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Wavelet Bases Adapted to Pseudodifferential Operators

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This paper is concerned with the numerical treatment of pseudodifferential equations in $\mathbb{R}^2$, employing wavelet Galerkin methods. We construct wavelet bases adapted to a given pseudodifferential operator in the sense that functions on different refinement levels are orthogonal with respect to a certain bilinear form induced by the operator.

1. INTRODUCTION

Lately, newly developed wavelet decompositions were employed for the numerical treatment of partial differential equations; see, e.g., [3, 4, 16–18, 23]. In general, a system of functions $\{\psi^i\}_{i=1,\dots,N}$ is called a family of (mother) wavelets if the scaled and integer translated versions of $\{\psi^i\}_{i=1,\dots,N}$ form an (orthonormal) basis of $L^2(\mathbb{R}^n)$. These functions can be utilized as basis functions for a Galerkin approach. Since the structure of the resulting stiffness matrix depends on the wavelets and the differential operator, it seems natural to try to construct wavelets adapted to a given differential operator in an appropriate way.

A classical finite element approach using a nodal basis gives rise to sparse stiffness matrices, which condition numbers typically exhibit a polynomial growth rate. To avoid this problem one can, for example, use the hierarchical basis preconditioner from Yserentant [32] or the Bramble–Pasciak–Xu preconditioner [5]. Then the condition numbers only grow logarithmically or they are uniformly bounded, respectively; see Yserentant [32] or Dahmen and Kunoth [11].

Both concepts are closely related to wavelet expansions, since, in both cases, one defines suitable projectors Q_j onto the approximation spaces V_j and tries to find a basis in a complement space W_j of V_j in V_{j+1} defined by the range of $Q_{j+1} - Q_j$.

The preconditioning methods mentioned above have the disadvantage that the stiffness matrices become less sparse. Therefore, one could try to find a wavelet basis such that, for a given problem, the stiffness matrices are sparse, simply structured and, moreover, have bounded condition numbers. The optimal shape would be a diagonal matrix. Unfortunately, this is difficult to realize by using wavelet expansions since; e.g., it is not possible to construct a generator for the approximation spaces V_j whose integer translates are orthogonal with respect to the bilinear form induced by the differential operator; see Dahlke and Weinreich [10]. Quite recently, it was shown by Amaratunga and Williams [1] that for special kinds of one-dimensional differential operators the adapted biorthogonal wavelet basis constructed in [10] gives rise to almost perfectly diagonal stiffness matrices.

Motivated by these problems we have tried to answer the following question. How can the potential advantages and the generality of refinable shift-invariant spaces be exploited in principle for the treatment of higher dimensional partial differential equations?

This paper illuminates one special aspect of this problem. We show that it is possible to adapt wavelets to a given differential operator in the sense that functions on different refinement levels are orthogonal with respect to the bilinear form induced by the differential operator. Then the stiffness matrix splits into blocks, implying that for the solution of the corresponding linear system there is a variety of efficient numerical algorithms which, in particular, are suitable for the implementation on parallel computers; see Ortega and Voigt [27]. In special cases, we can achieve uniformly bounded condition numbers as in [11]. Similar results were also obtained by Jaffard [18] for elliptic problems on open domains and by Meyer [26] for the case of vaguelettes.

Apart from the applications we have in mind, the problems studied here seem to be interesting from a theoretical point of view. According to this, we have formulated our results for the more general case of pseudodifferential operators.

When dealing with wavelet Galerkin methods, the treatment of the boundary conditions is a nontrivial problem which apparently has not yet been solved in a satisfactory way. We will not give a detailed exploration of this prob-

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lem here and confine our discussion to global problems concerning shift-invariant spaces. Especially, we will restrict ourselves to pseudodifferential operators of the subclass $S^m_{1,0}$ of Hörmander's class defined by formula (3.8) below whose symbols are independent of x. We think that this restriction is justified by the flexibility and generality of the techniques used here. However, the difficulties mentioned above become less serious in the case of periodic boundary conditions on rectangles $\Omega = [0, M]^2$.

The main objective of this paper can be described as follows. Given a bilinear form $a(u, v)$ induced by a differential operator or a pseudodifferential operator and given a multiresolution analysis with generator ϕ, we construct a wavelet basis $\{\psi_e\}_{e\in E\setminus\{0\}}$ such that

$$a\left(\phi(\cdot - k), \psi_e(\cdot - l)\right) = 0 \quad \forall k, l \in \mathbb{Z}^2, e \in E \setminus \{0\},$$

where E denotes the set of all vertices in the unit cube $[0, 1]^2$. We present two approaches. The first one yields compactly supported wavelets for special operators, e.g., for differential operators, provided the generator is compactly supported, but it is based on some restrictive assumptions on the symbol; see (3.10) and (3.11). The second approach makes use of the concept of biorthogonal wavelets and is more general. However, it does not guarantee compact support in all cases.

This paper is organized as follows: in Section 2 we briefly recall the construction of (orthogonal and biorthogonal) wavelet bases. In particular, we state an existence theorem on biorthogonal wavelets appropriate to the investigations in the following sections. In Section 3 we present our first approach and in Section 4 the biorthogonal approach.

For the special case of the Laplace operator, similar things have been done before by Battle [2], (the so-called "massless Sobolev ondelettes",) by applying a quite different method. Instead of using a multiresolution analysis and biorthogonal wavelet bases, he constructs his wavelets by solving a suitable minimization problem.

2. WAVELETS

Wavelet bases are discrete families of functions obtained by dilations and integer translations of a finite number of mother functions ("mother wavelets"). The best known are dyadic orthonormal bases of $L^2(\mathbb{R}^n)$, i.e.,

$$\psi_{j,e,k}(\cdot) := 2^{-jn/2}\psi_e\left(2^{-j}\cdot - k\right), \quad j \in \mathbb{Z}, k \in \mathbb{Z}^n, e \in E \setminus \{0\} \tag{2.1}$$

where as stated above E denotes the set of vertices in the unit cube $[0, 1]^n$ in $\mathbb{R}^n$. (In the following, we will use the abbreviation $E^* := E \setminus \{0\}$). A possible alternative which was quite recently developed is to replace the diagonal scaling by powers of two by some expanding integer scaling matrix; see Cohen and Daubechies [8]. The most important tool for the construction of such a basis is the multiresolution approximation of functions introduced by Mallat [24]; see also Meyer [25]. It is defined as follows.

DEFINITION 2.1. A *multiresolution analysis* of $L^2(\mathbb{R}^n)$ is a sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^n)$ such that

$$\forall j \in \mathbb{Z}, \quad V_j \subset V_{j+1}, \tag{2.2}$$

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\mathbb{R}^n), \quad \bigcap_{j=-\infty}^{\infty} V_j = \{0\}, \tag{2.3}$$

$$f(\cdot) \in V_j \iff f(2\cdot) \in V_{j+1}, \tag{2.4}$$

$$f(\cdot) \in V_0 \iff f(\cdot - k) \in V_0, \quad k \in \mathbb{Z}^n. \tag{2.5}$$

There exists a function ϕ whose integer translates form a Riesz basis of V_0, (2.6)

this means that V_0 is the closed linear span of $\phi(\cdot - k), k \in \mathbb{Z}^n$, and there exist constants $c_1, c_2 \in \mathbb{R}^+$ such that

$$c_1 \|\lambda\|_{l_2} \leq \left\|\sum_{k\in\mathbb{Z}^n} \lambda_k \phi(\cdot - k)\right\|_2 \leq c_2 \|\lambda\|_{l_2}, \quad \forall \lambda \in l_2(\mathbb{Z}^n). \tag{2.7}$$

The function ϕ is called the *generator* of the multiresolution analysis. Let W_0 denote the orthogonal complement of V_0 in V_1,

$$V_1 = V_0 \oplus W_0, \quad W_0 \perp V_0. \tag{2.8}$$

Then a natural way to construct an orthonormal family of wavelets in $L^2(\mathbb{R}^n)$ is to find $2^n - 1$ functions $\{\psi_e\}_{e\in E^*}$ whose translates are orthonormal with respect to the usual L^2-inner product $\langle\cdot,\cdot\rangle$,

$$\left\langle \psi_e(\cdot - k), \psi_{\tilde{e}}\left(\cdot - \tilde{k}\right)\right\rangle = \delta_{e\tilde{e}}\delta_{k\tilde{k}}, \tag{2.9}$$

and span W_0, i.e.,

$$W_0 = \oplus_{e\in E^*} W_{0,e}, \quad W_{0,e} = \overline{\text{span}\left\{\psi_e(\cdot - k), k \in \mathbb{Z}^n\right\}}. \tag{2.10}$$

Hence, defining

$$W_{j,e} := \left\{f \in L^2(\mathbb{R}^n) \,|\, f\left(2^{-j}\cdot\right) \in W_{0,e}\right\}, \tag{2.11}$$

one has

$$V_{j+1} = V_j \oplus_{e\in E^*} W_{j,e}, \quad L^2(\mathbb{R}^n) = \oplus_{j\in\mathbb{Z}} \oplus_{e\in E^*} W_{j,e}, \tag{2.12}$$

and the functions

$$\psi_{j,e,k} := 2^{-nj/2}\psi_e\left(2^j\cdot - k\right) \tag{2.13}$$

form a complete orthonormal system in $L^2(\mathbb{R}^n)$.

This concept can be generalized a little bit further. In many applications, it is convenient to work with so-called *prewavelets*. In this case, the orthogonality condition (2.9) is replaced by requiring l_2-*stability*, i.e.,

$$\left\| \sum_{e\in E^*} \sum_{k\in \mathbb{Z}^n} \lambda_k^e \psi_e(\cdot - k) \right\|_2 \geq c_3 \sum_{e\in E^*} \|\lambda^e\|_{l_2}. \quad (2.14)$$

This means, one only demands orthogonality of different refinement levels. This concept has the following advantage. If one starts with a compactly supported generator, then the prewavelets can be chosen that they are also compactly supported. This is in general not true for strictly orthonormal wavelets. To save the compact support in this case, one has to start with a compactly supported generator ϕ that satisfies $\langle \phi(\cdot), \phi(\cdot - k)\rangle = \delta_{0k}$. To find such a (sufficiently smooth) ϕ is a nontrivial problem which was solved by Daubechies [15] in the univariate case. Her construction can be carried over to the multivariate case only by using tensor products or by employing a more complicated scaling matrix; see Cohen and Daubechies [8]. When dealing with prewavelets, one does not have this restriction. For the general theory of multivariate prewavelets, multiresolution analysis and stability the reader is referred to Jia and Micchelli [19], Lemarié [20–22], and Meyer [25]. In the following, we will not distinguish between wavelets and prewavelets.

In this paper, we go again one step further. We construct wavelets such that the space W_0 is indeed a complement, but not necessarily the orthogonal complement of V_0 in V_1. The "angle" between V_0 and W_0 in our case is determined by the bilinear form induced by certain pseudodifferential operators. We still require stability in the sense of (2.14) and completeness in the sense of (2.12).

Another generalization of orthonormal wavelets are biorthogonal wavelet bases introduced by Cohen *et al.* [9]. In this case, one starts with *two* hierarchical sequences of approximation spaces

$$\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \cdots$$
$$\cdots \subset \tilde{V}_{-2} \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \tilde{V}_2 \cdots,$$

and the orthogonality condition (2.8) is replaced by the condition

$$V_0 \perp \tilde{W}_0, \;\; \tilde{V}_0 \perp W_0, \;\; V_1 = V_0 \oplus W_0, \;\; \tilde{V}_1 = \tilde{V}_0 \oplus \tilde{W}_0. \quad (2.15)$$

Then one looks for functions $\{\psi_e\}_{e\in E^*}, \{\tilde{\psi}_{\tilde{e}}\}_{\tilde{e}\in E^*}$ whose translates span $W_0, \tilde{W}_0$, respectively, and are *biorthogonal* in the following sense:

$$\left\langle \psi_{j,e,k}, \tilde{\psi}_{\tilde{j},\tilde{e},\tilde{k}} \right\rangle = \delta_{j,\tilde{j}} \delta_{e,\tilde{e}} \delta_{k,\tilde{k}}. \quad (2.16)$$

According to Definition 2.1, ϕ and $\tilde{\phi}$ have to satisfy the *two-scale-equations*

$$\phi(x) = \sum_{k\in \mathbb{Z}^n} a_k \phi(2x - k),$$
$$\tilde{\phi}(x) = \sum_{k\in \mathbb{Z}^n} b_k \tilde{\phi}(2x - k). \quad (2.17)$$

Applying Fourier transform yields

$$\hat{\phi}(\xi) = \frac{a(z)}{2^n} \hat{\phi}\left(\frac{\xi}{2}\right), \quad \hat{\tilde{\phi}}(\xi) = \frac{b(z)}{2^n} \hat{\tilde{\phi}}\left(\frac{\xi}{2}\right), \quad z := e^{-i(\xi/2)}, \quad (2.18)$$

where the *symbols* $a(z), b(z)$ are defined by

$$a(z) = \sum_{k\in \mathbb{Z}^n} a_k z^k, \quad b(z) = \sum_{k\in \mathbb{Z}^n} b_k z^k, \quad z = e^{-i(\xi/2)}. \quad (2.19)$$

Therefore, assuming that (2.17) has solutions ϕ and $\tilde{\phi}$, we can conclude from (2.18) that their Fourier transforms can be computed as

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} \frac{a\left(e^{-i(\xi/2^j)}\right)}{2^n},$$
$$\hat{\tilde{\phi}}(\xi) = \prod_{j=1}^{\infty} \frac{b\left(e^{-i(\xi/2^j)}\right)}{2^n}. \quad (2.20)$$

Conversely, if we define $\hat{\phi}$ and $\hat{\tilde{\phi}}$ by (2.20) and assume that the infinite products converge in L^2, then we obtain solutions of (2.17).

