = \widehat{\varphi}_0(t) = \sqrt{2} \pi^{\frac{1}{4}} \int_{-\infty}^{\infty} e^{-\frac{\omega^2}{2}} (-i \operatorname{sgn}(\omega)) e^{i\omega t} d\omega.$$

Since $\text{sgn}(\omega)e^{-\frac{\omega^2}{2}}$ is odd we have

$$\widehat{\varphi}_0(t) = 2\sqrt{2}\pi^{\frac{1}{4}} \int_0^\infty e^{-\frac{\omega^2}{2}} \sin(\omega t) d\omega, \quad (4.5)$$

which can be used in (4.4) to derive the rest of the Hilbert transforms.

It has been found that the error is large for higher order of the Hilbert transformed Hermite functions (e.g. $\widehat{\varphi}_5(t)$) when we use the recursive formula (4.4). This effect is caused by the finite accuracy in the calculation of the integral in (4.5) where the error increases in (4.4). It is therefore not suitable to use the recursive method to calculate the Hilbert transform for higher order Hermite functions.

Another method to calculate the Hilbert transform by the Hermite functions is to multiply the Hilbert transformer by the spectrum of the Hermite function and use the inverse Fourier transform. No infinite integrals is needed in the calculations of the Hermite functions. Therefore the error does not propagate as in (4.4).

The series expansion of $f(t)$ can be written as

$$f(t) = \sum_{n=0}^{\infty} a_n \varphi_n(t),$$

where

$$a_n = \int_{-\infty}^{\infty} f(t) \varphi_n(t) dt.$$

If the series expansion $f(t)$ is limited at infinity there will be an error $\varepsilon_N(t)$, that is

$$f(t) = \sum_{n=0}^{N-1} a_n \varphi_n(t) + \varepsilon_N(t).$$

This series expansion can also be used to calculate the Hilbert transform

$$Hf(t) = \sum_{n=0}^{N-1} a_n \widehat{\varphi}_n(t) + \widehat{\varepsilon}_N(t), \quad (4.6)$$

and we see that this kind of method is useful for functions where $\widehat{\varepsilon}_N(t)$ is small.

To calculate the Hilbert transform of $\varphi_n(t)$ by using the inverse Fourier transform on the product of the Hermite function spectra and the Hilbert transformer is a rather demanding method. However, the Hermite functions never change and we therefore only have to calculate their Hilbert transforms once while a_n , which depends on f , represents an easy integral to calculate.

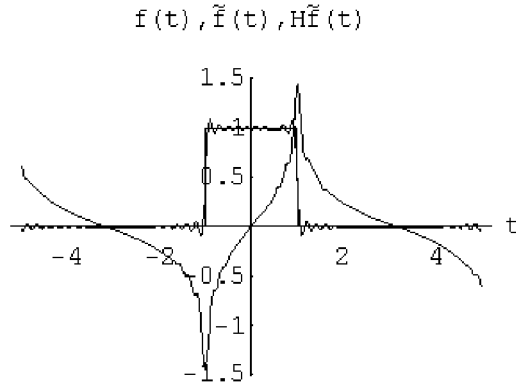


Figure 4.1: Approximation by a Fourier series $\tilde{f}(t)$ with its Hilbert transform $H\tilde{f}(t)$, $n = 30$.

4.1.3. Fourier series

If we create a Fourier series of a function $f(t)$ and changes the sine functions to cos functions and cos functions to sine functions we get the Hilbert transform of $f(t)$. This method gives us a the ability, in an easy way, to study the unavoidable truncation errors (a finite number of terms in the Fourier series) for the Hilbert transform of a periodic rectangular wave. In Figure 4.1 we see that the peaks of the Hilbert transform only are 1.5 in their maxima and minima while they should have been infinite. The implementation in computer code of the Fourier series in Figure 4.1 is found in Appendix A.2.

