WAVELET METHODS IN COMPUTATIONAL FLUID DYNAMICS

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ABSTRACT

We discuss in this paper the numerical solution of boundary value problems for partial differential equations by methods relying on compactly supported wavelet approximations. After defining compactly supported wavelets and stating their main properties we discuss their application to boundary value problems for partial differential equations, giving a particular attention to the treatment of the boundary conditions. Finally, we discuss application of wavelets to the solution of the Navier-Stokes equations for incompressible viscous fluids.

Introduction

Wavelets is a generic term denoting various mathematical objects which have been introduced mostly during the last decade. It is a topic in full evolution at the moment leading to a very active research effort, in the Mathematical and Electrical Engineering communities

M. Y. Hussaini et al. (eds.), *Algorithmic Trends in Computational Fluid Dynamics*

particularly. The limitations that we see concerning wavelet popularity in the Computation Fluid Dynamics community are essentially the following two:

(i) To enter the wavelet world requires a serious mathematical investment. Wavelets are considerably more complicated than finite element, finite difference, finite volume and spectral approximations.

(ii) A substantial effort has to be done to solve boundary value problems with complicated boundaries and/or boundary conditions. Indeed this difficulty is not specific to wavelets since we already encountered it for spectral approximations.

The content of this article is the following:

In Section 1 we define *compactly supported wavelets* à la Daubechies and state their basic properties. In Section 2 we briefly discuss some further properties of wavelet approximations which may be of interest for Scientific Computing. In Section 3 we briefly discuss the wavelet solution of boundary value problems for partial differential equations; a particular attention is given to the treatment of boundary conditions. Finally, we discuss in Section 4 the wavelet solution of the Navier-Stokes equations for incompressible viscous fluids.

1. Definition and Basic Properties of Compactly Supported Wavelets

1.1 Generalities

An inconvenient of traditional Fourier methods is that the sine and cosine functions have an unbounded support and even worst they do not vanish at infinity. On the other hand their spectra are very local consisting of a finite sum of Dirac measures. Conversely, if one uses approximations based on finite sum of Dirac measures such as (in one space dimension)

$$\sum_{j\in J} a_j \delta(x - x_j), \tag{1.1}$$

the spectrum of the basis "functions" $\delta(x - x_j)$, namely $s \to \exp(2i\pi x_j s)$ does not vanish at infinity in the frequency domain.

On the basis of the above observations it is then reasonable to look for orthonormal basis of $L^2(\mathbb{R})$, constructed from a *unique* generating function φ (the *scaling function*), via translation, dilation and linear combinations. We also want φ to be localized in x (space variable) and s (Fourier variable).

Example: Suppose that $\varphi(x) = e^{-x^2}$; concerning the Fourier transform $\hat{\varphi}$ of φ, we have then $\hat{\varphi}(s) = e^{-\pi^2 s^2}$; both functions vanish quickly for large values of their respective argument. Indeed wavelet families based on the above function φ are discussed in [1], [2]. □

More recently I. Daubechies has introduced in [3] wavelet families based on scaling functions φ such that:

(i) φ has a *bounded* support,

(ii) $\hat{\varphi}(s) \to 0$ *quickly* when $|s| \to +\infty$.

In the following parts of this paper the only wavelets that we shall consider are the Daubechies' ones. For an introduction to wavelets, intended for applied mathematicians and engineers we highly recommend [4].

1.2 The Daubechies Wavelets

Following [3] we require the *scaling function* φ to satisfy (N being a *positive integer*)

$$\varphi(x) = \sum_{k=0}^{2N-1} a_k \varphi(2x - k), \forall x \in \mathbb{R}, \tag{1.2}$$

$$\int_{\mathbf{R}} \varphi(x)dx = 1, \tag{1.3}$$

$$\int_{\mathbf{R}} \varphi(x - l)\varphi(x - m)dx = 0, \forall l, m \in \mathbb{Z}, l \neq m. \tag{1.4}$$

Relations (1.2) - (1.4) clearly imply

$$\sum_{k=0}^{2N-1} a_k = 2, \tag{1.5}$$

$$\sum_{k=0}^{2N-1} a_k a_{k-2m} = 0, \quad \forall m \in \mathbb{Z}, m \neq 0. \tag{1.6}$$