For the applications we have in mind, we need the following (two-dimensional) result concerning biorthogonal wavelet bases.

THEOREM 2.2. *Let* $\{a_k\}_{k\in\mathbb{Z}^2}, \{b_k\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2)$, *and let* $a(z), b(z)$ *satisfy the biorthogonality condition*

$$\sum_{e\in E} a((-1)^e z)\overline{b((-1)^e z)} = 2^4. \quad (2.21)$$

Let ϕ *and* $\tilde{\phi}$ *be defined by*

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} \frac{a\left(e^{-i(\xi/2^j)}\right)}{4},$$
$$\hat{\tilde{\phi}}(\xi) = \prod_{j=1}^{\infty} \frac{b\left(e^{-i(\xi/2^j)}\right)}{4}, \quad (2.22)$$

where we assume that the infinite products converge pointwise. Furthermore, let $\phi \in \mathscr{L}^2(\mathbb{R}^2) := \{f \in L^2(\mathbb{R}^2) | \int_{[0,1)^2}$

$(\sum_k |f(x-k)|)^2 dx < \infty\}$, and suppose that ϕ has l_2-stable integer translates in the sense of (2.14). Suppose that ϕ and $\tilde{\phi}$ form a dual pair, i.e.,

$$\langle \phi(\cdot), \tilde{\phi}(\cdot - k) \rangle = \delta_{0k}, \tag{2.23}$$

and satisfy the estimates

$$\begin{aligned} |\hat{\phi}(\xi)| &\leq c_4 (1 + \|\xi\|)^{-1-\varepsilon} \\ \left|\hat{\tilde{\phi}}(\xi)\right| &\leq c_4 (1 + \|\xi\|)^{-1-\varepsilon}, \quad \text{for some } \varepsilon > 0. \end{aligned} \tag{2.24}$$

Moreover, assume that there exist symbols $c^e(z), d^e(z), e \in E^*$, $\{c_k^e\}_{k\in\mathbb{Z}^2}, \{d_k^e\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2)$ related by the condition

$$\begin{pmatrix} a(z) & a((-1)^{e_1} z) & \cdots & a((-1)^{e_3} z) \\ c^{e_1}(z) & \cdots & \cdots & c^{e_1}((-1)^{e_3} z) \\ \vdots & & & \vdots \\ c^{e_3}(z) & \cdots & \cdots & c^{e_3}((-1)^{e_3} z) \end{pmatrix} \times \begin{pmatrix} \overline{b(z)} & \overline{d^{e_1}(z)} & \cdots & \overline{d^{e_3}(z)} \\ \overline{b((-1)^{e_1} z)} & \cdots & \cdots & \overline{d^{e_3}((-1)^{e_1} z)} \\ \vdots & & & \vdots \\ \overline{b((-1)^{e_3} z)} & \cdots & \cdots & \overline{d^{e_3}((-1)^{e_3} z)} \end{pmatrix} = 2^4 E. \tag{2.25}$$

Then the functions $\{\psi_e\}_{e\in E^*}, \{\tilde{\psi}_{\tilde{e}}\}_{\tilde{e}\in E^*}$ defined by

$$\hat{\psi}_e(\xi) = \frac{c^e(z)}{4} \hat{\phi}\left(\frac{\xi}{2}\right), \quad \hat{\tilde{\psi}}_e(\xi) = \frac{d^e(z)}{4} \hat{\tilde{\phi}}\left(\frac{\xi}{2}\right), \tag{2.26}$$

are biorthogonal in the sense that

$$\left\langle \psi_{j,e,k}, \tilde{\psi}_{\tilde{j},\tilde{e},\tilde{k}} \right\rangle = \delta_{j\tilde{j}} \delta_{e\tilde{e}} \delta_{k\tilde{k}}. \tag{2.27}$$

Furthermore, the series

$$f = \sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left\langle f, \tilde{\psi}_{j,e,k} \right\rangle \psi_{j,e,k} \tag{2.28}$$

converges strongly in L^2 and the system $\{\psi_{j,e,k}\}_{e\in E^*, j\in\mathbb{Z}, k\in\mathbb{Z}^2}$ forms a global Riesz basis, i.e., there exist $c_5 > 0, c_6 < \infty$ such that

$$\begin{aligned} c_5 \sum_{e\in E^*} \|\lambda^e\|_{l_2} &\leq \left\| \sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \lambda_{j,k}^e 2^j \psi_e(2^j \cdot -k) \right\|_2 \\ &\leq c_6 \sum_{e\in E^*} \|\lambda^e\|_{l_2}. \end{aligned} \tag{2.29}$$

Proof. This theorem can be proved by following the lines of the investigations of Dahmen and Micchelli in [12]. The only problem is that Dahmen and Micchelli consider compactly supported functions $\phi, \tilde{\phi}$ and finite sequences $\{a_k\}_{k\in\mathbb{Z}^2}, \{b_k\}_{k\in\mathbb{Z}^2}$. For our applications, it is not convenient to make these restrictions. Therefore, all arguments using compact support have to be revised by employing the decay estimates (2.24) and applying estimation techniques developed by Cohen *et al.* [9]. Let us sketch the most important steps.

First, using Theorem 2.2 in [19], it follows that ϕ is the generator of a multiresolution approximation in $L^2(\mathbb{R}^2)$. Condition (2.25) implies that one has full reconstruction, i.e.,

$$\begin{aligned} V_1 &= V_0 \oplus_{e\in E^*} W_{0,e}, \\ W_{0,e} &= \overline{\operatorname{span}\{\psi_e(\cdot - k), k \in \mathbb{Z}^2\}}. \end{aligned} \tag{2.30}$$

(This is a consequence of Theorem 3.4 stated below.) (2.27) can be proved directly. Let us attack (2.28). As shown by Dahmen and Micchelli [12, Thm. 5.1], it is sufficient to check

$$\|Q_j f\|_2 \leq c_7 \|f\|_2, \tag{2.31}$$

$$\lim_{j\to-\infty} \|Q_j f\| = 0, \tag{2.32}$$

where Q_j denotes the projector

$$\left(Q_j f\right)(x) = \sum_{k\in\mathbb{Z}^2} \left\langle f, \tilde{\phi}_{j,k} \right\rangle \phi_{j,k}(x), \quad \text{for } j \in \mathbb{Z}. \tag{2.33}$$

Employing the decay estimates (2.24), and following the lines of Cohen *et al.* [9, Thm. 3.2], one can show that

$$\begin{aligned} \sum_{k\in\mathbb{Z}^2} |\langle f, \phi_{0,k} \rangle|^2 &\leq c_8 \|f\|_2^2, \\ \sum_{k\in\mathbb{Z}^2} |\langle f, \tilde{\phi}_{0,k} \rangle|^2 &\leq c_8 \|f\|_2^2, \end{aligned} \tag{2.34}$$

$$\begin{aligned} \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \phi_{j,k} \right\rangle\right|^2 &\leq c_9 \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 |\hat{\phi}(2^{-j}\xi)|^{2\delta} d\xi, \\ \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \tilde{\phi}_{j,k} \right\rangle\right|^2 &\leq c_9 \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \left|\hat{\tilde{\phi}}(2^{-j}\xi)\right|^{2\delta} d\xi, \end{aligned} \tag{2.35}$$

where $\delta \in (0,1)$ is sufficiently small, and

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \psi_{j,e,k} \right\rangle\right|^2 &\leq c_{10} \|f\|_2^2, \\ \sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \tilde{\psi}_{j,e,k} \right\rangle\right|^2 &\leq c_{10} \|f\|_2^2. \end{aligned} \tag{2.36}$$

Therefore, by the l_2-stability of ϕ, we get

$$\|Q_0 f\|_2^2 = \left\| \sum_{k\in\mathbb{Z}^2} \langle f, \tilde{\phi}_{0,k}\rangle \phi_{0,k} \right\|_2^2 \leqslant c_{11} \sum_{k\in\mathbb{Z}^2} |\langle f, \tilde{\phi}_{0,k}\rangle|^2 \leqslant c_8 c_{11} \|f\|_2^2,$$

proving (2.31). Statement (2.32) can be shown analogously,

$$\begin{aligned}\lim_{j\to-\infty} \|Q_j f\|_2^2 &= \lim_{j\to-\infty} \left\| \sum_{k\in\mathbb{Z}^2} \left\langle f, \tilde{\phi}_{j,k}\right\rangle \phi_{j,k} \right\|_2^2 \\ &\leqslant \lim_{j\to-\infty} c_{11} \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \tilde{\phi}_{j,k}\right\rangle\right|^2 \\ &\leqslant c_9 c_{11} \int_{\mathbb{R}^2} |\hat{f}(\xi)|^2 \lim_{j\to\infty} \left|\hat{\tilde{\phi}}(2^{-j}\xi)\right|^{2\delta} d\xi = 0,\end{aligned}$$

where we used (2.35), the dominated convergence theorem, and (2.24).

It remains to show (2.29). It is easy to see that (2.28) and (2.36) imply that $\{\psi_{j,e,k}\}_{j\in\mathbb{Z}, e\in E^*, k\in\mathbb{Z}^2}$ constitute a *frame*, this means that there exist constants $c_{12} > 0, c_{13} < \infty$ such that

$$c_{12} \|f\|_2^2 \leqslant \sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \psi_{j,e,k}\right\rangle\right|^2 \leqslant c_{13} \|f\|_2^2. \tag{2.37}$$

The upper bound is exactly (2.36). The lower bound can be obtained as follows:

$$\begin{aligned}\|f\|_2 &= \sup_{\|g\|_2=1} |\langle f, g\rangle| \\ &= \sup_{\|g\|_2=1} \left| \left\langle f, \sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left\langle g, \tilde{\psi}_{j,e,k}\right\rangle \psi_{j,e,k} \right\rangle \right| \\ &\leqslant \sup_{\|g\|_2=1} \left(\sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \psi_{j,e,k}\right\rangle\right|^2 \right)^{1/2} \times \left(\sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left|\left\langle g, \tilde{\psi}_{j,e,k}\right\rangle\right|^2 \right)^{1/2} \\ &\leqslant c_{10}^{1/2} \left(\sum_{j\in\mathbb{Z}} \sum_{e\in E^*} \sum_{k\in\mathbb{Z}^2} \left|\left\langle f, \psi_{j,e,k}\right\rangle\right|^2 \right)^{1/2},\end{aligned}$$

where we used the Cauchy–Schwarz inequality and (2.36). But (2.27) implies that the system $\{\psi_{j,e,k}\}$ is linearly independent. By this it can be shown that (2.37) implies (2.29); see Young [31] for details. ■

When we have found $a(z), b(z)$ such that (2.21) is satisfied, we have to check that the infinite products (2.22) converge and that the conditions (2.23) and (2.24) are fulfilled. (Using the refinement equations (2.17) one can check that (2.21) is necessary for (2.23), but it is not sufficient. Furthermore, in contrary to the orthonormal case, there are no direct arguments that guarantee (2.24)). Concerning the problems mentioned above, one has the following three lemmata. They can be proved by following the lines of Daubechies [15, Lemma 3.1, Lemma 3.2] and Cohen *et al.* [9, Prop. 4.9], respectively.