4.2. Discrete Fourier transform

To derive the discrete Hilbert transform we need the definition of the discrete Fourier transform (*DFT*), that is

$$F[k] = \sum_{n=0}^{N-1} f[n]e^{-i\frac{2\pi}{N}kn}, \quad k = 0, 1, \dots, N-1, \quad (4.7)$$

and the inversion formula

$$f[n] = \frac{1}{N} \sum_{k=0}^{N-1} F[k]e^{i\frac{2\pi}{N}kn}, \quad n = 0, 1, \dots, N-1. \quad (4.8)$$

where k is the discrete frequency and n is the discrete time. It is easy to prove (4.8) by inserting (4.8) into (4.7). Note that (4.8) defines a periodic function with

period N . Let us expand (4.7) in its real and imaginary parts on both sides, thus

$$F[k] = F_{\text{Re}}[k] + iF_{\text{Im}}[k],$$

and

$$\sum_{n=0}^{N-1} f[n]e^{-i\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} f[n] \cos\left(\frac{2\pi}{N}kn\right) - i \sum_{n=0}^{N-1} f[n] \sin\left(\frac{2\pi}{N}kn\right).$$

The real and imaginary part is now identified as

$$\begin{aligned} F_{\text{Re}}[k] &= \sum_{n=0}^{N-1} f[n] \cos\left(\frac{2\pi}{N}kn\right), \\ F_{\text{Im}}[k] &= - \sum_{n=0}^{N-1} f[n] \sin\left(\frac{2\pi}{N}kn\right), \end{aligned}$$

and we conclude that $F_{\text{Im}} = 0$ when $k = 0$ and $k = N/2$. As we have seen before the Hilbert transform of the delta pulse $\delta(t)$ gives us the Hilbert transformer $1/(\pi t)$ and the Fourier transform of the Hilbert transformer gives us the sign shift function $-i\text{sgn}(\omega)$, that is

$$\delta(t) \stackrel{H}{\Leftrightarrow} \frac{1}{\pi t} \stackrel{F}{\Leftrightarrow} -i\text{sgn}(\omega). \quad (4.9)$$

The discrete analogue of the Hilbert transformer in (4.9) for even N is therefore given by

$$H[k] = \begin{cases} -i & \text{for } k = 1, 2, \dots, N/2 - 1 \\ 0 & \text{for } k = 0 \text{ and } N/2 \\ i & \text{for } k = N/2 + 1, \dots, N - 2, N - 1, \end{cases}$$

and $H[k]$ can be written on the form

$$H[k] = -i\text{sgn}\left(\frac{N}{2} - k\right)\text{sgn}(k). \quad (4.10)$$

Here we have used the convention that $\text{sgn}(0) = 0$. The discrete frequency k is called positive in the interval $0 < k < N/2$ and negative in the interval $N/2 < k < N$ and alternate sign at $N/2$.

The discrete inverse Fourier transform of the discrete Hilbert transformer in (4.10) gives us the discrete impulse response in the time domain, for even N , thus

$$\begin{aligned} h[n] &= \frac{1}{N} \sum_{k=0}^{N-1} H[k]e^{i\frac{2\pi}{N}kn} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} -i\text{sgn}\left(\frac{N}{2} - k\right)\text{sgn}(k)e^{i\frac{2\pi}{N}kn} \\ &= \frac{2}{N} \sum_{k=1}^{\frac{N}{2}-1} \sin\left(\frac{2\pi}{N}kn\right), \end{aligned} \quad (4.11)$$

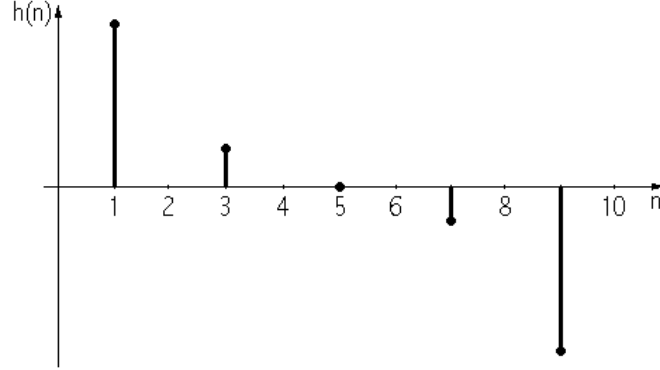


Figure 4.2: The discrete impulse response of the Hilbert transform for even N ($N = 10$) given by the cotangent function with every second sample ($n = 0, 2, 4, \dots$) erased by $\sin^2(\pi n/2)$.

