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# On Fourier and Zeta(s)

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**Abstract.** We study some of the interactions between the Fourier Transform and the Riemann zeta function (and Dirichlet-Dedekind-Hecke-Tate $L$-functions).

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## Contents



## 1 Introduction

### 1.1 A framework for this paper

The zeta function $\zeta(s)$ assumes in Riemann's paper quite a number of distinct identities: it appears there as a Dirichlet series, as an Euler product, as an integral transform, as an Hadamard product (rather, Riemann explains how $\log \zeta(s)$ may be written as an infinite sum involving the zeros)... We retain three such identities and use them as symbolic vertices for a triangle:

$$\sum_n \frac{1}{n^s}$$

$$\prod_\rho \left(1 - \frac{s}{\rho}\right) \quad — \quad \prod_p \frac{1}{1-p^{-s}}$$

These formulae stand for various aspects of the zeta function which, for the purposes of this manuscript, we may tentatively name as follows:

summations

zeros — primes

Even a casual reading of Riemann's paper will reveal how much Fourier analysis lies at its heart, on a par with the theory of functions of the complex variable. Let us enhance appropriately the triangle:

$$\sum_n \frac{1}{n^s}$$

Fourier Fourier

$$\prod_\rho \left(1 - \frac{s}{\rho}\right) \quad — \quad \text{Fourier} \quad — \quad \prod_p \frac{1}{1-p^{-s}}$$

Indeed, each of the three edges is an arena of interaction between the Fourier Transform, in various incarnations, and the Zeta function (and Dirichlet $L$-series, or even more general number theoretical zeta functions.) We thus specialize to a triangle which will be the framework of this manuscript:

$$\sum_n \frac{1}{n^s}$$

summations

Hilbert spaces and vectors ? Adeles and Ideles

$$\prod_\rho \left(1 - \frac{s}{\rho}\right) \quad — \quad \text{Explicit Formulae} \quad — \quad \prod_p \frac{1}{1-p^{-s}}$$

zeros primes

The big question mark serves as a reminder that we are missing the 2-cell (or 2-cells) which would presumably be there if the nature of the Riemann zeta function was really understood.

### 1.2 The contents of this paper

The paper contains in particular motivation, proofs, and developments related to a "fairly simple" (hence especially interesting) formula[1]:

$$\int_{\mathbb{R}}\left(\sum_{n\neq 0}\frac{g(t/n)}{|n|}-\int_{\mathbb{R}}\frac{g(1/x)}{|x|}\,dx\right)e^{2\pi iut}\,dt=\sum_{m\neq 0}\frac{g(m/u)}{|u|}-\int_{\mathbb{R}}g(y)\,dy$$

We call this the *co-Poisson intertwining formula*. The summations are over the non-zero relative integers. The formula applies, for example, to a function $g(t)$ of class $\mathscr{C}^\infty$ which is compactly supported on a closed set not containing the origin. Then the right-hand side is a function in the Schwartz class of smooth, rapidly decreasing functions, and the formula exhibits it as the Fourier Transform of another Schwartz function. These Schwartz functions have the peculiar property of being *constant, together with their Fourier transform*, in a neighborhood of the origin. A most interesting situation arises when the formula constructs square-integrable functions of this type and from our discussion of this it will be apparent that, although fairly simple, the *co-Poisson Formula* is related to a framework which is very far from being formal.

Once found, the formula is immediately proven, and in many different ways. Furthermore it is one among infinitely many such co-Poisson formulae (it is planned to discuss this further in [23]). This prototype is directly equivalent to the functional equation of the Riemann zeta function. It has implications concerning the problems of zeros.

We start with a discussion of our previous work [12] [13] [16] on the "Explicit Formulae" and the *conductor operators* $\log|x|_v + \log|y|_v$. We also include a description of our work on adeles, ideles, scattering and causality [14] [15], which is a first attempt to follow from local to global the idea of multiplicatively analysing the additive Fourier transform. This is necessary to explain the motivation which has led to a reexamination of the Poisson-Tate summation formula on adeles and to the discovery of the related but subtly distinct *co-Poisson intertwining*.

Both the *conductor operators* and the *co-Poisson intertwining* originated from an effort to move Tate's Thesis [43] towards the zeros and the *so-called Hilbert-Pólya idea*. It is notable that the zeros do not show up at all in Tate's Thesis: the conductor operator results in part from a continuation of the local aspects of Tate's Thesis; the co-Poisson formula results from a reexamination of the global aspects of Tate's Thesis. This reexamination, initially undergone during the fall of 1998, shortly after the discovery of the conductor operator, was also in part motivated by the preprint version of the work of Connes [25] (extending his earlier Note [24]) which had just appeared and where a very strong emphasis is put on the *so-called Hilbert-Pólya idea*.

[1] Note added in proofs (March 2003): The formula has in fact been discovered earlier by Duffin and Weinberger (Proc. Nat. Acad. Sci. **88** (1991), no. 16, 7348–7350; J. Fourier Anal. Appl. **3** (1997), 487–497) and should have been referred to here as the Duffin-Weinberger dualized Poisson formula. Our whole analysis, which relates it to the study of the Riemann zeta function and generalizations, is a novel contribution.

As we felt that the symmetries of the local conductor operators should have some bearing on global constructions we were very much interested by the constructions of Connes, and especially by the attempt to realize a cut-off simultaneously in position and momentum. This provides an indirect connection with our work, as reported upon here. But our cut-off is (or, perhaps better, appears to be) infrared, not ultraviolet. On our first encounter (on the adeles) with the formula we call here co-Poisson, we realized that we were constructing distributions for which it was easy to compute the Fourier Transform, and that these distributions were formally perpendicular to the zeros, but it was not immediately apparent to us that something beyond the usual use of the Poisson Formula was at work, as we did not at first understand that there was a temperature parameter, and that the Riemann zeta function is associated with a phase transition as we vary the temperature below a certain point. So, we left this aside for a while.

A key additional component to our effort came from the Theorem of Báez-Duarte, Balazard, Landreau and Saias [3] which is related to the Nyman-Beurling criterion [37, 6] for the validity of the Riemann Hypothesis. The link we have established ([18]) between the so-called Hilbert-Pólya idea and this important Theorem of Báez-Duarte, Balazard, Landreau and Saias leads under a further examination, which is reported upon here, to the consideration of certain functions which are meromorphic in the entire complex plane.

This then connects to the mechanism provided by the *co-Poisson intertwining* for the construction of Hilbert Spaces $HP_\lambda$ and Hilbert vectors $Z^\lambda_{\rho,k}$ associated with the non-trivial zeros of the Riemann zeta function. The method applies to Dirichlet $L$-series as well, and the last theorem of this paper is devoted to this. Some importance is ascribed by the author to this concluding result, not in itself of course (as many infinitely more subtle results than this one have been established on the zeta and $L$-functions since Riemann's paper), but rather as a clue which could provide inspiration for further endeavours. The light is extremely dim, but it has the merit of existence.

The discussion leading to this final result makes use in particular of an important theorem of Krein (on entire functions of finite exponential type [33]), and we relate the matter with the theory of Krein type spaces as exposed in the book [29] by Dym and McKean. An intriguing question arises on the properties of the *Krein string* which is thus associated with the Riemann zeta function. It seems that this Krein string is considered here for the first time, but we add immediately that we do not provide anything beyond mentioning it! Rather our technical efforts, which are not completely obvious, and not even fairly simple, lead to a realization of the Krein type spaces of this very special Krein string as subspaces of certain spaces $K_\lambda$ $(0 < \lambda < \infty)$ which are involved in a kind of multiplicative spectral (scattering) analysis of the Fourier cosine transform. The quotient spaces $HP_\lambda$ $(0 < \lambda < 1)$ are the spaces we propose as an approximation (getting better as $\lambda \to 0$) to an hypothetical so-called Hilbert-Pólya space.

The ambient spaces $K_\lambda$ have a realization as Hilbert Spaces of entire functions in the sense of de Branges [8]. The co-Poisson formula and the discussion ot the Nyman-Beurling criterion both suggest that it is useful to go beyond the framework of entire functions and consider more generally certain Hilbert spaces of meromorphic functions, but no general development has been attempted here.

The spaces $K_\lambda$ are among the *Sonine spaces* originally studied in the sixties by de Branges [7], V. Rovnyak [39] and J. Rovnyak and V. Rovnyak [40, 41] (the terminology "Sonine spaces of entire functions" was introduced in [41]). They are a special instance of the theory of Hilbert spaces of entire functions [8]. But it is only for the Sonine spaces associated to the Hankel transform of integer orders that the de Branges structure could be explicited in these papers. The theory of the Sonine spaces for the cosine and sine transforms is far less advanced. Recently though, the author has made some initial progress on this topic ([22]).

As de Branges has considered the use of the general Hankel-Sonine spaces in papers [10, 11] (and also in electronically available unpublished manuscripts) where the matter of the Riemann Hypothesis is mentioned, it is important to clarify that neither the co-Poisson formula, nor the spaces $HP_\lambda$ $(0 < \lambda < 1)$, $W_\lambda$ and $W'_\lambda$, nor the vectors $Z^\lambda_{\rho,k}$ for $k \geq 1$, have arisen in any of de Branges's investigations known to this author (this is said after having spent some time to investigate the demands of the situation created by these papers).

The circumstances of the genesis of this paper have led us to devote a special final section, which is very brief, to some speculations on the nature of the zeta function, the GUE hypothesis, and the Riemann hypothesis.

### 1.3 Acknowledgements

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## 2 Explicit Formulae, $\log|x| + \log|y|$, adeles, ideles, scattering, causality

Riemann discovered the zeros and originated the idea of counting the primes and prime powers (suitably weighted) using them. Indeed this was the main focus of his famous paper. Later a particularly elegant formula was rigorously proven by von Mangoldt:

$$\sum_{1<n<X} \Lambda(n) + \frac{1}{2}\Lambda(X) = X - \sum_{\rho} \frac{X^{\rho}}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2})$$

Here $X > 1$ (not necessarily an integer) and $\Lambda(Y) = \log(p)$ if $Y > 1$ is a positive power of the prime number $p$, and is 0 for all other values of $Y$. The $\rho$'s are the Riemann Zeros (in the critical strip), the sum over them is not absolutely convergent, even after pairing $\rho$ with $1-\rho$. It is defined as $\lim_{T\to\infty} \sum_{|\mathrm{Im}(\rho)|<T} X^{\rho}/\rho$.

In the early fifties Weil published a paper [46] on the Riemann-von Mangoldt Explicit Formula, and then another one [47] in the early seventies which considered non-abelian Artin (and Artin-Weil) $L$-functions. While elucidating already in his first paper new algebraic structure, he did this maintaining a level of generality encompassing in its scope the von Mangoldt formula (although it requires some steps to deduce this formula from the Weil explicit formula.) The analytical difficulties arising are an expression of the usual difficulties with Fourier inversion. The "test-function flavor" of the "Riemann-Weil explicit formula" had been anticipated by Guinand [31].

So in our opinion a more radical innovation was Weil's discovery that the local terms of the Explicit Formulae acquire a natural expression on the $\nu$-adics, and that this enables to put the real and complex places on a par with the finite places (clearly Weil was motivated by analogies with function fields, we do not discuss that here.) Quite a lot of algebraic number theory [48] is necessary in Weil's second paper to establish this for Artin-Weil $L$-functions.

We stay here at the simpler level of Weil's first paper and show how to put all places of the number field at the same level. It had first appeared in Haran's work [32] that it was possible to formulate the Weil's local terms in a more unified manner than had originally been done by Weil. We show that an operator theoretical approach allows, not only to formulate, but also to deduce the local terms in a unified manner. The starting point is Tate's Thesis [43]. Let $K$ be a number field and $K_\nu$ one of its completions. Let $\chi_\nu : K_\nu^\times \to S^1$ be a (unitary) multiplicative character. For $0 < \mathrm{Re}(s) < 1$ both $\chi_\nu(x)|x|^{s-1}$ and $\chi_\nu(x)^{-1}|x|^{-s}$ are tempered distributions on the additive group $K_\nu$ and the *Tate's functional equations* are the identities of distributions:

$$\mathscr{F}_\nu(\chi_\nu(x)|x|^{s-1}) = \Gamma(\chi_\nu, s)\chi_\nu(x)^{-1}|x|^{-s}$$

for certain functions $\Gamma(\chi_\nu, s)$ analytic in $0 < \mathrm{Re}(s) < 1$, and meromorphic in the complex plane. This is the local half of Tate's Thesis, from the point of view of distributions. See also [30]. Implicit in this equation is a certain normalized choice of additive Haar measure on $K_\nu$, and $\mathscr{F}_\nu$ is the corresponding additive Fourier transform.

Let us view this from a Hilbert space perspective. The quasi-characters $\chi_\nu(x)^{-1}|x|^{-s}$ are never square-integrable, but for $\mathrm{Re}(s) = 1/2$ they are the generalized eigenvectors arising in the spectral analysis of the unitary group of dilations (and contractions): $\phi(x) \mapsto \phi(x/t)/\sqrt{|t|_\nu}$, $x \in K_\nu$, $t \in K_\nu^\times$. Let $I_\nu$ be the unitary operator $\phi(x) \mapsto \phi(1/x)/|x|_\nu$, and let $\Gamma_\nu = \mathscr{F}_\nu \cdot I_\nu$. Then:

$$\Gamma_\nu(\chi_\nu(x)^{-1}|x|^{-s}) = \Gamma(\chi_\nu, s)\chi_\nu(x)^{-1}|x|^{-s}$$

and this says that the $\chi_v(x)^{-1}|x|_v^{-s}$, for $\mathrm{Re}(s) = 1/2$, are the generalized eigenvectors arising in the spectral analysis of the unitary scale invariant operator $\Gamma_v = \mathscr{F}_v \cdot I_v$.

The question [12] which leads from Tate's Thesis (where the zeros do not occur at all) to the topic of the Explicit Formulae is: *what happens if we take the derivative with respect to s in Tate's functional equations?* Proceeding formally we obtain:

$$-\Gamma_v(\log|x|_v\chi_v(x)^{-1}|x|_v^{-s}) = \Gamma'(\chi_v, s)\cdot\chi_v(x)^{-1}|x|_v^{-s}$$

$$- \Gamma(\chi_v, s)\log|x|_v \cdot \chi_v(x)^{-1}|x|_v^{-s}$$

$$\log(|x|_v)\cdot\chi_v(x)^{-1}|x|_v^{-s} - \Gamma_v\left(\log|x|_v\frac{\chi_v(x)^{-1}|x|_v^{-s}}{\Gamma(\chi_v, s)}\right)$$

$$= \left(\frac{d}{ds}\log\Gamma(\chi_v, s)\right)\cdot\chi_v(x)^{-1}|x|_v^{-s}$$

$$(\log|x|_v - \Gamma_v\cdot\log|x|_v\cdot\Gamma_v^{-1})\cdot(\chi_v(x)^{-1}|x|_v^{-s}) = \left(\frac{d}{ds}\log\Gamma(\chi_v, s)\right)\cdot\chi_v(x)^{-1}|x|_v^{-s}$$

Let $H_v$ be the scale invariant operator $\log|x|_v - \Gamma_v\cdot\log|x|_v\cdot\Gamma_v^{-1} = \log|x|_v + \mathscr{F}_v\cdot\log|x|_v\cdot\mathscr{F}_v^{-1}$, which we also write symbolically as:

$$H_v = \log|x|_v + \log|y|_v$$

Then we see that the conclusion is:

**2.1 Theorem ([12] [13]).** *The generalized eigenvalues of the conductor operator $H_v$ are the logarithmic derivatives of the Tate Gamma functions:*

$$H_v(\chi_v(x)^{-1}|x|_v^{-s}) = \left(\frac{d}{ds}\log\Gamma(\chi_v, s)\right)\cdot\chi_v(x)^{-1}|x|_v^{-s}$$

Let $g(u)$ be a smooth function with compact support in $\mathbb{R}_+^\times$. Let $\hat{g}(s) = \int g(u)u^{s-1}\,du$ be its Mellin transform. Let $\chi$ be a unitary character on the idele class group of the number field $K$, with local components $\chi_v$. Let $Z(g,\chi)$ be the sum of the values of $\hat{g}(s)$ at the zeros of the (completed) Hecke $L$-function $L(\chi, s)$ of $\chi$, minus the contribution of the poles when $\chi$ is a principal character ($t \mapsto |t|^{-i\tau}$). Using the calculus of residues we obtain $Z(g,\chi)$ as the integral of $\hat{g}(s)(d/ds)\log L(\chi, s)$ around the contour of the infinite rectangle $-1 \le \mathrm{Re}(s) \le 2$. It turns out that the compatibility between Tate's Thesis (local half) and Tate's Thesis (global half) allows to use the functional equation without having ever to write down explicitely all its details (such as the discriminant of the number field and the conductor of the character), and leads to:

**2.2 Theorem ([12]).** *The explicit formula is given by the logarithmic derivatives of the Tate Gamma functions:*

$$Z(g,\chi) = \sum_v \int_{\mathrm{Re}(s)=1/2} \left(\frac{d}{ds}\log\Gamma(\chi_v,s)\right)\hat{g}(s)\,\frac{|ds|}{2\pi}$$

At an archimedean place the values $\hat{g}(s)$ on the critical line give the multiplicative spectral decomposition of the function $g_{\chi,v} := x \mapsto \chi_v(x)^{-1}g(|x|_v)$ on (the additive group) $K_v$, and, after checking normalization details, one finds that the local term has exact value $H_v(g_{\chi,v})(1)$. At a non-archimedean place, one replaces the integral on the full critical line with an integral on an interval of periodicity of the Tate Gamma function, and applying Poisson summation (in the vertical direction) to $\hat{g}(s)$ to make it periodical as well it is seen to transmute into the multiplicative spectral decomposition of the function $g_{\chi,v} := x \mapsto \chi_v(x)^{-1}g(|x|_v)$ on $K_v$! So we jump directly from the critical line to the completions of the number field $K$, and end up with the following version of the explicit formula:

**2.3 Theorem ([12] [13]).** *Let at each place $v$ of the number field $K$:*

$$g_{\chi,v} = x \mapsto \chi_v(x)^{-1}g(|x|_v)$$

*on $K_v$ $(g_{\chi,v}(0) = 0)$. Then*

$$Z(g,\chi) = \sum_v H_v(g_{\chi,v})(1)$$

*where $H_v$ is the scale invariant operator* $\log|x|_v + \log|y|_v$ *acting on* $L^2(K_v, dx_v)$.

As we evaluate at 1, the "$\log|x|_v$" half of $H_v$ could be dropped, and we could sum up the situation as follows: *Weil's local term is the (additive) Fourier transform of the logarithm!* This is what Haran had proved ([32], for the Riemann zeta function), except that he formulated this in terms of Riesz potentials $|y|_v^{-s}$, and did a separate check for finite places and the infinite place that the Weil local terms may indeed be written in this way. The explicit formula as stated above with the help of the operator $\log|x|_v + \log|y|_v$ incorporates in a more visible manner the compatibility with the functional equations. Indeed we have

**2.4 Theorem ([12] [13]).** *The conductor operator $H_v$ commutes with the operator $I_v$:*

$$H_v \cdot I_v = I_v \cdot H_v$$

*or equivalently as* $I_v \cdot \log|x|_v \cdot I_v = -\log|x|_v$*:*

$$I_v \cdot \log|y|_v \cdot I_v = 2\log|x|_v + \log|y|_v$$

To see abstractly why this has to be true, one way is to observe that $H_v$ and $\Gamma_v$ are simultaneously diagonalized by the multiplicative characters, hence they commute. But obviously $H_v$ commutes with $\mathscr{F}_v$ so it has to commute with $I_v$. Later, when dealing with what we call "co-Poisson intertwining", we will see a similar argument in another context.