LEMMA 2.3. *Suppose that, for some* $\varepsilon > 0$

$$\sum_{k\in\mathbb{Z}^2} |a_k| \, \|k\|^\varepsilon < \infty \quad \textit{and} \quad \sum_{k\in\mathbb{Z}^2} |b_k| \, \|k\|^\varepsilon < \infty. \tag{2.38}$$

Then (2.22) *converges pointwise, for all* $\xi \in \mathbb{R}^2$. *The convergence is uniform on compact sets.*

LEMMA 2.4. *If*

$$a(z) = 4\left(\frac{1+e^{-i(\xi_1/2)}}{2}\right)^N \left(\frac{1+e^{-i(\xi_2/2)}}{2}\right)^N Q(z_1, z_2),$$

where $Q(z_1, z_2) = \sum_k q_k z^k$ *satisfies*

$$\sum_{k\in\mathbb{Z}^2} |q_k| \, \|k\|^\varepsilon < \infty \quad \textit{for some} \quad \varepsilon > 0,$$

$$Q(1,1) = 1, \tag{2.39}$$

and

$$\sup_{z_1,z_2} \left| Q(z_1,z_2) Q\left(z_1^2, z_2^2\right) \cdots Q\left(z_1^{2^{m-1}}, z_2^{2^{m-1}}\right) \right| =: B_m,$$

$$z = e^{-i(\xi/2)}, \tag{2.40}$$

then there exists $c_{14} > 0$ *such that, for all* $\xi \in \mathbb{R}^2$

$$\left| \prod_{j=1}^{\infty} \frac{a\left(e^{-i(\xi/2^j)}\right)}{4} \right| \leqslant c_{14} \left(1 + \|\xi\|\right)^{-N + ((\log B_m)/(m \log 2))} \tag{2.41}$$

LEMMA 2.5. *Assume that* $a(z)$ *and* $b(z)$ *satisfying* (2.21) *can be factored as*

$$a(z) = 4\left(\frac{1+e^{-i(\xi_1/2)}}{2}\right)^N \left(\frac{1+e^{-i(\xi_2/2)}}{2}\right)^N Q(z_1, z_2),$$

$$b(z) = 4\left(\frac{1+e^{-i(\xi_1/2)}}{2}\right)^{\tilde{N}} \left(\frac{1+e^{-i(\xi_2/2)}}{2}\right)^{\tilde{N}} \tilde{Q}(z_1, z_2),$$

and suppose that (2.39) *holds for* Q *and* $\tilde{Q}$. *Furthermore, suppose that, for some* $m, \tilde{m} > 0$,

$$B_m < 2^{m(N-1)} \quad \text{and} \quad \tilde{B}_{\tilde{m}} < 2^{\tilde{m}(\tilde{N}-1)} \tag{2.42}$$

where B_m, $\tilde{B}_{\tilde{m}}$ *are defined by* (2.40).

Then $\phi, \tilde{\phi} \in L^2(\mathbb{R}^2)$ *and* $\int_{\mathbb{R}^2} \phi(x)\tilde{\phi}(x-n)dx = \delta_{0n}$.

3. THE DIRECT APPROACH

After the preliminaries in the preceeding section we are now ready to construct wavelets adapted to a given pseudodifferential operator in two dimensions. Let us first briefly recall the definition of pseudodifferential operators. To do this consider the Fourier inversion formula

$$u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi, \quad u \in C_0^\infty(\mathbb{R}^n). \tag{3.1}$$

Differentiating this expression and employing the notation $D_j = (1/i)(\partial/\partial x_j)$ yields

$$D^\alpha u(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \xi^\alpha e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi. \tag{3.2}$$

Let $P = p(x, D) = \sum_{|\alpha|\leqslant k} a_\alpha(x) D^\alpha$ be a differential operator defined on a domain $\Omega \subset \mathbb{R}^n$. Then

$$(Pu)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} p(x,\xi)\, e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi. \tag{3.3}$$

Instead of the polynomial $p(x, \xi)$ one can take a function $\sigma(x, \xi)$ belonging to a more general class of functions. Then the operator L defined by

$$(Lu)(x) := \int_{\mathbb{R}^n} \sigma(x,\xi)\, e^{ix\cdot\xi} \hat{u}(\xi)\, d\xi \tag{3.4}$$

is called a *pseudodifferential operator* with *symbol* $\sigma(x, \xi)$. (For details we refer to the standard literature concerning pseudodifferential operators, e.g., Taylor [30].) Since we are dealing with shift-invariant spaces we will henceforth assume that the symbol σ is independent of x.

We are interested in the numerical solution of pseudodifferential equations of the type

$$Lu = f \tag{3.5}$$

on a sufficiently smooth domain $\Omega \subset \mathbb{R}^2$. We treat this equation by means of a Galerkin approach. For this we employ the spaces V_j of a multiresolution analysis as approximation spaces. Therefore the weak formulation of (3.5) is to find $u \in V_j$ such that

$$\langle Lu, v_k \rangle = \langle f, v_k \rangle, \tag{3.6}$$

where $\{v_k\}_{k\in\mathbb{Z}^2}$ is a basis of V_j. (In practice, one only deals with a finite collection of functions according to the domain Ω on which (3.5) is defined.)

The structure of the resulting stiffness matrix clearly depends on the pseudodifferential operator L and the basis of V_j. If we choose the basis in a way that the basis functions corresponding to different refinement levels are orthogonal with respect to the bilinear form induced by L, the matrix has block diagonal form. To obtain such a basis, we have to construct wavelets $\{\psi_e\}_{e\in E^*}$ such that

$$\langle L\phi(\cdot - k), \psi_e(\cdot - l)\rangle = 0 \quad \forall k, l \in \mathbb{Z}^2, e \in E^*, \tag{3.7}$$

holds, where ϕ is the generator of the multiresolution analysis. Theorem 3.1 below states the conditions for the existence of such a wavelet basis.

Contrary to classical elliptic boundary value problems, it is not clear that a Galerkin approach converges for our more general setting. Quite recently, Dahmen *et al.* [13, 14] studied this problem for the general case of Petrov–Galerkin schemes arising from pseudodifferential equations on the torus T^n. They present necessary and sufficient conditions for convergence and stability for pseudodifferential operators of the class $S^m_{1,0}$ which is the subclass of Hörmander's class with the property that

$$\left| D_x^\beta D_\xi^\alpha \sigma(x,\xi) \right| \leqslant c_{\alpha,\beta} (1 + \|\xi\|)^{m-|\alpha|} \tag{3.8}$$

holds for all multi-indices α, β. Their conditions are formulated with respect to the numerical symbol τ defined by

$$\tau(\omega) := \sum_{k\in\mathbb{Z}^n} \sigma(\omega + k)\hat{\phi}(\omega + k)\hat{\eta}(\omega + k), \tag{3.9}$$

where η is a fixed distribution according to the Petrov–Galerkin scheme. Following Dahmen *et al.* we will henceforth assume that σ satisfies (3.8).

For the construction of our wavelet-basis, it is furthermore necessary to claim the validity of the following estimation that relates the symbol σ with the generator ϕ. Suppose there exist constants $c_1, c_2 \in \mathbb{R}, 0 < c_1 \leqslant c_2 < \infty$, with

$$c_1 \leqslant \sum_{m\in\mathbb{Z}^2} \left|\hat{\phi}\left(\frac{\xi + 4\pi m}{2}\right)\right|^2 \sigma(\xi + 4\pi m) \leqslant c_2. \tag{3.10}$$

Furthermore, we will assume that the series in the expression above converges absolutely to a function in $L^1([0, 4\pi)^2)$ and that the Fourier coefficients of the limit are in $l_1(\mathbb{Z}^2)$, i.e.,

$$\sum_{m\in\mathbb{Z}^2} \left|\hat{\phi}\left(\frac{\xi + 4\pi m}{2}\right)\right|^2 \sigma(\xi + 4\pi m) \sim \sum_{m\in\mathbb{Z}^2} d_m e^{i4\pi m\cdot\xi}, \quad \{d_m\}_{m\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2). \tag{3.11}$$

THEOREM 3.1. *Let L be a pseudodifferential operator with real symbol σ such that* (3.10) *and* (3.11) *hold. Furthermore, let $\phi \in \mathcal{L}^2(\mathbb{R}^2)$, l_2-stable, and skew-symmetric about some point $c_\phi \in \mathbb{R}^2$, i.e.,*

$$\phi(c_\phi + x) = \overline{\phi(c_\phi - x)}, \quad x \in \mathbb{R}^2. \tag{3.12}$$

Suppose that ϕ is refinable,

$$\phi(x) = \sum_{k\in\mathbb{Z}^2} a_k \phi(2x-k), \quad \{a_k\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2). \tag{3.13}$$

Then ϕ generates a multiresolution analysis and there exist wavelets $\{\psi_e\}_{e\in E^*}$ such that

(i) $V_1 = V_0 \oplus W_0$, $W_0 = \overline{\operatorname{span}\{\psi_e(\cdot - k) \mid k \in \mathbb{Z}^2, e \in E^*\}}$,
(ii) $\langle L\phi(\cdot - k), \psi_e(\cdot - l)\rangle = 0 \ \forall k, l \in \mathbb{Z}^2, e \in E^*$.
(iii) *The system $\{\psi_e\}_{e\in E^*}$ is l_2-stable.*

Proof. From the conditions stated above it follows immediately that ϕ generates a multiresolution approximation of $L^2(\mathbb{R}^2)$; see Jia and Micchelli [19, Theorem 2.2]. We start the proof of the remaining parts by showing (ii). We do this by specifying an equivalent condition (see (3.16) below) for the validity of (ii). To solve equation (3.16) we apply a result of Riemenschneider and Shen presented in [29] (see Lemma 3.3 below). Second, we prove the l_2-stability of the system $\{\psi_e\}_{e\in E^*}$. For that we also employ a result from [29]. Third, to complete the proof, we have to show (i), for this purpose we use a theorem of Jia and Micchelli [19] (see Theorem 3.4 below).

For simplification we use the following notation

$$G(\xi) := \sum_{m\in\mathbb{Z}^2} \left| \hat{\phi}\left(\frac{\xi + 4\pi m}{2}\right)\right|^2 \sigma(\xi + 4\pi m). \tag{3.14}$$

Let ψ_e be defined by

$$\hat{\psi}_e(\xi) = \frac{c^e(z)}{4}\hat{\phi}\left(\frac{\xi}{2}\right), \quad \{c_k^e\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2). \tag{3.15}$$

First of all, we show that the system $\{\psi_e\}_{e\in E^*}$ satisfies (ii) if and only if

$$\sum_{\tilde{e}\in E} a\left((-1)^{\tilde{e}} z\right) \overline{c^e\left((-1)^{\tilde{e}} z\right)} G(\xi + 2\pi\tilde{e}) = 0, \qquad e \in E^*, \xi \in \mathbb{R}^2. \tag{3.16}$$

A direct computation employing the definition of L and the functional equations (3.13), (3.15) yields

$$\begin{aligned}
&\langle L\phi(\cdot - k), \psi_e(\cdot - l)\rangle \\
&= \left\langle \int_{\mathbb{R}^2} e^{ix\cdot\xi}\sigma(\xi)\, \widehat{\phi(\cdot - k)}(\xi)\, d\xi, \psi_e(\cdot - l)\right\rangle \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{ix\cdot\xi}\sigma(\xi)\, e^{-ik\cdot\xi}\hat{\phi}(\xi)\, d\xi \overline{\psi_e(x-l)} dx \\
&= \int_{\mathbb{R}^2} \sigma(\xi)\, e^{-ik\cdot\xi}\hat{\phi}(\xi) \overline{\int_{\mathbb{R}^2} e^{-ix\cdot\xi}\psi_e(x-l)\, dx}d\xi \\
&= \int_{\mathbb{R}^2} \sigma(\xi)\, e^{-ik\cdot\xi}\hat{\phi}(\xi)\cdot \overline{\widehat{\psi_e(\cdot - l)}(\xi)} d\xi \\
&= \int_{\mathbb{R}^2} \sigma(\xi)\, e^{-i(k-l)\cdot\xi}\hat{\phi}(\xi)\, \overline{\hat{\psi}_e(\xi)} d\xi \\
&= \frac{1}{16}\sum_m \int_{[0,2\pi)^2 + 2\pi m} e^{-i(k-l)\cdot\xi}\sigma(\xi) \\
&\quad \times a(z)\hat{\phi}\left(\frac{\xi}{2}\right)\overline{c^e(z)\hat{\phi}\left(\frac{\xi}{2}\right)} d\xi \\
&= \frac{1}{16}\int_{[0,2\pi)^2} e^{-i(k-l)\cdot\xi} \sum_{m\in\mathbb{Z}^2} \sigma(\xi + 2\pi m)\, a\left(e^{-i((\xi+2\pi m)/2)}\right) \\
&\qquad \times \overline{c^e\left(e^{-i((\xi+2\pi m)/2)}\right)} \cdot \left|\hat{\phi}\left(\frac{\xi + 2\pi m}{2}\right)\right|^2 d\xi.
\end{aligned}$$

