This cannot exceed $2S$ multiplied by the total variation of s from $A-\epsilon$ to infinity, and can be made uniformly small in both ϵ and η by taking A large enough. The same applies to the integral from $-\infty$ to $-A$. Let us make these two integrals together less than ηV. There only remains the integral from $-A$ to A. This cannot exceed

$$\max_{-A<u<A} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)| \frac{1}{2\epsilon}\int_{-A}^{A} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|\,du,$$

and we may take ϵ so small that this is less than

$$2\eta\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|\,du$$
$$=2\eta\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon}\sum_{n=-\infty}^{\infty} |s_\eta(u+(2n+1)\epsilon)-s_\eta(u+(2n-1)\epsilon)|\,du$$
$$\leqslant 2\eta\frac{1}{2\epsilon}\int_{-\epsilon}^{\epsilon} V\,du \leqslant 2\eta V.$$

This establishes (20·265).

It follows from (20·265) that if we choose ϵ small enough,

$$\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}}$$

differs from the square root of the sum of the squares of those jumps of $s(u)$ exceeding η in magnitude by less than $(3\eta V)^{\frac{1}{2}}$. If we call this sum Σ_η, we have

$$\overline{\lim_{\epsilon\to 0}}\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}} < \{\Sigma_\eta\}^{\frac{1}{2}}+(3\eta V)^{\frac{1}{2}},$$

$$\underline{\lim_{\epsilon\to 0}}\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}} > \{\Sigma_\eta\}^{\frac{1}{2}}-(3\eta V)^{\frac{1}{2}}.$$

It follows that

$$\overline{\lim_{\epsilon\to 0}}\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}}$$
$$-\underline{\lim_{\epsilon\to 0}}\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}} < 2(3\eta V)^{\frac{1}{2}},$$

and since η is arbitrarily small,

$$\lim_{\epsilon\to 0}\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}}$$

exists, and differs from $\{\Sigma_\eta\}^{\frac{1}{2}}$ by less than $(3\eta V)^{\frac{1}{2}}$. Hence

$$\lim_{\eta\to 0}\{\Sigma_\eta\}^{\frac{1}{2}} = \lim_{\epsilon\to 0}\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}},$$

and

$$\text{(20·27)}\quad \lim_{\eta\to 0}\Sigma_\eta = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du.$$

The left-hand side of (20·27) is however the sum of the squares of *all* the jumps of $s(u)$. Theorem 22 completes the proof of theorem 24. Of course, if $s(u)$ is of limited total variation and has no jumps,

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du = 0,$$

and hence
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |f(x)|^2\,dx = 0.$$

A corollary of theorem 22 is:

THEOREM 24. *Let $f(x)$ be a function for which* (20·01) *is bounded in T. Let $s_1(u)$ be defined as in* (20·04), *$s_2(u)$ as in* (20·05), *and $s(u)$ as in* (20·06). *Let $s(u)$ be of limited total variation over $(-\infty, \infty)$. Then*

$$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(x)|^2\,dx \tag{20·261}$$

exists and is equal to $1/(2\pi)$ times the sum of the squares of the jumps of $s(u)$ at all its points of discontinuity. In particular, if $s(u)$ is continuous, the expression (20·261) *is equal to* 0.

We may write

$$s(u) = s_\eta(u) + t_\eta(u),$$

where $t_\eta(u)$ is a step-function constant except at the jumps of $s(u)$ exceeding η in magnitude, and there having the same jump as $s(u)$. The function $s_\eta(u)$ will have no jump exceeding η in magnitude, and both $s_\eta(u)$ and $t_\eta(u)$ will have total variations not exceeding V, the total variation of $s(u)$.

By a form of the Minkowski inequality,

$$\left|\left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}} - \left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |t_\eta(u+\epsilon)-t_\eta(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}}\right| \leqslant \left\{\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|^2\,du\right\}^{\frac{1}{2}}.$$

Now, if ϵ is small enough,

$$\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |t_\eta(u+\epsilon)-t_\eta(u-\epsilon)|^2\,du$$

will be the sum of the squares of all the jumps of $s(u)$ exceeding η in magnitude. To see this, let us notice that

$$|t_\eta(u+\epsilon)-t_\eta(u-\epsilon)|$$

will vanish except in regions of length 2ϵ extending a distance ϵ on each side of each jump, in which it will be equal to the

The expression $|s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|$ can be made less than 2η by choosing ϵ small enough. Over any finite range of u, this choice of ϵ may be made independently of u. To show this, we appeal to the Heine-Borel theorem, and to the fact that about each point u we can place an interval in which the fluctuation of $s_\eta(u)$ is less than 2η. Hence we can cover any finite range of u by a finite set of overlapping intervals in which the fluctuation of $s_\eta(u)$ is less than 2η, and if we choose ϵ less than half the overlap of any two of these intervals,

$$|s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|$$

will be less than 2η for any u in the finite range.