Let us suppose $\chi_v$ to be ramified (which means not trivial when restricted to the $v$-adic units) and let $f(\chi_v)$ be its *conductor exponent*, $e_v$ the number field *differental exponent* at $v$, and $q_v$ the cardinality of the residue field. In Tate's Thesis [43], one finds for a ramified character

$$\Gamma(\chi_v, s) = w(\chi_v) q_v^{(f(\chi_v)+e_v)(s-1/2)}$$

where $w(\chi_v)$ is a certain non-vanishing complex number, quite important in Algebraic Number Theory, but not here. Indeed we take the logarithmic derivative and find:

$$\frac{d}{ds} \log \Gamma(\chi_v, s) = (f(\chi_v) + e_v) \log q_v$$

So that:

$$H_v(\chi_v^{-1}(x) \mathbf{1}_{|x|_v=1}(x)) = (f(\chi_v) + e_v) \log(q_v) \chi_v^{-1}(x) \mathbf{1}_{|x|_v=1}(x)$$

which says that ramified characters are eigenvectors of $H_v$ with eigenvalues $(f(\chi_v) + e_v) \log q_v$. Hence the name "conductor operator" for $H_v$. We note that this contribution of the differental exponent is there also for a non-ramified character and explains why in our version of the Explicit Formula there is no explicit presence of the discriminant of the number field. If we now go through the computation of the distribution theoretic additive Fourier transform of $\log|x|_v$ and compare with the above we end up with a proof [12] of the well-known Weil integral formula [46] [47] [48] (Weil writes $d^\times t$ for $\log(q_v)\, d^*t$):

$$f(\chi_v) \log q_v = \int_{K_v^\times} \mathbf{1}_{|t|_v=1}(t) \frac{1 - \chi_v(t)}{|1-t|_v} d^\times t$$

In Weil's paper [46] we see that this formula's rôle has been somewhat understated. Clearly it was very important to Weil as it confirmed that it was possible to express similarly all contributions to the Explicit Formula: from the infinite places, from finite unramified places, and from finite ramified places. Weil leaves establishing the formula to the attentive reader. In his second paper [47] he goes on to extend the scope to Artin $L$-function, and this is far from an obvious thing.

Let us now consider the zeta and $L$-functions from the point of view of Adeles and Ideles. Again a major input is Tate's Thesis. There the functional equations of the abelian $L$-functions are established in a unified manner, but the zeros do not appear at all. It is only recently that progress on this arose, in the work of Connes [25]. We have examined this question anew [14] [15], from the point of view of the study of the interaction between the additive and multiplicative Fourier Transforms [12] [14], which as we saw is a mechanism underlying the operator theoretic approach to the explicit formula. This led us to the scattering theory of Lax and Phillips [35] and to a formulation of the Riemann Hypothesis, simultaneous for all $L$-functions, as a property of *causality* [15]

A key theorem from the global half of Tate's Thesis is the following:

$$\sum_{q\in K}\mathscr{F}(\varphi)(qv)=\frac{1}{|v|}\sum_{q\in K}\varphi\left(\frac{q}{v}\right)$$

This was called the "Riemann-Roch Theorem" by Tate, but we prefer to call it the *Poisson-Tate formula* (which sounds less definitive, and more to the point). Let us explain the notations: $K$ is a number field, $\varphi(x)$ is a function on the adeles $\mathbb{A}$ of $K$ (satisfying suitable conditions), $q\in K$ is diagonally considered as an adele, $v\in\mathbb{A}^\times$ is an idele and $|v|$ is its module. Finally $\mathscr{F}$ is the adelic additive Fourier Transform (we refer to [43] for the details of the normalizations[1]). We note that it does not matter if we exchange the $\varphi$ on the right with the $\mathscr{F}(\varphi)$ on the left as $\mathscr{F}(\mathscr{F}(\varphi))(x)=\varphi(-x)$ and $-1\in K^\times$. A suitable class of functions stable under $\mathscr{F}$ for which this works is given by the Bruhat-Schwartz functions: finite linear combinations of infinite product of local factors, almost all of them being the indicator function of the local ring of integers, in the Schwartz class for the infinite places, locally constant with compact support at each finite place. To each such function and unitary character $\chi$ on the idele class group Tate associates an $L$-function $L(\chi,\varphi)(s)=\int_{\text{ideles}}\varphi(v)\chi(v)|v|^s\,d^*v$, and shows how to choose $\varphi$ so that this coincides exactly with the complete Hecke $L$-function with grossencharakter $\chi$.

Let us write $E_0$ (very soon we will switch to a related $E$) for the map which to the function $\varphi(x)$ on the *adeles* associates the function $\sum_{q\in K}\varphi(qv)\sqrt{|v|}$ on the *ideles* or even on the *idele class group* $\mathscr{C}_K$ (ideles quotiented by $K^\times$). This map $E_0$ plays an important rôle in the papers of Connes [24] [25] (where it is used under the additional assumption $\varphi(0)=0=\mathscr{F}(\varphi)(0)$, and then coincides with the $E$ we introduce next.) The Poisson-Tate formula tells us that $E_0$ intertwines the additive Fourier transform with the operator $I: g(v)\mapsto g(1/v)$.

$$(E_0\cdot\mathscr{F})(\varphi)=(I\cdot E_0)(\varphi)$$

If we ([15]) manipulate a little bit the Poisson-Tate formula into:

$$\sqrt{|v|}\sum_{q\in K^\times}\mathscr{F}(\varphi)(qv)-\frac{1}{\sqrt{|v|}}\varphi(0)=\frac{1}{\sqrt{|v|}}\sum_{q\in K^\times}\varphi\left(\frac{q}{v}\right)-\sqrt{|v|}\int_{\text{adeles}}\varphi(x)\,dx$$

we still have the intertwining property

$$(E\cdot\mathscr{F})(\varphi)=(I\cdot E)(\varphi)$$

where we have written $E$ for the map which to $\varphi(x)$ associates

$$u\mapsto\sqrt{|v|}\sum_{q\in K^\times}\varphi(qv)-\frac{\int_{\text{adeles}}\varphi(x)\,dx}{\sqrt{|u|}}$$

on the idele class group ($v$ in the class $u$).

[1] e.g., on $\mathbb{R}$ the Fourier transform is $\tilde{\varphi}(y)=\int_{\mathbb{R}}e^{2\pi iyx}\varphi(x)\,dx$

**2.5 Note.** Let $\phi(x)$ be an even Schwartz function on $\mathbb{R}$. Let $F(x) = \sum_{n\geq 1} \phi(nx)$. We have $\lim_{x\to 0}|x|F(x) = \int_0^\infty \phi(x)\,dx$. For $\mathrm{Re}(s) > 1$ we may intervert the integral with the summation and this gives $\int_0^\infty F(x)x^{s-1}\,dx = \zeta(s)\int_0^\infty \phi(x)x^{s-1}\,dx$. The analytic continuation of this formula to the critical strip ($0 < \mathrm{Re}(s) < 1$) requires a modification which is due to Müntz (as stated in Titchmarsh's book [45, II.11])

$$\int_0^\infty \left(F(x) - \frac{\int_0^\infty \phi(y)\,dy}{x}\right)x^{s-1}\,dx = \zeta(s)\int_0^\infty \phi(x)x^{s-1}\,dx$$

So it is in truth not the original Poisson summation but the Müntz-modified Poisson (where one takes out $\phi(0)$ and replaces it with $-(\int_{\mathbb{R}} \phi(y)\,dy)/|x|)$) which corresponds to $\zeta(s)$ as multiplier. The Müntz modification was used by the author in [15] unknowingly of its previous appearance in the literature. The author thanks Luis Báez-Duarte for pointing out the reference to the section of the book of Titchmarsh where the Müntz formula is discussed.

**2.6 Theorem ([15]).** *For $\varphi$ a Bruhat-Schwartz function $E(\varphi)$ is square-integrable on the idele class group $\mathscr{C}_K$ (for the multiplicative Haar measure $d^*u$), and its unitary multiplicative Fourier transform, as a function of the unitary characters, coincides up to an overall constant with the Tate L-function on the critical line* $\mathrm{Re}(s) = 1/2$. *The functions $E(\varphi)$ are dense in $L^2(\mathscr{C}_K, d^*u)$ and $E(\mathscr{F}(\varphi)) = I(E(\varphi))$.*

Connes [24] [25] had already considered the functions $\sqrt{|v|}\sum_{q\in K^\times}\varphi(qv)$ with $\varphi(0) = 0 = \mathscr{F}(\varphi)(0)$ and he had shown that they are dense in $L^2(\mathscr{C}_K, d^*u)$. Let $\mathscr{S}_1$ be the set of Bruhat-Schwartz functions $\varphi(x)$ which are supported in a parallelepiped $P(v) = \{\forall v |x|_\nu \leq |v|_\nu\}$ with $|v| \leq 1$. Let $\mathscr{D}_+ = E(\mathscr{S}_1)^\perp$ and let $\mathscr{D}_- = E(\mathscr{F}(\mathscr{S}_1))^\perp$. The following holds:

**2.7 Theorem ([15]).** *The subspaces $\mathscr{D}_+$ and $\mathscr{D}_-$ of the Hilbert space of square-integrable functions on the idele class group are outgoing and incoming subspaces for a Lax-Phillips scattering system, where the idele class group plays the rôle of time. The Riemann Hypothesis for all abelian L-functions of K holds if and only if the causality axiom $\mathscr{D}_+ \perp \mathscr{D}_-$ is satisfied.*

## 3 Poisson-Tate and a novel relative: co-Poisson

We have already mentioned the Tate $L$-functions (the integral is absolutely convergent for $\mathrm{Re}(s) > 1$):

$$L(\chi, \varphi)(s) = \int_{\text{ideles}} \varphi(v)\chi(v)|v|^s\,d^*v$$

Using the Poisson-Tate summation formula, Tate established the analytic continuation and the functional equations:

$$L(\chi, \mathscr{F}(\varphi))(s) = L(\chi^{-1}, \varphi)(1-s)$$

This follows from an integral representation

$$L(\chi,\varphi)(s) = C\delta_\chi\left(\frac{\mathcal{F}(\varphi)(0)}{s-1-i\tau} - \frac{\varphi(0)}{s-i\tau}\right) + \int_{|v|\geq 1}(\varphi(v)\chi(v)|v|^s + \mathcal{F}(\varphi)(v)\chi(v)^{-1}|v|^{1-s})\,d^*v$$

where $C$ is a certain constant associated to the number field $K$ (and relating the Haar measures $d^*v$ on $\mathbb{A}^\times$ and $d^*u$ on $\mathcal{C}_K$), and the Kronecker symbol $\delta_\chi$ is 1 or 0 according to whether $\chi(v) = |v|^{-i\tau}$ for a certain $\tau \in \mathbb{R}$ (principal unitary character) or not (ramified unitary character). The integral over the ideles (this is not an integral over the idele *classes*) with $|v| \geq 1$ is absolutely convergent for *all* $s \in \mathbb{C}$.

We recall that we associated to the Bruhat-Schwartz function $\varphi(x)$ on the adeles the square-integrable function $E(\varphi)(u)$ on the idele class group $\mathcal{C}_K$ (with $u \in \mathcal{C}_K$ the class of $v \in \mathbb{A}^\times$):

$$E(\varphi)(u) = \sqrt{|u|}\sum_{q\in K^\times}\varphi(qv) - \frac{\int_{\text{adeles}}\varphi(x)\,dx}{\sqrt{|u|}}$$

The precise relation [15] to the Tate $L$-functions is:

$$L(\chi,\varphi)(s) = C\int_{\mathcal{C}_K} E(\varphi)(u)\chi(u)|u|^{s-1/2}\,d^*u$$

This integral representation is absolutely convergent for $0 < \mathrm{Re}(s) < 1$ and we read the functional equations directly from it and from the intertwining property $E \cdot \mathcal{F} = I \cdot E$.

Let $\Delta(u)$ be the function of $u$ with values in the distributions on the adeles:

$$\Delta(u)(\varphi) = C\cdot\left(\sqrt{|u|}\sum_{q\in K^\times}\varphi(qv) - \frac{\int_{\text{adeles}}\varphi(x)\,dx}{\sqrt{|u|}}\right) = C\cdot E(\varphi)(u)$$

We will show that it is relevant to look at $\Delta(u)$ not as a *function* in $u$ (which it is from the formula above) but as a *distribution* in $u$ (so that $\Delta$ is a distribution with values in distributions ...) It takes time to explain why this is not a tautological change of perspective. Basically we shift the emphasis from the Poisson-Tate summation [43] [25] [15] which goes from adeles to ideles, to the *co-Poisson summation* which goes from ideles to adeles. The Poisson-Tate summation is a function with values in distributions, whereas the co-Poisson-Tate summation is a distribution whose values we try to represent as $L^2$-functions.

Let $g(v)$ be a compact Bruhat-Schwartz function on the idele group $\mathbb{A}^\times$. This is a finite linear combination of infinite products $v \mapsto \prod_\nu g_\nu(v_\nu)$, where almost each component is the indicator function of the $\nu$-adic units, the component $g_\nu(v_\nu)$ at an infinite place is a smooth compactly supported function on $K_v^\times$, and the components at finite places are locally constant compactly supported.

**3.1 Definition.** The *co-Poisson summation* is the map $E'$ which assigns to each compact Bruhat-Schwartz function $g(v)$ the distribution on the *adeles* given by:

$$E'(g)(\varphi) = \int_{\mathbb{A}^\times} \varphi(v) \sum_{q \in K^\times} g(qv)\sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} g(v)|v|^{-1/2}\, d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

**3.2 Note.** Clearly $E'(g)$ depends on $g(v)$ only through the function $R(g)$ on $\mathscr{C}_K$ given by $R(g)(u) = \sum_{q \in K^\times} g(qv)$, with $u \in \mathscr{C}_K$ the class of $v \in \mathbb{A}^\times$. However, for various reasons (among them avoiding the annoying constant $C$ in all our formulae), it is better to keep the flexibility provided by $g$. The function $R(g)$ has compact support. To illustrate this with an example, and explain why the integral above makes sense, we take $K = \mathbb{Q}$, $g(v) = \prod_p \mathbf{1}_{|v_p|_p=1}(v_p) \cdot g_\infty(v_\infty)$. Then $\sum_{q \in \mathbb{Q}^\times} g(qv) = g_\infty(|v|) + g_\infty(-|v|)$. We may bound this from above by a multiple of $|v|$ (as $g_\infty$ has compact support in $\mathbb{R}^\times$), and the integral $\int_{\mathbb{A}^\times} \varphi(v)|v|^{3/2}\, d^*v$ converges absolutely as $1 < 3/2$ (we may take $\varphi(x)$ itself to be an infinite product here.)

**3.3 Theorem.** *The co-Poisson summation intertwines the operator $I : g(v) \mapsto g(1/v)$ with the additive adelic Fourier Transform $\mathscr{F}$:*

$$\mathscr{F}(E'(g)) = E'(I(g))$$

*Furthermore it intertwines between the multiplicative translations $g(v) \mapsto g(v/w)$ on ideles and the multiplicative translations on adelic distributions $D(x) \mapsto D(x/w)/\sqrt{|w|}$. And the distribution $E'(g)$ is invariant under the action of the multiplicative group $K^\times$ on the adeles.*

*Proof.* We have:

$$E'(g)(\varphi)$$

$$= C\int_{\mathscr{C}_K} \sum_{q \in K^\times} \varphi(qv) R(g)(u)\sqrt{|u|}\, d^*u - \int_{\mathbb{A}^\times} g(v)|v|^{-1/2}\, d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

$$= C\int_{\mathscr{C}_K} \left(\frac{E(\varphi)(u)}{\sqrt{|u|}} + \frac{\int \varphi(x)\, dx}{|u|}\right) R(g)(u)\sqrt{|u|}\, d^*u - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

$$= C\int_{\mathscr{C}_K} E(\varphi)(u) R(g)(u)\, d^*u + \left(C \int_{\mathscr{C}_K} \frac{R(g)(u)}{\sqrt{|u|}} d^*u - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v\right) \int \varphi(x)\, dx$$

$$= C\int_{\mathscr{C}_K} E(\varphi)(u) R(g)(u)\, d^*u$$

Using the intertwinings $E \cdot \mathscr{F} = I \cdot E$ and $R(I(g))(u) = R(g)(1/u)$ we get:

$$E'(g)(\mathscr{F}(\varphi)) = C\int_{\mathscr{C}_K} E(\varphi)(u)R(g)\left(\frac{1}{u}\right)d^*u = E'(I(g))(\varphi)$$

which completes the proof of $\mathscr{F}(E'(g)) = E'(I(g))$. The compatibility with multiplicative translations is easy, and the invariance under the multiplication $x \mapsto qx$ follows. □

**3.4 Note.** The way the ideles have to act on the distributions on adeles for the intertwining suggests some Hilbert space properties of the distribution $E'(g)$ (more on this later).

**3.5 Note.** The invariance under $K^\times$ suggests that it could perhaps be profitable to discuss $E'(g)$ from the point of view of the Connes space $\mathbb{A}/K^\times$ [24] [25].

**3.6 Theorem.** *The following Riemann-Tate formula holds:*

$$\begin{aligned} E'(g)(\varphi) = &\int_{|v|\geq 1}\mathscr{F}(\varphi)(v)\sum_{q\in K^\times} g(q/v)\sqrt{|v|}\,d^*v + \int_{|v|\geq 1}\varphi(v)\sum_{q\in K^\times} g(qv)\sqrt{|v|}\,d^*v \\ &- \varphi(0)\int_{|v|\geq 1} g(1/v)|v|^{-1/2}\,d^*v - \int_{\mathbb{A}}\varphi(x)\,dx\cdot\int_{|v|\geq 1} g(v)|v|^{-1/2}\,d^*v \end{aligned}$$

*Proof.* From $E'(g)(\varphi) = C\int_{\mathscr{C}_K} E(\varphi)(u)R(g)(u)\,d^*u$ we get

$$\begin{aligned} &E'(g)(\varphi) \\ &= C\int_{|u|\leq 1} E(\varphi)(u)R(g)(u)\,d^*u + C\int_{|u|\geq 1} E(\varphi)(u)R(g)(u)\,d^*u \\ &= C\int_{|u|\geq 1} E(\mathscr{F}(\varphi))(u)R(g)(1/u)\,d^*u + C\int_{|u|\geq 1} E(\varphi)(u)R(g)(u)\,d^*u \\ &= C\int_{|u|\geq 1}\left(E(\mathscr{F}(\varphi))(u) + \frac{\varphi(0)}{\sqrt{|u|}}\right)R(g)(1/u)\,d^*u - \varphi(0)\int_{|v|\geq 1} g(1/v)|v|^{-1/2}\,d^*v \\ &\quad + C\int_{|u|\geq 1}\left(E(\varphi)(u) + \frac{\int_{\mathbb{A}}\varphi(x)\,dx}{\sqrt{|u|}}\right)R(g)(u)\,d^*u - \left(\int\varphi\right)\int_{|v|\geq 1} g(v)|v|^{-1/2}\,d^*v \\ &= \int_{|v|\geq 1}\mathscr{F}(\varphi)(v)\sum_{q\in K^\times} g(q/v)\sqrt{|v|}\,d^*v + \int_{|v|\geq 1}\varphi(v)\sum_{q\in K^\times} g(qv)\sqrt{|v|}\,d^*v \\ &\quad - \varphi(0)\int_{|v|\geq 1} g(1/v)|v|^{-1/2}\,d^*v - \left(\int_{\mathbb{A}}\varphi(x)\,dx\right)\int_{|v|\geq 1} g(v)|v|^{-1/2}\,d^*v \end{aligned}$$

which completes the proof. □

We note that if we replace formally $\sum_{q\in K^\times} g(qv)\sqrt{|v|}$ with $\chi(v)|v|^s$ we obtain exactly the Tate formula for $L(\chi,\varphi)(s)$. But some new flexibility arises with a "compact" $g(v)$:

**3.7 Theorem.** *The following formula holds:*

$$E'(g)(\varphi) = \int_{|v|\le 1} \mathscr{F}(\varphi)(v) \sum_{q\in K^\times} g(q/v)\sqrt{|v|}\, d^*v + \int_{|v|\le 1} \varphi(v) \sum_{q\in K^\times} g(qv)\sqrt{|v|}\, d^*v$$

$$- \varphi(0)\int_{|v|\le 1} g(1/v)|v|^{-1/2}\, d^*v - \int_{\mathbb{A}} \varphi(x)\, dx \cdot \int_{|v|\le 1} g(v)|v|^{-1/2}\, d^*v$$

*Proof.* Exactly the same as above exchanging everywhere $|u| \ge 1$ with $|u| \le 1$ (we recall that $R(g)$ has compact support and also that as we are dealing with a number field $|v| = 1$ has zero measure). This is not possible with a quasicharacter in the place of $R(g)(u)$. Alternatively one adds to the previous formula and checks that one obtains $2E'(g)(\varphi)$ (using $E'(I(g))(\mathscr{F}(\varphi)) = E'(g)(\varphi)$). □

To illustrate some Hilbert Space properties of the co-Poisson summation, we will assume $K = \mathbb{Q}$. The components $(a_v)$ of an adele $a$ are written $a_p$ at finite places and $a_r$ at the real place. We have an embedding of the Schwartz space of test-functions on $\mathbb{R}$ into the Bruhat-Schwartz space on $\mathbb{A}$ which sends $\psi(x)$ to $\varphi(a) = \prod_p \mathbf{1}_{|a_p|_p\le 1}(a_p)\cdot \psi(a_r)$, and we write $E'_{\mathbb{R}}(g)$ for the distribution on $\mathbb{R}$ thus obtained from $E'(g)$ on $\mathbb{A}$.

