

A new integral and series representation of Riemann's symmetrical functional equation to prove the Riemann Hypothesis

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Abstract: Riemann's symmetrical functional equation is given by

$$\xi^*(s) := \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{2} \int_1^\infty \psi(x) \left[x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \right] \frac{dx}{x} - \frac{1}{2s(1-s)}.$$

A new integral and series of $\xi^*(s)$ is given in the form

$$\xi^*(s) := \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi_\phi(s) + \xi_\pi(s).$$

It is enabled by the paper of M. S. Milgram, (MiM), providing a new integral and series representation of the zeta function. The convergent integral representation

$$\xi_\phi(s) := 2 \int_1^\infty x^{-\frac{1}{2}} \left[\sum_{n=1}^\infty e^{-\pi n^2 x^2} - \sum_{n=1}^\infty e^{-2\pi n x} \right] \cosh \left[\left(s - \frac{1}{2} \right) \log x \right] dx$$

is like a polynomial of infinite degree, which solves the RH. The corresponding entire Zeta function in the form

$$\xi^{**}(s) := \pi^{-\frac{s}{2}} \sin\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s\right) \Gamma\left(\frac{s}{2}\right) \zeta(s)$$

is accompanied by a corresponding product representation in the form

$$\xi^{**}(s) = \zeta(s) \cos\left(\frac{\pi}{2}s\right) \pi^{1-\frac{s}{2}} e^{-\frac{\pi s^2}{2}} \prod_{n=1}^\infty \left(1 - \frac{s}{2n}\right) \left(1 + \frac{s}{2n}\right) e^{\frac{s}{2n}}$$

enabling Riemann's method for deriving the formula for his prime number density function $J(x)$, (EdH) 1.11.

A new integral and series representation of Riemann's symmetrical functional equation

Riemann's symmetrical functional equation with poles (of first order) at $s = 0$ and $s = 1$ is given by, (EdH) 1.7,

$$(*) \quad \xi^*(s) := \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \frac{1}{2} \int_1^\infty \psi(x) \left[x^{\frac{s}{2}} + x^{\frac{1-s}{2}} \right] \frac{dx}{x} - \frac{1}{2s(1-s)} = \xi^*(1-s).$$

The integral converges for all s , because $\psi(x) = \sum_{n=1}^\infty e^{-\pi n^2 x}$ decreases more rapidly than any power of x as $x \rightarrow \infty$, (EdH) 1.7. In the critical stripe the function $\xi^*(s)$ has the same zeros as the zeta function $\zeta(s)$ (which has only a simple pole at $s = 1$ with residue 1), and, as a consequence of the functional equation, it is real on the critical line.

Multiplying (*) by $s(s-1)/2$ results into the Riemann's entire Zeta function

$$\xi(s) := \Gamma\left(1 + \frac{s}{2}\right) (s-1) \pi^{-\frac{s}{2}} \zeta(s) = s(s-1) \xi^*(s) = \xi(1-s),$$

from which Riemann derived his famous series representation (appendix)

$$\mathcal{E}(t) := \xi\left(\frac{1}{2} + it\right) = 4 \int_1^\infty x^{-1/4} \frac{d}{dx} \left[x^{3/2} \psi'(x) \right] \cos\left(\frac{t}{2} \log x\right) dx.$$

Riemann's „multiplying by $s(s-1)/2$ “ approach is in line with the „Mellin transform rules“ $M[xh'](s) = -sM[h](s)$, $M[(xg)'](s) = (1-s)M[g](s)$ resp. $M[(x^2h)'](s) = (-s)(1-s)M[h](s)$, provided that the considered integrals are convergent, (EdH) 10.3, 10.5. Correspondingly, the integral representation of $\mathcal{E}(t)$ is the result of a two times partial integration process accompanied by a series representation of $\mathcal{E}(t)$; without giving a proof Riemann claimed that this series is convergent, (EdH) 1.8.