It follows that by taking ϵ small enough, we shall have

$$\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|^2\,du < 3\eta V. \tag{20·265}$$

For let us put

$$\begin{aligned}\frac{1}{2\epsilon}\int_{-\infty}^{\infty} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|^2\,du &= \frac{1}{2\epsilon}\int_{-\infty}^{-A} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|^2\,du \\ &+ \frac{1}{2\epsilon}\int_{-A}^{A} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|^2\,du \\ &+ \frac{1}{2\epsilon}\int_{A}^{\infty} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|^2\,du.\end{aligned}$$

The function $|s_\eta|$ will have some finite upper bound, say S. Then

$$\begin{aligned}\frac{1}{2\epsilon}\int_{A}^{\infty} &|s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|^2\,du \\ &\leqslant 2S\frac{1}{2\epsilon}\int_{A}^{\infty} |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|\,du \\ &= 2S\frac{1}{2\epsilon}\left[\int_{A}^{A+2\epsilon} + \int_{A+2\epsilon}^{A+4\epsilon} + \dots\right] |s_\eta(u+\epsilon)-s_\eta(u-\epsilon)|\,du \\ &= 2S\frac{1}{2\epsilon}\int_{0}^{2\epsilon} \{|s_\eta(u+A+\epsilon)-s_\eta(u+A-\epsilon)| \\ &\qquad + |s_\eta(u+A+3\epsilon)-s_\eta(u+A+\epsilon)| \\ &\qquad + |s_\eta(u+A+5\epsilon)-s_\eta(u+A+3\epsilon)| \\ &\qquad + \dots\}\end{aligned}$$

This vanishes only for $u = 2n\pi/\epsilon$ and there is hence no one u for which this vanishes for every ϵ. Thus, by theorem 7, (20·19) is equivalent to the statement that for all ϵ

(20·22)

$$\begin{aligned}\lim_{\xi\to\infty}\int_{-\infty}^{\infty} K_1(\xi-\eta,\epsilon)\,dg(\eta) &= A\int_{-\infty}^{\infty} K_1(\xi,\epsilon)\,d\xi\\ &= \frac{A}{\epsilon}\int_{-\infty}^{\infty} d\xi\int_{\xi}^{\xi+\epsilon} K_1(\eta)\,d\eta\\ &= \frac{A}{\epsilon}\int_{-\infty}^{\infty} d\xi\int_0^{\epsilon} K_1(\xi+\eta)\,d\eta\\ &= \frac{A}{\epsilon}\int_0^{\epsilon} d\eta\int_{-\infty}^{\infty} K_1(\xi)\,d\xi.\end{aligned}$$

In lemma 7_5 no essential use was made of the continuity of $K(x)$. Thus we may establish in the same way that if (20·18) holds, (20·22) holds for every ϵ. Thus (20·18) implies (20·19). If we can now show that the validity of (20·22) for every ϵ implies that of (20·18), we shall have completed the proof of theorem 21 and hence of theorem 22.

To this end, let us notice that, by (20·21),

$$\frac{1-e^{-\epsilon}}{\epsilon} K_1(\xi) \leqslant K_1(\xi,\epsilon) \leqslant \frac{e^{\epsilon}-1}{\epsilon} K_1(\xi+\epsilon).$$

Thus, as $g(\eta)$ is monotone increasing,

$$\begin{aligned}\frac{1-e^{-\epsilon}}{\epsilon}\overline{\lim_{\xi\to\infty}}\int_{-\infty}^{\infty} K_1(\xi-\eta)\,dg(\eta) &\leqslant \overline{\lim_{\xi\to\infty}}\int_{-\infty}^{\infty} K_1(\xi-\eta,\epsilon)\,dg(\eta) = A\\ &\leqslant \frac{e^{\epsilon}-1}{\epsilon}\underline{\lim_{\xi\to\infty}}\int_{-\infty}^{\infty} K_1(\xi-\eta+\epsilon)\,dg(\eta)\\ &= \frac{e^{\epsilon}-1}{\epsilon}\underline{\lim_{\xi\to\infty}}\int_{-\infty}^{\infty} K_1(\xi-\eta)\,dg(\eta).\end{aligned}$$