and $h[n]$ can be expressed in closed form as

$$h[n] = \frac{2}{N} \sin^2\left(\frac{\pi n}{2}\right) \cot\left(\frac{\pi n}{N}\right). \quad (4.12)$$

The function is given by the cotangent function with every second sample ($n = 0, 2, 4, \dots$) erased by $\sin^2(\pi n/2)$, see Figure 4.2.

The same derivation for odd N is given by

$$H[k] = \begin{cases} -i & \text{for } k = 1, 2, \dots, (N-1)/2 \\ 0 & \text{for } k = 0 \\ i & \text{for } k = (N+1)/2, \dots, N-2, N-1. \end{cases}$$

It is written on the same closed form as in the even case with the difference that there is no cancellation for $\text{sgn}(N/2 - k)$ that is

$$H[k] = -i \text{sgn}\left(\frac{N}{2} - k\right) \text{sgn}(k). \quad (4.13)$$

The discrete impulse response for odd N of the Hilbert transformer in (4.9) is given by the discrete inverse Fourier transform of $H[k]$ in (4.13), thus

$$h[n] = \frac{2}{N} \sum_{k=1}^{\frac{N-1}{2}} \sin\left(\frac{2\pi}{N} kn\right),$$

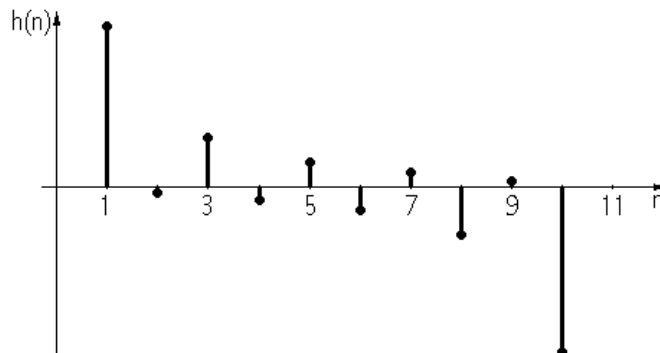


Figure 4.3: The discrete impulse response of the Hilbert transform for odd N ($N = 11$) with every second sample positive with next sample negative.

where the closed form of $h[n]$ can be expressed as

$$h[n] = \frac{1}{N} \left(\cot\left(\frac{\pi n}{N}\right) - \frac{\cos(\pi n)}{\sin\left(\frac{\pi n}{N}\right)} \right). \quad (4.14)$$

As we mentioned before we do not have the same cancellation for odd N (4.14) as for even N (4.12), instead $h[n]$ is changing sign by odd and even values of n , see Figure (4.3).

The discrete Hilbert transform of the sequence $\hat{f}[n]$ is defined by convolution on the form

$$\hat{f}[n] = \sum_{m=0}^{N-1} h[n-m] f[m]. \quad (4.15)$$

If we instead choose to use the DFT algorithm we then have the following relations

$$f[n] \xrightarrow{DFT} F[k] \xrightarrow{DHT} \hat{F}[k] = -i \operatorname{sgn}\left(\frac{N}{2} - k\right) \operatorname{sgn}(k) F[k] \xrightarrow{DFT^{-1}} \hat{f}[n], \quad (4.16)$$

where DFT denotes the discrete Fourier transform, DHT denotes the discrete Hilbert transform and DFT^{-1} denotes the discrete inverse Fourier transform.