We shall apply the following abbreviations

- i) $\zeta^*(s) := \zeta(s) \sin\left(\frac{\pi}{2}(1-s)\right)$
- ii) $\zeta_n^*(s) := \frac{\zeta(2n)}{2n-s}$
- iii) $\xi_{\pi_1}(s) := \frac{\zeta^*(s)+\zeta^*(1-s)}{\sin(\pi s)} = \frac{1}{2} \left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\cos(\frac{\pi}{2}s)} \right], \quad s \neq v, v \in Z$
- iv) $\xi_{\pi_2}(s) := \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n [\zeta_n^*(s) + \zeta_n^*(1-s)], \quad s \neq 2n, 1-2n, n \in N$
- v) $\xi_{\pi}(s) := \xi_{\pi_1}(s) + \xi_{\pi_2}(s)$
- vi) $\varphi(x) := \frac{1}{2} \frac{e^{-\pi x}}{\sinh(\pi x)} = \frac{1}{e^{2\pi x} - 1} = \sum_{n=1}^{\infty} e^{-2\pi n x}, \quad (*)$
- vii) $\Phi(x) := \psi(x^2) - \varphi(x) = \sum_{n=1}^{\infty} (e^{-\pi(n x)^2} - e^{-2\pi n x}), \quad x \geq 1$
- viii) $\xi_{\Phi}(s) := \int_1^{\infty} \Phi(x) [x^s + x^{1-s}] \frac{dx}{x} = 2 \int_1^{\infty} \sqrt{x} \Phi(x) \cosh\left[\left(s - \frac{1}{2}\right) \log x\right] \frac{dx}{x}, \quad (**)$
- ix) $\xi_{\Phi}\left(\frac{1}{2} + iz\right) = 2 \int_0^{\infty} \Psi(t) \cos(zt) dt \quad \text{with} \quad \Psi(t) := e^{t/2} \sum_{n=1}^{\infty} (e^{-\pi(n e^t)^2} - e^{-2\pi n e^t}).$

Lemma:

$$-\frac{1}{2s(1-s)} = -\int_1^{\infty} [x^s + x^{1-s}] \varphi(x) + \xi_{\pi}(s), \quad s \neq v, v \in Z.$$

The proof of the lemma is given in the section below. It is based on a new integral representation of Riemann's zeta function as provided in (MiM).

From the lemma it follows

$$(**) \quad \xi^*(s) = \frac{1}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi_{\Phi}(s) + \xi_{\pi}(s), \quad s \neq v, v \in Z.$$

Multiplication by $\sin(\pi s) = 2 \sin\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s\right)$ governs the singularities at $s = v$ resulting into an entire function in the form

$$\xi^{**}(s) := \sin(\pi s) \xi^*(s) = \pi^{-\frac{s}{2}} \sin\left(\frac{\pi}{2}s\right) \cos\left(\frac{\pi}{2}s\right) \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Applying the product representations of $\sin(\pi s)$ and $\Gamma\left(1 + \frac{s}{2}\right)$ (***) results into, (****)

$$\xi^{**}(s) = \zeta(s) \cos\left(\frac{\pi}{2}s\right) \pi^{1-\frac{s}{2}} e^{-\gamma \frac{s}{2}} \prod_{n=1}^{\infty} \left(1 - \frac{s}{2n}\right) \left(1 + \frac{s}{2n}\right) e^{\frac{s}{2n}}.$$

The principle term of Riemann's density function $J(x)$ corresponds to the term $-\log(s-1)$, (EdH) 1.14. The representation in the form $(0 < \text{Re}(s) < 1)$ (*****)

$$\begin{aligned} \xi^{**}(s) &= (1-s) \sin(\pi s) M \left[{}_1F_1 \left(\frac{1}{2}; \frac{3}{2}, -\pi x^2 \right) \right] (s) \zeta(s) \\ &= \pi(s-1) \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right) M \left[x {}_1F_1' \left(\frac{1}{2}; \frac{3}{2}, -\pi x^2 \right) \right] (s) \zeta(s) \end{aligned}$$

provides the link to a Kummer function based zeta function theory.