As ϵ is arbitrarily small, we may make $(1-e^{-\epsilon})/\epsilon$ and $(e^{\epsilon}-1)/\epsilon$ as near as we wish to 1, and we have

$$\overline{\lim_{\xi\to\infty}}\int_{-\infty}^{\infty} K_1(\xi-\eta)\,dg(\eta) \leqslant A \leqslant \underline{\lim_{\xi\to\infty}}\int_{-\infty}^{\infty} K_1(\xi-\eta)\,dg(\eta),$$

which is equivalent to (20·18). This completes the proof of theorem 21.

A closely related theorem is:

THEOREM 23. *Let $\phi(x)$ be a bounded function, and let either of the limits*

(20·23)
$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^{\epsilon}\phi(x)\,dx$$

or

(20·24)
$$\lim_{T\to\infty}\frac{2}{\pi T}\int_0^{\infty}\phi(x)\frac{\sin^2 Tx}{x^2}\,dx$$

exist. Then the other limit exists and assumes the same value.

In this theorem the rôles of 0 and ∞ are reversed in comparison with theorem 21. We put

$$x = e^{-\xi};\quad T = e^{\eta} = \epsilon^{-1};\quad \phi(x) = \psi(\xi).$$

The expression (20·23) becomes

(20·25)
$$\lim_{\eta\to\infty}\int_{\eta}^{\infty} e^{\eta-\xi}\psi(\xi)\,d\xi,$$

and (20·24) becomes

(20·26)
$$\lim_{\eta\to\infty}\int_{-\infty}^{\infty}\frac{2}{\pi}\frac{\sin^2 e^{\eta-\xi}}{e^{\eta-\xi}}\psi(\xi)\,d\xi.$$

As we have assumed ψ to be bounded, we may apply theorem 4 to show the complete equivalence of (20·25) and (20·26). We have only to show that neither

$$\int_{-\infty}^{0} e^{\xi(1-iu)}\,d\xi$$

nor

$$\int_{-\infty}^{\infty} e^{-\xi(1+iu)}\sin^2 e^{\xi}\,d\xi$$

vanishes for any real u. The first is $1/(1-iu)$ and does not vanish, while the second is equal to

$$\int_{-\infty}^{\infty} e^{\xi(1+iu)}\sin^2 e^{-\xi}\,d\xi,$$

and this, by (20·19, 20), is equal to

$$\frac{\Gamma(1-iu)(e^{\pi u/2}-e^{-\pi u/2})\,2^{iu-1}}{u(iu+1)},$$

which vanishes for no real u. Thus theorem 23 is established.

To show the equivalence of (20·18) and (20·19), we invoke theorem 5. The kernel K_2 clearly belongs to M_1, and has a Fourier transform differing only by a constant factor from

$$\int_{-\infty}^{\infty} e^{\xi(1-iu)} \sin^2 e^{-\xi}\, d\xi$$
$$= \int_0^{\infty} x^{iu-2} \sin^2 x \, dx$$
$$= \int_0^{\infty} x^{iu-2}\left(\frac{1}{2} - \frac{e^{2ix}}{4} - \frac{e^{-2ix}}{4}\right) dx$$
$$= \lim_{\epsilon\to 0} \int_{\epsilon}^{\infty} x^{iu-2}\left(\frac{1}{2} - \frac{e^{2ix}}{4} - \frac{e^{-2ix}}{4}\right) dx$$
$$= \lim_{\epsilon\to 0}\left[\int_{\epsilon}^{\infty} x^{iu-2}\frac{1-e^{2ix}}{4}\, dx + \int_{\epsilon}^{\infty} x^{iu-2}\frac{1-e^{-2ix}}{4}\, dx\right].$$