The discrete convolution algorithm (4.15) is faster than the DFT algorithm (4.16) because it involves only a single summation. However the DFT algorithm may be replaced by a fast Fourier transform algorithm (FFT).

The accuracy of the discrete Fourier transform when applied to continuous signals is depending on how many samples we have in our calculation (A sample is the same as one single value of a function for one specific time). It is important that the sample frequency is higher than double the signal frequency, that is the Nyquist frequency.

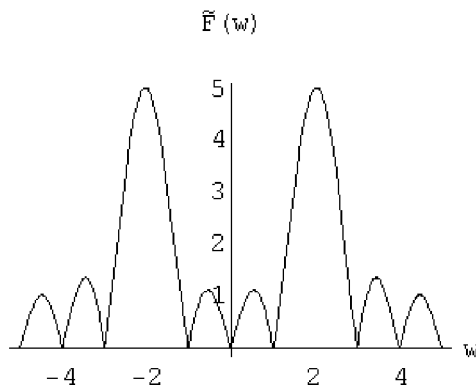


Figure 4.4: The Fourier transform $\tilde{F}(w)$ by $N = 10$ terms of the function $f(t) = \cos(2t)$.

Example 4.2. Assume that we want to calculate the discrete Hilbert transform of a sine-wave $f(t) = \cos(2t)$ with $N = 10$ samples using the DFT algorithm according to (4.16) where the sampling frequency is $5/\pi$ and the signal frequency is $1/\pi$. First we need to calculate the discrete Fourier transform (4.7) of $f(t)$. Then we need to use (4.11) (N even) to apply the inverse discrete Fourier transform on the product of the discrete Hilbert transform operator $H[k]$ in (4.13) and the discrete Fourier transform of $f(t)$. The definition of the discrete Fourier transform gives us (see Figure 4.4)

$$\tilde{F}[k] = \sum_{n=0}^{N-1} f[n] e^{-i\frac{2\pi}{N}kn} = \sum_{n=0}^9 \cos\left(\frac{2\pi}{5}n\right) e^{-i\frac{\pi}{5}kn}.$$

By using (4.11) we get the Hilbert transform $Hf[n]$ in the time domain (N even)

$$\begin{aligned} Hf[n] &= \frac{2}{N} \sum_{k=1}^{\frac{N}{2}-1} \tilde{F}[k] \sin\left(\frac{2\pi}{N}kn\right) \\ &= \frac{1}{5} \sum_{k=1}^4 \left(\sum_{n=0}^9 \cos\left(\frac{2\pi}{5}n\right) e^{-i\frac{\pi}{5}kn} \right) \sin\left(\frac{\pi}{5}kn\right), \end{aligned}$$

The fact that this is an harmonic periodic signal (of sin and cos) and that the sampling rate is more than double the signal frequency gives us the exact answer, see Figure 4.5. The implementation in computer code is found in Appendix A.3.

Example 4.3. We want to calculate the Hilbert transform of $f(t) = 1/(t^2 + 1)$ with $N = 2$ terms of the Hermitian polynomial in (4.6) and use $n = 20$ terms of

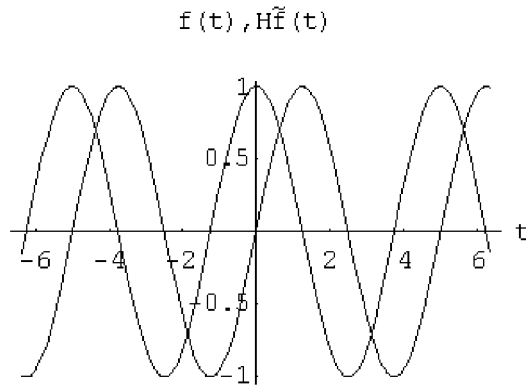


Figure 4.5: $f(t)$ and its Hilbert transform $H\tilde{f}(t)$.

the discrete Fourier transform in (4.7) and (4.11).