(*) ; $\frac{1}{r-1} = \sum_{n=1}^{\infty} r^{-n}$; the integral $\int_1^{\infty} x^{2m} \varphi(x) \frac{dx}{x}$ is related to the Bernoulli numbers by the formula, (Grl) 3.552: $\int_1^{\infty} x^{2m} \varphi(x) \frac{dx}{x} = \frac{|B_{2m}|}{2m}$

(**) (PoG) Part V, 173: $f(t) > 0, f'(t) < 0, f''(t) < 0$ for $0 \leq t \leq 1$, then the even function $F(z) = \int_0^1 f(t) \cos(zt) dt$ has infinite many, only real zeros

(***) (LeB) p. 32: $\sin\left(\frac{\pi}{2}s\right) = \frac{\pi}{2} s \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{4n^2}\right) = \frac{\pi}{2} s \prod_{n \neq 0}^{\infty} \left(1 - \frac{s}{2n}\right) e^{s/2n}, \quad \Gamma\left(1 + \frac{s}{2}\right) = e^{-\gamma \frac{s}{2}} \prod_{n=1}^{\infty} \left(1 + \frac{s}{2n}\right)^{-1} e^{s/(2n)}$

(****) (ZyA) p. 5: $-\frac{1}{2\pi} \log|2 \sin(\pi x)| = \sum_{k=1}^{\infty} \frac{\cos(2\pi k x)}{2\pi k}, \quad 0 < x < 1$, (Grl) 1.518: $-\log\left(\cos\left(\frac{\pi}{2}x\right)\right) = -\sum_{k=1}^{\infty} \frac{(-1)^k [2^{2k}-1]}{2k} B_{2k}(\pi x)^{2k} = \sum_{k=1}^{\infty} \frac{\sin^{2k}\left(\frac{\pi}{2}x\right)}{2k}, \quad |x| < 1.$

(***** (BrK): $M \left[{}_1F_1 \left(\frac{1}{2}; \frac{3}{2}, -\pi x \right) \right] \left(\frac{s}{2}\right) = \int_0^{\infty} x^{\frac{s}{2}} {}_1F_1 \left(\frac{1}{2}; \frac{3}{2}, -\pi x \right) \frac{dx}{x} = 2 \int_0^{\infty} x^s {}_1F_1 \left(\frac{1}{2}; \frac{3}{2}, -\pi x^2 \right) \frac{dx}{x} = \pi^{-s/2} \frac{\Gamma\left(\frac{s}{2}\right)}{1-s}, \quad 0 < \text{Re}(s) < 1$

(LeN) 9.13: $\text{Erf}(x) = x {}_1F_1 \left(\frac{1}{2}; \frac{3}{2}, -x^2 \right) = \int_0^x e^{-t^2} dt.$

Proof of the lemma

Lemma:

$$-\frac{1}{s(1-s)} = -\int_1^{\infty} [x^s + x^{1-s}] \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x} + \left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\cos(\frac{\pi}{2}s)} \right] + \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s} \right].$$

The proof of the lemma is based on a novel integral and series representation of the Riemann zeta function $\zeta(s)$ as provided in (MiM):

By application of the residue theorem the convergent Dirichlet series representation $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$, $\text{Re}(s) > 1$, of the Riemann Zeta function can be reproduced from the contour integral representation

$$\zeta(s) = \pi \sum_{n=1}^{\infty} \text{Res}_{t=k} \oint_{t=s_k} x^{1-s} \frac{e^{i\pi x}}{\sin(\pi x)} \frac{dx}{x}, \quad \text{Re}(s) > 1 \quad (\text{MiM}) (2.4)$$

Convert this representation into a contour integral representation enclosing the positive integers on the x -axis in a clockwise direction, and, provided that $\text{Re}(s) > 1$ so that contributions from infinity vanish, the contour may then be opened such that it stretches vertically in the complex x -plane, given,

$$\zeta(s) = \frac{i}{2} \int_{c-i\infty}^{c+i\infty} x^{1-s} \cot(\pi x) \frac{dx}{x}, \quad 0 < c < 1, c \text{ real} \quad (\text{MiM}) (2.6)$$

$$= -\frac{1}{2} \int_{-\infty}^{\infty} (c+ix)^{-s} \cot(\pi(c+ix)) dx \quad (\text{MiM}) (2.7)$$