In the first integral, we now deform the path of integration into the half-ordinate extending from ϵ to $\epsilon + i\infty$, and in the second, into the half-ordinate extending from ϵ to $\epsilon - i\infty$. The integrand will in each case be of the order of x^{-2} at infinity, so that the integral along the quadrant from a large positive value of x to a large imaginary value will tend to 0 as the radius becomes infinite, and the deformation of the path is legitimate. We have

$$\int_{-\infty}^{\infty} e^{\xi(1-iu)} \sin^2 e^{-\xi}\, d\xi$$
$$= \lim_{\epsilon\to 0}\left[i\int_0^{\infty} (\epsilon+iy)^{iu-2}\frac{1-e^{2(i\epsilon-y)}}{4}\, dy - i\int_0^{\infty}(\epsilon-iy)^{iu-2}\frac{1-e^{2(-i\epsilon-y)}}{4}\, dy\right]$$
$$= \lim_{\epsilon\to 0}\int_0^{\infty}\left[i(\epsilon+iy)^{iu-2}\frac{1-e^{2(i\epsilon-y)}}{4} - i(\epsilon-iy)^{iu-2}\frac{1-e^{2(-i\epsilon-y)}}{4}\right] dy.$$

Upon integration by parts, this becomes

(20·20)
$$\lim_{\epsilon\to 0}\left\{\int_0^{\infty}\left[-\frac{(\epsilon+iy)^{iu-1}}{iu-1}\frac{e^{2(i\epsilon-y)}}{2} - \frac{(\epsilon-iy)^{iu-1}}{iu-1}\frac{e^{2(-i\epsilon-y)}}{2}\right] dy - \frac{\epsilon^{iu-1}}{iu-1}\sin^2\epsilon\right\},$$

which, similarly,

$$= \lim_{\epsilon\to 0}\left\{\int_0^{\infty}\left[\frac{(\epsilon+iy)^{iu}}{u(iu-1)}e^{2(i\epsilon-y)} - \frac{(\epsilon-iy)^{iu}}{u(iu-1)}e^{2(-i\epsilon-y)}\right] dy + \frac{\epsilon^{iu}}{iu(iu-1)}\sin 2\epsilon - \frac{\epsilon^{iu-1}}{iu-1}\sin^2\epsilon\right\}.$$

We may now proceed to the limit by the principle of dominated convergence, and (20·20) becomes

$$\int_0^{\infty}\frac{y^{iu}}{u(iu-1)}(i^{iu}-(-i)^{iu})\, e^{-2y}\, dy$$
$$= \frac{\Gamma(iu+1)}{u(iu-1)}(e^{-\frac{\pi u}{2}} - e^{\frac{\pi u}{2}})\, 2^{-iu-1},$$

which does not vanish for any real u.

The function K_1 is discontinuous, and hence does not belong to M_1. The functions

(20·21)
$$K_1(\xi, \epsilon) = \frac{1}{\epsilon}\int_{\xi}^{\xi+\epsilon} K_1(\eta)\, d\eta$$
$$\begin{cases} = 0; & (\xi < -\epsilon) \\ = \dfrac{1-e^{-\xi-\epsilon}}{\epsilon}; & (-\epsilon \leqslant \xi < 0) \\ = e^{-\xi}\dfrac{1-e^{-\epsilon}}{\epsilon}; & (0 \leqslant \xi) \end{cases}$$

do however belong to M_1. Their Fourier transforms are

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iu\xi}\frac{d\xi}{\epsilon}\int_{\xi}^{\xi+\epsilon} K_1(\eta)\, d\eta$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-iu\xi}\frac{d\xi}{\epsilon}\int_0^{\epsilon} K_1(\xi+\eta)\, d\eta$$
$$= \frac{1}{\epsilon\sqrt{2\pi}}\int_0^{\epsilon} d\eta\int_{-\eta}^{\infty} e^{-\xi(1+iu)-\eta}\, d\xi$$
$$= \frac{1}{\epsilon\sqrt{2\pi}}\int_0^{\epsilon}\frac{e^{iu\eta}}{1+iu}\, d\eta$$
$$= \frac{1}{\epsilon\sqrt{2\pi}}\,\frac{e^{iu\epsilon}-1}{iu(1+iu)}.$$

or

$$\lim_{\epsilon\to 0} \frac{2}{\pi\epsilon}\int_0^\infty \phi(x)\,\frac{\sin^2 \epsilon x}{x^2}\,dx \tag{20·12}$$

exists, then the other limit exists and assumes the same value.

