PROOF. We shall use the formula

$$[\log f(z)]^{(\rho)} = \frac{\rho!}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})| \frac{2Re^{i\psi}}{(Re^{i\psi}-z)^{\rho+1}}\,d\psi + \sum_{|a_n|<R} \frac{(\rho-1)!\overline{a}_n^{\rho}}{(R^2-\overline{a}_n z)^{\rho}} - \sum_{|a_n|<R} \frac{(\rho-1)!}{(a_n-z)^{\rho}}$$

proved in Section 4.2. If we set $z=0$, we obtain

$$[\log f(z)]^{(\rho)}_{z=0} + (\rho-1)! \sum_{|a_n|<R} a_n^{-\rho} = \frac{2\rho!}{2\pi R^{\rho}}\int_0^{2\pi} \log|f(Re^{i\psi})| e^{-i\rho\psi}\,d\psi + \sum_{|a_n|<R} \frac{(\rho-1)!\overline{a}_n^{\rho}}{R^{2\rho}}$$

or

(10)
$$\left|[\log f(z)]^{(\rho)}_{z=0} + (\rho-1)! \sum_{|a_n|<R} a_n^{-\rho}\right| \le \frac{2\rho!}{R^{\rho}}\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f(Re^{i\psi})|\big|\,d\psi + (\rho-1)!\frac{n(R)}{R^{\rho}}\,.$$

It is easily seen that the logarithm of a primary factor has a root at $z=0$ of multiplicity $\rho+1$. Hence,

$$[\log(\Pi(z))]^{(\rho)}_{z=0} = 0$$

and

(11)
$$[\log f(z)]^{(\rho)}_{z=0} = \rho! a_{\rho}\,.$$

n addition,

$$\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f(Re^{i\psi})|\big|\,d\psi = m(R,f) + m\Big(R,\frac{1}{f}\Big)\,.$$

'he Jensen formula yields

$$m\Big(R,\frac{1}{f}\Big) = m(R,f) - N(R,0) + O(1) < m(R,f) + O(1)\,,$$

ıd

2)
$$\frac{1}{2\pi}\int_0^{2\pi} \big|\log|f(Re^{i\psi})|\big|\,d\psi \le 2m(R,f) + O(1) \le 2\log M_f(R) + O(1)\,.$$

ıe Jensen formula yields the estimate

3)
$$n(R) \le \log M_f(eR) + O(1)\,.$$

bstituting (11)–(13) in (10) we obtain

$$\left|\rho! a_{\rho} + (\rho-1)! \sum_{|a_n|<R} a_n^{-\rho}\right|$$

or

$$\delta_f(R) \overset{\text{as}}{<} C\,\frac{\log M_f(eR)}{R^{\rho}}\,,$$

with a constant C independent of the function f. It follows from this inequality that $\overline{\delta}_f \le Ce^{\rho}\sigma_f$. Since by virtue of (13) we have $\overline{\Delta}_f \le e^{\rho}\sigma_f$, we obtain finally

(14)
$$\gamma_f = \max(\overline{\delta}_f, \overline{\Delta}_f) \le C_1\sigma_f\,.$$

To estimate σ_f from above via γ_f we shall write the Hadamard representation of $f(z)$ in the form

$$f(z) = \exp\Big[\Big(a_{\rho} + \frac{1}{\rho}\sum_{|a_n|<r} a_n^{-\rho}\Big)z^{\rho}\Big]\exp P_{\rho-1}(z) \times \prod_{|a_n|<r} G\Big(\frac{z}{a_n},\rho-1\Big)\prod_{|a_n|\ge r} G\Big(\frac{z}{a_n},\rho\Big)\,,\quad r=|z|\,,$$

where $P_{\rho-1}$ is a polynomial of degree at most $\rho-1$. Using the estimate of the primary factor $G(u,p)$ given by Lemma 1, Section 4.3, we obtain

$$\log|f(z)| \le \delta_f(r)r^{\rho} + A_{\rho}\Big[r^{\rho}\int_0^r \frac{dn(t)}{t^{\rho-1}(t+r)} + r^{\rho+1}\int_r^{\infty}\frac{dn(t)}{t^{\rho}(t+r)}\Big] + o(r^{\rho})\,.$$