For the special case $c = 0$ the integral

$$\zeta(s) = -\pi^{s-1} \frac{\sin(\frac{\pi}{2}s)}{s-1} \int_0^{\infty} \frac{x^{1-s}}{\sinh^2(x)} dx, \quad \text{Re}(s) < 0 \quad (\text{MiM}) (4.1)$$

can be broken into two parts $\zeta(s) = \zeta_0(s) + \zeta_1(s)$ where

$$\zeta_1(s) = \frac{\sin(\frac{\pi}{2}s)}{s-1} + \sin(\frac{\pi}{2}s) \int_1^{\infty} x^{1-s} \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x} \quad (\text{MiM}) (4.6)$$

$$\zeta_0(s) = -\frac{2}{\pi} \sin(\frac{\pi}{2}s) \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{2n-s} \quad (*) \quad (\text{MiM}) (4.8)$$

which are both valid for all s .

The lemma follows from the related identities

$$\begin{aligned} \frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} &= \frac{1}{s-1} - \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{2n-s} + \int_1^{\infty} x^{1-s} \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x} \\ \frac{\zeta(1-s)}{\sin(\frac{\pi}{2}(1-s))} &= \frac{1}{-s} - \frac{2}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{(2n-1)+s} + \int_1^{\infty} x^s \frac{e^{-\pi x}}{\sinh(\pi x)} \frac{dx}{x}. \end{aligned}$$

(*) The reduction of the related integral $\zeta_0(s) = \frac{i}{2} \oint_{-t}^t x^{1-s} \cot(\pi x) \frac{dx}{x}$ is enabled by the formula $x \cot(\pi x) = -\frac{2}{\pi} \sum_{n=0}^{\infty} x^{2n} \zeta(2n)$, $|x| < 1$, (MiM) (4.7).

References

- (BrK) Braun K., A toolbox to solve the RH and to build a non-harmonic Fourier series based two-semicircle method
- (EdH) Edwards H. M., Riemann's Zeta Function, Dover Publications, Inc., Mineola, New York, 2001
- (GrI) Gradshteyn I. S., Ryzhik I. M., Table of Integrals Series and Products, Fourth Edition, Academic Press, New York, San Francisco, London, 1965
- (LeB) Levin B. Ya., Lectures on Entire Functions, American Mathematical Society, Providence, Rhode Island, 1996
- (LeN1) Lebedev N. N., Special functions & their applications, Dover Publications Inc., New York, 1972
- (MiM) Milgram M. S., Integral and Series Representations of Riemann's Zeta Function, Dirichlet's Eta Function and a Medley of Related Results, Journal of Mathematics, Vol. 2013, pp. 1-17
- (PoG) Pólya G., Szegő G., Problems and Theorems in analysis, Volume II, Springer-Verlag, New York, Heidelberg, Berlin, 1976
- (ZyA) Zygmund A., Trigonometric Series, Cambridge University Press, Cambridge, 1959

Appendix

Some more details about Riemann's $\xi(s)$ Function

For $\psi(x) := \sum_{n=1}^{\infty} e^{-n^2 \pi x}$ the function

$$(1) \quad \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \int_0^{\infty} x^{s/2} \psi(x) \frac{dx}{x} = \int_1^{\infty} \psi(x) [x^{s/2} + x^{(1-s)/2}] \frac{dx}{x} - \frac{1}{s(1-s)},$$

which occurs in the symmetrical form of the functional equation, has poles at $s = 0$ and $s = 1$ ^(*). The integral converges for all s because $\psi(x)$ decreases more rapidly than any power of x as $x \rightarrow \infty$, (EdH) 1.7.