Splitting up the sum into summations over the coarser lattices $\tilde{e} + 2\mathbb{Z}^2, \tilde{e} \in E$, leads us to

$$\begin{aligned}
&\langle L\phi(\cdot - k), \psi_e(\cdot - l)\rangle \\
&= \frac{1}{16}\int_{[0,2\pi)^2} e^{-i(k-l)\cdot\xi} \sum_{\tilde{e}\in E} a\left((-1)^{\tilde{e}} z\right) \overline{c^e\left((-1)^{\tilde{e}} z\right)} \\
&\quad \times \sum_{m\in\mathbb{Z}^2} \left|\hat{\phi}\left(\frac{\xi + 2\pi(\tilde{e} + 2m)}{2}\right)\right|^2 \\
&\quad \times \sigma(\xi + 2\pi(\tilde{e} + 2m))\, d\xi \\
&= \frac{1}{16}\int_{[0,2\pi)^2} e^{-i(k-l)\cdot\xi} \sum_{\tilde{e}\in E} a\left((-1)^{\tilde{e}} z\right) \\
&\qquad \times \overline{c^e\left((-1)^{\tilde{e}} z\right)} G(\xi + 2\pi\tilde{e})\, d\xi.
\end{aligned}$$

It follows that

$$\sum_{\tilde{e}\in E} a\left((-1)^{\tilde{e}} z\right) \overline{c^e\left((-1)^{\tilde{e}} z\right)} G(\xi + 2\pi\tilde{e}) = 0$$

is a necessary and sufficient condition for (ii) to hold. (One can check that all the operations performed above are justified by our decay and integrability conditions.)

Therefore, to construct the wavelet basis, we have to solve the equation (3.16). Then, for the obtained solutions we have to check that the corresponding wavelets satisfy the conditions (i) and (iii).

To solve (3.16) let us introduce a function $\eta : E \to E$, which is required to satisfy the conditions

$$\begin{gathered}
\eta(0) = 0 \quad \text{and} \\
(\eta(e_1) + \eta(e_2))(e_1 + e_2) \quad \text{is odd, when } e_1 \neq e_2.
\end{gathered} \tag{3.17}$$

Remark 3.2. In [28], Riemenschneider and Shen give an example of a function η that satisfies (3.17)

$$\eta(0,0) = (0,0), \quad \eta(0,1) = (0,1),$$
$$\eta(1,0) = (1,1), \quad \eta(1,1) = (1,0).$$

Using such a function, we can apply the following result, proved by Riemenschneider and Shen [29, Lemma 2.12]. Similar things were also developed by Chui *et al.* [7].

LEMMA 3.3. *Suppose K is a 2π-periodic function that satisfies*

$$\overline{K(\xi)} = e^{i2c\cdot\xi} K(\xi) \quad \text{for some } c \in \mathbb{Z}^2/2. \tag{3.18}$$

Then for η defined by (3.17) *the functions*

$$K_e(\xi) = \begin{cases} e^{i\eta(e)\cdot\xi} K(\cdot + \pi e), & \text{if } 2c\cdot e \text{ is even}, \\ e^{i\eta(e)\cdot\xi}\overline{K(\cdot + \pi e)}, & \text{if } 2c\cdot e \text{ is odd}, \end{cases} \tag{3.19}$$

where $e \in E$, satisfy

$$\sum_{\tilde{e}\in E} K_{e_1}(\cdot + \pi\tilde{e})\overline{K_{e_2}(\cdot + \pi\tilde{e})} = 0, \quad e_1, e_2 \in E. \tag{3.20}$$

Now, let us return to the proof of Theorem 3.1, i.e., to the solution of (3.16). The equalities (3.16) are equivalent to

$$\sum_{\tilde{e}\in E} a\left((-1)^{\tilde{e}} e^{-i\xi}\right) \overline{c^e\left((-1)^{\tilde{e}} e^{-i\xi}\right)} G\left(2\left(\xi + \pi\tilde{e}\right)\right) = 0, \quad e \in E^*. \tag{3.21}$$

To solve (3.21) we want to apply Lemma 3.3 with

$$K(\xi) := a\left(e^{-i\xi}\right) G(2\xi) \quad \text{and} \quad c = c_\phi. \tag{3.22}$$

The function $a(e^{-i\xi})G(2\xi)$ is obviously 2π-periodic. Furthermore, condition (3.12) implies

$$\overline{a\left(e^{-i\xi}\right)} = e^{i2c_\phi\xi} a\left(e^{-i\xi}\right). \tag{3.23}$$

Therefore, since G is real, the conditions of Lemma 3.3 are satisfied and we get the following solutions

$$c^e(z) = \begin{cases} e^{i\eta(e)\cdot(\xi/2)} a\left((-1)^e z\right) G\left(\xi + 2\pi e\right) & \text{if } 2c_\phi \cdot e \text{ is even}, \\ e^{i\eta(e)\cdot(\xi/2)} \overline{a\left((-1)^e z\right)} G\left(\xi + 2\pi e\right) & \text{if } 2c_\phi \cdot e \text{ is odd}. \end{cases} \tag{3.24}$$

Now we have to check that $\{\psi_e\}_{e\in E^*}$ is an l_2-stable system. According to Riemenschneider and Shen [29, Prop. 3.6] we have to show that the matrix

$$\mathcal{U} := \left(c^e\left((-1)^{\tilde{e}} e^{-i\xi}\right)\right)_{e\in E^*, \tilde{e}\in E}$$

has full rank. It follows from (3.20) that $\mathcal{U}\bar{\mathcal{U}}^{\mathrm{T}}$ is a diagonal matrix with diagonal entries

$$\left(\mathcal{U}\bar{\mathcal{U}}^{\mathrm{T}}\right)_{ee} = \sum_{\tilde{e}\in E} |K_e(\cdot + \pi\tilde{e})|^2 = \sum_{\tilde{e}\in E} \left|a(-1)^{\tilde{e}} e^{-i\xi})\right|^2 \left|G\left(2\xi + 2\pi\tilde{e}\right)\right|^2.$$

Using (3.10), all we have to show is

$$\sum_{\tilde{e}\in E} \left|a\left((-1)^{\tilde{e}} e^{-i\xi}\right)\right|^2 > 0. \tag{3.25}$$

This is an easy consequence of the fact that ϕ is l_2-stable; see [29] for details.

To complete the proof of Theorem 3.1 it remains to show (i). It will be convenient to define the semidiscrete convolution product $\phi *' a$ by

$$\phi *' a := \sum_{k\in\mathbb{Z}^s} a_k \phi(\cdot - k)$$

for a given function $\phi \in \mathcal{L}^2(\mathbb{R}^s)$ and a sequence $a \in l_\infty$. We will employ the following theorem, proved by Jia and Micchelli [19, Thm. 4.3].

THEOREM 3.4. *Suppose that $\phi_1, \ldots, \phi_n \in \mathcal{L}^2(\mathbb{R}^s)$ have stable integer translates. Let $\psi_1, \ldots, \psi_n$ be functions given by*

$$\psi_j = \sum_{k=1}^{n} \phi_k *' a_{jk}, \quad j = 1, \ldots, n, \quad a_{jk} \in l_1. \tag{3.26}$$

Then the following conditions are equivalent:

(i) *$\psi_1, \ldots, \psi_n$ have stable integer translates.*
(ii) *The matrix $A(z) := (a_{jk}(z))$ is nonsingular for every $z \in T^s$.*
(iii) $\overline{\operatorname{span}\{\phi_1(\cdot - k_1), \ldots, \phi_n(\cdot - k_n), k_i \in \mathbb{Z}^s\}} = \overline{\operatorname{span}\{\psi_1(\cdot - k_1), \ldots, \psi_n(\cdot - k_n), k_i \in \mathbb{Z}^s\}}$.

Remark 3.5. "Stable integer translates" in the sense of Jia and Micchelli means that either (2.14) or the inequality

$$\left\|\sum_{j=1}^{n} \psi_1 *' \lambda_j\right\|_1 \geqslant c \sum_{j=1}^{n} \|\lambda_j\|_{l_1}$$

holds, i.e., $\psi_1, \ldots, \psi_n$ are l_1- or l_2-stable. Jia and Micchelli have shown that these two conditions are equivalent for functions in $\mathcal{L}^2$; see [19, Thm. 4.2]. This fact can be used to say something more about our wavelet system $\{\psi_e\}_{e\in E^*}$. Since we assume that $\{c_k^e\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2), e \in E^*$, it is easy to check that $\psi_e \in \mathcal{L}^2, e \in E^*$. (See [19, Thm. 2.1].) Therefore the l_2-stability of $\{\psi_e\}_{e\in E^*}$ implies that $\{\psi_e\}_{e\in E^*}$ is also l_1-stable.

We apply Theorem 3.4 to the functions

$$\phi_e(\cdot) = \phi(2\cdot - e), \quad e \in E \tag{3.27}$$

and to the sequences

$$a_{e\tilde{e}} = \begin{cases} \{c^e_{\tilde{e}+2k}\}_{k\in\mathbb{Z}^2}, & e \in E^* \\ \{a_{\tilde{e}+2k}\}_{k\in\mathbb{Z}^2}, & e = (0,0). \end{cases} \tag{3.28}$$

Since $V_1 = \overline{\operatorname{span}\{\phi_e(\cdot - k), e \in E, k \in \mathbb{Z}^2\}}$, it follows from the equivalence of the statements (i) and (iii) in Theorem 3.4 that we have to show:

The system $(\phi, \{\psi_e\}_{e\in E^*})$ has l_2-stable integer translates. (3.29)

To prove (3.29) it is sufficient to show that there exist constants c_3, c_4 such that

$$c_3 \|f\|_2^2 \leqslant \langle Lf, f\rangle \leqslant c_4 \|f\|_2^2 \tag{3.30}$$

for functions $f \in V_1$ that can be represented in the form

$$f = \sum_{k\in\mathbb{Z}^2} b_k \phi(2x-k), \quad \{b_k\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2).$$

Assuming for the moment that (3.30) holds, the statement (3.29) can be proved as follows. Exploiting the l_2-stability of ϕ and $\{\psi_e\}_{e\in E^*}$ as well as the condition (3.30), part (ii) and the fact that assuming $\sigma(\xi)$ to be real implies that L is self-adjoint, we obtain for finite sequences $\{\lambda^e\}_{e\in E}$

$$\begin{aligned}
\sum_{e\in E} \|\lambda^e\|_{l_2}^2 &\leqslant \sum_{e\in E^*} \|\lambda^e\|_{l_2}^2 + c_5 \left\| \sum_{k\in\mathbb{Z}^2} \lambda_k^0 \phi(\cdot - k)\right\|_2^2 \\
&\leqslant c_6 \left\| \sum_{e\in E^*}\sum_{k\in\mathbb{Z}^2} \lambda_k^e \psi_e(\cdot - k)\right\|_2^2 + c_5 \left\| \sum_{k\in\mathbb{Z}^2} \lambda_k^0 \phi(\cdot - k)\right\|_2^2 \\
&\leqslant \frac{\max\{c_5, c_6\}}{c_3} \left\{ \left\langle L\left(\sum_{e\in E^*} \psi_e *' \lambda^e\right), \sum_{e\in E^*} \psi_e *' \lambda^e \right\rangle \right. \\
&\quad \left. + \langle L(\phi *' \lambda^0), \phi *' \lambda^0\rangle \right\} \leqslant \frac{\max\{c_5, c_6\}}{c_3} \\
&\quad \times \left\langle L\left(\sum_{e\in E^*} \psi_e *' \lambda^e + \phi *' \lambda^0\right), \sum_{e\in E^*} \psi_e *' \lambda^e + \phi *' \lambda^0 \right\rangle \\
&\leqslant \frac{\max\{c_5, c_6\} c_4}{c_3} \left\| \sum_{e\in E^*} \psi_e *' \lambda^e + \phi *' \lambda^0 \right\|_2^2
\end{aligned}$$

and (3.29) is proved.