Much as in the proof of Theorem 2, Section 4.3, integration by parts yields

$$\log M_f(r) \le \delta_f(r)r^{\rho} + K_{\rho}\Big\{r^{\rho-1}\int_0^r \frac{n(t)}{t^{\rho}}\,dt + r^{\rho+1}\int_r^{\infty}\frac{n(t)}{t^{\rho+2}}\,dt\Big\} + o(r^{\rho})\,.$$

Now we apply the inequality $n(t) \overset{\text{as}}{<} (\overline{\Delta}+\varepsilon)t^{\rho}$, $\varepsilon>0$, and obtain

$$\log M_f(r) \le \delta_f(r)r^{\rho} + 2K_{\rho}(\overline{\Delta}_f+\varepsilon)r^{\rho}\,.$$

Therefore,

$$\limsup_{r\to\infty}\frac{\log M_f(r)}{r^{\rho}} \le \overline{\delta}_f + 2K_{\rho}\overline{\Delta}_f \le C_{\rho}\gamma_f\,,$$

with a constant C_{ρ}, which proves $\sigma_f \le C_{\rho}\gamma_f$. Together with (14) this proves Theorem 4.

PROBLEM 1 (Valiron). Prove the following statement.
If ρ is not an integer, then convergence of the integral

(15)
$$\int^{\infty}\frac{\log M_f(r)}{r^{\rho+1}}\,dr$$

is equivalent to convergence of the series

(16)
$$\sum_{n=1}^{\infty}\frac{1}{|a_n|^{\rho}}\,.$$

If ρ is an integer, then convergence of the integral (15) is equivalent to convergence of both the series (16) and the integral

$$\int^{\infty}\frac{\delta_f(r)dr}{r}\,.$$

and by Hadamard's representation

$$\log M_f(r) \overset{\text{as}}{<} a_0 r^q + C_\rho(\overline{\Delta}_f + \varepsilon) r^\rho \overset{\text{as}}{<} C_\rho(\overline{\Delta}_f + r\varepsilon) r^\rho$$

or

(2) $$\sigma_f \le C_\rho \overline{\Delta}_f ,$$

which proves Theorem 2.

5.2. Functions of integer order

An entire function of integer order ρ may not have zeros at all. It is possible that $\rho = q$, q being the degree of the polynomial in Hadamard's representation, and the order of the canonical product ρ_1 is less than ρ. But another feature of entire functions of integer order is more essential. It turns out that, for an integer order, inequality (2) may fail even for a canonical product. The upper density of the zero set may be finite while the canonical product may be of maximal type.

Consider, for example, the entire functions

$$\sin\frac{\pi}{2}z = \frac{\pi}{2}z \prod_{n=1}^{\infty}\Big(1 - \frac{z^2}{4n^2}\Big) = \frac{\pi}{2}z \prod_{n=-\infty}^{\infty}{}' \Big(1 - \frac{z}{2n}\Big) e^{z/2n}$$

and

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty}\Big(1 + \frac{z}{n}\Big) e^{-z/n} ,$$

where γ is the Euler constant. It is evident that, for both functions, $n(t) \sim t$ and the convergence exponent of zeros is $\rho_1 = 1$. Since the functions differ inessentially from the canonical products, they are of order one. For the former function we have

$$\frac{1}{2}(e^{\frac{\pi}{2}|y|} - e^{-\frac{\pi}{2}|y|}) \le \Big|\sin\frac{\pi}{2}z\Big| < e^{\frac{\pi}{2}|y|} ,$$
$$\frac{1}{2}(e^{\frac{\pi}{2}r} - e^{-\frac{\pi}{2}r}) \le M(r) < e^{\frac{\pi}{2}r} .$$

Hence, $\log M(r) \sim (\pi/2)r$ and $\sigma = \pi/2$. For the latter function, according to the Stirling formula,

$$\log\frac{1}{\Gamma(z)} = -\Big(z - \frac{1}{2}\Big)\log z + z - \frac{1}{2}\log 2\pi + O\Big(\frac{1}{|z|}\Big) = -z(1 + o(1))\log z ,$$

where the plane is assumed to be cut along the negative real axis, and $|\arg z| < \pi$. Hence

$$\log\frac{1}{|\Gamma(z)|} = -(1 + o(1))(\cos\varphi) r \log r , \quad z = re^{i\varphi} , \quad \frac{\pi}{2} < |\varphi| < \pi ,$$

and therefore $\log M(r) \ge Cr\log r$. This means that $f = 1/\Gamma$ is of maximal type.