(EdH) 1.8: Riemann multiplies (1) by $s(s-1)/2$, ^(**), and defines

$$(2) \quad \xi(s) := \Gamma\left(1 + \frac{s}{2}\right) (s-1) \pi^{-s/2} \zeta(s).$$

Then $\xi(s)$ is an entire function – that is an analytic function of s which is defined for all values of s – and the functional equation of the zeta function is equivalent to $\xi(s) = \xi(1-s)$. In combination with (1) this results into the representation in the form

$$\xi(s) = \frac{1}{2} - \frac{s(1-s)}{2} \int_0^{\infty} [x^{s/2} + x^{(1-s)/2}] \psi(x) \frac{dx}{x}$$

from which Riemann derived his final formula by two times partial integration in the form, (EdH) 1.8,

$$(3) \quad \xi(s) = 4 \int_1^{\infty} \frac{d}{dx} \left[x^{\frac{3}{2}} \psi'(x) \right] x^{-\frac{1}{4}} \cosh \left[\frac{1}{2} \left(s - \frac{1}{2} \right) \log x \right] dx = \sum_{n=0}^{\infty} a_{2n} \left(s - \frac{1}{2} \right)^{2n}$$

with

$$a_{2n} := 4 \int_1^{\infty} x^{-1/4} \frac{\left(\frac{1}{2} \log x\right)^{2n}}{(2n)!} \frac{d}{dx} \left[x^{3/2} \psi'(x) \right] dx.$$

Remark: In the notion of Riemann the integral form (3) is denoted in the form

$$\Xi(t) := \xi\left(\frac{1}{2} + it\right) = 4 \int_1^{\infty} x^{-1/4} \frac{d}{dx} \left[x^{3/2} \psi'(x) \right] \cos\left(\frac{t}{2} \log x\right) dx.$$

Remark (EdH) 1.8: “Riemann states that this series representation of ξ as an even function of $s - \frac{1}{2}$ “converges very rapidly,” but he gives no explicit estimates ... the statements that this series “converges very rapidly,” is also a statement that $\xi(s)$ is like a polynomial of infinite degree – a finite number of terms gives a very good approximation in any finite part of the plane. Hadamard proved that the rapid decrease of the coefficients a_{2n} is necessary and sufficient to the validity of Riemann's infinite product formula of $\xi(s)$ ”.

Based on this statement ^(***), in combination with the approximation function $= \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi}$ for the number of roots of $\Xi(t) = 0$ whose real parts lie between 0 and T , Riemann conjectured that all roots are real.

^(*) For $Re(s) < 0$ the zeta function is holomorph with (trivial) zeros (of first order) for $s = -2n$, $n \geq 1$. The non-trivial zeros lie in the critical stripe and are identical to the zeros of $\xi(s)$, while the functions $\Gamma(1-s)$ and $\sin\left(\frac{\pi s}{2}\right)$ have no zeros in the critical stripe. As the middle point of the zeros s_0 and $1-s_0$ is given by $\frac{s_0+(1-s_0)}{2} = \frac{1}{2}$ it follows $\xi\left(\frac{1}{2}+s\right) = \xi\left(\frac{1}{2}-s\right)$ and $\xi\left(\frac{1}{2}+s\right) = \xi\left(\frac{1}{2}-\bar{s}\right)$. Because of $\sum_{n=1}^{\infty} r^{-n} = \frac{1}{r-1}$ for $\varphi(x) := \sum_{n=1}^{\infty} e^{-nx}$ and $Re(s) > 1$ one gets the representation $\Gamma(s)\zeta(s) = \int_0^{\infty} x^s \varphi(x) \frac{dx}{x} = \int_0^{\infty} \frac{x^s}{e^x-1} \frac{dx}{x}$. Riemann's extension in the form $\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x-1} \frac{dx}{x}$ is valid for $s \in \mathbb{C}$ and is equal to Dirichlet's function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for $Re(s) > 1$, (EdH) 1.4. We note Riemann's original contour integral representation in the form $\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \oint \frac{x^s}{e^x-1} \frac{dx}{x}$, where the contour of integration encloses the negative x-axis, looping from $x = -\infty - i0$ to $x = -\infty + i0$ enclosing the point $x = 0$.

^(**) We note the rules $M[xh'](s) = -sM[h](s)$, $M[(xg)'](s) = (1-s)M[g](s)$ resp. $M[(x^2h)'](s) = (-s)(1-s)M[h](s)$ provided that the considered integrals are convergent; multiplication with s resp. with $(1-s)$ „governs“ a singularity of first order at $s = 0$ resp. $s = 1$.