If the above relations hold, then the set

$$\bigcup_{j=0}^{+\infty} \bigcup_{l \in \mathbb{Z}} \varphi_{jl} \quad \left(with\ \varphi_{jl}(x) = 2^{j/2}\varphi(2^j x - l) \right) \tag{1.7}$$

is an *orthogonal* basis of $L^2(\mathbb{R})$; also, the fact that the set of the coefficients, i.e. $\{a_k\}_{k=0}^{2N-1}$, is *finite* implies that φ has a *compact support.*

Example: Take $N = 1$ and $a_o = a_1 = 1$. We then have φ defined by

$$\varphi(x) = 1 \ if\ 0 \leq x \leq 1,\ \varphi(x) = 0 \ elsewhere. \tag{1.8}$$

The corresponding family (1.7) (namely the Haar functions family; cf. [5]) is an orthogonal basis of $\mathbb{R}$; its elements however are *discontinuous.* □

To "force" the *smoothness* of the scaling function φ we may require, for example, the monomials $1, x, \ldots x^{N-1}$ to be linear combination of the $\varphi(x - l)$; this implies the additional relations

$$\sum_{k=0}^{2N-1} (-1)^k k^m a_k = 0,\ m = 0, 1, \ldots, N-1. \tag{1.9}$$

From the coefficients a_k we can use the *Fast Fourier Transform* (FFT) to construct φ via $\hat{\varphi}$ (see, e.g., [6] for this construction). Indeed, since $\mathrm{supp}(\varphi) = [0, 2N-1]$, it suffices to know φ at $0, 1, \ldots, 2N-1$ and use the scaling relation (1.2) to compute φ at the *dyadic values* of x (in practice everywhere since the set of the dyadic numbers is *dense* in $\mathbb{R}$).

Once the coefficients a_k are known, we define the *wavelet function* ψ by

$$\psi(x) = \sum_{k=2-2N}^{1} (-1)^k a_{1-k} \varphi(2x - k), \tag{1.10}$$

and then the functions ψ_{jl} by

$$\psi_{jl}(x) = 2^{j/2}\psi(2^j x - l). \tag{1.11}$$

Let us denote by V_n (resp. W_n) the closure of the vector space span by $\{\varphi_{nl}\}_{l\in\mathbb{Z}}$ (resp. $\{\psi_{nl}\}_{l\in\mathbb{Z}}$); we have then the following properties

$$V_n \subset V_{n+1}, \tag{1.12}$$

$$\text{closure } (\bigcup_n V_n) = L^2(\mathbb{R}), \tag{1.13}$$

$$\{\varphi_{nl}\}_{l\in\mathbb{Z}} \textit{ is an orthogonal basis for } V_n, \tag{1.14}$$

$$\{\psi_{nl}\}_{l\in\mathbb{Z}} \textit{ is an orthogonal basis for } W_n, \tag{1.15}$$

$$\textit{the orthogonal of } W_n \textit{ in } V_{n+1} \textit{ is } V_n. \tag{1.16}$$

The functions φ_{nl} and ψ_{nl} have *compact* supports; they also verify

$$\int_{\mathbb{R}} \varphi_{nl}(x)dx = 2^{-n/2}, \int_{\mathbb{R}} \psi_{nl}(x)dx = 0. \tag{1.17}$$

Finally we have for $L^2(\mathbb{R})$ the following decomposition properties

$$L^2(\mathbb{R}) = V_n \oplus (\bigoplus_{j\geq n} W_j) = V_o \oplus (\bigoplus_{j\geq 0} W_j). \tag{1.18}$$

The potential of wavelets for multiscale analysis is related to relation (1.18).

2. Further Properties of Wavelets and Generalizations

From the following decomposition property

$$V_{n+1} = V_n \oplus W_n \tag{2.1}$$

we can expect wavelets to be well suited to multilevel solution methodologies and to the implementation of methods such as nonlinear Galerkin's (see [7]).

Remark 2.1: Several authors (see, e.g., [8]) have introduced *hierarchical finite element* bases leading to decomposition properties close to those mentioned above. These bases are well suited to adaptive mesh refinement. □

The wavelet functions considered so far are single variable functions. Concerning generalization to $\mathbb{R}^d$, with $d \geq 2$, we see at the moment the two following options:

(i) Use tensor products of one variable wavelet function spaces.