**3.8 Theorem.** *Let $g$ be a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The co-Poisson summation $E'_{\mathbb{R}}(g)$ is a square-integrable function (with respect to the Lebesgue measure). The $L^2(\mathbb{R})$ function $E'_{\mathbb{R}}(g)$ is equal to the constant $-\int_{\mathbb{A}^\times} g(v)|v|^{-1/2}\, d^*v$ in a neighborhood of the origin.*

*Proof.* We may first, without changing anything to $E'_{\mathbb{R}}(g)$, replace $g$ with its average under the action of the finite unit ideles, so that it may be assumed invariant. Any such compact invariant $g$ is a finite linear combination of suitable multiplicative translates of functions of the type $g(v) = \prod_p \mathbf{1}_{|v_p|_p=1}(v_p)\cdot f(v_r)$ with $f(t)$ a smooth compactly supported function on $\mathbb{R}^\times$, so that we may assume that $g$ has this form. We claim that:

$$\int_{\mathbb{A}^\times} |\varphi(v)| \sum_{q\in\mathbb{Q}^\times} |g(qv)|\sqrt{|v|}\, d^*v < \infty$$

Indeed $\sum_{q\in\mathbb{Q}^\times}|g(qv)| = |f(|v|)| + |f(-|v|)|$ is bounded above by a multiple of $|v|$. And $\int_{\mathbb{A}^\times}|\varphi(v)|\,|v|^{3/2}\, d^*v < \infty$ for each Bruhat-Schwartz function on the adeles (basically, from $\prod_p(1-p^{-3/2})^{-1} < \infty$). So

$$E'(g)(\varphi) = \sum_{q\in\mathbb{Q}^\times} \int_{\mathbb{A}^\times} \varphi(v)g(qv)\sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

$$E'(g)(\varphi) = \sum_{q\in\mathbb{Q}^\times} \int_{\mathbb{A}^\times} \varphi(v/q)g(v)\sqrt{|v|}\, d^*v - \int_{\mathbb{A}^\times} \frac{g(v)}{\sqrt{|v|}} d^*v \int_{\mathbb{A}} \varphi(x)\, dx$$

Let us now specialize to $\varphi(a) = \prod_p \mathbf{1}_{|a_p|_p \le 1}(a_p) \cdot \psi(a_r)$. Each integral can be evaluated as an infinite product. The finite places contribute 0 or 1 according to whether $q \in \mathbb{Q}^\times$ satisfies $|q|_p < 1$ or not. So only the inverse integers $q = 1/n$, $n \in \mathbb{Z}$, contribute:

$$E'_{\mathbb{R}}(g)(\psi) = \sum_{n \in \mathbb{Z}^\times} \int_{\mathbb{R}^\times} \psi(nt) f(t) \sqrt{|t|} \frac{dt}{2|t|} - \int_{\mathbb{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbb{R}} \psi(x)\, dx$$

We can now revert the steps, but this time on $\mathbb{R}^\times$ and we get:

$$E'_{\mathbb{R}}(g)(\psi) = \int_{\mathbb{R}^\times} \psi(t) \sum_{n \in \mathbb{Z}^\times} \frac{f(t/n)}{\sqrt{|n|}} \frac{dt}{2\sqrt{|t|}} - \int_{\mathbb{R}^\times} \frac{f(t)}{\sqrt{|t|}} \frac{dt}{2|t|} \int_{\mathbb{R}} \psi(x)\, dx$$

Let us express this in terms of $\alpha(y) = (f(y) + f(-y))/2\sqrt{|y|}$:

$$E'_{\mathbb{R}}(g)(\psi) = \int_{\mathbb{R}} \psi(y) \sum_{n \ge 1} \frac{\alpha(y/n)}{n} dy - \int_0^\infty \frac{\alpha(y)}{y} dy \int_{\mathbb{R}} \psi(x)\, dx$$

So the distribution $E'_{\mathbb{R}}(g)$ is in fact the even smooth function

$$E'_{\mathbb{R}}(g)(y) = \sum_{n \ge 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y} dy$$

As $\alpha(y)$ has compact support in $\mathbb{R}\setminus\{0\}$, the summation over $n \ge 1$ contains only vanishing terms for $|y|$ small enough. So $E'_{\mathbb{R}}(g)$ is equal to the constant $-\int_0^\infty \frac{\alpha(y)}{y} dy = -\int_{\mathbb{R}^\times} \frac{f(y)}{\sqrt{|y|}} \frac{dy}{2|y|} = -\int_{\mathbb{A}^\times} g(t)/\sqrt{|t|}\, d^*t$ in a neighborhood of 0. To prove that it is $L^2$, let $\beta(y)$ be the smooth compactly supported function $\alpha(1/y)/2|y|$ of $y \in \mathbb{R}$ ($\beta(0) = 0$). Then ($y \ne 0$):

$$E'_{\mathbb{R}}(g)(y) = \sum_{n \in \mathbb{Z}} \frac{1}{|y|} \beta\left(\frac{n}{y}\right) - \int_{\mathbb{R}} \beta(y)\, dy$$

From the usual Poisson summation formula, this is also:

$$\sum_{n \in \mathbb{Z}} \gamma(ny) - \int_{\mathbb{R}} \beta(y)\, dy = \sum_{n \ne 0} \gamma(ny)$$

where $\gamma(y) = \int_{\mathbb{R}} \exp(i2\pi yw)\beta(w)\, dw$ is a Schwartz rapidly decreasing function. From this formula we deduce easily that $E'_{\mathbb{R}}(g)(y)$ is itself in the Schwartz class of rapidly decreasing functions, and in particular it is square-integrable. □

It is useful to recapitulate some of the results arising in this proof:

**3.9 Theorem.** *Let $g$ be a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The* *co-Poisson summation $E'_{\mathbb{R}}(g)$ is an even function on $\mathbb{R}$ in the Schwartz class of rapidly decreasing functions. It is constant, as well as its Fourier Transform, in a neighborhood of the origin. It may be written as*

$$E'_{\mathbb{R}}(g)(y) = \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y}\,dy$$

*with a function $\alpha(y)$ smooth with compact support away from the origin, and conversely each such formula corresponds to the co-Poisson summation $E'_{\mathbb{R}}(g)$ of a compact Bruhat-Schwartz function on the ideles of $\mathbb{Q}$. The Fourier transform $\int_{\mathbb{R}} E'_{\mathbb{R}}(g)(y)\exp(i2\pi wy)\,dy$ corresponds in the formula above to the replacement $\alpha(y) \mapsto \alpha(1/y)/|y|$.*

Everything has been obtained previously.

## 4 More proofs and perspectives on co-Poisson

The intertwining property was proven as a result on the adeles and ideles, but obviously the proof can be written directly on $\mathbb{R}$. It will look like this, with $\varphi(y)$ an even Schwartz function (and $\alpha(y)$ as above):

*Proof.* From $\int_{\mathbb{R}} \sum_{n\geq 1} |\varphi(ny)|\,|\alpha(y)|\,dy < \infty$:

$$\int_{\mathbb{R}} \sum_{n\geq 1} \varphi(ny)\alpha(y)\,dy = \sum_{n\geq 1} \int_{\mathbb{R}} \varphi(ny)\alpha(y)\,dy$$

$$= \sum_{n\geq 1} \int_{\mathbb{R}} \varphi(y)\frac{\alpha(y/n)}{n}\,dy = \int_{\mathbb{R}} \varphi(y)\sum_{n\geq 1}\frac{\alpha(y/n)}{n}\,dy$$

On the other hand applying the usual Poisson summation formula:

$$\int_{\mathbb{R}} \sum_{n\geq 1} \varphi(ny)\alpha(y)\,dy$$

$$= \int_{\mathbb{R}} \left( \sum_{n\geq 1} \frac{\mathscr{F}(\varphi)(n/y)}{|y|} - \frac{\varphi(0)}{2} + \frac{\mathscr{F}(\varphi)(0)}{2|y|} \right) \alpha(y)\,dy$$

$$= \int_{\mathbb{R}} \left( \sum_{n\geq 1} \mathscr{F}(\varphi)(ny) \right) \frac{\alpha(1/y)}{|y|}\,dy - \varphi(0)\int_0^\infty \alpha(y)\,dy + \mathscr{F}(\varphi)(0)\int_0^\infty \frac{\alpha(y)}{|y|}\,dy$$

$$= \int_{\mathbb{R}} \mathscr{F}(\varphi)(y)\sum_{n\geq 1}\frac{\alpha(n/y)}{|y|}\,dy - \varphi(0)\int_0^\infty \alpha(y)\,dy + \mathscr{F}(\varphi)(0)\int_0^\infty \frac{\alpha(y)}{|y|}\,dy$$

The conclusion being:

$$\int_{\mathbb{R}} \varphi(y) \sum_{n\geq 1} \frac{\alpha(y/n)}{n}\,dy - \mathscr{F}(\varphi)(0) \int_0^\infty \frac{\alpha(1/y)}{y}\,dy$$

$$= \int_{\mathbb{R}} \mathscr{F}(\varphi)(y) \sum_{n\geq 1} \frac{\alpha(n/y)}{|y|}\,dy - \varphi(0) \int_0^\infty \alpha(y)\,dy$$

which, after exchanging $\varphi$ with $\mathscr{F}(\varphi)$, is a distribution theoretic formulation of the intertwining property:

$$\mathscr{F}\left( \sum_{n\geq 1} \frac{\alpha(y/n)}{n} - \int_0^\infty \frac{\alpha(y)}{y}\,dy \right) = \sum_{n\geq 1} \frac{\alpha(n/y)}{|y|} - \int_0^\infty \alpha(y)\,dy \qquad \square$$

It is useful to have a version of co-Poisson intertwining without compactness nor smoothness conditions:

**4.1 Lemma.** *Let $g(u)$ be an even measurable function with*

$$\int_0^\infty \frac{|g(u)|}{u}\,du < \infty$$

*The sum $\sum_{n\geq 1} \frac{g(t/n)}{n}$ is Lebesgue almost-everywhere absolutely convergent. It is a locally integrable function of $t$. It is a tempered distribution.*

*Proof.* Let $\varphi(t)$ be an arbitrary even measurable function. One has:

$$\int_0^\infty \sum_{n\geq 1} |\varphi(nt)|\,|g(t)|\,dt = \int_0^\infty |\varphi(t)| \sum_{n\geq 1} \left| \frac{g(t/n)}{n} \right| dt$$

If we take $\varphi(t)$ to be 1 for $|t| \leq \Lambda$, 0 for $|t| > \Lambda$, we have $\sum_{n\geq 1} |\varphi(nt)| = [\Lambda/|t|] \leq \Lambda/|t|$. From this:

$$\int_0^\Lambda \sum_{n\geq 1} \left| \frac{g(t/n)}{n} \right| dt = O(\Lambda)$$

and this implies that $A(t) = \sum_{n\geq 1} \frac{g(t/n)}{n}$ is almost everywhere absolutely convergent, that it is locally integrable and also that $\int_0^u A(t)\,dt$ is $O(u)$. So the continuous function $\int_0^u A(t)\,dt$ is a tempered distribution. Hence its distributional derivative $A(u)$ is again a tempered distribution. $\square$

**4.2 Theorem.** *Let $g(t)$ be an even measurable function with*

$$\int_0^\infty \frac{|g(t)|}{t}\,dt + \int_0^\infty |g(t)|\,dt < \infty$$

*Then the co-Poisson intertwining*

$$\mathscr{F}\left(\sum_{n\geq 1}\frac{g(t/n)}{n}-\int_0^\infty\frac{g(u)}{u}du\right)=\sum_{n\geq 1}\frac{g(n/t)}{t}-\int_0^\infty g(u)\,du$$

*holds as an identity of tempered distributions.*

*Proof.* The proof 3.9 given above applies identically. To get it started one only has to state trivially for $\varphi(y)$ a Schwartz function that $\sum_{n\geq 1}|\varphi(ny)|$ is $O(1/|y|)$. □

So the conclusion is that as soon as the two integrals are absolutely convergent the full co-Poisson intertwining makes sense and holds true. If one manages to get more information, the meaning of the $\mathscr{F}$ will be accordingly improved. For example if one side is in $L^2$ then the other side has to be too and the equality holds with $\mathscr{F}$ being the Fourier-Plancherel (cosine) transform.

**4.3 Note.** In this familiar $\mathbb{R}$ setting our first modification of the Poisson formula was very cosmetic: the Poisson formula told us the equality of two functions and we exchanged a term on the left with a term on the right. This was to stay in a Hilbert space, but it remained a statement about the equality of two functions (and in the adelic setting, the equality of two functions with values in the distributions on the adeles). With the co-Poisson if we were to similarly exchange the integral on the left with the integral on the right, we would have to use Dirac distributions, and the nature of the identity would change. So the co-Poisson is more demanding than the (modified) Poisson. Going from Poisson to co-Poisson can be done in many ways: conjugation with $I$, or conjugation with $\mathscr{F}$, or Hilbert adjoint, or more striking still and at the same time imposed upon us from adeles and ideles, the switch from viewing a certain quantity as a function (Poisson) to viewing it as a distribution (co-Poisson). The co-Poisson is a distribution whose values we try to understand as $L^2$-functions, whereas the (modified) Poisson is a function with values in distributions (bad for Hilbert space).

We state again the important intertwining property as a theorem, with an alternative proof:

**4.4 Theorem.** *Let $\alpha(y)$ be a smooth even function on $\mathbb{R}$ with compact support away from the origin. Let $P'(\alpha)$ be its co-Poisson summation:*

$$P'(\alpha)(y)=\sum_{n\geq 1}\frac{\alpha(y/n)}{n}-\int_0^\infty\frac{\alpha(y)}{y}dy$$

*Then the additive Fourier Transform of $P'(\alpha)$ is $P'(I(\alpha))$ with $I(\alpha)(y)=\alpha(1/y)/|y|$.*

*Proof.* Let $P$ be the (modified) Poisson summation on even functions:

$$P(\alpha)(y) = \sum_{n\geq 1} \alpha(ny) - \frac{\int_0^\infty \alpha(y)\,dy}{|y|}$$

Obviously $P' = I \cdot P \cdot I$. And we want to prove $\mathcal{F}P' = P'I$. Let us give a formal operator proof:

$$\mathcal{F}P' = \mathcal{F}IPI = P\mathcal{F}II = P\mathcal{F} = IP = P'I$$

Apart from the usual Poisson summation formula $P\mathcal{F} = IP$ the crucial step was the commutativity of $\mathcal{F}I$ and $P$. This follows from the fact that both operators commute with the multiplicative action of $\mathbb{R}^\times$, so they are simultaneously diagonalized by multiplicative characters, hence they have to commute.

To elucidate this in a simple manner we extend our operators $I$, $\mathcal{F}$, $P$ and $P'$ to a larger class of functions, a class stable under all four operators. It is not difficult [15] to show that for each Schwartz function $\beta$ (in particular for $\beta = I(\alpha)$) the Mellin Transform of $P(\beta)(y)$ is:

$$\int_0^\infty P(\beta)(y)\,y^{s-1}\,dy = \zeta(s)\int_0^\infty \beta(y)\,y^{s-1}\,dy$$

initially at least for $0 < \mathrm{Re}(s) < 1$. Let us consider the class of functions $k(1/2 + i\tau)$ on the critical line which decrease faster than any inverse power of $\tau$ when $|\tau| \to \infty$. On this class of functions we define $I$ as $k(s) \mapsto k(1-s)$, $P$ as $k(s) \mapsto \zeta(s)k(s)$, $P'$ as $k(s) \mapsto \zeta(1-s)k(s)$, and $\mathcal{F}I$ as $k(s) \mapsto \gamma_+(s)k(s)$ with $\gamma_+(s) = \pi^{-(s-1/2)}\Gamma(s/2)/\Gamma((1-s)/2)$. The very crude bound (on $\mathrm{Re}(s) = 1/2$) $|\zeta(s)| = O(|s|)$ (for example, from $\zeta(s)/s = 1/(s-1) - \int_0^1 \{1/t\}t^{s-1}\,dt$) shows that it is a multiplier of this class (it is also a multiplier of the Schwartz class from the similar crude bounds on its derivatives one obtains from the just given formula). And $|\gamma_+(s)| = 1$, so this works for it too (and also for the Schwartz class, see [12]). The above formal operator proof is now not formal anymore (using, obviously, that the Mellin transform is one-to-one on our $\alpha$'s). The intertwining property for $P'$ is equivalent to the intertwining property for $P$, because both are equivalent, but in different ways, to the functional equation for the zeta function. □

As was stated in the previous proof a function space which is stable under all four operators $I$, $\mathcal{F}$, $P$ and $P'$ is the space of inverse Mellin transforms of Schwartz functions on the critical line: these are exactly the even functions on $\mathbb{R}$ with the form $k(\log|y|)/\sqrt{|y|}$ where $k(a)$ is a Schwartz function of $a \in \mathbb{R}$. We pointed out the stability under Fourier Transform in [12]. We note that although $P$ and $P'$ make sense when applied to $k(\log|y|)/\sqrt{|y|}$ and that they give a new function of this type, this can not always be expressed as in their original definitions, for example because the integrals involved have no reason to be convergent (morally they correspond to evaluations away from the critical line at 0 and at 1.)

We may also study $I$, $\mathcal{F}$, $P$ and $P'$ as operators on $L^2$, but some minimal care has to be taken because $P$ and its adjoint $P'$ are not bounded. Nevertheless they are closed operators and they commute with the Abelian von Neumann algebra of bounded operators commuting with $\mathbb{R}^\times$ (see [16]). This gives one more method to establish the co-Poisson intertwining as a corollary to the Poisson intertwining, as we may go from $P\mathcal{F} = IP$ (*modified* Poisson) to $\mathcal{F}P' = P'I$ (co-Poisson) simply by taking Hilbert adjoints: $P' = P^*$ (and here $\mathcal{F}$ is the cosine transform, so $\mathcal{F}^* = \mathcal{F}$):

$$P\mathcal{F} = IP \Rightarrow (P\mathcal{F})^* = (IP)^* \Rightarrow \mathcal{F}P' = P'I$$

It is important to be aware that for this to be correct it is crucial that we are using $P$ to denote, not the *original* Poisson sums, but the *Müntz-modified* Poisson sums.

Another perspective on co-Poisson comes from a re-examination of the use of the multiplicative version (now called Mellin) of the Fourier Transform in Riemann's paper. To establish the functional equation, Riemann uses the *left* Mellin transform $\int_0^\infty f(u)u^{s-1}\,du$, but when he relates the zeros to the primes with an explicit formula he uses the *right* Mellin transform $\int_0^\infty f(u)u^{-s}\,du$. After all the zeta-function itself is most simply expressed as the right Mellin transform of the sum of the Dirac at the positive integers. We said earlier that the Müntz-modified sums corresponded under Mellin Transform to the zeta function $\zeta(s)$, but this is using the *left* convention. If, rather, we use the *right* convention we are bound to associate to $\zeta(s)$ the *co-Poisson sums*! If we now ask what the functional equation tells us, then the answer is: in particular the co-Poisson intertwining ... It should be clear from this discussion that the co-Poisson intertwining is a formula of the nineteenth century which was discovered at the close of the twentieth century.