^(***) “This function $\Xi(t)$ is finite for all finite t , and allows itself to be developed in powers of t^2 as very rapidly converging series“.

The connection between $\zeta(s)$ and the primes (EdH) 1.11

The essence of the relationship between $\zeta(s)$ and the prime numbers is the Euler product formula

$$(1) \quad \zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \prod_p \frac{p^s}{p^s-1}, \quad \text{Re}(s) > 1,$$

in which the product on the right is over all prime numbers p . The Möbius inverse transform is simply the inverse transform of the product formula, (EdH) 10.9,

$$\zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right) = 1 \quad \text{with} \quad \prod_p \left(1 - \frac{1}{p^s}\right) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}.$$

Taking the log of both sides of (1) and using the series $-\log(1-x) = \log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{1}{n} x^n$ puts this in the form

$$\log \zeta(s) = \sum_p \left[\sum_{n=1}^{\infty} \frac{1}{n} p^{-ns} \right], \quad \text{Re}(s) > 1.$$

Since the double series on the right is absolute convergent for $\text{Re}(s) > 1$, the order of summation is unimportant and the sum can be written simply

$$(2) \quad \log \zeta(s) = \sum_p p^{-s} + \frac{1}{2} \sum_p p^{-2s} + \frac{1}{3} \sum_p p^{-3s} + \frac{1}{4} \sum_p p^{-4s} + \dots, \quad \text{Re}(s) > 1.$$

The terms p^{-ns} are replaced by $s \int_{p^n}^{\infty} x^{-s-1} dx$ leading to

$$\frac{\log \zeta(s)}{s} = \int_0^{\infty} x^{-s-1} J(x) dx$$

It will be convenient in what follows to write this sum as a Stieltjes integral

$$(3) \quad \log \zeta(s) = \int_0^{\infty} x^{-s} dJ(x), \quad \text{Re}(s) > 1,$$

where $J(x)$ is the function which begins at 0 for $x = 0$ and increases by a jump of 1 at the primes p , by a jump $\frac{1}{2}$ at prime squares p^2 , by a jump of $\frac{1}{3}$ at prime cubes, etc. As it is usual in the theory of Stieltjes integrals, the value of $J(x)$ at each jump is defined to be halfway between its new value and its old value. Thus $J(x)$ is zero for $0 \leq x < 2$, is 1 for $2 < x < 3$, is $\frac{3}{2}$ for $x = 3$, is 2 for $3 < x < 4$, is $2\frac{1}{4}$ for $x = 4$, is $2\frac{1}{2}$ for $4 < x < 5$, is 3 for $x = 5$, is $3\frac{1}{2}$ for $5 < x < 7$, etc. A formula for $J(x)$ is

$$J(x) = \frac{1}{2} \left[\sum_{p^n < x} \frac{1}{n} + \sum_{p^n \leq x} \frac{1}{n} \right].$$

Riemann stated the formula (3) in the slightly different form (obtained by integration of parts)

$$(4) \quad \log \zeta(s) = s \int_0^{\infty} x^{-s} J(x) \frac{dx}{x}, \quad \text{Re}(s) > 1.$$

This integral can be considered to be an ordinary Riemann integral and the formula itself can be derived without using Stieltjes integration by setting $p^{-ns} = s \int_{p^n}^{\infty} x^{-s} \frac{dx}{x}$, $\text{Re}(s) > 1$ in (1), which is Riemann's derivation of (3). Formulas (2)-(4) should be thought of as minor variations of the Euler product formula, which is the basic idea connecting $\zeta(s)$ and primes.

Note: We note that the distribution function $J(x)$ is zero for $0 \leq x < 2$ in line with the Chebyshev approximation function $\pi(x) \sim \int_2^x \frac{dt}{\log t}$, defining the $\log \zeta(s)$ Fourier inverse function for $\text{Re}(s) > 1$ while the extended $\zeta(s)$ function to the critical stripe containing all non-trivial zeros of the zeta function, governing any appropriately defined prime number density function.