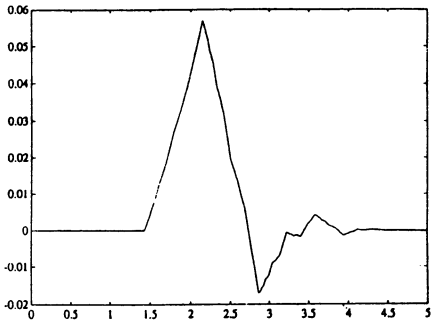
(ii) Use nonfactorable wavelets; such object exist, unfortunately they are not easy to handle (the support of the scaling function can be a region of $\mathbb{R}^d$ with a fractral boundary; see [9] for more details).

Finally, it can be shown that if f is sufficiently smooth then one has the following approximation property

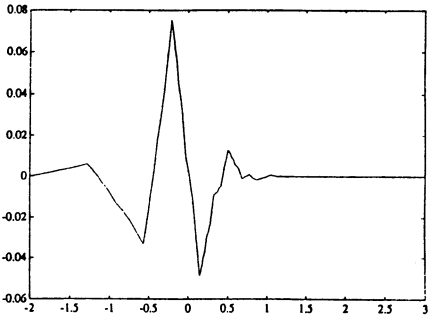
$$\|f - P_n(f)\|_{H^m(\mathbb{R})} = 0(2^{-n(N-m)}),$$

where $P_n : L^2(\mathbb{R}) \to V_n$ is the orthogonal projector from $L^2(\mathbb{R})$ into V_n.

On Figure 2.1, we have shown the graphs of the scaling and wavelet functions corresponding to $N = 3$. These graphs suggest that φ and ψ are close to piecewise linear functions; in fact, it can be shown that for $N = 3$, φ and ψ are C^1 functions. We observe the asymmetry of the graphs of φ and ψ.



Daubechies-3 scaling function



Daubechies-3 wavelet

Figure 2.1

Remark 2.2: Independently of the hierarchical structure of wavelet spaces, their double *localization* property (in the physical and Fourier spaces) make wavelets interesting for the solution of partial differential equations. □

3. Application to the Solution of Boundary Value Problems for Partial Differential Equations

3.1 Generalities

Wavelet based *Galerkin* methods have been applied, in [6], to the solution of boundary and initial/boundary value problems for linear and nonlinear elliptic, parabolic and hyperbolic equations in *one space variable*, with Dirichlet, Neumann and periodic boundary conditions. Indeed, as shown in [6] and [10], wavelet based Galerkin methods have proved quite efficient to solve the Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{3.1}$$

with $\nu \geq 0$.

Concerning multidimensional applications, let us make the following statement: *Everything you can do with (standard) spectral methods for periodic boundary conditions you can do better, for the same cost, with wavelet methods.* This (strong) statement is supported by the calculations, and comparisons to spectral methods, done by J. Weiss (see ref. [11]) for the incompressible Euler and Navier-Stokes equations, with periodic boundary conditions; wavelet based methods give in particular an excellent resolution of *shear layers.*

Concerning more general domains and/or boundary conditions the following approaches can be considered:

(i) Use boundary fitted wavelets as in [12]; their practical implementation seems quite complicated.

(ii) Use geometrical transformations to transform the multidimensional region in a box shaped region (or in a patch of such boxes). Indeed non periodic boundary conditions are complicated to implement even for a box shaped domain.

(iii) Extend to wavelet approximations the *domain imbedding methods* discussed in, e.g., [13] - [15].

Some domain imbedding methods will be discussed in Sections 3.2 (for Dirichlet boundary conditions) and 3.3 (for Neumann boundary conditions); we have chosen to discuss the solution of Dirichlet and Neumann problems since these problems are basic to many issues of Mechanics and Physics.

3.2 Domain Embedding Treatment of Dirichlet Boundary Conditions

3.2.1 Problem formulation

For simplicity, we shall concentrate on a simple *Dirichlet problem*, namely

$$\alpha u - \nabla^2 u = f \text{ in } \omega, \; u = g \text{ on } \gamma, \tag{3.2}$$

where, in (3.2), α is a positive number, ω is a bounded open connected region of $\mathbb{R}^d$ $(d \geq 1)$, γ is the boundary of ω and $\nabla = \left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^{d}$; both functions f and g are given. We imbed ω in Ω, where Ω is a box shaped domain, as shown in Figure 3.1, below, where n denotes the unit outward normal vector at γ.