Thus (20·09) and (20·10) are completely equivalent, and we have as a corollary (in view of 20·08):

THEOREM 22. *Let $f(x)$ be a function for which* (20·01) *is bounded in T. Let $s_1(u)$ be defined as in* (20·04), *$s_2(u)$ as in* (20·05), *and $s(u)$ as in* (20·06). *Then we shall have*

(20·13)
$$\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty} |s(u+\epsilon)-s(u-\epsilon)|^2\,du = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx,$$

in the sense that if either side of (20·13) *exists, the other side exists and assumes the same value.*

We now proceed to the proof of theorem 21. There will clearly be no restriction in supposing that $\phi(x)=0$ for $x<1$, inasmuch as over this region $\phi(x)/T$ and $2\phi(x)\sin^2 \epsilon x/(\pi\epsilon x^2)$ tend monotonely to zero (the latter for sufficiently small values of ϵ), so that, by "monotone convergence,"

$$\lim_{T\to\infty}\frac{1}{T}\int_0^1 \phi(x)\,dx = \lim_{\epsilon\to 0}\frac{2}{\pi\epsilon}\int_0^1 \phi(x)\,\frac{\sin^2\epsilon x}{x^2}\,dx = 0.$$

We shall accordingly suppose $\phi(x)$ to vanish over (0, 1).

Let us now put

$$x = e^\xi;\quad T = e^\eta = \epsilon^{-1};\quad \phi(x) = \psi(\xi).$$

The expression (20·11) becomes

$$\lim_{\eta\to\infty}\int_{-\infty}^{\eta} e^{\xi-\eta}\,\psi(\xi)\,d\xi, \tag{20·14}$$

and (20·12) becomes

$$\lim_{\eta\to\infty}\int_{-\infty}^{\infty}\frac{2}{\pi}\,\frac{\sin^2 e^{\xi-\eta}}{e^{\xi-\eta}}\,\psi(\xi)\,d\xi. \tag{20·15}$$

If either (20·14) or (20·15) is finite, we may conclude by a reasoning similar to the argument of (17·16) that

$$\overline{\lim_{n\to\infty}}\int_n^{n+1}\psi(\xi)\,d\xi < \infty.$$

The function $\psi(\xi)$ is summable over any finite range, and vanishes for negative arguments. Thus, as in (17·17),

$$\int_n^{n+1}\psi(\xi)\,d\xi$$

is bounded. Hence if we put

$$g(\xi) = \int_0^\xi \psi(\eta)\,d\eta,$$

since ψ is non-negative, we have

$$\int_n^{n+1}|dg(\xi)| = g(n+1)-g(n) = \int_n^{n+1}\psi(\xi)\,d\xi < \text{const.}$$

We wish to show that the two propositions

$$\lim_{\xi\to\infty}\int_{-\infty}^{\infty} K_1(\xi-\eta)\,dg(\eta) = A\int_{-\infty}^{\infty} K_1(\xi)\,d\xi \tag{20·16}$$

and

$$\lim_{\xi\to\infty}\int_{-\infty}^{\infty} K_2(\xi-\eta)\,dg(\eta) = A\int_{-\infty}^{\infty} K_2(\xi)\,d\xi \tag{20·17}$$

are equivalent, where

$$K_1(\xi) = 0\ \ (-\infty<\xi<0);\quad K_1(\xi) = e^{-\xi}\ \ (0<\xi<\infty)$$

and
$$K_2(\xi) = \frac{2}{\pi}\,e^\xi \sin^2 e^{-\xi}.$$

Let us notice that

$$\int_{-\infty}^{\infty} K_1(\xi)\,d\xi = \int_0^\infty e^{-\xi}\,d\xi = 1$$

and that
$$\int_{-\infty}^{\infty} K_2(\xi)\,d\xi = \frac{2}{\pi}\int_{-\infty}^{\infty} e^\xi \sin^2 e^{-\xi}\,d\xi = \frac{2}{\pi}\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = 1.$$

Thus, if (20·16) and (20·17) are equivalent, the two propositions

$$\lim_{\xi\to\infty}\int_{-\infty}^{\infty} K_1(\xi-\eta)\,dg(\eta) = A \tag{20·18}$$

and

$$\lim_{\xi\to\infty}\int_{-\infty}^{\infty} K_2(\xi-\eta)\,dg(\eta) = A \tag{20·19}$$

will be equivalent, and we shall have established theorem 21.

§ 20. The Mean Square Modulus of a Function.