It remains to prove (3.30). First we show the upper bound. Assume that $f \in V_1$ can be represented in the form

$$f(x) = \sum_{k\in\mathbb{Z}^2} b_k \phi(2x-k), \quad \{b_k\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2).$$

Applying Fourier transform yields

$$\hat{f}(\xi) = \frac{1}{4} b(z) \hat{\phi}(\xi/2), \quad z = e^{-i(\xi/2)}.$$

A similar calculation as in the proof of (3.16) shows that

$$\begin{aligned}
\langle Lf, f\rangle &= \frac{1}{16}\int_{\mathbf{R}^2} b(z)\overline{b(z)}\sigma(\xi)\, |\hat{\phi}(\xi/2)|^2\, d\xi \\
&= \frac{1}{16}\sum_{m\in\mathbb{Z}^2} \int_{[0,4\pi)^2 + 4\pi m} |b(z)|^2 \sigma(\xi)\, |\hat{\phi}(\xi/2)|^2\, d\xi \\
&= \frac{1}{16}\int_{[0,4\pi)^2} |b(z)|^2 \sum_{m\in\mathbb{Z}^2} \sigma(\xi + 4\pi m) \\
&\quad \times \left| \hat{\phi}\left(\frac{\xi + 4\pi m}{2}\right)\right|^2 d\xi \\
&= \frac{1}{16}\int_{[0,4\pi)^2} |b(z)|^2 G(\xi)\, d\xi.
\end{aligned}$$

With (3.10) and (3.14) we obtain

$$\langle Lf, f\rangle \leqslant \frac{c_2}{16}\int_{[0,4\pi)^2} \left| b\left(e^{-i\xi/2}\right)\right|^2 d\xi.$$

A change of variables followed by Parseval's equality yields

$$\begin{aligned}
\langle Lf, f\rangle &\leqslant \frac{c_2}{4}\int_{[0,2\pi)^2} \left| \sum_{k\in\mathbb{Z}^2} b_k e^{-i\xi\cdot k}\right|^2 d\xi \\
&= \frac{c_2}{4}\sum_{k\in\mathbb{Z}^2} |b_k|^2 = \frac{c_2}{4}\|b\|_{l_2}^2.
\end{aligned}$$

Finally, using the stability of ϕ, we get the upper bound in (3.30),

$$\langle Lf, f\rangle \leqslant \frac{c_2}{4}\|b\|_{l_2}^2 \leqslant \tilde{c}\left\| \sum_{k\in\mathbb{Z}^2} b_k \phi(\cdot - k)\right\|_2^2 = \tilde{c}\|f\|_2^2.$$

The lower bound in (3.30) can be proved analogously by employing the lower estimate in (3.10). ∎

Theorem 3.1 enables us to construct the whole adapted wavelet basis. This construction illuminates a new aspect appearing in our setting. In contrary to wavelets orthonormal with respect to the usual L^2-inner product one can not work with a fixed wavelet basis. To get full orthogonality, one has to solve a new equation on each refinement level. This implies that the conditions (3.10) and (3.11) have to be also satisfied for scaled versions of the symbol.

COROLLARY 3.6. *Suppose that* (3.10) *and* (3.11) *hold for all symbols*

$$ {}_j\sigma(\xi) = \sigma(2^j\xi) \quad j \in \mathbb{N}_0. \tag{3.31}$$

Let $\{{}_j\psi_e\}_{e\in E^*}$ *be the wavelet basis constructed by Theorem* 3.1 *with respect to* ${}_j\sigma$. *Then the system*

$$\left\{\phi(\cdot - k),\, {}_j\psi_e\left(2^j\cdot - l\right)\right\}_{j\in\mathbb{N}_0, k,l\in\mathbb{Z}^2 e\in E^*}, \tag{3.32}$$

provides a whole adapted wavelet basis in the sense that functions on different scales are orthogonal with respect to the bilinear form induced by the pseudodifferential operator L.

Corollary 3.6 follows immediately from Theorem 3.1 by a change of variables. Since we do not claim to obtain a global Riesz basis in the sense of (2.29), the constants c_1 and c_2 may depend on j.

The result presented above can, for example, be applied to the Helmholtz-equation

$$Lu = -\Delta u + \lambda u = f, \quad \lambda > 0, \tag{3.33}$$

for then L possesses the symbol

$$\sigma(\xi) = \xi_1^2 + \xi_2^2 + \lambda > 0, \tag{3.34}$$

and the l_2-stability of ϕ implies that $\sum_m |\hat{\phi}(\xi/2 + 2\pi m)|^2$ cannot vanish; see Jia and Micchelli [19, Thm. 3.3]. However, the above construction cannot be used to treat the Poisson-equation

$$-\Delta u = f. \tag{3.35}$$

This can be seen as follows. If $\phi \in L^1$ is refinable, then

$$\hat{\phi}(2\pi m) = 0, \quad m \in \mathbb{Z}^2, m \neq 0. \tag{3.36}$$

This result was proved by Cavaretta *et al.* [6, Thm. 8.4], see also Jia and Micchelli [19, Thm. 2.4]. But since $\sigma(0) = 0$ for the Laplace operator, we see that (3.36) has the consequence

$$G(0) = \sum_{m\in\mathbb{Z}^2} |\hat{\phi}(2\pi m)|^2 \sigma(4\pi m) = 0. \tag{3.37}$$

Therefore, to construct a wavelet basis adapted to the Laplace operator, we have to discard the singularity at 0. A quite natural way to do this is using a biorthogonal wavelet approach, which will be described in Section 4.

Remark 3.7. The results presented above can be generalized to problems in three dimensions. (Although everything becomes more complicated from the technical point of view.) However, our approach cannot be extended to more than three dimensions. The crucial point is the function η. As remarked by Riemenschneider and Shen [28], such a function only exists for dimension 1, 2 and 3.

EXAMPLE 3.8. As stated above, our results can be applied to the Helmholtz-equation (3.33). (Using partial integration, we study this problem in the form $\int_{\mathbb{R}^2} \nabla u \cdot \nabla v dx + \lambda \int_{\mathbb{R}^2} uv dx = \int_{\mathbb{R}^2} fv dx$. This leads us to the same formulas for the symbols $c^e(z)$, but we can work with a generator of lower smoothness.) Let us take the Courant finite element as the generator, i.e., the box spline $M(u|X_\mu), X_\mu = \binom{101}{011}$, which is a piecewise linear function whose graph looks like a hexagonal pyramid. To simplify the computations, we use the centralized version of $M(u|X_\mu)$, i.e.,

$$\tilde{M}\left(u\,|X_\mu\right) := M\left(u + \begin{pmatrix}1\\1\end{pmatrix}|X_\mu\right). \tag{3.38}$$

Then $\tilde{M}(u|X_\mu)$ is skew-symmetric about the origin, so (3.12) holds with $c_\phi = 0$. First of all, we have to compute

$$\begin{aligned} G(\xi) &= \sum_{k\in\mathbb{Z}^2} \left|\hat{\phi}\left(\frac{\xi + 4\pi k}{2}\right)\right|^2 \sigma(\xi + 4\pi k) \\ &= \sum_{k\in\mathbb{Z}^2} \left|\hat{\phi}\left(\frac{\xi + 4\pi k}{2}\right)\right|^2 \Big((\xi_1 + 4\pi k_1)^2 \\ &\quad + (\xi_2 + 4\pi k_2)^2\Big) + \lambda \sum_{k\in\mathbb{Z}^2} \left|\hat{\phi}\left(\frac{\xi + 4\pi k}{2}\right)\right|^2 \\ &=: H(\xi) + \lambda L(\xi). \end{aligned} \tag{3.39}$$

With

$$\widehat{\tilde{M}\left(\cdot\,|X_\mu\right)}(\xi) = \frac{8\sin(\xi_1/2)\sin(\xi_2/2)\sin((\xi_1 + \xi_2)/2)}{\xi_1\xi_2(\xi_1 + \xi_2)}, \tag{3.40}$$

the function $H(\xi)$ can easily be computed as

$$H(\xi) = 16\sin^2\left(\frac{\xi_1}{4}\right) + 16\sin^2\left(\frac{\xi_2}{4}\right). \tag{3.41}$$

The function $L(\xi)$ is given by Riemenschneider and Shen in [28],

$$L(\xi) = \frac{1}{2} + \frac{1}{6}\left(\cos\left(\frac{\xi_1}{2}\right) + \cos\left(\frac{\xi_2}{2}\right) + \cos\left(\frac{\xi_1+\xi_2}{2}\right)\right). \quad (3.42)$$

Furthermore, (3.40) provides

$$a(z) = 4\cos\left(\frac{\xi_1}{4}\right)\cos\left(\frac{\xi_2}{4}\right)\cos\left(\frac{\xi_1+\xi_2}{4}\right), \quad (3.43)$$

so that we finally obtain

$$a(z)G(\xi) = 4\cos\left(\frac{\xi_1}{4}\right)\cos\left(\frac{\xi_2}{4}\right)\cos\left(\frac{\xi_1+\xi_2}{4}\right) \times\left(16\sin^2\left(\frac{\xi_1}{4}\right) + 16\sin^2\left(\frac{\xi_2}{4}\right) + \frac{\lambda}{2} + \frac{\lambda}{6}\left(\cos\left(\frac{\xi_1}{2}\right) + \cos\left(\frac{\xi_2}{2}\right) + \cos\left(\frac{\xi_1+\xi_2}{2}\right)\right)\right). \quad (3.44)$$

According to (3.24), the symbols of our wavelets can be computed from (3.44). (In this case, the symbols $c^e(z)$ are Laurent polynomials, so the resulting wavelets are in fact compactly supported.) For example, one gets:

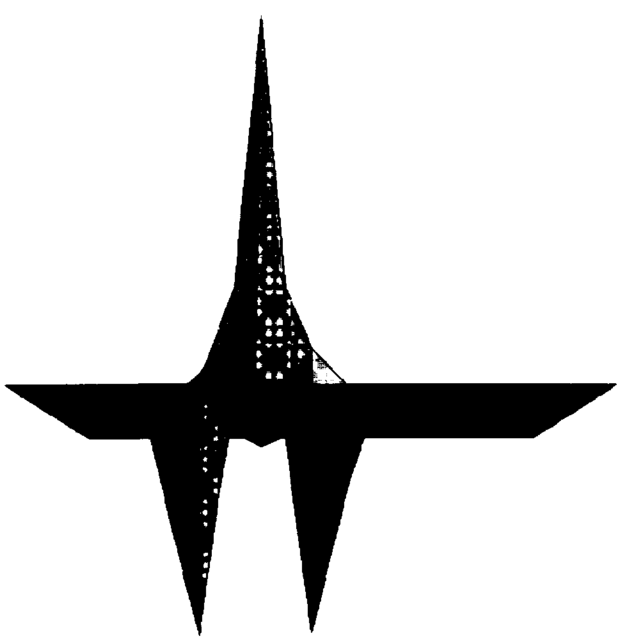

FIG. 1. Wavelet $\psi_{(1,0)}$ for $\lambda = 1$.