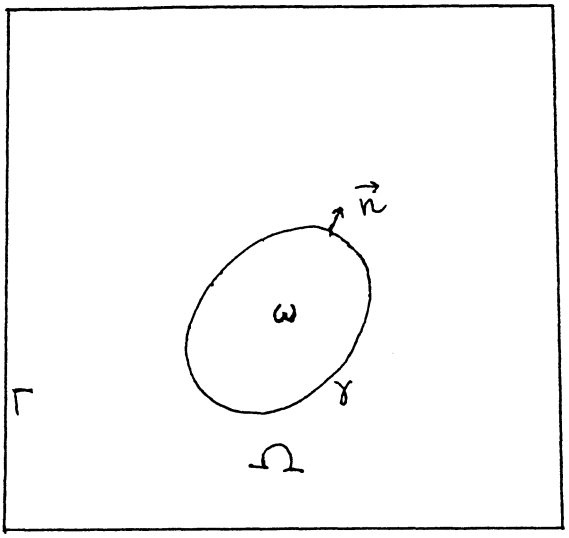


Figure 3.1

Two classical ways to implement the domain imbedding concept are *penalty* and the use of a *Lagrange multiplier*, respectively.

Both approaches rely on the fact that problem (3.2) is equivalent to

$$\begin{cases} \textit{Find } \tilde{u} \in W_g \textit{ such that} \\ \\ \displaystyle\int_\Omega (\alpha \tilde{u} v + \nabla \tilde{u} \cdot \nabla v) dx = \int_\Omega \tilde{f} v dx, \ \forall v \in W_o, \end{cases} \tag{3.3}$$

where, in (3.3), $\tilde{f}$ is an extension of f to Ω, and where W_g and W_o are defined by

$$W_g = \{v | v \in V, v = g \textit{ on } \gamma\}, \quad W_o = \{v | v \in V, v = 0 \textit{ on } \gamma\}; \tag{3.4}$$

in (3.4), V is either $H^1(\Omega)$ or $H^1_o(\Omega)$, or $H^1_P(\Omega)$ (the space of the functions of $H^1(\Omega)$, periodic at Γ). If $g \in H^{1/2}(\gamma)$, then W_g is a closed nonempty affine subspace of V, implying that problem (3.3) has a *unique* solution; this solution clearly coincides with u on ω, justifying therefore the imbedding approach.

3.2.2 A penalty method

From now on we drop the $\sim$ in $\tilde{u}$ and $\tilde{f}$. We approximate then problem (3.3) by

$$\begin{cases} \textit{Find } u_\varepsilon \in V \textit{ such that, } \forall v \in V, \textit{ we have} \\ \\ \displaystyle\int_\Omega (\alpha u_\varepsilon v + \nabla u_\varepsilon \cdot \nabla v) dx + \frac{1}{\varepsilon} \int_\gamma u_\varepsilon v d\gamma = \int_\Omega f v dx + \frac{1}{\varepsilon} \int_\gamma g v d\gamma, \end{cases}' \tag{3.5}$$

where, in (3.5), ε is a positive parameter. It can be easily shown (see, e.g., Appendix 1 in [16]) that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{H^1(\Omega)} = 0, \tag{3.6}$$

where u is the solution of (3.3).

From a practical point of view we shall have to replace V by a finite dimensional subspace of it; also, in order to handle the integrals over γ, we have found convenient to proceed as follows:

Suppose that we want to evaluate $\int_\gamma g v d\gamma$, then

(i) Introduce G defined over Ω such that $g = G \cdot n$ on γ.

(ii) Observe that from the *divergence theorem* we have

$$\int_\gamma gvd\gamma = \int_\gamma G \cdot nvd\gamma = \int_\omega \nabla \cdot (vG)dx = \int_\Omega \chi_\omega \nabla \cdot (vG)dx, \tag{3.7}$$

where χ_ω is the *characteristic function* of ω, i.e., $\chi_\omega(x) = 1$ if $x \epsilon \omega$, $\chi_\omega(x) = 0$ elsewhere.