Immediately after being communicated the co-Poisson formula, Luis Báez-Duarte replied that an application of Euler-McLaurin summation establishes the co-Poisson intertwining in a more elementary manner, inasmuch as his method uses neither distributions nor Mellin transforms, and does not appeal directly to either the Poisson summation formula, nor to the functional equation of the Riemann zeta function. Here is the proof emerging from this discussion:

**4.5 Theorem (proven jointly with Luis Báez-Duarte).** *Let $g(t)$ be an (even) function of class $C^2$ which has compact support away from the origin. Then both $\sum_{n\geq 1}\frac{g(t/n)}{n} - \int_0^\infty \frac{g(u)}{u}\,du$ and $\sum_{n\geq 1}\frac{g(n/t)}{t} - \int_0^\infty g(u)\,du$ are continuous $L^1$-functions and the co-Poisson intertwining formula*

$$\mathcal{F}\left(\sum_{n\geq 1}\frac{g(t/n)}{n} - \int_0^\infty \frac{g(u)}{u}\,du\right) = \sum_{n\geq 1}\frac{g(n/t)}{t} - \int_0^\infty g(u)\,du$$

*holds as a pointwise equality and may be established as a corollary to the Euler-McLaurin summation formulae.*

*Proof.* Let $B(u)$ be the periodic funtion which on $[0,1)$ has values $\frac{u^2}{2} - \frac{u}{2} + \frac{1}{12}$. It is a continuous even function, which as is well-known is also expressed as:

$$B(u) = \sum_{n\geq 1} \frac{\cos(2\pi nu)}{2\pi^2 n^2}$$

Let $f(t)$ be a $C^2$ function with compact support away from 0. We have:

$$\sum_{n\geq 1} f(n) - \int_0^\infty f(u)\,du = -\int_0^\infty B(u)\left(\frac{d}{du}\right)^2 f(u)\,du$$

If we apply this to the function $u \to f(u/w)/w$ for $w > 0$, we obtain:

$$\sum_{n\geq 1} \frac{f(n/w)}{w} - \int_0^\infty f(u)\,du = -\int_0^\infty B(u)\left(\frac{d}{du}\right)^2 \frac{f(u/w)}{w}\,du$$

$$= -\frac{1}{w^2}\int_0^\infty B(wv)\left(\frac{d}{dv}\right)^2 f(v)\,dv$$

The left-hand side is locally a finite sum, hence of class $C^2$ (and when $|w|$ is sufficiently small it reduces to the constant $-\int_0^\infty f(u)\,du$) and the formula above shows that it is $O(1/w^2)$ when $w \to \infty$. If we only assume $f(u)$ to be bounded but still with compact support away from the origin then obviously $\sum_{n\geq 1} \frac{f(n/w)}{w}$ is at any rate bounded (the number of non-vanishing terms being $O(w)$). We now apply the above formula to $g(t) = f(1/t)/t$:

$$\sum_{n\geq 1} \frac{g(t/n)}{n} - \int_0^\infty \frac{g(1/u)}{u}\,du = -\int_0^\infty B(u)\left(\frac{d}{du}\right)^2 \frac{g(t/u)}{u}\,du$$

$$= -\sum_{n\geq 1}\int_0^\infty \frac{\cos(2\pi nu)}{2\pi^2 n^2}\left(\frac{d}{du}\right)^2 \frac{g(t/u)}{u}\,du = -\sum_{n\geq 1}\int_0^\infty \frac{\cos(2\pi tw)}{2\pi^2 t^2}\left(\frac{d}{dw}\right)^2 \frac{g(n/w)}{w}\,dw$$

At this stage we expand the second derivative and using that $\sum_{n\geq 1} \frac{k(n/w)}{w}$ is bounded with $k(t) = |g(t)|, |tg'(t)|, |t^2 g''(t)|$ we see that dominated convergence applies. So:

$$= -\int_0^\infty \frac{\cos(2\pi tw)}{2\pi^2 t^2}\left(\frac{d}{dw}\right)^2 \left(\sum_{n\geq 1} \frac{g(n/w)}{w}\right) dw$$

$$= -\int_0^\infty \frac{\cos(2\pi tw)}{2\pi^2 t^2}\left(\frac{d}{dw}\right)^2 \left(\sum_{n\geq 1} \frac{g(n/w)}{w} - \int_0^\infty g(\alpha)\,d\alpha\right) dw$$

$$= +\int_0^\infty 2\cos(2\pi tw)\left(\sum_{n\geq 1} \frac{g(n/w)}{w} - \int_0^\infty g(\alpha)\,d\alpha\right) dw$$

The first integration by parts at the end is justified by $\lim_{w\to\infty} \frac{d}{dw}\sum_{n\geq 1} \frac{g(n/w)}{w} = 0$, using that $\sum_{n\geq 1}\frac{k(n/w)}{w}$ is bounded with $k(t) = |g(t)|, |tg'(t)|$. So far we have $t \neq 0$, but the integral being dominated we may let $t = 0$ in the final formula. If we now replace $g(t)$ with $g(1/t)/t$ we get the co-Poisson intertwining as a pointwise equality. □

The most direct attack when first confronted with the co-Poisson formula is presumably this: $\mathscr{F}(\sum g(t/n)/n) = \sum \mathscr{F}(g)(n\xi) = \sum g(m/\xi)/|\xi|$, where we first intervert and then apply Poisson. This has a number of pitfalls (for whose unraveling the language of distributions is very useful), but it is possible to make it work to prove something correct. This proof is a joint-work with Bernard Candelpergher, and as in the previous approach it uses only elementary tools and especially the (simplest cases of) Euler-McLaurin summation.

**4.6 Theorem (proven jointly with Bernard Candelpergher).** *Let $g(t)$ be an even function of class $C^2$ which has compact support away from the origin. Then both $\sum_{n\geq 1}\frac{g(t/n)}{n} - \int_0^\infty \frac{g(u)}{u}du$ and $\sum_{n\geq 1}\frac{g(n/t)}{t} - \int_0^\infty g(u)\,du$ are continuous $L^1$-functions and the co-Poisson intertwining formula*

$$\mathscr{F}\left(\sum_{n\geq 1}\frac{g(t/n)}{n} - \int_0^\infty \frac{g(u)}{u}du\right) = \sum_{n\geq 1}\frac{g(n/t)}{t} - \int_0^\infty g(u)\,du$$

*holds as a pointwise equality and may be established as a corollary to the Euler-McLaurin summation formulae.*

*Proof.* We work on $(0, \infty)$. We have $(t > 0)$:

$$\sum_{n\geq 1}\frac{g(t/n)}{n} - \int_0^\infty \frac{g(1/u)}{u}du = \sum_{n\geq 1} A_n(t)$$

with:

$$A_n(t) = \frac{1}{2}\left(\frac{g(t/n)}{n} + \frac{g(t/(n-1))}{n-1}\right) - \int_{n-1}^{n} \frac{g(t/u)}{u}du$$

where the term with $n-1$ is dropped for $n = 1$. With $k(t) = g(1/t)/t$ one has:

$$A_n(t) = \frac{1}{2}\left(\frac{k(n/t)}{t} + \frac{k((n-1)/t)}{t}\right) - \int_{n-1}^{n} \frac{k(u/t)}{t}du$$

$$= -\int_{n-1}^{n} C_2(u)\frac{d^2}{du^2}\frac{k(u/t)}{t}du$$

where $C_2(u) = (\{u\}^2 - \{u\})/2$. From this:

$$\sum_{n\geq 1}|A_n(t)| \leq \frac{1}{t^2}\int_0^\infty |C_2(u)|\,|k''(u/t)|\frac{du}{t} = O(1/t^2)$$

On the other hand for $t$ small enough one has $A_n(t) \equiv 0$ for $n \geq 2$. So the sum $\sum_{n\geq 1} A_n(t)$ is absolutely convergent to an $L^1$-function and also we can compute its Fourier transform termwise. This gives, with $\tilde{g} = \mathscr{F}(g)$:

$$\mathscr{F}(\sum_{n\geq 1} A_n(t))(\xi) = \frac{-\tilde{g}(0)}{2} + \sum_{n\geq 1}\left(\frac{\tilde{g}(n\xi)+\tilde{g}((n-1)\xi)}{2} - \int_{n-1}^{n} \tilde{g}(u\xi)\,du\right)$$

The appearance of $-\tilde{g}(0)/2$ is from the fact that this time the $n-1$ term with $n=1$ is to be counted in the sum, so we have to compensate for this. The formula holds for all $\xi$ ($\xi \geq 0$), in particular for $\xi = 0$ it reads:

$$\int_{\mathbb{R}}\left(\sum_{n\geq 1}\frac{g(t/n)}{n} - \int_0^\infty \frac{g(1/u)}{u}du\right)dt = -\int_0^\infty g(v)\,dv$$

For $\xi > 0$ we are one step away from co-Poisson, it only remains to apply Poisson to our sum (note that $\int_0^\infty \tilde{g}(u\xi)\,du = 0$). But we can also base this on Euler-McLaurin. Indeed with $B_1(v) = \{v\} - \frac{1}{2}$:

$$\frac{\tilde{g}(n\xi)+\tilde{g}((n-1)\xi)}{2} - \int_{n-1}^{n}\tilde{g}(u\xi)\,du = \int_{n-1}^{n} B_1(v)\frac{d}{dv}\tilde{g}(v\xi)\,dv$$

Hence

$$\mathscr{F}(\sum_{n\geq 1} A_n(t))(\xi) = -\int_0^\infty g(v)\,dv + \int_0^\infty B_1(v)\frac{d}{dv}\tilde{g}(v\xi)\,dv$$

Now, from the fact that $g$ is $C^2$ with compact support, its Fourier transform and all derivatives of it are $O(1/|\xi|^2)$. We may thus use the boundedly convergent expression:

$$B_1(v) = \sum_{m\geq 1}\frac{-\sin(2\pi m v)}{m\pi}$$

and then

$$\int_0^\infty B_1(v)\frac{d}{dv}\tilde{g}(v\xi)\,dv = \lim_{M\to\infty}\sum_{1\leq m\leq M}\int_0^\infty \frac{-\sin(2\pi m v)}{m\pi}\frac{d}{dv}\tilde{g}(v\xi)\,dv$$

$$= \lim_{M\to\infty}\sum_{1\leq m\leq M} 2\int_0^\infty \cos(2\pi m v)\tilde{g}(v\xi)\,dv = \lim_{M\to\infty}\sum_{1\leq m\leq M}\frac{g(m/\xi)}{\xi}$$

where Fourier inversion was used ($\xi > 0$). This completes the proof of co-Poisson. It is also interesting to prove in another manner the special formula ($\xi = 0$):

$$\int_0^\infty \left( \sum_{n\geq 1} \frac{g(t/n)}{n} - \int\limits_0^\infty \frac{g(1/u)}{u} du \right) dt = -\frac{1}{2}\int_0^\infty g(v)\, dv$$

This may be done as follows: first,

$$\int_0^\Lambda \sum_{n\geq 1} \frac{g(t/n)}{n} dt = \sum_{n\geq 1} \int_0^{\Lambda/n} g(t)\, dt = \int_0^\infty \left[\frac{\Lambda}{t}\right] g(t)\, dt$$

so we are looking at

$$\lim_{\Lambda\to\infty} - \int_0^\infty \left\{\frac{\Lambda}{t}\right\} g(t)\, dt = -\lim_{\Lambda\to\infty} \int_0^\infty \{\Lambda v\} h(v)\, dv$$

with the $L^1$-function $h(v) = g(1/v)/v^2$. It is obvious that

$$0 \leq A \leq B \text{ implies } \lim_{\Lambda\to\infty} \int_0^\infty \{\Lambda v\} \mathbf{1}_{A\leq v\leq B}(v)\, dv = \frac{B-A}{2}$$

so using the density argument from the usual proof of the Riemann-Lebesgue lemma one deduces that

$$\lim_{\Lambda\to\infty} \int_0^\infty \{\Lambda v\} h(v)\, dv = \frac{1}{2}\int_0^\infty h(v)\, dv$$

for all $L^1$-functions. One last remark is that at the level of (left) Mellin, co-Poisson is like multiplication by $\zeta(1-s)$, so the special formula is just another instance of $\zeta(0) = -\frac{1}{2}$. □

Here is one last approach to co-Poisson (extracted from a manuscript in preparation [23]). The Poisson summation identity is

$$\sum_{n\in\mathbb{Z}} \mathscr{F}(f)(n) = \sum_{m\in\mathbb{Z}} f(m)$$

It applies in particular to Schwartz functions, and may be written as:

$$\mathscr{F}\left(\sum_{n\in\mathbb{Z}} \delta_n(x)\right) = \sum_{m\in\mathbb{Z}} \delta_m(y)$$

We take this identity of tempered distributions seriously, on its own, and do not completely identify it with the Poisson summation identity. Let $t \neq 0$ and let us replace $x$ by $tx$. One has $\delta_n(tx) = \delta_{n/t}(x)/|t|$. So

$$\mathscr{F}\left(\sum_{n\in\mathbb{Z}} \frac{\delta_{n/t}(x)}{|t|}\right) = \frac{1}{|t|}\sum_{m\in\mathbb{Z}} \delta_m\left(\frac{y}{t}\right) = \sum_{m\in\mathbb{Z}} \delta_{tm}(y)$$

We average these tempered distributions with an integrable weight $g(t)$ (which is compactly supported away from the origin) to obtain an identity of tempered distributions:

$$\mathscr{F}\left(\sum_{n\in\mathbb{Z}}\int g(t)\frac{\delta_{n/t}(x)}{|t|}\,dt\right)=\sum_{m\in\mathbb{Z}}\int g(t)\delta_{tm}(y)\,dt$$

But for $n\neq 0$ (resp. $m\neq 0$) and as distributions in $x$ (resp. $y$):

$$\int g(t)\frac{\delta_{n/t}(x)}{|t|}\,dt=\int g(n/\alpha)\delta_{\alpha}(x)\frac{d\alpha}{|\alpha|}=\frac{g\left(\frac{n}{x}\right)}{|x|}$$

$$\int g(t)\delta_{tm}(y)\,dt=\int g(\beta/m)\delta_{\beta}(y)\frac{d\beta}{|m|}=\frac{g\left(\frac{y}{m}\right)}{|m|}$$

whereas $\int g(t)\frac{\delta_0(x)}{|t|}\,dt=\left(\int g(t)\frac{dt}{|t|}\right)\delta_0(x)$ and $\int g(t)\delta_0(y)\,dt=(\int g(t)\,dt)\delta_0(y)$. If we exchange sides for the contributions of $n=0$ and $m=0$ we end up with the co-Poisson intertwining as an identity of tempered distributions. This method of multiplicative convolution applies to situations where the discreteness of the support of the original distributions applies only at the origin. A general discussion is planned in [23].

There is reason to believe that the problem of understanding the spaces of $L^2$-functions which are vanishing together with their Fourier Transform in a neighborhood of the origin, is important simultaneously for Analysis and Arithmetic. It is a remarkable ancient discovery of de Branges [7] [8] that these spaces have the rich structure which he developed generally in his theory of Hilbert Spaces of entire functions: they are among the "Sonine spaces". The co-Poisson summations have the (extended) property of being constant together with their Fourier transform, in some neighborhood of the origin, and we will show later (see also [21]) that the zeros of the Riemann zeta function are the obstructions for (the square-integrable among) the co-Poisson sums to fill up the full spaces of such square-integrable functions.

## 5 Impact of the Báez-Duarte, Balazard, Landreau and Saias theorem on the so-called Hilbert-Pólya idea

How could it be important to replace $\zeta(s)$ with $\zeta(1-s)$? Clearly only if we leave the critical line and start paying attention to the difference between the right half-plane $\mathrm{Re}(s)>1/2$ and the left half-plane $\mathrm{Re}(s)<1/2$. Equivalently if we switch from the full group of contractions-dilations $C_\lambda$, $0<\lambda<\infty$, which acts as $\phi(x)\mapsto\phi(x/\lambda)/\sqrt{|\lambda|}$ on even functions on $\mathbb{R}$, or as $Z(s)\mapsto\lambda^{s-1/2}Z(s)$ on their Mellin Transforms, to its sub-semi-group of contractions ($0<\lambda\leq 1$). The contractions act as isometries on the Hardy space $\mathbb{H}^2(\mathrm{Re}(s)>1/2)$ or equivalently on its inverse Mellin transform the space $L^2((0,1),dt)$.

It is a theme contemporaneous to Tate's Thesis and Weil's first paper on the explicit formula that it is possible to formulate the Riemann Hypothesis in such a semi-group set-up: this is due to Nyman [37] and Beurling [6] and builds on the Beurling [5] (and later for the half-plane) Lax [34] theory of invariant subspaces of the Hardy spaces. The criterion of Nyman reads as follows: the linear combinations of functions $t \mapsto \{1/t\} - \{a/t\}/a$, for $0 < a < 1$, are dense in $L^2((0,1), dt)$ if and only if the Riemann Hypothesis holds. It is easily seen that the smallest closed subspace containing these functions is stable under contractions, so to test the closure property it is only necessary to decide whether the constant function $\mathbf{1}$ on $(0,1)$ may be approximated. The connection with the zeta function is established through $(0 < \mathrm{Re}(s) < 1)$:

$$\int_0^\infty \{1/t\} t^{s-1}\, dt = -\frac{\zeta(s)}{s}$$

This gives for our functions, with $0 < a < 1$ and $0 < \mathrm{Re}(s)$:

$$\int_0^1 \left(\{1/t\} - \frac{1}{a}\{a/t\}\right) t^{s-1}\, dt = (a^{s-1} - 1)\frac{\zeta(s)}{s}$$

The question is whether the invariant (under contractions) subspace of $\mathbb{H}^2(\mathrm{Re}(s) > 1/2)$ of linear combinations of these Mellin transforms is dense or not. Each zero $\rho$ of $\zeta(s)$ in $\mathrm{Re}(s) > 1/2$ is an obvious obstruction as (the complex conjugate of) $t^{\rho-1}$ belongs to $L^2((0,1), dt)$. The Beurling-Lax theory describing the structure of invariant subspaces allows the conclusion in that case that there are no other obstructions (we showed [17] as an addendum that the norm of the orthogonal projection of $\mathbf{1}$ to the Nyman space is $\prod_{\mathrm{Re}(\rho)>1/2} |(1-\rho)/\rho|$, where the zeros are counted with their multiplicities). Recently ([2]) Luis Báez-Duarte has shown that the appeal to the Beurling-Lax invariant subspace theory could be completely avoided, and furthermore he has put the Nyman-Beurling criterion in the stronger form where one applies to the fractional part function only integer-ratio contractions.

One could think from our description of the original proof of the Nyman-Beurling criterion that the zeros *on the critical line* are out of its scope, as they don't seem to play any rôle. So it has been a very novel thing when Báez-Duarte, Balazard, Landreau and Saias asked the right question and provided a far from obvious answer [3]. First a minor variation is to replace the Nyman criterion with the question whether the function $\mathbf{1}_{0<t<1}$ can be approximated in $L^2(0,\infty)$ with linear combinations of the contractions of $\{1/t\}$. Let $D(\lambda)$ be the Hilbert space distance between $\mathbf{1}_{0<t<1}$ and linear combinations of contractions $C_\theta(\{1/t\})$, $\lambda \leq \theta \leq 1$. The Riemann Hypothesis holds if and only if $\lim D(\lambda) = 0$.