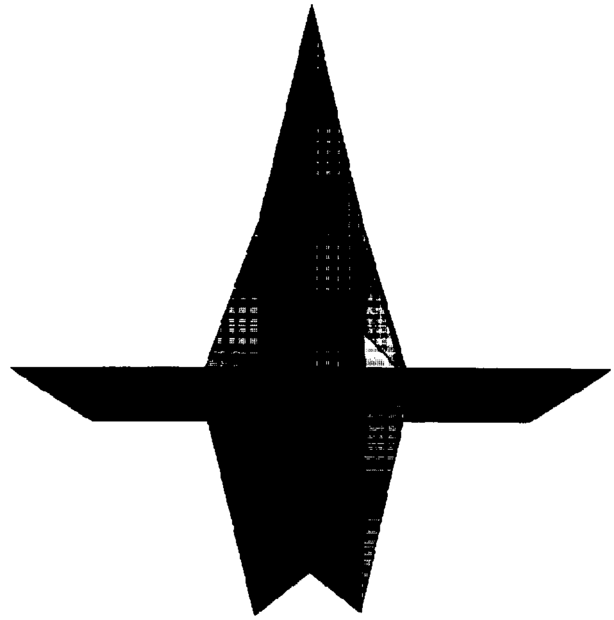

FIG. 2. Wavelet $\psi_{(0,1)}$ for $\lambda = 1$.

$$\begin{pmatrix} c^{\binom{1}{0}}_{-1,-1} & \cdots & c^{\binom{1}{0}}_{-1,3} \\ \vdots & & \vdots \\ c^{\binom{1}{0}}_{3,-1} & \cdots & c^{\binom{1}{0}}_{3,3} \end{pmatrix} = \begin{pmatrix} \frac{\lambda}{24} & -\frac{1}{8}+\frac{\lambda}{12} & -\frac{1}{8}+\frac{\lambda}{24} & 0 & 0 \\ \frac{1}{8}-\frac{\lambda}{12} & -\frac{1}{4}-\frac{5\lambda}{12} & -\frac{1}{8}-\frac{5\lambda}{12} & \frac{1}{4}-\frac{\lambda}{12} & 0 \\ -\frac{1}{8}+\frac{\lambda}{24} & \frac{1}{8}+\frac{5\lambda}{12} & \frac{1}{2}+\frac{3\lambda}{4} & \frac{1}{8}+\frac{5\lambda}{12} & -\frac{1}{8}+\frac{\lambda}{24} \\ 0 & \frac{1}{4}-\frac{\lambda}{12} & -\frac{1}{8}-\frac{5\lambda}{12} & -\frac{1}{4}-\frac{5\lambda}{12} & \frac{1}{8}-\frac{\lambda}{12} \\ 0 & 0 & -\frac{1}{8}+\frac{\lambda}{24} & -\frac{1}{8}+\frac{\lambda}{12} & \frac{\lambda}{24} \end{pmatrix}.$$

The symbols of the remaining wavelets have almost the same form. In Figs. 1–6, we plotted the wavelets for some special values of λ. One sees that, for fixed λ, all three wavelets can be obtained from one function by rotation and translation. This is simply a consequence of (3.24).

4. THE BIORTHOGONAL APPROACH

The applicability of the construction in the last chapter is restricted by the condition (3.10). To overcome this difficulty, we want to employ a biorthogonal approach. In the biorthogonal setting, W_0 is a complement of V_0 in V_1, but (in general) not an orthogonal complement. Therefore, we try to find a biorthogonal wavelet basis such that the "angle" between W_0 and V_0 is determined by the bilinear form induced by a pseudodifferential operator.

Using a biorthogonal approach, we do not have to worry about exact reconstruction, since this is always true on account of Theorem 2.2; see also Dahmen and Micchelli [12]

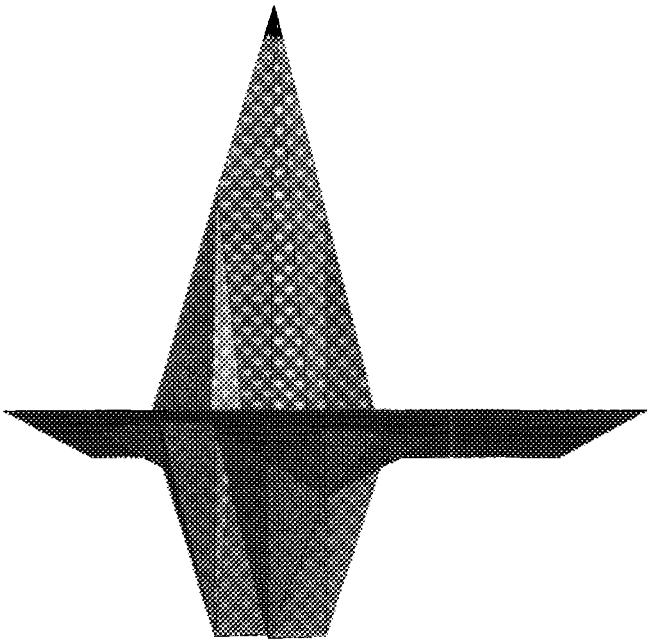

FIG. 3. Wavelet $\psi_{(1,1)}$ for $\lambda = 1$.

for details. In contrast to the wavelets constructed in Section 3, our biorthogonal wavelet basis in general will not have compact support, even if the operator is of simple structure. Let us start with the following lemma.

LEMMA 4.1. *Let $\phi \in \mathcal{L}^2(\mathbb{R}^2)$, l_2-stable, refinable with symbol $a(z)$, $\{a_k\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2)$ and skew-symmetric about a point $c_\phi \in \mathbb{Z}^2/2$. Let $\tilde{\phi}$ be a biorthogonal generator, i.e.,*

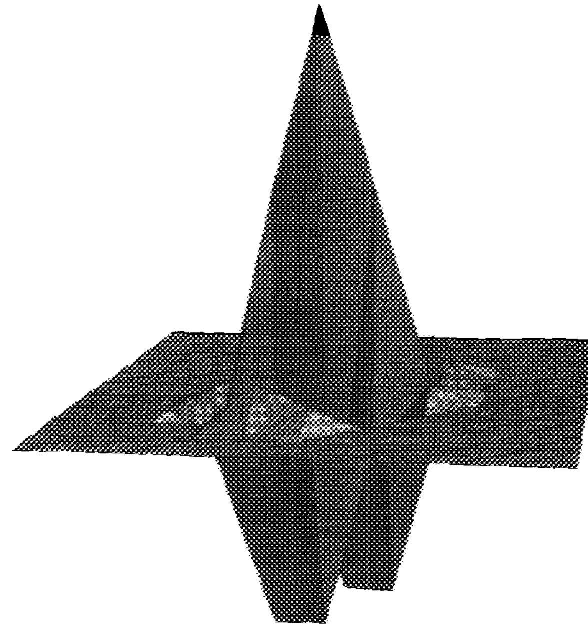

FIG. 5. Wavelet $\psi_{(0,1)}$ for $\lambda = 30$.

$$\langle \phi(\cdot), \tilde{\phi}(\cdot - k)\rangle = \delta_{0k}, \quad k \in \mathbb{Z}^2, \tag{4.1}$$

and suppose that $\tilde{\phi}$ is refinable with symbol $b(z)$, $\{b_k\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2)$, and skew-symmetric about the same point c_ϕ. Furthermore, suppose that $\hat{\phi}$ and $\hat{\tilde{\phi}}$ satisfy the estimates

$$\begin{aligned} |\hat{\phi}(\xi)| &\leq c_1 (1 + \|\xi\|)^{-1-\varepsilon} \\ |\hat{\tilde{\phi}}(\xi)| &\leq c_1 (1 + \|\xi\|)^{-1-\varepsilon}, \quad \textit{for some } \varepsilon > 0. \end{aligned} \tag{4.2}$$

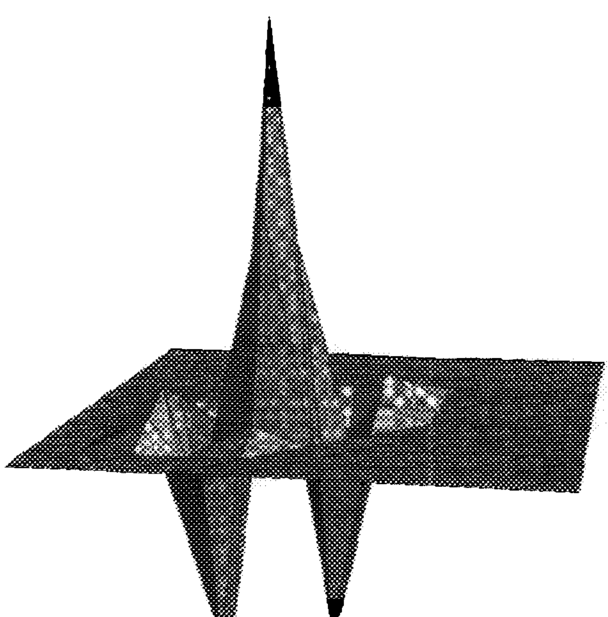

FIG. 4. Wavelet $\psi_{(1,0)}$ for $\lambda = 30$.

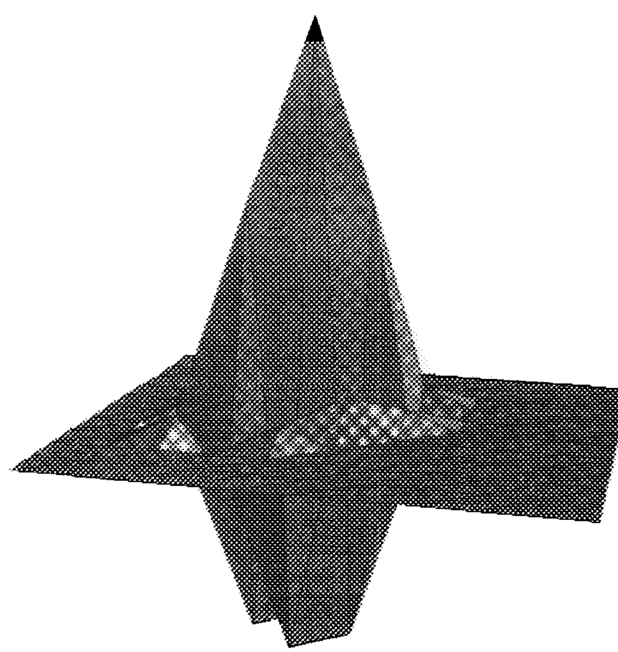

FIG. 6. Wavelet $\psi_{(1,1)}$ for $\lambda = 30$.

Then the system $\{\psi_e\}_{e\in E*}$ *defined by*

$$\hat{\psi}_e(\xi) = \begin{cases} \frac{1}{4} z^{\eta(e)} b((-1)^e z)\hat{\phi}\left(\frac{\xi}{2}\right), & \text{if } 2c_\phi \cdot e \text{ is even,} \\ \frac{1}{4} z^{\eta(e)} \overline{b((-1)^e z)}\hat{\phi}\left(\frac{\xi}{2}\right), & \text{if } 2c_\phi \cdot e \text{ is odd,} \end{cases} \tag{4.3}$$

gives rise to a biorthogonal wavelet basis in the sense of Theorem 2.2.

Proof. We have to check that there exist $(d^e(z))_{e\in E*}$, $\{d_k^e\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2)$, such that (2.25) holds with

$$c^e(z) = \begin{cases} z^{\eta(e)} b((-1)^e z), & \text{if } 2c_\phi \cdot e \text{ is even,} \\ z^{\eta(e)} \overline{b((-1)^e z)}, & \text{if } 2c_\phi \cdot e \text{ is odd.} \end{cases} \tag{4.4}$$

To do this, we have to convince ourselves that

$$\det \begin{pmatrix} a(z) & \cdots & a((-1)^{e_3} z) \\ c^{e_1}(z) & \cdots & c^{e_1}((-1)^{e_3} z) \\ \vdots & & \vdots \\ c^{e_3}(z) & \cdots & c^{e_3}((-1)^{e_3} z) \end{pmatrix} = \det \begin{pmatrix} a(z) & \cdots & a((-1)^{e_3} z) \\ z^{\eta(e_1)} b((-1)^{e_1} z) & \cdots & (-1)^{e_3\cdot\eta(e_1)} z^{\eta(e_1)} b((-1)^{e_1+e_3} z) \\ \vdots & & \vdots \\ z^{\eta(e_3)} b((-1)^{e_3} z) & \cdots & (-1)^{e_3\cdot\eta(e_3)} z^{\eta(e_3)} b(z) \end{pmatrix} \neq 0.$$

It follows from Lemma 3.3 that the matrix

$$\mathcal{U} = \left((-1)^{\tilde{e}\cdot\eta(e)} z^{\eta(e)} b\left((-1)^{e+\tilde{e}} z\right)\right)_{e\in E*, \tilde{e}\in E}$$

has full rank since $\mathcal{U}\bar{\mathcal{U}}^{\mathrm{T}}$ is a diagonal matrix with diagonal entries

$$\left(\mathcal{U}\overline{\mathcal{U}}^{\mathrm{T}}\right)_{ee} = \sum_{\tilde{e}\in E} \left|b\left((-1)^{\tilde{e}} z\right)\right|^2.$$

According to the biorthogonality condition (2.21), which is necessary for (4.1), the sum on the right-hand side cannot vanish. This means that it remains to show that

$$(a(z), \ldots, a((-1)^{e_3} z)) \notin \operatorname{span}\{(c^{e_1}(z), \ldots, c^{e_1}((-1)^{e_3} z)), \ldots, (c^{e_3}(z), \ldots, c^{e_3}((-1)^{e_3} z))\}.$$

Suppose the opposite is true. Then Lemma 3.3 would imply

$$\sum_{e\in E} a((-1)^e z)\overline{b((-1)^e z)} = 0,$$

which is a contradiction to the biorthogonality condition (2.21).