(iii) Approximate χ_ω by χ_ω^a, where χ_ω^a is *differentiable* over Ω and substitute to χ_ω in the last integral of (3.7). We have then (assuming that $\chi_\omega^a|_\Gamma = 0$ which is reasonable)

$$\int_\gamma gvd\gamma \sim \int_\Omega \chi_\omega^a \nabla \cdot (vG)dx = -\int_\Omega vG \cdot \nabla \chi_\omega^a dx. \tag{3.8}$$

Concerning the approximate characteristic function χ_ω^a, numerical experiments done at Rice University suggest the following rule of thumb: *To approximate χ_ω by χ_ω^a use a "grid" four times finer than the one which will be used to compute u.*

3.2.3 A Lagrange multiplier method

Following [17], [18], we associate to the relation $u = g$ on γ a *Lagrange multiplier* λ defined over γ. Solving problem (3.3) is then equivalent to find a pair $\{\tilde{u}, \lambda\}$ such that

$$\begin{cases} \int_\Omega (\alpha \tilde{u} v + \nabla \tilde{u} \cdot \nabla v)dx = \int_\Omega \tilde{f} v dx + \int_\gamma \lambda v d\gamma, \ \forall v \in V, \\ \\ \int_\gamma (\tilde{u} - g)\mu d\gamma = 0, \ \forall \mu \in \Lambda; \tilde{u} \in V, \ \lambda \in \Lambda. \end{cases} \tag{3.9}$$

For sufficiently smooth data, we can take $\Lambda = L^2(\gamma)$; however, the *natural choice* for Λ is $H^{-1/2}(\gamma)$. Incidentally the multiplier λ in (3.9) is nothing but the jump of the normal derivative $\frac{\partial \tilde{u}}{\partial n}$ at γ.

The *conjugate gradient* solution of problem (3.9), together with the description of a finite element implementation is discussed in [17], [18]; numerical experiments show the validity of this approach; its wavelet implementation is currently taking place.

Remark 3.1: In principle one can combine the penalty and Lagrange multiplier methods via an *Augmented Lagrangian* approach, as shown

in, e.g., [19], [20]. This new methodology still has to be investigated when applied to the solution of problem (3.3) (for multidimensional problems, at least, since we have already applied it in [6] for the wavelet solution of elliptic boundary value problems in one variable).

3.3 Domain Embedding Treatment of Neumann Boundary Conditions

3.3.1 Problem formulation

We keep the notation of Section 3.3. The problem that we consider is defined by

$$\alpha u - \nabla^2 u = f \text{ in } \omega, \quad \frac{\partial u}{\partial n} = g \text{ on } \gamma. \tag{3.10}$$

Problem (3.10) has the following variational formulation:

$$\begin{cases} u \in H^1(\omega); \\ \int_\omega (\alpha uv + \nabla u \cdot \nabla v) dx = \int_\omega f v dx + \int_\gamma g v d\gamma, \ \forall v \in H^1(\omega). \end{cases} \tag{3.11}$$

The embedding methods to be described here are variants of those used for solving the Dirichlet problem; they are based on *regularization* and *Lagrange multiplier*, respectively.

Both methods rely on the following result:

With Ω, Γ, and V as in Section 3.2.1, consider the variational problem defined by

$$\begin{cases} \tilde{u} \in W, \\ \int_\Omega (\alpha \tilde{u} v + \nabla \tilde{u} \cdot \nabla v) dx = 0, \ \forall v \in W_o, \end{cases} \tag{3.12}$$

where

$$W = \{v | v \in V, \int_\omega (\alpha vw + \nabla v \cdot \nabla w) dx - \int_\omega f w dx - \int_\omega g w d\gamma = 0, \ \forall w \epsilon V\}, \tag{3.13}_1$$

and

$$W_o = \{v | v \in V, \int_\omega (\alpha vw + \nabla v \cdot \nabla w) dx = 0, \ \forall w \epsilon V\}; \tag{3.13}_2$$

problem (3.12) has clearly a unique solution and it coincides with the solution u of (3.10), (3.11) over ω. In fact, $\tilde{u}$ is the extension of u over Ω which *minimizes* $\int_\Omega(\alpha v^2 + |\nabla v|^2)dx$.

3.3.2 A regularization method

Let ε be a *positive* number and consider the following variational problem

$$\begin{cases} u_\varepsilon \in V; \forall v \in V \text{ one has} \\ \varepsilon \int_\Omega(\alpha u_\varepsilon v + \nabla u_\varepsilon \cdot \nabla v)dx + \int_\omega(\alpha u_\epsilon v + \nabla u_\epsilon \cdot \nabla v)dx \\ \quad = \int_\omega f v dx + \int_\gamma g v d\gamma. \end{cases} \tag{3.14}$$

It can be shown that

$$\lim_{\epsilon \to 0} \|u_\epsilon - u\|_{H^1(\Omega)} = 0, \tag{3.15}$$

where u is the solution of (3.12). The wavelet implementation of the above regularization method is under investigation.