**Theorem of Báez-Duarte, Balazard, Landreau and Saias [3].** *One has the lower bound:*

$$\liminf |\log(\lambda)| D(\lambda)^2 \geq \sum_\rho \frac{1}{|\rho|^2}$$

*where the sum is over all non-trivial zeros of the zeta function, counted only once independently of their multiplicity.*

The authors of [3] conjecture that equality holds (also with lim in place of lim inf) when one counts the zeros with their multiplicities: our next result shows that their conjecture not only implies the Riemann Hypothesis but it also implies the simplicity of all the zeros:

**5.1 Theorem ([18]).** *The following lower bound holds:*

$$\liminf |\log(\lambda)| D(\lambda)^2 \geq \sum_\rho \frac{m_\rho^2}{|\rho|^2}$$

Our proof relies on the link we have established between the study originated by Báez-Duarte, Balazard, Landreau and Saias of the distance function $D(\lambda)$ and the so-called Hilbert-Pólya idea. This idea will be taken here in the somewhat vague acceptation that the zeros of $L$-functions may have a natural interpretation as Hilbert space vectors, eigenvectors for a certain self-adjoint operator. If we had such vectors in $L^2((0,\infty), dt)$, perpendicular to $\{1/t\}$ and to its contractions $C_\theta(\{1/t\})$, $\lambda \leq \theta \leq 1$ then we would be in position to obtain a lower bound for $D(\lambda)$ from the orthogonal projection of $\mathbf{1}_{0<t<1}$ to the space spanned by the vectors. This lower bound would be presumably easily expressed as a sum indexed by the zeros from the fact that eigenspaces of a self-adjoint operator are mutually perpendicular. The first candidates are $t \mapsto t^{-\rho}$: they satisfy formally the perpendicularity condition to $\{1/t\}$ and its contractions, but they do not belong to $L^2$. Nevertheless we could be in a position to approximately implement the idea if we used instead the square integrable vectors $t \mapsto t^{-(\rho-\varepsilon)}\mathbf{1}_{0<t<1}$, $\varepsilon > 0$, $\mathrm{Re}(\rho) = 1/2$. The authors of [3] followed more or less this strategy, but as they did not benefit from exact perpendicularity, they had to provide not so easily obtained estimates. It appears that the $\varepsilon > 0$ does not seem to allow to take easily into account the multiplicities of the zeros. For their technical estimates the authors of [3] used to great advantage a certain scale invariant operator $U$, which had been introduced by Báez-Duarte in an earlier paper [1] discussing the Nyman-Beurling problem.

How is it possible that the use of this Báez-Duarte operator $U$ (whose definition only relies on some ideas of harmonic analysis, and some useful integral formulae, with at first sight no arithmetic involved) allows Báez-Duarte, Balazard, Landreau and Saias to make progress on the Nyman-Beurling criterion? We related the mechanism underlying the insightful Báez-Duarte construction [1] of the operator $U$ to a construction quite natural in scattering theory and it appeared then that it was possible to use it (or a variant $V$) to construct true Hilbert space vectors $Y_{s,k}^\lambda$ indexed by $s$ on the critical line, and $k \in \mathbb{N}$, having the property of expressing the values of the Riemann zeta function and its derivatives on the critical line as Hilbert space scalar products:

$$(A, Y_{s,k}^\lambda) = \left(-\frac{d}{ds}\right)^k \frac{s-1}{s} \frac{\zeta(s)}{s}$$

The additional factors are such that $\frac{s-1}{s}\frac{\zeta(s)}{s}$ belongs to the Hardy space of the right half-plane and $A(t)$ is its inverse Mellin transform, an element of $L^2((0,1),dt)$. What is more, we can replace $A$ with its contractions $C_\theta(A)$ as long as $\lambda \le \theta \le 1$:

$$\lambda \le \theta \le 1 \Rightarrow (C_\theta(A), Y^\lambda_{s,k}) = \left(-\frac{d}{ds}\right)^k \theta^{s-1/2}\frac{s-1}{s}\frac{\zeta(s)}{s}$$

So the only ones among the $Y^\lambda_{s,k}$ which are (exactly, not approximately) perpendicular to $A$ and its contractions up to $\lambda$ are the $Y^\lambda_{\rho,k}$, $\zeta(\rho) = 0$, $k < m_\rho$ ($\lambda < 1$). This connects the Nyman-Beurling criterion with the so-called Hilbert-Pólya idea. From the explicit integral formulae of Báez-Duarte for his operator $U$ one can write formulae for the vectors $Y^\lambda_{s,k}$ from which the following asymptotic behavior emerges:

$$\lim_{\lambda\to 0} |\log(\lambda)|^{-1-k-l} \cdot (Y^\lambda_{s_1,k}, Y^\lambda_{s_2,l}) = 0 \quad (s_1 \ne s_2)$$

$$\lim_{\lambda\to 0} |\log(\lambda)|^{-1-k-l} \cdot (Y^\lambda_{s,k}, Y^\lambda_{s,l}) = \frac{1}{k+l+1}$$

In particular the rescaled vectors $X^\lambda_{s,0} = Y^\lambda_{s,0}/\sqrt{|\log(\lambda)|}$ become orthonormal in the limit when $\lambda \to 0$. The limit can not work inside $L^2((0,\infty))$ because this is a separable space! In fact one shows without difficulty that the vectors $X^\lambda_{s,k}$ weakly converge to 0 as $\lambda \to 0$. The theorem 5.1 is easily deduced from the above estimates.

We have not explained yet what is $U$ and how the $Y^\lambda_{s,k}$ are constructed with the help of it. The Báez-Duarte operator is the unique scale invariant operator with sends $\{1/t\}$ to its image under the *time reversal* $J : \varphi(t) \mapsto \overline{\varphi(1/t)/t}$, which is here the function $\{t\}/t$. Equivalently at the level of the Mellin Transforms, $U$ acts as a multiplicator with multiplier on the critical line

$$U(s) = \frac{\overline{\zeta(s)/s}}{\zeta(s)/s} = \frac{\zeta(1-s)}{\zeta(s)}\frac{s}{1-s}$$

So this construction is extremely general: as soon as the function $A \in L^2((0,\infty),dt)$ has an almost everywhere non-vanishing Mellin transform $Z(s)$ on the critical line (which by a theorem of Wiener is equivalent to the fact that the multiplicative translates $C_\theta(A)$, $0 < \theta < \infty$, span $L^2$) then we may associate to it the scale invariant operator $V$ with acts as $\overline{Z(s)}/Z(s)$, and sends $A$ to its time reversal $J(A)$. We see that $V$ is necessarily unitary. Let us in particular suppose that $A$ is in $L^2((0,1),dt)$: then $J(A)$ has support in $[1,\infty)$ and the images under $V$ of the contractions $C_\theta(A)$, $\lambda \le \theta \le 1$, being contractions of $J(A)$, will be in $L^2((\lambda,\infty),dt)$. In the case at hand we have:

$$Z(s) = \frac{s-1}{s}\frac{\zeta(s)}{s} \qquad V(s) = \frac{\zeta(1-s)}{\zeta(s)}\left(\frac{s}{1-s}\right)^3$$

The Báez-Duarte operator $U$ and its cousin $V$ depend on $\zeta(s)$ only through its functional equation, which means that they are associated with the (even) Fourier transform $\mathscr{F}_+$ (the cosine transform). We can use (almost) the same operators for Dirichlet $L$-series with an even character (with due attention paid to the conductors $q > 1$), and there are other operators we would use for odd characters, associated with the sine transform $\mathscr{F}_-$.

The vectors $Y^\lambda_{s,k}$ are obtained as follows: we start from $|\log(t)|^k t^{-(s-\varepsilon)}\mathbf{1}_{0<t<1}$, apply $V$, restrict to $[\lambda, +\infty)$, take the limit which now exists in $L^2$ as $\varepsilon \to 0$, and apply $V^{-1}$. What happens is that $V(t^{-1/2-i\tau}\mathbf{1}_{0<t<1})$ does not belong to $L^2$ but this is entirely due to its singularity at 0, which, it turns out, is $V(1/2+i\tau)t^{-1/2-i\tau}$. We would not expect it to be possible that a localized singularity would remain localized after the action of $\mathscr{F}_+$ but the point is that the operator with multiplier $\zeta(1-s)/\zeta(s)$ is the composite $\mathscr{F}_+ \cdot I$ with $I(\phi)(x) = \phi(1/x)/|x|$. So the singularity is first sent to infinity and $\mathscr{F}_+$ puts it back at the origin.

**5.2 Note.** Let us denote by $L$ the scale invariant unitary operator with spectral function $L(s) = s/(s-1)$. One has $L = 1 - M$ where $M$ is the Hardy averaging $\phi(t) \mapsto (\int_0^t \phi(u)\,du)/t$. The operator $V$ is $(-L)^3\mathscr{F}_+ I = \mathscr{F}_+ I(-L)^3$. One has $ILI = L^{-1}$ and $LL^* = 1$. The operator $L$ is "real", meaning that it commutes with the anti-unitary complex conjugation $\phi(t) \mapsto \overline{\phi(t)}$. One has $IV = I\mathscr{F}_+ I(-L)^3$. We will write $\mathscr{G}_+ = I\mathscr{F}_+ I$, so that $IV = \mathscr{G}_+(-L)^3 = (-L)^{-3}\mathscr{G}_+$. The operator $\mathscr{G}_+$ is unitary and satisfies $\mathscr{G}_+^2 = 1$. The operator $\mathscr{G}_+$ is real. The $U$ operator of Báez-Duarte is $\mathscr{F}_+ I(-L)$, so $V = UL^2 = L^2 U$. The operators $I$, $\mathscr{F}_+$, $\mathscr{G}_+$, $U$ and $V$ are real.

We now proceed with a more detailed study of the vectors $Y^\lambda_{s,k}$, and of their use to express values of Mellin transforms and their derivatives on and off the critical line as Hilbert space scalar products. We established

$$(B, Y^\lambda_{s,k}) = \left(-\frac{d}{ds}\right)^k \hat{B}(s)$$

for the Hardy functions $C_\theta(A)$, $\lambda \le \theta \le 1$. Their Mellin Transforms $\int_0^\infty C_\theta(A)(t)t^{s-1}\,dt = \theta^{s-1/2}\frac{s-1}{s}\frac{\zeta(s)}{s}$ are analytic in the entire complex plane except for a double pole at $s = 0$. The vectors $Y^\lambda_{s,k}$ are the analytic continuation to $\mathrm{Re}(s) = 1/2$ of vectors with the same definition $Y^\lambda_{w,k}$, $\mathrm{Re}(w) < 1/2$. The above equation has thus its right hand side analytic in $s$ but its left-hand side seemingly anti-analytic, as our scalar product is linear in its first factor and conjugate linear in its second factor. So we will use rather the *euclidean bilinear form* $[B, C] = \int_0^\infty B(t)C(t)\,dt$. The spaces we consider are stable under complex conjugation $B(t) \mapsto \overline{B(t)}$, and the operators we use are real, so statements of perpendicularity may equivalently be stated using either $[\cdot,\cdot]$ or $(\cdot,\cdot)$. The identity can then be restated for all finite linear combinations $B$ of our $C_\theta(A)$'s, $\lambda \le \theta \le 1$, as

$$[B, Y^\lambda_{w,k}] = \left(+\frac{d}{dw}\right)^k \int_0^1 t^{-w}B(t)\,dt = \left(\frac{d}{dw}\right)^k (\hat{B}(1-w))$$

for $\mathrm{Re}(w) \leq 1/2$. If we look at the proof of the main theorem in [18] we see that the only thing that matters about $B$ is that it should be supported in $[0,1]$ and that $V(B)$ should be supported in $[\lambda, \infty)$, equivalently that $(IV)(B)(t)$ has support in $[0, \Lambda]$, $\Lambda = 1/\lambda$. Let us note the following:

**5.3 Theorem.** *The real unitary operator $IV$ satisfies* $(IV)^2 = 1$.

**5.4 Note.** In particular $IV$ is what Báez-Duarte calls "a skew-root" [1].

*Proof.* This is clear from the spectral representation

$$V(s) = \frac{\zeta(1-s)}{\zeta(s)} \left(\frac{s}{1-s}\right)^3$$

which shows that $IVI = V^*$. □

We let $\Lambda = 1/\lambda$ and $\mathcal{G}_\Lambda = IVC_\Lambda = C_\lambda IV$. We note that $(\mathcal{G}_\Lambda)^2 = 1$. We also note $V\mathcal{G}_\Lambda = C_\lambda VIV = C_\lambda I$. Let $M_\Lambda = \mathbb{H}^2 \cap \mathcal{G}_\Lambda(\mathbb{H}^2)$, where we use the notation $\mathbb{H}^2 = L^2((0,1), dt)$. Obviously $\mathcal{G}_\Lambda(M_\Lambda) = M_\Lambda$. The function $A$ as well as its contractions $C_\theta(A)$, $\lambda \leq \theta \leq 1$ belong to $M_\Lambda$. Indeed $\mathcal{G}_\Lambda(A) = C_\lambda(A)$. We note that the Mellin transform $\int_0^1 B(t)t^{w-1}\, dt$ of $B \in M_\Lambda$ is analytic at least in $\mathrm{Re}(w) > 1/2$. Let $Q_\lambda$ be the orthogonal projection to $L^2(\lambda, \infty)$.

**5.5 Theorem.** *The vectors $Y^\lambda_{w,k}$, originally defined for* $\mathrm{Re}(w) < 1/2$ *as*

$$V^{-1} Q_\lambda V(|\log(t)|^k t^{-w} \mathbf{1}_{0<t<1})$$

*have (inside $L^2$) an analytic continuation in $w$ to the entire complex plane $\mathbb{C}$ except at $w = 1$. The Mellin Transform of $B \in M_\Lambda$ has an analytic continuation to $\mathbb{C} \backslash \{0\}$, with at most a pole of order 2 at $w = 0$. One has for $w \neq 1$ and $k \in \mathbb{N}$:*

$$[B, Y^\lambda_{w,k}] = \left(\frac{d}{dw}\right)^k (\hat{B}(1-w))$$

*The following functional equation holds:*

$$\widehat{\mathcal{G}_\Lambda(B)}(w) = \lambda^{w-1/2} V(1-w)\hat{B}(1-w)$$

*One has*

$$\forall B \in M_\Lambda \quad \hat{B}(-2) = \hat{B}(-4) = \cdots = 0$$

*Proof.* We leave the details of the case $k > 0$ to the reader. Let first $\mathrm{Re}(w) > 1/2$. We have for $B \in M_\Lambda$:

$$\hat{B}(w) = \int_0^1 B(t)t^{w-1}\, dt = [B, t^{w-1}\mathbf{1}_{0<t<1}] = [V(B), V(t^{w-1}\mathbf{1}_{0<t<1})]$$

Writing $B = \mathscr{G}_\Lambda(C)$, with $C \in \mathbb{H}^2$, we get $V(B) = C_\lambda I(C)$. So $V(B)$ has its support in $[\lambda, \infty)$ and:

$$\hat{B}(w) = \int_\lambda^\infty V(B)(u)\, V(t^{w-1}\mathbf{1}_{0<t<1})(u)\, du$$

We will show that $V(t^{w-1}\mathbf{1}_{0<t<1})(u)$ is analytic, for fixed $u$, in $w \in \mathbb{C}\setminus\{0\}$ and that it is $O((1+|\log(u)|)/u)$ on $[\lambda, \infty)$, uniformly when $w$ is in a compact subset of $\mathbb{C}\setminus\{0\}$ (this is one logarithm better than the estimate in [18] for $\mathrm{Re}(1-w) < 1$). We will thus have obtained the analytic continuation of the vectors $Y^\lambda_{1-w,0}$ from $\mathrm{Re}(w) > 1/2$ and at the same time the analytic continuation of $\hat{B}(w)$ as well as the formula:

$$w \neq 0 \Rightarrow [B, Y^\lambda_{1-w,0}] = \hat{B}(w)$$

So the problem is to study the analytic continuation of $V(t^{-z}\mathbf{1}_{0<t<1})(u)$ from $\mathrm{Re}(z) < 1/2$. If we followed the method of [18], we would write $V = (1-M)^2 U$, compute some explicit formula for $U(t^{-z}\mathbf{1}_{0<t<1})(u)$ and work with it. This works fine for the continuation to $\mathrm{Re}(z) < 1$, but for $\mathrm{Re}(z) \geq 1$ there is a problem with applying $M$ (which we must do before $Q_\lambda$) as the singularity at 0 is of the kind $u^{-z}$ and is not integrable anymore. So we apply first $L^2 = (1-M)^2$ and only later $U$.

We compute:

$$M(t^{-z}\mathbf{1}_{0<t<1})(u) = \frac{\int_0^{\min(1,u)} t^{-z}\, dt}{u} = \frac{u^{-z}\mathbf{1}_{u\leq 1}(u)}{1-z} + \frac{1}{1-z}\,\frac{\mathbf{1}_{u>1}(u)}{u}$$

$$M^2(t^{-z}\mathbf{1}_{0<t<1})(u) = \frac{u^{-z}\mathbf{1}_{u\leq 1}(u)}{(1-z)^2} + \frac{1}{(1-z)^2}\,\frac{\mathbf{1}_{u>1}(u)}{u} + \frac{1}{1-z}\,\frac{\log(u)\mathbf{1}_{u>1}(u)}{u}$$

$$L^2(t^{-z}\mathbf{1}_{0<t<1})(u) = \frac{z^2}{(z-1)^2}u^{-z}\mathbf{1}_{u\leq 1} + \left(\frac{z^2}{(z-1)^2} - 1\right)\frac{\mathbf{1}_{u>1}(u)}{u} + \frac{1}{1-z}\,\frac{\log(u)\mathbf{1}_{u>1}(u)}{u}$$

We note that $U(\mathbf{1}_{u>1}/u) = UI(\mathbf{1}_{u<1}) = (M-1)\mathscr{F}_+(\mathbf{1}_{u<1}) = (M-1)(\sin(2\pi u)/(\pi u))$ is $O(1/u)$ (from the existence of the Dirichlet integral) for $u > \lambda$ and then that $U(\log(u)\mathbf{1}_{u>1}/u) = UMI(\mathbf{1}_{u<1}) = MUI(\mathbf{1}_{u<1}) = M(M-1)(\sin(2\pi u)/(\pi u)$ is $O((1+|\log(u)|)/u)$. Clearly this reduces the problem of $V(t^{-z}\mathbf{1}_{0<t<1})(u)$ to the problem of the analytic continuation and estimation of $U(t^{-z}\mathbf{1}_{0<t<1})(u)$. From [3], proof of Lemme 6, one has

$$U(t^{-z}\mathbf{1}_{0<t<1})(u) = \frac{\sin(2\pi u)}{\pi u} + \frac{z}{\pi u}\int_1^\infty t^{z-1}\sin(2\pi ut)\frac{dt}{t}$$

and (for example) from [19] we know that the integral is an entire function of $z$ which is $O(1/u)$ on $[\lambda, \infty)$, uniformly in $z$ when $|z|$ is bounded. We also see from this and from the integral representation of $\hat{B}(w)$ that it has at most a pole of order 2 at $w = 0$ (which is $z = 1$).

The functional equation holds on the critical line from the spectral representation of $\mathcal{G}_\Lambda = C_\lambda IV$, hence it holds on $\mathbb{C}$ by analytic continuation. As $V(1-w)$ has poles at $1 - w = -2, -4, \ldots$, and the left hand side is regular at these values of $w$ it follows that $\hat{B}(1-w)$ has to vanish for $1 - w = -2, -4, \ldots$. □

**5.6 Note.** The distance function $D(\lambda)^2$ has two components: one corresponding to the distance to the subspace $M_\Lambda$ in $\mathbb{H}^2$ and then another one corresponding to the additional distance inside this space to the translates $C_\theta(A)$, $\lambda \le \theta \le 1$. The first step has absolutely no arithmetic, it is a problem of analysis. In the second step the orthogonal projections to $M_\Lambda$ of the vectors $Y^\lambda_{\rho,k}$, $\zeta(\rho) = 0$, $k < m_\rho$ are obstructions. When $\lambda \to 0$ $(\Lambda \to \infty)$ the first contribution is presumably much smaller than the second, and the original vectors $Y^\lambda_{\rho,k}$ will not themselves differ much from their orthogonal projections to $M_\Lambda$. This seems to suggest as a plausible thing that the estimate $(\sum_\rho m_\rho^2/|\rho|^2)/|\log(\lambda)|$ gives the exact asymptotic decrease of $D(\lambda)^2$ (under assumption of the Riemann Hypothesis).

## 6 Sonine spaces of de Branges, novel spaces $HP_\lambda$, vectors $Z^\lambda_{\rho,k}$, Krein string of the zeta function

Let $K$ be the Hilbert space $L^2((0,\infty), dt)$ of complex-valued square-integrable functions on $(0,\infty)$ with Hilbertian scalar product $(f,g) = \int_0^\infty f(t)\overline{g(t)}\, dt$. We also use the "Euclid" bilinear form $[f,g] = \int_0^\infty f(t)g(t)\, dt$. A vector $Z(t)$ is "Euclid-perpendicular" to a subspace $H$ for the bilinear form $[f,g]$ if only and if $\overline{Z(t)}$ is ("Hilbert")-perpendicular to $H$ for the scalar product $(f,g)$ if and only if $Z(t)$ is Hilbert-perpendicular to the complex-conjugated space $\overline{H}$. We also consider the functions in $K$ as even functions with the definition $f(t) = f(|t|)$ for $t < 0$.