Therefore, we can define $d^e(z)$ by the equation

$$\begin{pmatrix} a(z) & \cdots & a((-1)^{e_3} z) \\ \vdots & & \vdots \\ c^{e_3}(z) & \cdots & c^{e_3}((-1)^{e_3} z) \end{pmatrix} \times \begin{pmatrix} \overline{b(z)} & \cdots & \overline{d^{e_3}(z)} \\ \vdots & & \vdots \\ \overline{b((-1)^{e_3} z)} & \cdots & \overline{d^{e_3}((-1)^{e_3} z)} \end{pmatrix} = \begin{pmatrix} 2^4 & & 0 \\ & \ddots & \\ 0 & & 2^4 \end{pmatrix}.$$

Then this definition makes sense by Lemma 3.3 and the fact that the first matrix is nonsingular. Moreover, it is a consequence of Wiener's lemma that the coefficients of $\{d^e(z)\}_{e\in E*}$ are in $l_1(\mathbb{Z}^2)$. ■

Using Lemma 4.1, we can now prove the following theorem.

THEOREM 4.2. *Let* ϕ *and* $\tilde{\phi}$ *satisfy the conditions of Lemma* 4.1. *Furthermore, let L be a pseudodifferential operator with real symbol* $\sigma(\xi)$ *that satisfies condition* (3.11). *Then* $\{\psi_e\}_{e\in E*}$ *according to* (4.3) *satisfies*

$$\langle L\phi(\cdot - l), \psi_e(\cdot - k)\rangle = 0, \quad l, k \in \mathbb{Z}^2, e \in E*, \tag{4.5}$$

if and only if $a(z)$ *and* $b(z)$ *are related by*

$$b(z) = \frac{2^4 a(z) G(\xi)}{\sum_{e\in E} |a((-1)^e z)|^2 G(\xi + 2\pi e)}, \tag{4.6}$$

where $G(\xi)$ *is defined by* (3.14).

Proof. Let ζ_e be defined by

$$\zeta_e(\xi) = \frac{c^e(z)}{4}\hat{\phi}\left(\frac{\xi}{2}\right), \quad \{c_k^e\}_{k\in\mathbb{Z}^2} \in l_1(\mathbb{Z}^2). \tag{4.7}$$

In the proof of Theorem 3.1 we have already shown that (4.5) is equivalent with

$$\sum_{\tilde{e}\in E} a\left((-1)^{\tilde{e}} z\right)\overline{c^e\left((-1)^{\tilde{e}} z\right)} G(\xi + 2\pi\tilde{e}) = 0. \tag{4.8}$$

We want to construct $\{\psi_e\}_{e\in E*}$ in such a way that (4.5) is satisfied. To this end, we have to insert our expression (4.3) into (4.8). To simplify the computations, let us first assume that $2c_\phi \cdot e$ is even for all $e \in E*$. Then we have

$$\sum_{\tilde{e}\in E} a\left((-1)^{\tilde{e}} z\right)(-1)^{\tilde{e}\cdot\eta(e)}\,\overline{z^{\eta(e)} b\left((-1)^{\tilde{e}+e} z\right)} \times G\left(\xi + 2\pi\tilde{e}\right) = 0, \quad e \in E^*. \quad (4.9)$$

From (4.9) we see that we have to solve a linear system to determine the symbol $b(z)$. In fact, using the abbreviation

$$I(z) := a(z) G(\xi), \quad (4.10)$$

it follows from (4.9) that we have to find a solution of

$$\begin{pmatrix} a(z) & a((-1)^{e_1} z) & a((-1)^{e_2} z) & a((-1)^{e_3} z) \\ -z_1^{-1}z_2^{-1} I((-1)^{e_1} z) & z_1^{-1}z_2^{-1} I(z) & z_1^{-1}z_2^{-1} I((-1)^{e_3} z) & -z_1^{-1}z_2^{-1} I((-1)^{e_2} z) \\ -z_2^{-1} I((-1)^{e_2} z) & -z_2^{-1} I((-1)^{e_3} z) & z_2^{-1} I(z) & z_2^{-1} I((-1)^{e_1} z) \\ -z_1^{-1} I((-1)^{e_3} z) & z_1^{-1} I((-1)^{e_2} z) & -z_1^{-1} I((-1)^{e_1} z) & z_1^{-1} I(z) \end{pmatrix} \cdot \begin{pmatrix} \overline{b(z)} \\ \vdots \\ \overline{b((-1)^{e_3} z)} \end{pmatrix} = \begin{pmatrix} 2^4 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.11)$$

Denoting the matrix on the left-hand side of (4.11) by $A(z)$ we obtain the formula

$$\overline{b(z)} = 2^4 \left(A^{-1}(z)\right)_{11} = \frac{2^4 \left(\operatorname{adj} A(z)\right)_{11}}{\det A(z)}, \quad (4.12)$$

where $\operatorname{adj}A(z)$ is the adjoint matrix of $A(z)$.

Because of the symmetry of the matrix $A(z)$, the above expression can be evaluated directly,

$$\det A(z) = z_1^{-2} z_2^{-2} \sum_{e\in E} a((-1)^e z) \times I((-1)^e z) \left[\sum_{\tilde{e}\in E} I^2\left((-1)^{\tilde{e}} z\right)\right] \quad (4.13)$$

and

$$\left(\operatorname{adj} A(z)\right)_{11} = z_1^{-2} z_2^{-2} I(z) \left(\sum_{\tilde{e}\in E} I^2\left((-1)^{\tilde{e}} z\right)\right). \quad (4.14)$$

This yields

$$\overline{b(z)} = \frac{2^4 I(z)}{\sum_{e\in E} a((-1)^e z) I((-1)^e z)} = \frac{2^4 a(z) G(\xi)}{\sum_{e\in E} \left(a((-1)^e z)\right)^2 G(\xi + 2\pi e)}. \quad (4.15)$$

The general case can be studied in almost the same way. Employing the relation

$$\overline{I(z)} = e^{ic_\phi\cdot\xi} I(z), \quad (4.16)$$

we see that some entries in the matrix $A(z)$ change sign, but the symmetry of $A(z)$ remains essentially the same. A direct check shows that

$$\overline{b(z)} = \frac{2^4 a(z) G(\xi)}{\sum_{e\in E} e^{i2\pi c_\phi\cdot e} a((-1)^e z) I((-1)^e z)}. \quad (4.17)$$

Therefore, we obtain

$$\begin{aligned} b(z) &= \frac{2^4 \overline{a(z) G(\xi)}}{\sum_{e\in E} e^{-i2\pi c_\phi\cdot e}\, \overline{a((-1)^e z)}^2\, \overline{G(\xi + 2\pi e)}} \\ &= \frac{2^4 e^{ic_\phi\xi} a(z)\overline{G(\xi)}}{\sum_{e\in E} e^{-i2\pi c_\phi\cdot e} e^{ic_\phi\cdot(\xi+2\pi e)} \left|a((-1)^e z)\right|^2 \overline{G(\xi + 2\pi e)}} \\ &= \frac{2^4 a(z)\overline{G(\xi)}}{\sum_{e\in E} \left|a((-1)^e z)\right|^2 \overline{G(\xi + 2\pi e)}}. \end{aligned} \quad (4.18)$$

The assumption that σ is real and the fact that ϕ is skew-symmetric imply that $G(\xi)$ is real. This finishes the proof of Theorem 4.2. ■

For a given generator ϕ and a given pseudodifferential operator L, one therefore has to compute the function $G(\xi)$, and to check that the infinite product $\prod_{j=1}^{\infty}(b(e^{-i(\xi/2^j)})/4)$ converges to the Fourier transform of a function $\tilde{\phi}$ such that (4.1) and (4.2) are satisfied. To this end, one can, for example, apply Lemma 2.4. and Lemma 2.5., see the B-spline example below. However, even if (4.1) and (4.2) do not hold, the wavelets constructed by the procedure presented above can be used as basis functions, since one always has full reconstruction in the biorthogonal setting. But then the projectors Q_j according to (2.33) are no longer well-defined and one does not have the expedient expressions (2.28), (2.29). This has, among other disadvantages, the consequence that the facts mentioned in Remark 4.5 below are no longer true.

As stated in Corollary 3.6 the whole adapted wavelet basis is given by the system

$$\left\{\phi(\cdot - k),\ {}_j\psi_e\left(2^j \cdot - l\right)\right\}_{j\in\mathbb{N}_0, e\in E^*, k, l, \in \mathbb{Z}^2}, \quad (4.19)$$

where ${}_j\psi_e$ denotes the wavelet basis constructed by Theorem 4.2 with respect to the symbol

$${}_j\sigma(\xi) := \sigma\left(2^j \xi\right), \quad (4.20)$$

provided that (3.11) holds for all symbols ${}_j\sigma$.

In [10], we performed a similar calculation for the simple one-dimensional model problem

$$-u^{(2m)} = f.$$

We got the solution

$$b(z) = \frac{4a(z)g(\xi)}{|a(z)|^2 g(\xi) + |a(-z)|^2 g(\xi + 2\pi)}, \tag{4.21}$$

where $g(\xi)$ was defined by

$$g(\xi) = \sum_{n\in\mathbb{Z}} (\xi + 4\pi n)^{2m} \left|\hat{\phi}\left(\frac{\xi}{2} + 2\pi n\right)\right|^2. \tag{4.22}$$

According to these formulas, Theorem 4.2 can be interpreted as a natural generalization of the result presented in [10]. In special cases, the one-dimensional construction in [10] gives rise to compactly supported wavelets. In general, they have at least exponential decay. This is not necessarily true for the two-dimensional construction presented here, since in higher dimensions the symbols do not split into linear factors with respect to their roots. The decay of the wavelets depends on the decay of the symbol $b(z)$ and therefore on the smoothness of the expression in (4.6).

Remark 4.3. If σ is homogeneous of degree n in ξ, i.e.,

$$\sigma(r\xi) = r^n \sigma(\xi), \tag{4.23}$$

then formula (4.6) reduces to

$$b(z) = \frac{2^n a(z) G(\xi)}{G(2\xi)}. \tag{4.24}$$

This can be seen as follows:

$$\begin{aligned}
G(2\xi) &= \sum_{k\in\mathbb{Z}^2} |\hat{\phi}(\xi + 2\pi k)|^2 \sigma(2\xi + 4\pi k) \\
&= \sum_{k\in\mathbb{Z}^2} 2^{-4} \left|a\left(e^{-i((\xi/2)+\pi k)}\right)\right|^2 \\
&\quad \times \left|\hat{\phi}\left(\frac{\xi}{2} + \pi k\right)\right|^2 2^n \sigma(\xi + 2\pi k) \\
&= 2^{n-4} \sum_{e\in E} \sum_{k\in\mathbb{Z}^2} \left|a\left(e^{-i((\xi/2)+e\pi+2\pi k)}\right)\right|^2 \\
&\quad \times \left|\hat{\phi}\left(\frac{\xi}{2} + e\pi + 2\pi k\right)\right|^2 \sigma(\xi + 2\pi e + 4\pi k) \\
&= 2^{n-4} \sum_{e\in E} |a((-1)^e z)|^2 \sum_{k\in\mathbb{Z}^2} \left|\hat{\phi}\left(\frac{\xi}{2} + e\pi + 2\pi k\right)\right|^2 \\
&\quad \times \sigma(\xi + 2\pi e + 4\pi k) \\
&= 2^{n-4} \sum_{e\in E} |a((-1)^e z)|^2 G(\xi + 2\pi e).
\end{aligned}$$

EXAMPLE 4.4. We have applied Theorem 4.2 to the Poisson equation

$$-\Delta u = f. \tag{4.25}$$

In this case, one has

$$\sigma(\xi) = \xi_1^2 + \xi_2^2. \tag{4.26}$$

According to Remark 4.3 we obtain the formula

$$b(z) = \frac{4a(z)G(\xi)}{G(2\xi)}. \tag{4.27}$$

We have calculated $b(z)$ for tensor-product cardinal B-splines

$$\tilde{N}_m \otimes \tilde{N}_m(x, y) = \tilde{N}_m(x)\tilde{N}_m(y), \quad m \text{ even}.$$