3.3.3 A Lagrange multiplier method

We associate to (3.11) a *Lagrange multiplier* λ defined over Ω. Indeed, solving problem (3.12) (and therefore problem (3.11)) is equivalent to find a pair $\{\tilde{u}, \lambda\} \in V \times V$ such that

$$\begin{cases} \int_\Omega(\alpha\tilde{u}v + \nabla\tilde{u} \cdot \nabla v)dx = \int_\omega(\alpha\lambda v + \nabla\lambda \cdot \nabla v)dx, \ \forall v \in V, \\ \int_\omega(\alpha\tilde{u}\mu + \nabla\tilde{u} \cdot \nabla\mu)dx = \int_\omega f\mu dx + \int_\gamma g\mu d\gamma, \ \forall \mu \in V. \end{cases} \tag{3.16}$$

The *conjugate gradient* solution of problem (3.16), together with the description of finite element and wavelet implementations will be described elsewhere. Remark 3.1 still holds here, i.e., we can combine the regularization and Lagrange multiplier methods.

4. Application to the Solution of the Incompressible Navier-Stokes Equation

4.1 Generalities

The Navier-Stokes equations that we consider are

$$\frac{\partial u}{\partial t} - \nu \nabla^2 u + (u \cdot \nabla) u + \nabla p = 0 \text{ in } \Omega \tag{4.1}$$

$$\nabla \cdot u = 0 \text{ in } \Omega \tag{4.2}$$

completed by initial and boundary conditions.

Concerning the wavelet solution of the above Navier-Stokes equations, we see immediately three sources of potential difficulties, namely:

(i) The treatment of the *incompressibility condition* $\nabla \cdot u = 0$.

(ii) The treatment of the *boundary conditions.*

(iii) The simulation of flow at large Reynolds numbers.

In this section we shall focus on (i); however, the two other issues deserve some comments:

Concerning *boundary conditions*, the *periodic case* is quite easy to implement; on the other hand, other boundary conditions such as Dirichlet and Neumann yield serious difficulties, the main reasons being that in a wavelet expansion the coefficients are not *pointwise* values of the function or of its derivatives, as it is the case with finite elements or finite differences. Among the possible cures let us mention *boundary fitted wavelets* like the ones developed by S. Jaffard and Y. Meyer in [12], or *fictitious domain methods*, in the spirit of Section 3; we are currently investigating the second approach. Another possibility is to couple wavelet approximations (used away from the boundary) with finite elements (used in the neighborhood of the boundary), but the matching problems (at least for nonoverlapping couplings) are essentially as difficult to implement as are boundary conditions.

Concerning now the simulation of flow at large Reynold's numbers we can predict, on the basis of preliminary numerical experiments done with the Daubechies wavelets, that for an equivalent amount of computational work, wavelet based methods *are more stable and accurate than finite element, finite difference and spectral*

methods. The above experiments involved the solution of the *Burgers equation* $u_t + u_{xx} = \nu u_{xx}$ (cf. [6], [10]) and of the *Navier-Stokes equations* with *periodic boundary conditions* (cf. [11]). A key property of wavelet based solution methods is that they seem to require much less (if not at all) artificial viscosity for highly advective flow; a possible explanation of this behavior is that it is a consequence of the orthogonality of the basis functions and of their localization properties in the spatial *and* spectral domains.

The treatment of the incompressibility seems to be eventually fairly simple and will be addressed in the next paragraph.

4.2 Wavelet Treatment of the Incompressibility Condition

Operator splitting techniques applied to the solution of the Navier-Stokes equations (4.1), (4.2) lead to the following Stokes equations

$$\alpha u - \nu \Delta u + \nabla p = f \text{ in } \Omega, \tag{4.3}$$

$$\nabla \cdot u = 0 \text{ in } \Omega. \tag{4.4}$$

We suppose that the boundary conditions are defined by

$$u = g \text{ on } \Gamma (\text{with } \int_\Gamma g \cdot n d\Gamma = 0), \tag{4.5}$$

i.e. are of the *Dirichlet* type.