The Mellin transform (which is taken in the $L^2$-sense for $\mathrm{Re}(s) = \frac{1}{2}$)

$$f(t) \mapsto \hat{f}(s) = \int_0^\infty f(t) t^{-s}\, dt$$

isometrically identifies $K$ with the Hilbert space $L^2\big(s = \frac{1}{2} + i\tau, d\tau/2\pi\big)$. The cosine transform $\mathcal{F}_+$ acts (in the $L^2$ sense) on $K$ as $\mathcal{F}_+(f)(t) = 2\int_0^\infty \cos(2\pi t u) f(u)\, du$. It is a real operator. One has $\mathcal{F}_+^2 = 1$, so $K$ is the orthogonal sum of the subspaces of invariant functions ("self-reciprocal") under $\mathcal{F}_+$ and the subspaces of anti-invariant ("skew-reciprocal") functions. The operator $I$ is $f(t) \mapsto f(1/t)/|t|$. The composite $\Gamma_+ = \mathcal{F}_+ I$ is scale invariant so it is diagonalized by the Mellin transform: $\widehat{\Gamma_+(f)}(s) = \chi_+(s)\hat{f}(s)$. This is also written as

$$\widehat{\mathcal{F}_+(f)}(s) = \chi_+(s)\hat{f}(1-s)$$

The function $\chi_+(s)$ is a meromorphic function in the complex plane which is related to the Tate Gamma function $\gamma_+(s)$ through $\chi_+(s) = \gamma_+(1-s) = \gamma_+(s)^{-1}$. One has:

$$\chi_+(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)} = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s) = \frac{\zeta(s)}{\zeta(1-s)}$$

So an even function is self-reciprocal under the cosine transform if and only if its *right* Mellin transform satisfies the zeta-functional equation. Under the *left* Mellin transform $f(t) \mapsto \int_0^\infty f(t)t^{s-1}\,dt$, which is the one usually used in discussing the functional equation, self-reciprocal functions under the cosine transform satisfy the functional equation of $\zeta(1-s)$. The Müntz formula [45, II.11] shows that, for suitably regular functions $f(t)$, the scale-invariant operator corresponding to $\zeta(s)$ when using the *left* Mellin transform is given explicitely as a modified Poisson summation. So the action of the scale invariant operator corresponding to $\zeta(s)$ under the *right* Mellin transform is expressed as the co-Poisson summation (when applied to suitably regular functions; a more detailed analysis will be given in [23]).

The intertwining property for co-Poisson summations ($t > 0$):

$$\mathscr{F}_+\left(\sum_{n\geq 1}\frac{\alpha(t/n)}{n} - \int_0^\infty \frac{\alpha(1/t)}{t}\,dt\right) = \sum_{n\geq 1}\frac{\alpha(n/t)}{t} - \int_0^\infty \alpha(t)\,dt$$

is an equivalent expression of the zeta-functional equation. It shows how to give examples of even functions $f(t)$ which vanish identically in a neighborhood $(-\lambda, \lambda)$ of the origin and such that their Fourier cosine transform has the same property. For this we take $\alpha(t)$, smooth with support in $[\lambda, \Lambda]$ $(\Lambda = 1/\lambda)$ and such that $\int_0^\infty \alpha(t)\,dt = 0 = \int_0^\infty (\alpha(1/t)/t)\,dt$. A non zero function $f(t)$ may be obtained this way only for $0 < \lambda < 1$.

Nevertheless there exists for arbitrary $\lambda > 0$ non-zero square integrable even functions $f(t)$ vanishing in $(0, \lambda)$, and such that $\mathscr{F}_+(f)(t)$ also vanishes in $(0, \lambda)$. To the best of the author's knowledge this was first put forward as a fact of special importance in Analysis by de Branges in [7]. There, a beautiful isometric expansion of self- and skew-reciprocal functions for the Hankel transform of zeroth order is proven. The cosine transform is (essentially) the Hankel transform of order $-\frac{1}{2}$. The sine transform is (essentially) the Hankel transform of order $+\frac{1}{2}$.

**6.1 Definition.** We let $K_\lambda \subset K$ be the Hilbert space of square-integrable (even) functions $f(t)$ vanishing in $(0, \lambda)$, and such that $\mathscr{F}_+(f)(t)$ also vanishes in $(0, \lambda)$.

An explicit example of a function having this property, but which is not square-integrable, arises from an integral formula of Sonine concerning Bessel functions [42, p. 38]. This example is in Titchmarsh's book on Fourier integrals [44, 9.12.(8)]. The analogous example which is associated to the sine transform is a square-integrable (odd) function. The proof given by de Branges for existence of an even square-integrable $f(t)$ with the Sonine property appears in [7] on top of page 449. He constructs directly its Mellin transform from a trick which makes use of the already known non-triviality of the spaces with the Sonine property for the Hankel transform of positive order. Here is another trick: we take a non-zero square-integrable odd function $g(t)$ which works for the sine transform but with $\lambda' = \lambda + 1$. Let $f(t) = g(t-1) - g(t+1)$. Then $f(t)$ is even, and non-trivial. It vanishes on $(-\lambda, +\lambda)$ and its Fourier (cosine) transform vanishes on $(-\lambda', +\lambda')$.

Actually the simplest method leading to explicit examples of Sonine functions (in the Schwartz class), with arbitrarily large $\lambda$, was communicated to the author by Professor Kahane[1]: the first observation is that it is enough to regularize a tempered *distribution* having the Sonine property to a Schwartz function: additive convolution with a test-function supported in $(-\varepsilon, +\varepsilon)$, and multiplication with its Fourier (this replaces $\lambda$ with $\lambda - \varepsilon$). The second is that it is easy to obtain such distributions from the Poisson distribution $\sum_{n \in \mathbb{Z}} \delta(x-n)$. As an example of how to proceed one may take

$$x^3 \prod_{1 \le j \le N} (x^2 - j^2)^2 \sum_{n \in \mathbb{Z}} \delta'(x-n)$$

and replace $x$ by $\sqrt{N}x$. This gives an even tempered distribution with the Sonine property for $\lambda < \sqrt{N}$. In [23] we use the *multiplicative convolution* to regularize such distributions and thus obtain more general co-Poisson intertwining formulae.

An existence proof of Sonine square-integrable functions is straightforward: it suffices to say that $L^2(0,\lambda) + \mathscr{F}_+(L^2(0,\lambda))$ is a closed (obviously proper) subspace of $L^2(0,\infty)$. The (thus non trivial) perpendicular complement is the space $K_\lambda$. That the space sum is closed follows readily (see [28, sect 2.9, p. 126–127]) from the fact that the compact operator $P_\lambda \mathscr{F}_+ P_\lambda$ (where $P_\lambda$ is orthogonal projection to $L^2(0,\lambda)$) has operator bound strictly less than one (as no function can be compactly supported and with its Fourier compactly supported). The next step is to actually write down explicitely the associated orthogonal projection. This has led the author recently to advances in the theory of the de Branges Sonine spaces ([22]).

De Branges proves that for each $f \in K_\lambda$ its completed (right) Mellin Transform

$$M(f)(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\int_\lambda^\infty f(t)t^{-s}\,dt$$

is an entire function. The proof (up to a change of variable) appears on page 447 of [7]. We gave in [19] another, more elementary, proof. The space of entire functions thus associated to $K_\lambda$ is among the "Sonine spaces" from [8], also studied in [39, 40, 41]. The Sonine spaces are Hilbert spaces of entire functions satisfying the axioms of [8]. We will also call $K_\lambda \subset L^2(0,\infty)$ a Sonine space.

[1] Letter to the author, March 22, 2002

**6.2 Note.** In [8] as well as in other cited references it is the horizontal axis which is the axis of symmetry. Comparison with our conventions requires a change of variable (such as $s = \frac{1}{2} - 2iz$), as it is the critical line which we use as the axis of symmetry for the Hilbert spaces of entire functions.

So $K_\lambda = L^2((\lambda, \infty), dt) \cap \mathscr{F}_+(L^2((\lambda, \infty), dt))$. The convention in force in this chapter will be to use the *right* Mellin transform:

$$\hat{f}(s) = \int_0^\infty f(t) t^{-s}\, dt$$

**6.3 Theorem.** *The spaces $K_\lambda$ are all non-reduced to $\{0\}$. The Mellin transforms of elements $f(t)$ from $K_\lambda$ are entire functions with trivial zeros at $s = -2n$, $n \in \mathbb{N}$. The entire functions*

$$M(f)(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\hat{f}(s)$$

*satisfy the functional equations*

$$M(\mathscr{F}_+(f))(s) = M(f)(1-s)$$

*For each $w \in \mathbb{C}$, each $k \in \mathbb{N}$, the linear forms $f \mapsto M(f)^{(k)}(w)$ are continuous and correspond to (unique) vectors $Z^\lambda_{w,k} \in K_\lambda$: $\forall f \in K_\lambda\ [f, Z^\lambda_{w,k}] = M(f)^{(k)}(w)$.*

*Proof.* As we said, most of this is, up to a change of variable, from [7]. The $L^2$-boundedness of the evaluations of the derivatives ($k \geq 1$) follow by the Banach-Steinhaus theorem from the case $k = 0$. We provide elementary proofs of all statements in [19]. □

**6.4 Note.** Evaluators such as $Z^\lambda_{w,k}$ for $k \geq 1$, which are associated to derivatives, do not seem to have been put to use so far either in the general theory [8], or in the special theory of Sonine spaces [7, 10, 11].

**6.5 Note.** We have changed our conventions from [19] where we were studying the functions in $L^2(0,\Lambda) \cap I\mathscr{F}_+ I(L^2(0,\Lambda)) = I(K_\lambda)$ using left Mellin transforms. Here we study the functions from $K_\lambda$ using the right Mellin transforms. So we deal with exactly the same entire functions in the complex plane.

**6.6 Proposition.** *One has $K_\lambda = \bigcap_{\mu<\lambda} K_\mu$ and $K_\lambda = \overline{\bigcup_{\mu>\lambda} K_\mu}$. Furthermore $L^2((0,\infty), dt) = \overline{\bigcup_{\lambda>0} K_\lambda}$.*

*Proof.* Directly from the definition, the $K_\lambda$'s form a decreasing chain as $\lambda \to \infty$ and the first statement holds. Let $\phi$ be perpendicular to each $K_\mu$ for $\mu > \lambda$. Then $\phi \in L^2(0,\mu) + \mathscr{F}_+ L^2(0,\mu)$ and the decomposition as $f^\mu + g^\mu$ is unique. The entire function $g^\mu$ must then not depend on $\mu > \lambda$, as the difference between two such will be compactly supported, hence zero. Then $f^\mu$ does not depend on $\mu$ either and is in $L^2(0,\lambda)$ and $g^\mu \in \mathscr{F}_+ L^2(0,\lambda)$. So $K_\lambda = \overline{\bigcup_{\mu>\lambda} K_\mu}$. The same proof shows the last statement. □

**6.7 Proposition.** *One has* $\mathscr{F}_+(Z^\lambda_{w,k}) = (-1)^k Z^\lambda_{1-w,k}$.

*Proof.* Using the Euclid bilinear form: $[\mathscr{F}_+(Z^\lambda_{w,k}), f] = [Z^\lambda_{w,k}, \mathscr{F}_+(f)] = M(\mathscr{F}_+(f))^{(k)}(w) = (-1)^k M(f)^{(k)}(1-w)$ from $M(\mathscr{F}_+(f))(w) = M(f)(1-w)$. □

We use the bilinear form $[f, Z^\lambda_{w,k}]$ so that the vectors $Z^\lambda_{w,k}$ depend analytically on $w$. Evaluators ($k=0$) associed to points off the symmetry axis are always non-zero vectors in de Branges spaces. Using our elementary techniques we proved a stronger statement in the case at hand:

**6.8 Theorem ([19]).** *Any finite collection of vectors* $Z^\lambda_{w,k}$ *is a linearly independent system. In particular the vectors* $Z^\lambda_{w,k}$ *are all non-vanishing.*

**6.9 Note.** If we take an arbitrary sequence of distinct complex numbers having an accumulation point the corresponding evaluators $Z^\lambda_{w,0}$ will span $K_\lambda$. Remarkable orthogonal bases consisting of evaluators $Z^\lambda_{w,0}$ exist as a general fact from [8]. Some other non-trivial examples of infinite and minimal collection of evaluators are also known [21].

It is useful to have at our disposal "augmented Sonine" spaces $L_\lambda \supset K_\lambda$, whose elements' (Gamma-completed) Mellin Transforms may have poles at 0 and 1. Let $N$ be the unitary invariant operator which, under the right Mellin transform, has spectral function $s/(s-1)$. Explicitely:

$$N(f)(t) = f(t) - \int_t^\infty \frac{f(u)}{u}\,du$$

Let $L_\lambda = N \cdot L^2((\lambda,\infty), dt) \cap \mathscr{F}_+ \cdot N \cdot L^2((\lambda,\infty), dt)$.

**6.10 Theorem.** *Let* $\lambda > 0$. *Let* $f \in L_\lambda$. *The Mellin transform* $\hat{f}(s) = \int_0^\infty f(t)t^{-s}\,dt$ *is an analytic function in* $\mathbb{C}\backslash\{1\}$ *with at most a pole of order* 1 *at* $s=1$. *It has trivial zeros at* $s=-2n$, $n \geq 1$. *The function* $M(f)(s) = \pi^{-s/2}\Gamma(s/2)\hat{f}(s)$, *analytic in* $\mathbb{C}\backslash\{0,1\}$, *satisfies the functional equation*

$$M(\mathscr{F}_+(f))(s) = M(f)(1-s)$$

*For each* $w \in \mathbb{C}\backslash\{0,1\}$, *each* $k \in \mathbb{N}$, *the linear forms* $f \mapsto M(f)^{(k)}(w)$ *are continuous.*

*Proof.* As an intersection $L_\lambda$ is a closed subspace of $L^2((0,\infty), dt)$ hence a Hilbert space. The square integrable function $f(t)$ is a constant $\alpha(f)$ for $0 < t < \lambda$ (which is a continuous linear form in $f$). So $\int_0^\infty f(t)t^{-s}\,dt$ is absolutely convergent and analytic at least for $1/2 < \mathrm{Re}(s) < 1$. In this strip we may write it as:

$$\int_0^\infty f(t)t^{-s}\,dt = \frac{\alpha(f)\lambda^{1-s}}{1-s} + \int_\lambda^\infty f(t)t^{-s}\,dt$$

which gives its analytic continuation to the right half-plane $\mathrm{Re}(s) > 1/2$ with at most a pole at $s = 1$. Let us also note that the evaluation at these points are clearly continuous for the Hilbert structure. We have:

$$\int_\lambda^\infty f(t)t^{-s}\,dt = \int_0^\infty \mathscr{F}_+(f)(u)\mathscr{F}_+(\mathbf{1}_{t>\lambda}t^{-s})(u)\,du$$

We known from [19, Lemme 1.3] that the function $\mathscr{F}_+(\mathbf{1}_{t>\lambda}t^{-s})(u)$ is an entire function of $s$, which is (uniformly for $|s|$ bounded) $O(1/u)$ on $(\lambda, \infty)$, and also that it is $\chi_+(s)u^{s-1} + O(1)$ on $(0, \lambda)$ (uniformly for $\mathrm{Re}(s) \leq 1 - \varepsilon < 1$). Moreover $\mathscr{F}_+(f)(u)$ is a constant in the interval $(0, \lambda)$ (from $f \in L_\lambda$). Combining these informations we get that the above displayed equation has an analytic continuation to the critical strip $0 < \mathrm{Re}(s) < 1$. In this critical strip we have the functional equation:

$$\hat{f}(s) = \chi_+(s)\widehat{\mathscr{F}_+(f)}(1-s)$$

as it holds on the critical line. From this we get the analytic continuation of $\hat{f}(s)$ to $\mathrm{Re}(s) < 1$. We note that $\chi_+(s)$ vanishes at $s = 0$ and that this counterbalances the (possible) pole of $\widehat{\mathscr{F}_+(f)}(1-s)$. Also this functional equation shows that $\hat{f}(s)$ vanishes at $s = -2n$, $n \geq 1$. The evaluations at points strictly to the right of the critical line are continuous, hence also at points to the left, hence everywhere (except of course at $s = 0$, $s = 1$, where instead one may consider the residues) from the Banach-Steinhaus theorem. □

There are (for $w \neq 0, 1$) evaluators $W_{w,k}^\lambda \in L_\lambda$ which project orthogonally to the evaluators $Z_{w,k}^\lambda \in K_\lambda$. The augmented Sonine spaces $L_\lambda \supset K_\lambda$ are natural for discussing properties of the zeta-function along the lines involving the co-Poisson formula. Here we will stay in the realm of the spaces $K_\lambda$, using the spaces $L_\lambda$ as an auxiliary help.

Let $0 < \lambda < 1$ and let $\Lambda = 1/\lambda$. Let $\alpha(t)$ be a smooth function with support in $[\lambda, \Lambda]$ and let $f(t)$ be the co-Poisson summation $\sum_{n\geq 1}\alpha(t/n)/n - \int_\lambda^\Lambda \alpha(t)\,dt/t$. The function $f(t)$ is a Schwartz function, hence square-integrable. From the co-Poisson formula it belongs to $L_\lambda$. If we impose the conditions that $\hat{\alpha}(0) = 0 = \hat{\alpha}(1)$ then $f(t)$ belongs to the Sonine space $K_\lambda$. At the level of Mellin transform, we have $\hat{f}(s) = \zeta(s)\hat{\alpha}(s)$. So $f(t)$ is (Euclid)-perpendicular to the evaluators $Z_{\rho,k}^\lambda$, $k < m_\rho$ associated with the non-trivial zeros of the Riemann zeta function and with their (eventual) multiplicities. And conversely it follows ([19]) from $\hat{f}(s) = \zeta(s)\hat{\alpha}(s)$ that an evaluator $Z_{w,k}^\lambda$ is (Euclid or Hilbert) perpendicular to all functions $\sum_{n\geq 1}\alpha(t/n)/n$ with $\alpha$ smooth function with support in $[\lambda, \Lambda]$ and $\hat{\alpha}(0) = 0 = \hat{\alpha}(1)$ if and only if $w$ is a non-trivial zero of the zeta function with multiplicity strictly bigger than $k$.

**6.11 Definition.** Let $0 < \lambda < 1$ and $\Lambda = 1/\lambda$. We let $W_\lambda$ be the closure in $K$ of the functions $\sum_{n\geq 1} \alpha(t/n)/n$, with $\alpha(t)$ *smooth* with support in $[\lambda, \Lambda]$, and such that $\hat{\alpha}(0) = 0 = \hat{\alpha}(1)$.

**6.12 Definition.** Let $0 < \lambda < 1$ and $\Lambda = 1/\lambda$. We let $W'_\lambda$ be the sub-vector space of $K$ comprising the square-integrable functions $f(t)$ which may be written as $\sum_{n\geq 1} \alpha(t/n)/n$, where $\alpha(t) \in L^1(\lambda, \Lambda)$ and $\hat{\alpha}(0) = 0 = \hat{\alpha}(1)$.

**6.13 Definition.** Let $0 < \lambda < \infty$. We let $Z_\lambda$ be the closed subspace of $K_\lambda \subset K$ spanned by the evaluators $Z^\lambda_{\rho,k}$, $0 \leq k < m_\rho$ associated with the non-trivial zeros of the Riemann zeta function and with their (eventual) multiplicities.

The main theorem (whose proof takes up the next pages) is:

**6.14 Theorem. 1.** *Let* $0 < \lambda < 1$. *One has* $W_\lambda \subset W'_\lambda \subset K_\lambda$. *The subspace* $W'_\lambda$ *is closed and equals* $\bigcap_{0<\mu<\lambda} W_\mu = \bigcap_{0<\mu<\lambda} W'_\mu$. *One has* $W_\lambda = \overline{\bigcup_{\lambda<\mu<1} W_\mu} = \overline{\bigcup_{\lambda<\mu<1} W'_\mu}$. *One has* $K_\lambda = W'_\lambda \perp Z_\lambda$.