THEOREM 20. *Let $f(x)$ be a measurable function for which*

$$\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx \tag{20·01}$$

is bounded in T. Then

$$\int_{-\infty}^{\infty}\frac{|f(x)|^2}{1+x^2}\,dx<\infty. \tag{20·02}$$

For if we integrate by parts,

$$\int_{-A}^{A}\frac{|f(x)|^2}{1+x^2}\,dx \tag{20·03}$$

$$=\int_{-A}^{A}\frac{dx}{1+x^2}\frac{d}{dx}\int_0^x|f(\xi)|^2\,d\xi$$

$$=\frac{1}{1+A^2}\int_0^A|f(\xi)|^2\,d\xi-\frac{1}{1+A^2}\int_0^{-A}|f(\xi)|^2\,d\xi$$

$$-\int_{-A}^{A}\frac{d}{dx}\left(\frac{1}{1+x^2}\right)\left[\int_0^x|f(\xi)|^2\,d\xi\right]dx$$

$$=\frac{2A}{1+A^2}\frac{1}{2A}\int_{-A}^{A}|f(\xi)|^2\,d\xi+\int_0^A\frac{4x^2\,dx}{(1+x^2)^2}\frac{1}{2x}\int_{-x}^{x}|f(\xi)|^2\,d\xi$$

$$\leqslant\frac{2A}{1+A^2}\frac{1}{2A}\int_{-A}^{A}|f(\xi)|^2\,d\xi+4\int_0^A\frac{dx}{1+x^2}\frac{1}{2x}\int_{-x}^{x}|f(\xi)|^2\,d\xi$$

$$\leqslant\left\{\frac{2A}{1+A^2}+4\tan^{-1}A\right\}\limsup\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx$$

$$\leqslant(1+2\pi)\limsup\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx.$$

Thus if (20·01) is bounded, the function which is $f(x)/x$ for $|x|>1$ and 0 for $|x|\leqslant 1$ belongs to L_2, and, by the Plancherel theorem,

$$s_1(u)=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\left[\int_1^A+\int_{-A}^{-1}\right]\frac{f(x)\,e^{-iux}}{-ix}\,dx \tag{20·04}$$

exists. Furthermore, $f(x)$ is summable over $(-1,1)$, and

$$s_2(u)=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}f(x)\frac{e^{-iux}-1}{-ix}\,dx \tag{20·05}$$

exists for every u. Thus the function

$$s(u)=s_1(u)+s_2(u) \tag{20·06}$$

exists for almost every u.

We have

$$s(u+\epsilon)-s(u-\epsilon) \tag{20·07}$$

$$=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{e^{ix\epsilon}-e^{-ix\epsilon}}{ix}e^{-iux}\,dx$$

$$=\underset{A\to\infty}{\mathrm{l.i.m.}}\frac{1}{\sqrt{2\pi}}\int_{-A}^{A}f(x)\frac{2\sin\epsilon x}{x}e^{-iux}\,dx.$$

Thus $s(u+\epsilon)-s(u-\epsilon)$ is the Fourier transform of

$$f(x)\frac{2\sin\epsilon x}{x},$$

and, by the Plancherel theorem,

$$\frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty}|s(u+\epsilon)-s(u-\epsilon)|^2\,du=\frac{1}{\pi\epsilon}\int_{-\infty}^{\infty}|f(x)|^2\frac{\sin^2\epsilon x}{x^2}\,dx. \tag{20·08}$$

The right-hand side of this expression may be regarded as a weighted average of $|f(x)|^2$, for

$$\frac{1}{\pi\epsilon}\int_{-\infty}^{\infty}\frac{\sin^2\epsilon x}{x^2}\,dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}\,dx=1.$$

As $\epsilon\to 0$, the weighting becomes more nearly uniform over the interval $(-\infty,\infty)$. It becomes natural to inquire as to the relation between

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(x)|^2\,dx \tag{20·09}$$

and

$$\lim_{\epsilon\to 0}\frac{1}{\pi\epsilon}\int_{-\infty}^{\infty}|f(x)|^2\frac{\sin^2\epsilon x}{x^2}\,dx. \tag{20·10}$$

We shall prove the Tauberian theorem:

THEOREM 21. *If $\phi(x)\geqslant 0$ for $0\leqslant x<\infty$, and either of the limits*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T\phi(x)\,dx \tag{20·11}$$

§20 The Mean Square Modulus of a Function

THE FOURIER INTEGRAL AND CERTAIN OF ITS APPLICATIONS

BY

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