A *variational formulation* of problem (4.3) - (4.5) is given by

$$\begin{cases} u \in V_g;\ \forall v \in V_o \text{ we have} \\ \int_\Omega (\alpha u \cdot v + \nu \nabla u \cdot \nabla v) dx - \int_\Omega p \nabla \cdot v dx = \int_\Omega f \cdot v dx, \end{cases} \tag{4.6}$$

$$\int_\Omega q \nabla \cdot u dx = 0,\ \forall q \in L^2(\Omega); p \in L^2(\Omega). \tag{4.7}$$

In (4.6), (4.7), we have $v \cdot w = \sum_{i=1}^d v_i w_i, \forall v = \{v_i\}_{i=1}^d, w = \{w_i\}_{i=1}^d; \nabla v \cdot \nabla w = \sum_{i=1}^d \sum_{j=1}^d \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j}; V_o = (H_o^1(\Omega))^d$ and $V_g = \{v | v \in (H^1(\Omega))^d, v = g \text{ on } \Gamma\}$.

It follows from (4.6), (4.7) that the two fundamental spaces in the variational formulation of (4.3), (4.4) are $L^2(\Omega)$ (for the pressure) and $(H^1(\Omega))^d$ (for the velocity). We discuss now the wavelet approximation of the *variational problem* (4.6), (4.7):

From now on we shall denote by φ^N the *scaling function* associated to the positive integer N (the precise definition of the scaling function has been given in Section 1.2); the parameter N plays clearly the role of a polynomial degree. We define next φ_{jl}^N and $\Phi_n^N(\mathbb{R})$ by

$$\varphi_{jl}^N(x) = 2^{j/2}\varphi^N(2^j x - l), \forall x \in \mathbb{R},$$

$$\Phi_n^N(\mathbb{R}) = \textit{ closure of the linear space span by } \{\varphi_{nl}^N\}_{l\in\mathbb{Z}},$$

respectively.

In order to apply wavelets to the solution of *multidimensional problems* an obvious approach is to use *tensor products* of one variable function spaces to define the multidimensional ones. We define therefore the spaces $\mathcal{V}_n^N(\mathbb{R}^d)$ and $V_n^N(\mathbb{R}^d)$ by

$$\mathcal{V}_n^N(\mathbb{R}^d) = \bigotimes_{i=1}^{d} \Phi_n^N(\mathbb{R}_{x_i}),$$

$$V_n^N(\mathbb{R}^d) = (\mathcal{V}_n^N(\mathbb{R}^d))^d,$$

respectively.

By restricting to Ω the elements of the two above spaces, we obtain $\mathcal{V}_n^N(\Omega)$ and $V_n^N(\Omega)$; if Ω is bounded, these two spaces are *finite dimensional.* On the basis of the analysis done in [21], concerning the finite difference and finite element approximations of the Stokes/Dirichlet problem (4.3) - (4.5) we shall approximate the *velocity spaces* V_o and V_g by appropriate subspaces of $V_n^N(\Omega)$ (taking into account, in some way or another, the boundary conditions $v = 0$ and $v = g$, respectively), and then the *pressure* space by $V_{n-1}^N(\Omega)$; in order to have $V_n^N(\Omega) \subset H^1(\Omega)$, we have to take $N \geq 3$ (cf., e.g., [3], [6] for this result). We then substitute to V_o, V_g and $L^2(\Omega)$, their wavelet analogues in (4.6), (4.7) to obtain a wavelet/Galerkin approximation of the Stokes problem (4.3) - (4.5).

Conclusion

On the basis of few experiments we can expect compactly supported wavelets to have a most interesting potential for the numerical solution of fluid flow problems. However the practical implementation of this new kind of approximation gives rise to highly nontrivial difficulties which have to be successfully addressed if one wishes to

see wavelet-based approximations successfully competing with finite difference and finite element approximations.

Acknowledgment

We would like to acknowledge the helpful comments and suggestions of the following individuals: L. C. Cowsar, C. De la Foye, G. H. Golub, Y. Kuznetsov, P. Lallemand, A. Latto, W. Lawton, P. Le Tallec, J. L. Lions, P. L. Lions, G. Meurant, J. Pasciak, H. Resnikoff, J. Weiss, M. F. Wheeler, O. B. Widlund.

The support of the following corporations or institutions is also acknowledged: AWARE, Dassault Aviation, INRIA, University of Houston, Université Pierre et Marie Curie. We also benefited from the support of DARPA (Contracts AFOSR F49620-89-C-0125 and AFOSR-90-0334), DRET and NSF (Grants INT 8612680 and DMS 8822522). Finally, we would like to thank J. A. Wilson for the processing of this article.

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