**2.** *The set of* $\lambda$*'s for which* $W_\lambda \subset W'_\lambda$ *is a strict inclusion is at most countable.*

**3.** *Let* $1 \leq \lambda < \infty$. *One has* $K_\lambda = Z_\lambda$.

**6.15 Definition.** Let $0 < \lambda < 1$. We let $HP_\lambda$ be the perpendicular complement in $K_\lambda$ of $W_\lambda$.

We thus have $HP_\lambda \supset Z_\lambda$ and the question whether this may be strict is interesting (equivalently whether $W_\lambda \subset W'_\lambda$ may be a strict inclusion). This question is related to the properties of the Krein spaces of entire functions of finite exponential type which are associated with the measure $\left|\zeta\left(\frac{1}{2} + i\tau\right)\right|^2 d\tau/2\pi$ on the critical line. From $W'_\lambda = \bigcap_{\mu<\lambda} W_\mu$ a strict inclusion may happen only for a countable set of $\lambda$'s.

As usual our axis of symmetry is the critical line, not the real axis, and we use the Mellin transform to define Paley-Wiener functions, not the additive Fourier transform. Let $\mu$ be a measure on the critical line, and let $H = L^2\left(s = \frac{1}{2} + i\tau, d\mu\right)$. We suppose $1/s \in H$ and $d\mu\left(\frac{1}{2} + i\tau\right) = d\mu\left(\frac{1}{2} - i\tau\right)$. Let $\Lambda \geq 1$ and let $I^\Lambda$ be the subspace of $H$ of ($\mu$-equivalence classes of) functions $F(s)$ which are also entire functions of exponential type at most $\log(\Lambda)$. Let $J^\Lambda$ be the subspace of $H$ of functions $F(s)$ which are also entire functions of exponential type strictly less than $\log(\Lambda)$ (for $\Lambda = 1$ this means $J^1 = \{0\}$). It is proven in [29] that $I^\Lambda$, if it does not span $H$, is a closed subspace. It will then contain the closure of $J^\Lambda$, and the question whether it may be strictly larger is subtle. An isometric representation exists, the *Krein string*, where, if the description of the string is complete enough, one may read the answer to the question. We do not go into more details and refer the reader to the book [29] which is devoted to the theory of the Krein string, and which also contains an introduction to the de Branges theory. The following theorem is due to Krein and is also fundamental in the general de Branges theory.

**6.16 Theorem (Krein, [33]).** *Let $F(z)$ be an entire function which is in the Nevanlinna class separately in the half-plane* $\mathrm{Im}(z) > 0$ *and in the half-plane* $\mathrm{Im}(z) < 0$. *Then $F(z)$ has finite exponential type which is given by the formula*

$$\max\left(\limsup_{\sigma\to+\infty}\frac{\log|F(i\sigma)|}{\sigma}, \limsup_{\sigma\to+\infty}\frac{\log|F(-i\sigma)|}{\sigma}\right)$$

We recall that one possible definition of the Nevanlinna class of a half-plane is as the space of quotients of bounded analytic functions. For example it is known that any function in the Hardy space of a half-plane is a Nevanlinna function. Krein's theorem is more complete but we only need the result given here. Of course we will be using this theorem with the critical line replacing the horizontal axis.

For the following steps we let $0 < \lambda \le 1$, $\Lambda = 1/\lambda$, and the notations $H$, $I^\Lambda$, $J^\Lambda$ are relative to the measure $d\mu(s) = |\zeta(s)|^2\, d\tau/2\pi$ on the critical line $\left(s = \frac{1}{2} + i\tau\right)$.

$$H = L^2\left(\mathrm{Re}(s) = \frac{1}{2}, |\zeta(s)|^2\frac{d\tau}{2\pi}\right)$$

Unfortunately we are unable to describe the associated Krein string. Rather we will explain how to isometrically identify the co-Poisson spaces $W_\lambda$ (resp. $W'_\lambda$) (here $0 < \lambda < 1$) with subspaces of codimension 2 of $\overline{J^\Lambda}$ (resp. $I^\Lambda$). This will be used in the proof of the main theorem 6.14.

**6.17 Lemma.** *A function $G \in H$ is perpendicular to $J^\Lambda$ if and only if it is perpendicular to all functions $(u^s - 1)/s$ for $\lambda \le u \le \Lambda$. Hence the closure $\overline{J^\Lambda}$ is also the closure of the finite linear combinations $(u^s - 1)/s$ for $\lambda \le u \le \Lambda$.*

*Proof.* One direction is obvious. Let us now assume that $G \perp (u^s - 1)/s$ for $\lambda \le u \le \Lambda$. Let $F \in J^\Lambda$ and let $\varepsilon > 0$ be such that the type of $F$ is $<\log(\Lambda) - \varepsilon$. We consider

$$\int F(s)\frac{e^{\varepsilon s} - 1}{s}\overline{G(s)}|\zeta(s)|^2\, d\tau$$

If we take $F(s) = u^s$ with $e^\varepsilon\lambda \le u \le e^{-\varepsilon}\Lambda$ this integral vanishes. Using the Pollard-de Branges-Pitt "lemma" (*sic*) from [29, 4.8., p. 108], we deduce that the integral with the original $F(s)$ vanishes too. Then from $|(e^{\varepsilon s} - 1)/\varepsilon s| \le 2(e^{\varepsilon/2} - 1)/\varepsilon$ and dominated convergence we get the desired conclusion. □

**6.18 Lemma.** *Let $F(s) \in I^\Lambda$. One has $F(s)\zeta(s) \in N \cdot \Lambda^s\mathbb{H}^2$ and also $F(1-s)\zeta(s) \in N \cdot \Lambda^s\mathbb{H}^2$ (we write $\mathbb{H}^2$ for the Hardy space of the right half-plane and we recall that $N$ is the operator of multiplication with $s/(s-1)$.)*

*Proof.* The product $F(s)\zeta(s)$ belongs to $L^2(\mathrm{Re}(s) = 1/2, d\tau/2\pi)$. As $I^\Lambda \subset J^{\Lambda\exp(\varepsilon)}$ for $\varepsilon > 0$, $F(s)\zeta(s)$ is in the closure of finite sums of functions $(u^s - 1)\zeta(s)/s$ for $e^{-\varepsilon}\lambda \le$

$u \leq e^{+\varepsilon}\Lambda$. It belongs to the closed space $N \cdot (e^{\varepsilon}\Lambda)^{s}\mathbb{H}^2$ as $\zeta(s)/s$ itself belongs to $N \cdot \mathbb{H}^2$. We note that this space is the image under $N$ of the Mellin transform of $L^2((e^{-\varepsilon}\lambda, \infty), dt)$ so after letting $\varepsilon \to 0$ we obtain that $F(s)\zeta(s)$ belongs to $N \cdot \Lambda^s\mathbb{H}^2$. We note that $F(s) \to F(1-s)$ is an isometry of $I^\Lambda$ and the conclusion then follows. □

**6.19 Theorem.** *An entire function $F(s)$ belongs to $I^\Lambda$ (i.e. it is in $H$ and of exponential type at most* $\log(\Lambda)$*) if and only if $F(s)\zeta(s)$ is the Mellin transform of an element in $L_\lambda$. The space $I^\Lambda$ is a closed subspace of $H$ and is isometric through $F(s) \to \zeta(s)F(s)$ to the subspace of $L_\lambda$ of functions whose Mellin transform vanish at the zeros of the zeta function with at least the same multiplicities. For each complex number $w$ the evaluations $F \mapsto F(w)$ are continuous linear forms on $I^\Lambda$.*

*Proof.* From the lemma $\zeta(s)F(s)$ is the Mellin transform of an element of $N \cdot L^2((\lambda, \infty), dt)$ whose image under $\mathscr{F}_+$ also belongs to $N \cdot L^2((\lambda, \infty), dt)$ (as this corresponds to the replacement $F(s) \mapsto F(1-s)$). So the map $F(s) \to \zeta(s)F(s)$ is an isometric embedding into $\widehat{L_\lambda}$. If an element $G(s)$ from $\widehat{L_\lambda}$ vanishes at the non-trivial zeros of the zeta function (taking into account the multiplicities) then it factorizes as $G(s) = F(s)\zeta(s)$ with an entire function $F(s)$ (as $G(s)$ also vanishes at the trivial zeros and has at most a pole of order 1 at $s = 1$). From this, $F(s)$ is in the right half-plane in the Nevanlinna class (of quotients of bounded analytic functions) because both $F(s)\zeta(s)$ and $\zeta(s)$ are meromorphic functions in this class. And the same holds in the left half-plane, as $\mathscr{F}_+(G)(s) = F(1-s)\zeta(s)$. We now use the theorem of Krein 6.16 which tells us that the entire function $F(s)$ has finite exponential type given by

$$\max\left(\limsup_{\sigma\to+\infty}\frac{\log|F(\sigma)|}{\sigma}, \limsup_{\sigma\to+\infty}\frac{\log|F(1-\sigma)|}{\sigma}\right)$$

From this formula, and from $F(s)\zeta(s) \in N \cdot \Lambda^s\mathbb{H}^2$, $F(1-s)\zeta(s) \in N \cdot \Lambda^s\mathbb{H}^2$, and from the fact that elements of $\mathbb{H}^2$ are bounded in $\mathrm{Re}(s) \geq 1/2 + \varepsilon > 1/2$, we deduce that the exponential type of $F(s)$ is at most $\log(\Lambda)$. So $I^\Lambda$ is isometrically identified with the functions in $\widehat{L_\lambda}$ vanishing at least as $\zeta(s)$ does. This space is closed because the evaluators are continuous linear forms on $L_\lambda$. From this we see that the evaluators $F \mapsto F(s)$ are continuous linear forms except possibly at the zeros and poles of $\zeta(s)$, and the final statement then follows from this and the Banach-Steinhaus theorem (as $I^\Lambda$ is a Hilbert space from the preceding). □

**6.20 Theorem.** *Let* $0 < \lambda < \infty$.

**1.** *The vectors $Z^\lambda_{\rho,k}$, $k < m_\rho$, span $K_\lambda$ if and only if $\lambda \geq 1$.*

**2.** *A function $\alpha(s)$ on* $\mathrm{Re}(s) = 1/2$ *is the Mellin transform of an element of $K_\lambda$ perpendicular to $Z_\lambda$ if and only if:*

- **a.** *It is square integrable on the critical line for $d\tau/2\pi$.*
- **b.** *One has $\alpha(s) = \zeta(s)s(s-1)\beta(s)$ with $\beta(s)$ an entire function of finite exponential type at most* $\log(1/\lambda)$.

*Proof.* From the existence of $W_\lambda$ the vectors $Z^\lambda_{\rho,k}$, $k < m_\rho$, do not span $K_\lambda$ if $\lambda < 1$. Let $f \in (Z_\lambda)^\perp \cap K_\lambda$. By definition its Mellin transform vanishes at the non-trivial zeros of $\zeta$. It also vanishes at the trivial zeros and at 0 so it may be written

$$\hat{f}(s) = s(s-1)\zeta(s)\theta(s)$$

with an entire function $\theta(s)$. In the right half-plane $\theta(s)$ is in the Nevanlinna class (of quotients of bounded analytic functions) because both $\hat{f}(s)$ and $s(s-1)\zeta(s)$ are meromorphic in this class. From the functional equation one has

$$\widehat{\mathscr{F}_+(f)}(s) = s(s-1)\zeta(s)\theta(1-s)$$

So $\theta(s)$ is in the Nevanlinna class of the left half-plane. We now use the theorem of Krein 6.16 and conclude that the entire function $\theta(s)$ has finite exponential type which is given as

$$\max\left(\limsup_{\sigma\to+\infty}\frac{\log|\theta(\sigma)|}{\sigma}, \limsup_{\sigma\to+\infty}\frac{\log|\theta(1-\sigma)|}{\sigma}\right)$$

This formula (elements of $\mathbb{H}^2$ are bounded in $\mathrm{Re}(s) \geq 1$) shows that the exponential type of $\theta(s)$ is at most $\log(1/\lambda)$. This shows $Z_\lambda = K_\lambda$ for $\lambda > 1$. Let us prove this also for $\lambda = 1$: on the line $\mathrm{Re}(s) = +2$ one has $\hat{f}(s) = O(1)$ (as it belongs to $\mathbb{H}^2(\mathrm{Re}(s) > 1/2)$) hence $\theta(s)$ is $O(1/s(s-1))$. So it is square integrable on this line and by the Paley-Wiener theorem it vanishes identically as it is of minimal exponential type.

Conversely, let $F(s) = s(s-1)\beta(s)$ be an entire function of finite exponential type at most $\log(1/\lambda)$ which is such that $\alpha(s) = \zeta(s)F(s)$ is square-integrable on the critical line. From the previous theorem $F(s)$ is in the closed subspace $I^\Lambda$ of $H$ and $\alpha(s)$ is the Mellin transform of an element $f(t)$ of $L_\lambda$. As $\alpha(s)$ is analytic at $s = 1$ and vanishes at $s = 0$ one has in fact $f \in K_\lambda$. And $\alpha(s)$ vanishes at the zeros of zeta with at least the same multiplicities, in other words $f$ is perpendicular to $Z_\lambda$. □

**6.21 Lemma.** *Any function $F$ in $I^\Lambda$ is $O(1)$ in the closed strip $-1 \leq \mathrm{Re}(s) \leq 2$, in particular on the critical line.*

*Proof.* From the fact that $F(s)\zeta(s)\frac{s-1}{s}\Lambda^{-s}$ is bounded on the line $\mathrm{Re}(s) = 2$ (as it belongs to the Hardy space $\mathbb{H}^2(\mathrm{Re}(s) > \frac{1}{2})$) one deduces that $F(s)$ is bounded on $\mathrm{Re}(s) = 2$ , hence also on $\mathrm{Re}(s) = -1$ (as $F(1-s)$ also belongs to $I^\Lambda$.) As it has finite exponential type we may apply the Phragmen-Lindelöf theorem to deduce that $F(s)$ is $O(1)$ on this closed vertical strip. □

**6.22 Lemma.** *Let $\Lambda > 1$. Let $K^\Lambda$ be the closure of $J^\Lambda$ in $H$. Let $K_0^\Lambda$ be the subspace of functions in $K^\Lambda$ vanishing at* 0 *and at* 1*, and similarly let $J_0^\Lambda$ be the subspace of $J^\Lambda$ of functions vanishing at* 0 *and at* 1*. Then $K_0^\Lambda$ is the closure of $J_0^\Lambda$.*

*Proof.* Let for $1 < \mu < \Lambda$:

$$A_\mu(s) = \frac{(\mu^{s/2} - 1)(\mu^{s/2} - \mu^{1/2})}{\log(\mu)(1 - \mu^{1/2})/2} \frac{1}{s}$$

This is an entire function of exponential type $\log(\mu)$, in $H$ and with $A_\mu(0) = 1$, $A_\mu(1) = 0$. Let also $B_\mu(s) = A_\mu(1 - s)$. Let $F \in K_0^\Lambda$ and let us write $F = \lim F_\mu$ with $F_\mu \in J^\mu$, $\mu < \Lambda$. One has $F_\mu(0) \to F(0) = 0$ and $F_\mu(1) \to F(1) = 0$, because evaluations are continuous linear forms on $I^\Lambda$. So $F = \lim(F_\mu - F_\mu(0)A_\mu - F_\mu(1)B_\mu)$ (clearly the norms of $A_\mu$ and $B_\mu$ are bounded as $\mu \to \Lambda$). □

**6.23 Theorem.** *Let* $0 < \lambda < 1$. *A function* $\alpha(s)$ *on* $\mathrm{Re}(s) = 1/2$ *is the Mellin transform of an element of* $W_\lambda$ *if and only if:*

1. *It is square integrable on the critical line for* $d\tau/2\pi$.
2. *It is in the closure of the square integrable functions* $\alpha(s) = \zeta(s)s(s-1)\beta(s)$ *with* $\beta(s)$ *an entire function of finite exponential type* strictly less *than* $\log(1/\lambda)$.

*Proof.* Let $0 < \lambda < 1$. We have to show $\widehat{W_\lambda} = K_0^\Lambda \cdot \zeta(s)$. First let us prove the inclusion $\widehat{W_\lambda} \subset K_0^\Lambda \cdot \zeta(s)$: let $\phi(u)$ be a smooth function with support in $[\lambda, \Lambda]$ with $\hat{\phi}(1) = 0 = \hat{\phi}(0)$. It is elementary that there exists $\psi(u)$ smooth with support in $[\lambda, \Lambda]$ and with $\hat{\phi}(s) = s(s-1)\hat{\psi}(s)$. If we now consider for $a \to 1^-$ the smooth functions $\phi_a(u)$ with support in $[\lambda^a, \Lambda^a]$ such that $\widehat{\phi_a}(s) = s(s-1)\hat{\psi}(as)$ then $\widehat{\phi_a}(s)\zeta(s)$ belongs to $K_0^\Lambda \cdot \zeta(s)$ and converge to $\hat{\phi}(s)\zeta(s)$ in $L^2$ norm on the critical line as $a \to 1^-$.

For the converse inclusion $K_0^\Lambda \cdot \zeta(s) \subset \widehat{W_\lambda}$ let $F \in K_0^\Lambda$. We may approximate $F$ with an element of $J_0^\Lambda$, so we may assume $F$ itself to be of positive exponential type $\log(\mu) < \log(\Lambda)$. Let $\theta$ be a smooth function with support in $[1/e, e]$, with $\hat{\theta}(1/2) = 1$. Let $\widehat{\theta_\varepsilon}(s) = \hat{\theta}(\varepsilon(s - 1/2) + 1/2)$. Let $F_\varepsilon = \widehat{\theta_\varepsilon} F$. In $H$ the functions $F_\varepsilon$ converge to $F$. From 6.21 we know that $F$ is $O(1)$ on the critical line so the functions $F_\varepsilon$ are $O(|s|^{-N})$ for any $N \in \mathbb{N}$. From the Paley-Wiener theorem they are the Mellin transforms of $L^2$ functions $f_\varepsilon(t)$ with support in $[e^{-\varepsilon}\mu^{-1}, e^{\varepsilon}\mu]$. For $\varepsilon$ small enough this will be included in $[\lambda, \Lambda]$. From the decrease on the critical line the functions $f_\varepsilon(t)$ are smooth. As $\widehat{f_\varepsilon}(0) = 0 = \widehat{f_\varepsilon}(1)$ this tells us that $F_\varepsilon(s)\zeta(s)$ is the Mellin transform of a co-Poisson summation of a *smooth* function, and this implies that $F(s)\zeta(s)$ belongs to the (Mellin transform of) $W_\lambda$, as $W_\lambda$ is defined as the closure of the co-Poisson summations of smooth functions whose Mellin transforms vanish at 0 and at 1. □

**6.24 Theorem.** *Let* $F(s)$ *be an entire function of finite exponential type. Then*

$$\int_{\mathrm{Re}(s)=1/2} |F(s)|^2 |\zeta(s)|^2 |ds| < \infty \Rightarrow \int_{\mathrm{Re}(s)=1/2} |F(s)|^2 |ds| < \infty$$

*Proof.* We want to prove that any function $F(s)$ in $I^\Lambda$ is square-integrable for the Lebesgue measure on the critical line. We know from 6.21 that it is $O(1)$ in the closed strip $-1 \le \mathrm{Re}(s) \le 2$. From this, if for $s$ in this open strip we express $F(s)$ as a Cauchy integral with contributions from the two vertical sides and two horizontal segments, the contribution of the horizontal segments will vanish when they go to infinity. So:

$$F(s) = \int_{\mathrm{Re}(s)=2} \frac{F(z)}{z-s}\frac{|dz|}{2\pi} - \int_{\mathrm{Re}(s)=-1} \frac{F(z)}{z-s}\frac{|dz|}{2\pi}$$

On $\mathrm{Re}(s) = 2$, $F(s)$ is square integrable because $F(s)\zeta(s)\frac{s-1}{s}\Lambda^{-s}$ is, as it belongs to $\mathbb{H}^2\left(\mathrm{Re}(s) > \frac{1}{2}\right)$. It is an important fact that Cauchy integrals of $L^2$ functions on vertical line realize the orthogonal projection to the Hardy space of the corresponding half-plane. Hence the first integral above defines a function square-integrable on each vertical line $\mathrm{Re}(s) < 2$. And the second integral similarly for $\mathrm{Re}(s) > -1$ ($F(1-s)$ satisfies the same hypotheses as $F(s)$). So $F(s)$ is square-integrable on the critical line (and in fact on each vertical line in the complex plane.) □

The next theorem establishes $K_\lambda = W'_\lambda \perp Z_\lambda$:

**6.25 Theorem.** *Let $\lambda < 1$. The functions $A(u)$ in $Z_\lambda^\perp \cap K_\lambda$ are exactly the square-integrable functions which may be written $\sum_{n\ge1} g(u/n)/n$, with an integrable function $g(u)$ supported in $[\lambda, \Lambda]$ and such that $\int_0^\infty g(u)\,du = \hat{g}(0) = 0$. The function $g(u)$ is necessarily square-integrable and necessarily satisfies $\hat{g}(1) = \int_0^\infty \frac{g(u)}{u}\,du = 0$.*

**6.26 Note.** By a variant on the Mœbius inversion formula from $A(u) = \sum_{n\ge1} g(u/n)/n$ one has $g(u) = \sum_{n\ge1} \mu(m)A(u/m)/m$ (and this is a finite sum for each $u > 0$) in case $A$ (hence $g$ and conversely) has support in $(\lambda, \infty)$. It involves then in a neighborhood of each $u > 0$ only finitely many terms. If $g(u)$ has support in $[\lambda, \Lambda]$ we can express it on this interval as a finite combination of $A(u/m)/m$'s. So if $A$ is $L^2$ then $g$ had to be $L^2$ to start with. Also we will see that if $A$ is $L^2$ then $\int_0^\infty \frac{g(u)}{u}\,du$ necessarily vanishes.