First we need to calculate $\varphi_0(t), \dots, \varphi_2(t)$ by (4.2) or (4.3) and then $\hat{\varphi}_0(t), \dots, \hat{\varphi}_2(t)$ as in Example 4.2. a_n is possible to calculate with a numerical integration method because the integral converges sufficiently rapid at infinity. In Figure 4.6 we can see the function $f(t)$ and its approximated Hilbert transform $H_y(t)$ compared to the known Hilbert transform $Hf(t) = t/(t^2+1)$. The implementation in computer code is found in Appendix A.1.

5. An application

In signal systems we often make use of modulated signals. One big problem when we modulate a signal is that its spectrum mirrors itself on both sides of the carrier frequency. Only one of these spectra, or sidebands, is needed to demodulate the signal and therefore we have redundant information. There are different techniques to remove one of the sidebands and thereby create a single sideband signal (SSB). The most simple one is to use a low-pass filter with the middle of the transitionband on the carrier frequency. This technique demands that we have a narrow transitionband and a good suppression which is hard to obtain. Another technique is to use the Hilbert transform where we remove one of the sidebands by adding the signal to itself. In theory we will get a transitionband that is zero and a suppression that is infinite.

Let us use the Hilbert transform to create an amplitude modulated SSB signal (AM-SSB) with a Phase/Hilbert modulator, see Figure 5.1.

The incoming signal

$$f_m(t) = \sin(\omega_m t), \quad (5.1)$$

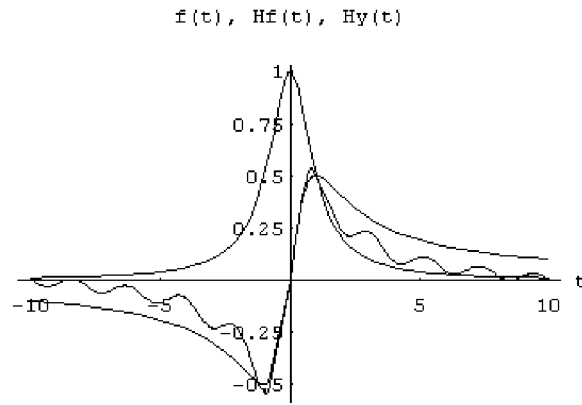


Figure 4.6: The function $f(t)$ and its Hilbert transform $Hf(t)$ and $H_y(t)$ as the approximated Hilbert transform.

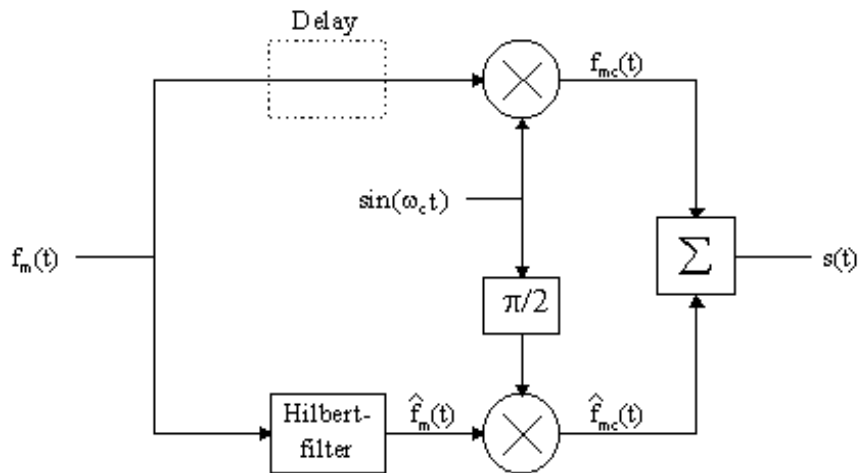


Figure 5.1: Phase/Hilbert modulator.