(We always work with the centralized versions, i.e., $\tilde{N}_m(x) = N_m(x + m/2)$, yielding a real symbol $a(z)$.) From

$$\widehat{\tilde{N}_m \otimes \tilde{N}_m}(\xi_1, \xi_2) = \frac{\sin^m(\xi_1/2)\sin^m(\xi_2/2)}{(\xi_1/2)^m(\xi_2/2)^m}, \tag{4.28}$$

one easily deduces that

$$a(z) = 4\cos^m\left(\frac{\xi_1}{4}\right)\cos^m\left(\frac{\xi_2}{4}\right). \tag{4.29}$$

As an example, we have studied the case $m = 8$. A long-winded but standard computation shows that

$$G(\xi_1, \xi_2) := H(\xi_1)L(\xi_2) + H(\xi_2)L(\xi_1), \tag{4.30}$$

where the functions H and L are defined by

$$\begin{aligned}
H(2y) := (&6081075\cos^{12}(y)\sin^2(y) \\
&+ 22297275\cos^{10}(y)\sin^4(y) + 31621590\cos^8(y)\sin^6(y) \\
&+ 21531510\cos^6(y)\sin^8(y) + 7012005\cos^4(y)\sin^{10}(y) \\
&+ 907725\cos^2(y)\sin^{12}(y) + 21844\sin^{14}(y))/16,
\end{aligned} \tag{4.31}$$

$$\begin{aligned}
L(2y) := (&638512875\cos^{14}(y) \\
&+ 2766889125\cos^{12}(y)\sin^2(y) \\
&+ 4838508675\cos^{10}(y)\sin^4(y) \\
&+ 4339860525\cos^8(y)\sin^6(y) \\
&+ 2087700615\cos^6(y)\sin^8(y) \\
&+ 507350025\cos^4(y)\sin^{10}(y) \\
&+ 50307087\cos^2(y)\sin^{12}(y) + 929569\sin^{14}(y))/32.
\end{aligned} \tag{4.32}$$

We used Lemma 2.4 and Lemma 2.5 to convince ourselves that (4.1) and (4.2) are satisfied. For that purpose, we checked numerically that (2.42) holds with $\tilde{N} = 2, \tilde{m} = 8$. This means that setting

$$b(z) = \frac{4a(z)G(\xi)}{G(2\xi)} = 4\cos^2\left(\frac{\xi_1}{4}\right)\cos^2\left(\frac{\xi_2}{4}\right) \times \left(4\cos^6\left(\frac{\xi_1}{4}\right)\cos^6\left(\frac{\xi_2}{4}\right)\frac{G(\xi)}{G(2\xi)}\right) = 4\left(\frac{1+e^{-i(\xi_1/2)}}{2}\right)^2\left(\frac{1+e^{-i(\xi_2/2)}}{2}\right)^2 \times \left(4e^{i(\xi_1/2)}e^{i(\xi_2/2)}\cos^6\left(\frac{\xi_1}{4}\right)\cos^6\left(\frac{\xi_2}{4}\right)\frac{G(\xi)}{G(2\xi)}\right)$$

and defining

$$\tilde{Q}(z_1,z_2) = 4e^{i(\xi_1/2)}e^{i(\xi_2/2)}\cos^6\left(\frac{\xi_1}{4}\right) \times \cos^6\left(\frac{\xi_2}{4}\right)G(\xi)\,(G(2\xi))^{-1},$$

one has

$$\sup_{z_1,z_2}\left|\tilde{Q}(z_1,z_2)\cdot\ldots\cdot\tilde{Q}\left(z_1^{2^7},z_2^{2^7}\right)\right| =: \tilde{B}_8 < 2^8.$$

This yields

$$-\tilde{N} + \frac{\log\tilde{B}_{\tilde{m}}}{\tilde{m}\log 2} < -1.$$

Remark 4.5. The results presented above can obviously be applied to more general elliptic differential operators. There one studies problems of the form

$$P(D) = f \quad \text{on } \Omega, \quad Bu = g \quad \text{on } \partial\Omega, \tag{4.33}$$

where P is a polynomial of degree $2k, \Omega \subset \mathbb{R}^2$ sufficiently smooth, and B expresses the boundary conditions. Denoting by $||\cdot||_E$ the energy norm induced by the problem (4.33), one requires that there exist constants K_1, K_2 such that

$$K_1\,||f||_{k,2}(\Omega) \leqslant ||f||_E \leqslant K_2\,||f||_{k,2}(\Omega), \quad f \in W^{k,2}(\Omega), \tag{4.34}$$

where $||\cdot||_{k,2}$ denotes the Sobolev norm for the space $W^{k,2}(\Omega)$. Let us suppose that the symbol of the differential operator is of the form (4.23). This case has the advantage that we do not have to change the symbol when going to a higher refinement level (see (4.19)), so that we can work with a "fixed" biorthogonal wavelet basis. Under the above assumptions, it turns out that the condition numbers of the corresponding stiffness matrices are uniformly bounded. In special cases, this can be proved directly by using the orthogonality (for example, by following the lines of the calculations in [10, Sect. 7]). We also can apply the more general results concerning multilevel preconditioning presented by Dahmen and Kunoth [11]. A special case of their investigations is the following.

Let $\phi \in C_0^k$ be refinable and l_2-stable, $V_j = \text{span}\{\phi(2^j\cdot -k), k \in \mathbb{Z}^n\}$. Furthermore, let $\{Q_j\}_{j\in\mathbb{N}_0}$ be a sequence of projectors onto V_j which is uniformly bounded, i.e.,

$$||Q_j|| \leqslant C \quad \forall j \in \mathbb{N}_0. \tag{4.35}$$

Then there exist constants K_1, K_2 such that

$$K_1\sum_j 2^{2kj}\left\|\left(Q_j - Q_{j-1}\right)f\right\|_2^2 \leqslant ||f||_{W^{k,2}}^2 \leqslant K_2\sum_j 2^{2kj}\left\|\left(Q_j - Q_{j-1}\right)f\right\|_2^2. \tag{4.36}$$

If we use the projectors

$$\left(Q_jf\right)(x) = \sum_{k\in\mathbb{Z}^2}\left\langle f,\tilde{\phi}_{j,k}\right\rangle\phi_{j,k} \tag{4.37}$$

and recall the proof of Theorem 2.2, we see that this sequence is uniformly bounded. From (2.28), it is obvious that every $f \in L^2(\mathbb{R}^2)$ has a unique representation

$$f = \sum_{k\in\mathbb{Z}^2}\lambda_k\phi(x-k) + \sum_{j=0}^{\infty}\sum_{e\in E^*}\sum_{k\in\mathbb{Z}^2}\lambda_{j,k}^e\psi_{j,e,k}(x). \tag{4.38}$$

Combining (4.36), (4.37), and (4.38) and using the fact that the global stability condition (2.29) implies stability on each refinement level, we obtain the following relation:

$$K_1c_5\sum_j 2^{2kj}\,||\lambda^{(j)}||_{l_2}^2 \leqslant ||f||_{W^{k,2}}^2 \leqslant K_2c_6\sum_j 2^{2kj}\,||\lambda^{(j)}||_{l_2}^2. \tag{4.39}$$

Denoting by A_j the stiffness matrix relative to level j, it is easy to see that (4.39) implies that the condition number of $L^{-1}A_jL^T$, where L is essentially a diagonal matrix, is uniformly bounded; see, e.g., Dahmen and Kunoth [11].

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REFERENCES

1. K. Amaratunga and J. R. Williams, "$O(n)$ Multiscale Wavelet Solvers with Arbitrarily High Accuracy, IESL Technical Report No. 93-14, 1993.
2. G. Battle, A block spin construction of ondelettes. Part II: The QFT connection, *Comm. Math. Phys.* **114** (1988), 93–102.
3. G. Beylkin, On wavelet-based algorithms for solving differential equations, preprint, 1992.
4. E. Bacry, S. Mallat, and G. Papnicolaou, A wavelet based space-time adaptive numerical method for partial differential equations, *Math. Modelling Numer. Anal.* **26,** No. 7 (1992), 793–834.
5. J. H. Bramble, J. E. Pasciak, and J. Xu, Parallel multilevel preconditioners, *Math. Comp.* **55** (1990), 1–22.
6. A. S. Cavaretta, W. Dahmen, and C. A. Micchelli, Stationary Subdivision, *Mem. Amer. Math. Soc.* **93,** No. 453 (1991).
7. C. K. Chui, J. Stöckler, and J. D. Ward, "Compactly Supported Box-Spline Wavelets," CAT Report 230, 1990, *Approx. Theory and Appl.* **8** (1992), 77–100.
8. A. Cohen and I. Daubechies, Non-separable bidimensional wavelet bases, *Rev. Mat. Iberoamericana* **9** (1993), 51–137.
9. A. Cohen, I. Daubechies, and J. C. Feauveau, Biorthogonal bases of compactly supported wavelets, *Comm. Pure Appl. Math.* **45** (1992), 485–560.
10. S. Dahlke and I. Weinreich, Wavelet–Galerkin methods: An adapted biorthogonal wavelet basis, *Constr. Approx.* **9** (1993), 237–262.
11. W. Dahmen and A. Kunoth, Multilevel preconditioning, *Numer. Math.* **63** (1992), 315–344.
12. W. Dahmen and C. A. Micchelli, Dual wavelet expansions for general scalings, in preparation.
13. W. Dahmen, S. Prössdorf, and R. Schneider, Wavelet approximation methods for pseudodifferential equations I: Stability and convergence, preprint No. 7, Institut für Angewandte Analysis und Stochastik, Berlin 1992; *Math. Z.,* to appear.
14. W. Dahmen, S. Prössdorf, and R. Schneider, Wavelet approximation methods for pseudodifferential equations II: Matrix compression and fast solution, *Adv. Comp. Math.* **1** (1993), 259–335.
15. I. Daubechies, Orthonormal bases of wavelets with compact support, *Comm. Pure Appl. Math.* **41** (1987), 909–996.
16. B. Engquist, S. Osher, and S. Zhong, "Fast Wavelet Based Algorithms for Linear Evolution Equations," ICASE Report 92-14, 1992.
17. R. Glowinski, W. Lawton, M. Ravachol, and E. Tenenbaum, Wavelet solutions of linear and nonlinear elliptic, parabolic and hyperbolic problems in one space dimension, preprint, 1989, Aware, Inc., Cambridge, MA.
18. S. Jaffard, Wavelet methods for fast resolution of elliptic problems, *SIAM J. Numer. Anal.* **29** (1992), 965–986.
19. R. Q. Jia and C. A. Micchelli, Using the refinement equations for the construction of pre-wavelets II: Powers of two, *in* "Curves and Surfaces" (P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, Eds.), Academic Press, New York, 1991.
20. P. G. Lemarié, Fonctions à support compact dans les analyses multirésolutions, *Rev. Mat. Iberoamericana* **7** (1991), 157–182.
21. P. G. Lemarié-Rieusset, Sur l'existence des analyses multirésolutions en théorie des ondelettes, *Rev. Mat. Iberoamericana* **8** (1992), 457–474.
22. P. G. Lemarié, Ondelettes à localisation exponentielle, *J. Math. Pures Appl.* **67** (1988), 227–236.
23. J. Liandrat and Ph. Tchamitchian, Elliptic operators, adaptivity and wavelets, submitted for publication.
24. S. Mallat, Multiresolution approximation and wavelet orthonormal bases of L^2, *Trans. Amer. Math. Soc.* **315** (1989), 69–88.
25. Y. Meyer, "Ondelettes et Opérateurs I, Ondelettes," Hermann, Paris, 1990.
26. Y. Meyer, "Ondelettes et Opérateurs II, Opérateurs de Caldéron–Zygmund," Hermann, Paris, 1990.
27. J. M. Ortega and R. G. Voigt, "Solution of Partial Differential Equations on Vector and Parallel Computers," SIAM, Philadelphia, 1985.
28. S. D. Riemenschneider and Z. W. Shen, Box splines, cardinal series, and wavelets, *in* "Approximation Theory and Functional Analysis" (C. K. Chui, Ed.), Academic Press, New York, 1991.
29. S. D. Riemenschneider and Z. W. Shen, Wavelets and pre-wavelets in low dimensions, *J. Approx. Theory* **71** (1991), 18–38.
30. M. E. Taylor, "Pseudo Differential Operators," Princeton Univ. Press, Princeton, NJ, 1981.
31. R. M. Young, "An Introduction to Nonharmonic Fourier Series," Academic Press, New York, 1980.
32. H. Yserentant, On the multilevel splitting of finite element spaces, *Numer. Math.* **49** (1986), 379–412.