*Proof.* Let $A \in Z_\lambda^\perp \cap K_\lambda$ and $\alpha(s) = \hat{A}(s)$. We know that $\alpha(s) = \zeta(s)F(s)$ with $F(s)$ an entire function vanishing at 0 and 1 and of exponential type at most $\log(\Lambda)$. From 6.24 we know that $F(s)$ is square-integrable on the critical line for the Lebesgue measure. So the Paley-Wiener theorem implies $F(s) = \hat{g}(s)$ with $g(u) \in L^2([\lambda, \Lambda])$. We have our function $g(u)$ in $L^2(\lambda, \Lambda)$ and we want to show that $A(u)$ is equal to $B(u) = \sum_{n\ge1} g(u/n)/n$. From Fubini $\int_\lambda^\infty B(u)u^{-s}\,du = \zeta(s)\hat{g}(s) = \hat{A}(s) = \int_\lambda^\infty A(u)u^{-s}\,du$ for $\mathrm{Re}(s) > 1$ and so $B(u) = A(u)$ (almost everywhere from the unicity theorem for Fourier transforms of $L^1$-functions). We have shown that each $A \in Z_\lambda^\perp \cap K_\lambda$ may be written (uniquely) as $\sum_{n\ge1} g(u/n)/n$ with $g \in L^2((\lambda, \Lambda), du)$, $\hat{g}(0) = \hat{g}(1) = 0$. So it belongs to $W'_\lambda$.

For the converse let $g \in L^2((\lambda, \Lambda), du)$ be such that $A(u) = \sum_{n\ge1} g(u/n)/n$ is square-integrable. Its distribution theoretic Fourier transform is (from 4.2):

$$\mathscr{F}\left(\sum_{n\ge1} g(u/n)/n\right) = \sum_{n\ge1} g(n/u)/u - \hat{g}(0) + \hat{g}(1)\delta_0$$

This distribution must coincide with the function which is the $L^2$-Fourier transform of $A(u)$ and so the square-integrability of $A(u)$ implies the vanishing of $\hat{g}(1)$.

If we impose $\hat{g}(0) = 0$ the Fourier transform of $A(u)$ is the function $\sum_{n\geq 1} g(n/u)/u$ which again vanishes on $(0, \lambda)$. So $A$ belongs to $K_\lambda$. Its Mellin transform is an entire function which by Fubini for $\mathrm{Re}(s) > 1$ equals $\zeta(s)\hat{g}(s)$ hence also everywhere. The vector $A(u)$ is thus perpendicular to the vectors $Z^\lambda_{\rho,k}$, which means that $A \in Z_\lambda^\perp$. This completes the proof of $K_\lambda = W'_\lambda \perp Z_\lambda$. □

We also take note of:

**6.27 Proposition.** *The map "$\frac{\cdot}{\zeta(s)}$" from $W'_\lambda$ to $L^2(\lambda, \Lambda)$ is bounded.*

*Proof.* Each $g(u)$ is expressed (on $(\lambda, \Lambda)$) as a finite Mœbius sum in terms of the $A(u/m)/m$'s, with a number of summands independent of $A$. □

The main theorem sums up almost everything that preceded:

**6.28 Theorem (6.14). 1.** *Let $0 < \lambda < 1$. One has $W_\lambda \subset W'_\lambda \subset K_\lambda$. The subspace $W'_\lambda$ is closed and equals $\bigcap_{0<\mu<\lambda} W_\mu = \bigcap_{0<\mu<\lambda} W'_\mu$. One has $W_\lambda = \overline{\bigcup_{\lambda<\mu<1} W_\mu} = \overline{\bigcup_{\lambda<\mu<1} W'_\mu}$. One has $K_\lambda = W'_\lambda \perp Z_\lambda$.*

**2.** *The set of $\lambda$'s for which $W_\lambda \subset W'_\lambda$ is a strict inclusion is at most countable.*

**3.** *Let $1 \leq \lambda < \infty$. One has $K_\lambda = Z_\lambda$.*

*Proof.* The basic inclusions $W_\lambda \subset W'_\lambda \subset K_\lambda$ are a corollary to the co-Poisson intertwining formula. One has $K_\lambda = Z_\lambda$ for $\lambda \geq 1$ from Theorem 6.20. Let $0 < \lambda < 1$. From Theorem 6.25 we have identified $W'_\lambda$ as the perpendicular component in $K_\lambda$ of $Z_\lambda$. From Theorem 6.20 $W'_\lambda$ is isometrically identified with the closed subspace of $L^2\left(\mathrm{Re}(s) = \frac{1}{2}, |\zeta(s)|^2\, d\tau/2\pi\right)$ of (restrictions) of entire functions $F(s)$ of exponential type at most $\log(1/\lambda)$ and vanishing at 0 and at 1. Hence $W'_\lambda = \bigcap_{0<\mu<\lambda} W'_\mu$. From Theorem 6.23 $W_\lambda$ is isometrically identified with the closure in $L^2\big(\mathrm{Re}(s) = \frac{1}{2}$, $|\zeta(s)|^2\, d\tau/2\pi\big)$ of entire functions $F(s)$ of exponential type *strictly less than* $\log(1/\lambda)$ and vanishing at 0 and at 1. Hence $W_\lambda = \overline{\bigcup_{\lambda<\mu<1} W_\mu}$. Also $\lambda < \mu < 1 \Rightarrow W_\lambda \supset W'_\mu \supset W_\mu$ (the last inclusion as $W'_\mu$ is known to be closed). Hence $W_\lambda = \overline{\bigcup_{\lambda<\mu<1} W'_\mu}$. Also $W'_\lambda \subset \bigcap_{0<\mu<\lambda} W_\mu$. We know $K_\lambda = \bigcap_{0<\mu<\lambda} K_\mu$, hence an element $f(t)$ in $\bigcap_{0<\mu<\lambda} W_\mu$ belongs to $K_\lambda$ and has its Mellin transform vanishing at least as the zeta function does. From $K_\lambda = W'_\lambda \perp Z_\lambda$ it belongs to $W'_\lambda$. Hence $W'_\lambda = \bigcap_{0<\mu<\lambda} W_\mu$. A non-countable set of exceptional $\lambda$'s contradicts the separability of $K$. □

We briefly explain how some of the considerations extend to Dirichlet $L$-series. For an odd character the cosine transform $\mathscr{F}_+$ is replaced with the sine transform $\mathscr{F}_-$, so we will stick with an even (primitive) Dirichlet character: $\chi(-1) = 1$. Let us recall the functional equation of $L(s, \chi) = \sum_{n\geq 1} \chi(n) n^{-s}$:

$$L(s, \chi) = w_\chi q^{-s+1/2} \chi_+(s) L(1 - s, \overline{\chi})$$

where $w_\chi$ is a certain complex number of modulus 1 and $q$ is the conductor (=period) of the primitive character $\chi$. One has $\overline{w_\chi} = w_{\bar\chi}$. Tate's Thesis [43] gives a unified manner of deriving all these functional equations as a corollary to the one-and-only Poisson-Tate intertwining formula on adeles and ideles (and additional local computations). A reference for the more classical approach is, for example, [27] (for easier comparison with the classical formula, we have switched from $L(\chi, s)$ to $L(s, \chi)$). The Poisson-Tate formula specializes to twisted Poisson summation formulae on $\mathbb{R}$, or rather on the even functions on $\mathbb{R}$ as we are dealing only with even characters.

Let $\phi(t)$ be an even Schwartz function, and let:

$$P_\chi(\phi)(t) = \sum_{n\geq 1} \chi(n)\phi(nt)$$

We suppose here that $\chi$ is not the principal character so there is no term $-(\int_0^\infty \phi(u)\,du)/|t|$ (which was engineered to counterbalance the pole of the Riemann zeta function at $s = 1$). At the level of (right) Mellin transforms $P_\chi$ corresponds to multiplication by $L(1-s, \chi)$.

So the composite $P_\chi \mathscr{F}_+ = P_\chi \mathscr{F}_+ II$ acts on right Mellin transforms as:

$$\hat\phi(s) \mapsto L(1-s, \chi)\chi_+(s)\hat\phi(1-s) = w_\chi q^{s-1/2} L(s, \bar\chi)\hat\phi(1-s)$$

and this gives the $\chi$-Poisson intertwining:

$$P_\chi \mathscr{F}_+ = w_\chi D_q I P_{\bar\chi}$$

where $D_q$ is the contraction of ratio $q$ which acts through multiplication by $q^{s-1/2}$ on Mellin transforms and as $f(t) \mapsto \sqrt{q}f(qt)$ on $L^2(0, \infty)$. Let us define the $\chi$-co-Poisson $P'_\chi$ on smooth even functions compactly supported away from 0 as:

$$P'_\chi(\phi)(t) = \sum_{n\geq 1} \overline{\chi(n)}\frac{\phi(t/n)}{n}$$

We have $P'_\chi = IP_{\bar\chi}I$, and $P'_\chi$ is the scale invariant operator with multiplier (under the right Mellin transform) $L(s, \bar\chi)$. From the commutativity of $P_{\bar\chi}$ with $\Gamma_+ = \mathscr{F}_+ I$ and the $\chi$-Poisson intertwining we get the $\chi$-co-Poisson intertwining:

$$\mathscr{F}_+ P'_\chi = \mathscr{F}_+ IP_{\bar\chi} I = P_{\bar\chi}\mathscr{F}_+ = w_{\bar\chi} D_q I P_\chi = w_{\bar\chi} D_q P'_{\bar\chi} I$$

The placement of the operator $D_q$ on the right-side of the Intertwining equation is very important! If the even function $\alpha(t)$ is supported in $(0, \infty)$ on $[\lambda_1, \lambda_2]$ then its $\chi$-co-Poisson summation $f(t)$ will be supported in $[\lambda_1, \infty)$ and the Fourier cosine transform of $f(t)$ will be supported in $[1/(q\lambda_2), \infty)$. The product of the lower ends of these two intervals is strictly less than $1/q$ (if $\alpha$ is not identically zero). So this means that we obtain (non-zero) functions which together with their cosine transform are supported in $[\lambda, \infty[$ only for $\lambda < 1/\sqrt{q}$.

We let $W_\lambda^\chi \subset K_\lambda$ be the closure of such $\chi$-twisted co-Poisson summations. The Mellin transforms of the functions in $W_\lambda^\chi$ are the functions $L(s,\bar\chi)\hat\alpha(s)$ where $\alpha(t)$ is a smooth function compactly supported in $[\lambda, \Lambda/q]$ ($\Lambda = 1/\lambda, \Lambda > \sqrt{q}$). A vector $Z_{\rho,k}^\lambda$ is (Hilbert-)perpendicular to $W_\lambda^\chi$ if and only if $Z_{\bar\rho,k}^\lambda$ is Euclid-perpendicular to $W_\lambda^\chi$ if and only if $\bar\rho$ is a (non-trivial) zero of $L(s,\bar\chi)$ of multiplicity strictly greater than $k$, if and only $\rho$ is a (non-trivial) zero of $L(s,\chi)$ of multiplicity strictly greater than $k$. So:

**6.29 Theorem.** *Let $\lambda < 1/\sqrt{q}$. A vector $Z_{w,k}^\lambda \in K_\lambda$ is perpendicular to $W_\lambda^\chi$ if and only if $w$ is a non-trivial zero $\rho$ of the Dirichlet L-function $L(s,\chi)$ of multiplicity $m_\rho > k$.*

We conclude with a statement whose analog we have already stated and proven for the Riemann zeta function. The proof is only slightly more involved, but as the statement is so important we retrace the steps here.

**6.30 Theorem.** *The vectors $Z_{\rho,k}^\lambda \in K_\lambda$, $L(\rho,\chi) = 0$, $0 \le k < m_\rho$, associated with the non-trivial zeros of the Dirichlet L-function (and with their multiplicities) span $K_\lambda$ if and only if $\lambda \ge 1/\sqrt{q}$.*

*Proof.* They can not span if $\lambda < 1/\sqrt{q}$ from the existence of $W_\lambda^\chi$. Let us suppose $\lambda \ge 1/\sqrt{q}$. Let $\hat f(s)$ be the Mellin Transform of an element of $K_\lambda$ which is (Hilbert)-perpendicular to all $Z_{\rho,k}^\lambda$, $L(\rho,\chi) = 0$ (non-trivial), $0 \le k < m_\rho$. This says that $f(s)$ vanishes at the $\bar\rho$'s. We know already that $f(s)$ vanishes at the trivial zeros. So one has:

$$\hat f(s) = L(s,\bar\chi)\theta_1(s)$$

with an entire function $\theta_1(s)$. The image of $f$ under the unitary $\mathscr{F}_+$ will be Hilbert-perpendicular to $\mathscr{F}_+(Z_{\rho,k}^\lambda) = (-1)^k Z_{1-\rho,k}^\lambda$ and so $\mathscr{F}_+(f)$ is Euclid-perpendicular to the vectors associated to the $\overline{1-\rho}$, which are the zeros of $L(s,\chi)$, hence:

$$\widehat{\mathscr{F}_+(f)}(s) = L(s,\chi)\theta_2(s)$$

with an entire function $\theta_2(s)$. From the functional equation:

$$\widehat{\mathscr{F}_+(f)}(s) = \chi_+(s)\hat f(1-s)$$

we get

$$\chi_+(s)L(1-s,\bar\chi)\theta_1(1-s) = L(s,\chi)\theta_2(s)$$

and combining with

$$L(s,\chi) = w_\chi q^{-s+1/2}\chi_+(s)L(1-s,\bar\chi)$$

this gives:

$$\theta_1(1-s) = w_\chi q^{-s+1/2}\theta_2(s)$$

Using Krein's theorem [33] we deduce that $F(s) = q^{-(s-1/2)/2}\theta_1(s)$ has finite exponential type which is equal to

$$\max\left(\limsup_{\sigma\to+\infty}\frac{\log|F(\sigma)|}{\sigma}, \limsup_{\sigma\to+\infty}\frac{\log|F(1-\sigma)|}{\sigma}\right)$$

and from $L(\sigma,\chi) \to_{\sigma\to+\infty} 1$ we see that the exponential type of $F(s)$ is at most

$$\max(\log(\Lambda) - \log(\sqrt{q}), \log(\Lambda) - \log(\sqrt{q})) = \log(\Lambda) - \log(\sqrt{q})$$

This concludes the proof when $\lambda > 1/\sqrt{q}$. When $\lambda = 1/\sqrt{q}$, we see that $q^{-(s-1/2)/2}\theta_1(s)$ has minimal exponential type. But from $\hat{f}(s) = L(s,\bar{\chi})\theta_1(s)$ we deduce that $\theta_1(s)$ is square-integrable on the line $\mathrm{Re}(s) = 2$. By the Paley-Wiener theorem it thus vanishes identically. □

## 7 Speculations on the zeta function, the renormalization group, duality

We turn now to some speculative ideas concerning the zeta function, the GUE hypothesis and the Riemann hypothesis. When we wrote our (unpublished) manuscript "The Explicit formula and a propagator" we had already spent some time trying to think about the nature of the zeta function. Our conclusion, which had found some kind of support with the conductor operator $\log|x| + \log|y|$, stands today. The spaces $HP_\lambda$ and especially Theorem 6.30 have given us for the first time a quite specific signal that it may hold some value. What is more Theorem 6.30 has encouraged us into trying to encompass in our speculations the GUE hypothesis[1], and more daring and distant yet, the Riemann Hypothesis Herself.

We are mainly inspired by the large body of ideas associated with the Renormalization Group, the Wilson idea of the statistical continuum limit, and the unification it has allowed of the physics of second-order phase transitions with the concepts of quantum field theory. Our general philosophical outlook had been originally deeply framed through the Niels Bohr idea of complementarity, but this is a topic more distant yet from our immediate goals, so we will leave this aside here.

We believe that the zeta function is analogous to a multiplicative wave-field renormalization. We expect that there exists some kind of a system, in some manner rather alike the Ising models of statistical physics, but much richer in its phase diagram, as each of the $L$-function will be associated to a certain universality domain. That is we do not at all attempt at realizing the zeta function as a partition function. No the zeta function rather corresponds to some kind of symmetry pattern[2] appearing at low

[1] i.e. the "Montgomery-Dyson proposal" [36] or "Montgomery-Odlyzko law" [38].

[2] Of course in statistical physics, symmetry is restored at high temperature and broken at low temperature. But this is from a point of view where a continuum is considered more symmetric than a lattice as it has a larger symmetry group. So here we are using the word "symmetry" under a more colloquial acceptation.

temperature. But the other $L$-functions too may themselves be the symmetry where the system gets frozen at low temperature.

Renormalization group trajectories flow through the entire space encompassing all universality domains, and perhaps because there are literally fixed points, or another more subtle mechanism, this gives rise to sets of critical exponents associated with each domain: the (non-trivial) zeros of the $L$-functions. So there could be some underlying quantum dynamics, but the zeros arise at a more classical level[3], at the level of the renormalization group flow.

The Fourier transform as has been used constantly in this manuscript will correspond to a simple symmetry, like exchanging all spins up with all spins down. The functional equations reflect this simple-minded symmetry and do not have a decisive significance in the phase picture.

But we do believe that some sort of a much more hidden thing exist, a Kramers-Wannier like duality exchanging the low temperature phase with a single hot temperature phase, not number-theoretical. If this were really the case, some universal properties would hold across all phases, reflecting the universality examplified by the GUE hypothesis. Of course the hot phase is then expected to be somehow related with quantities arising in the study of random matrices. In the picture from Theorem 6.30, $\lambda$ seems to play the rôle of a temperature (inverse of coupling constant).

We expect that if such a duality did reign on our space it would interact in such a manner with the renormalization group flow that this would give birth to scattering processes. Indeed the duality could be used to compare incoming to outgoing (classical) states. Perhaps the constraints related with this interaction would result in a property of causality equivalent to the Riemann Hypothesis.

Concerning the duality at this time we can only picture it to be somehow connected with the Artin reciprocity law, the ideas of class field theory and generalizations thereof. So here our attempt at being a revolutionary ends in utmost conservatism.

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N. Bohr, *The Philosophical Writings of Niels Bohr*, Ox Bow Press, Woodbridge, Connecticut.
Volume I: *Atomic Theory and the Description of Nature* (1934)
Volume II: *Essays 1932–1957 on Atomic Physics and Human Knowledge* (1958)
Volume III: *Essays 1958–1962 on Atomic Physics and Human Knowledge* (1963)
Volume IV: *Causality and Complementarity*, Supplementary papers edited by Jan Faye and Henry J. Folse, (1999).

[3] "classical" in its kinematics: understanding the flow of coupling constants of a quantum theory with infinitely many degrees of freedom has become almost synonymous with understanding its "quantum physics".

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