is divided in two parts where one is preserved and the other one is passing through a Hilbert filter, that is

$$\hat{f}_m(t) = \cos(\omega_m t).$$

We modulate the preserved signal $f_m(t)$ with a sine-signal $\sin(\omega_c t)$ with the carrier frequency ω_c , thus

$$f_{mc}(t) = \sin(\omega_m t) \sin(\omega_c t).$$

To modulate $\hat{f}_m(t)$ we need a $\pi/2$ phase lag on the carrier frequency which gives us a $\cos(\omega_c t)$ function, see Figure 5.1, that is

$$\hat{f}_{mc}(t) = \cos(\omega_m t) \cos(\omega_c t).$$

With some help from the trigonometry we get that

$$f_{mc}(t) = \frac{1}{2} (\cos((\omega_m - \omega_c)t) - \cos((\omega_m + \omega_c)t)),$$

and

$$\hat{f}_{mc}(t) = \frac{1}{2} (\cos((\omega_m - \omega_c)t) + \cos((\omega_m + \omega_c)t)).$$

Finally we add f_{mc} and \hat{f}_{mc} which gives us

$$s(t) = \cos((\omega_m - \omega_c)t),$$

and we have a lower AM-SSB signal. If we instead subtract f_{mc} and \hat{f}_{mc} we get an upper AM-SSB signal.

The delay-box in Figure 5.1 has the same delay as the Hilbert filter. The delay is needed in real applications to synchronize the two signals $f_m(t)$ and $\hat{f}_m(t)$.

A. Appendix

A.1. Implementation of the Hermite polynomial

Implemented in Mathematica.

```
t:=t;
x:=x;
k:=k;
Hy=0;
f=1/(t^2+1);
Hf=t/(t^2+1);
a=-Infinity;
b=Infinity;
m=2;
M=20;
Do[
  Fi=E^(-t^2/2)*HermiteH[n,t]/Sqrt[2^n*n!*Sqrt[Pi]];
  FFi=Sum[Fi*E^(-I*2*Pi*k*t/M),{t,0,M-1}];
  HFi=Re[2/M*Sum[FFi*Sin[(2*Pi*k*t)/M],{k,1,M/2-1}]];
  Hy+=Integrate[f*Fi,{t,a,b}]*HFi,
{n,0,m}
Plot[{f,Hf,Hy},{t,-10,10},AxesLabel->{"t","f(t), Hf(t), Hy(t)"}];
```

A.2. Implementation of the Fourier series

Implemented in Mathematica.

```
t:=t;
k:=k;
y=Hy=0;
f=1;
a=-1;
b=1;
m=30;
Do[
  If[k<1,
    an=1/(2*Pi)*Integrate[f,{t,a,b}],
    an=1/Pi*Integrate[f*Cos[k*t],{t,a,b}];
    bn=1/Pi*Integrate[f*Sin[k*t],{t,a,b}];
    y+=an*Cos[k*t]+bn*Sin[k*t],
  {k,0,m}]
Do[
  Han=1/Pi*Integrate[f*Sin[k*t],{t,a,b}];
  Hbn=1/Pi*Integrate[f*Cos[k*t],{t,a,b}];
  Hy+=Han*Cos[k*t]+Hbn*Sin[k*t],
{k,1,m}]
Plot[{y,Hy,f*(Sign[b-t]+Sign[t-a])/2},{t,-5,5},AxesLabel -> {"t","f(t)"}];
Print["f(t)=",f," from ",a," to ",b," calculated by ",m," terms."]
```

A.3. Implementation of the Fourier transform

Implemented in Mathematica.

```
n:=n;
k:=k;
M=10;
Freq=2;
f=Cos[2*Pi/M*n*Freq];
F=Sum[f*E^(-I*2*Pi*k*n/M),{n,0,M-1}];
Hf=2/M*Sum[F*Sin[(2*Pi*k*n)/M],{k,1,M/2-1}];
Plot[Abs[F],{k,-M/2,M/2},AxesLabel->{"w","F(w)"}];
Plot[{f,Hf},{n,-2*M/Pi,2*M/Pi},AxesLabel->{"t","f(t),Hf(t)"}];
```

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