We shall see that the "root of all evil" is the presence of a symmetry in the distribution of zeros of the first function and its absence for the second function.

We remind the reader that p denotes the smallest integer for which the series converges. By virtue of the Hadamard and Borel theorems from the previous lecture we have $p \le \rho \le p+1$. Two cases are possible if ρ is an integer: either $\rho = p+1$ and the series $\sum |a_n|^{-\rho}$ converges, or $\rho = p$ and the same series diverges.

In what follows we denote by a_ρ the coefficient of z^ρ of the polynomial $P(z)$ in Hadamard's representation.

THEOREM 3 (Lindelöf). *If $\rho = p + 1$, then $f(z)$ is an entire function of minimal type for $a_\rho = 0$ and of mean type for $a_\rho \ne 0$.*[5]

PROOF. According to the Hadamard theorem, we have

(3) $$\log|f(z)| \le \mathrm{Re}(a_\rho z^\rho) + \log|\Pi(z)| + O(|z|^{\rho-1}) , \quad z \to \infty .$$

It follows from inequality (5), Section 4.3, that if $\rho = p+1$, then

(4) $$\log M_\Pi(r) \overset{\text{as}}{<} \varepsilon r^\rho , \quad \varepsilon > 0 ,$$

and inequality (3) yields

(5) $$\log M_f(r) \overset{\text{as}}{<} (|a_\rho| + 3\varepsilon) r^\rho .$$

To estimate $\log M_f(r)$ from below we start from the evident relation

(6) $$m(r, \exp(a_\rho z^\rho + \cdots + a_0)) = m\Big(r, \frac{f}{\Pi}\Big) ,$$

where m is the Nevanlinna proximity function (see Section 2.4). It follows that

(7) $$\frac{|a_\rho|}{2\pi} r^\rho \overset{\text{as}}{<} m(r, \exp(a_\rho z^\rho + \cdots + a_0)) < m(r,f) + m\Big(r, \frac{1}{\Pi}\Big) + \log 2 .$$

By virtue of Jensen's formula

$$N(r,0) = m(r,\Pi) - m\Big(r, \frac{1}{\Pi}\Big) - \log|\Pi(0)| ,$$

whence

(8) $$m\Big(r, \frac{1}{\Pi}\Big) \le m(r,\Pi) .$$

Taking into account (7), (8) and (4) we obtain

(9) $$\frac{|a_\rho|}{2\pi} r^\rho \overset{\text{as}}{<} m(r,f) + m(r,\Pi) + O(1) \overset{\text{as}}{<} \log M_f(r) + 3\varepsilon r^\rho , \quad \varepsilon > 0 .$$

The statement of Theorem 3 follows from (9) and (5).

THEOREM 4 (Lindelöf). *Let $\rho = p$. Set*

$$\delta_f(r) = \Big|a_\rho + \frac{1}{\rho}\sum_{|a_n|<r} a_n^{-\rho}\Big| , \quad \overline{\delta}_f = \limsup_{r\to\infty} \delta_f(r) ,$$

and $\gamma_f = \max(\overline{\Delta}_f, \overline{\delta}_f)$. Then σ_f and γ_f simultaneously are equal either to zero, or to infinity, or to positive numbers.

$$|\Pi(z)| \le \sum_{n=1}^{\infty} \log\left(1+\frac{r}{|a_n|}\right) = \int_0^{\infty} \log\left(1+\frac{r}{t}\right) dn(t)$$

$$= r\int_0^{\infty} \frac{n(t)}{t(t+r)}\, dt \le \int_0^{r} \frac{n(t)}{t}\, dt + r\int_r^{\infty} \frac{n(t)}{t^2}\, dt\,.$$

proved.

3 (Borel). *The growth order ρ of a canonical product is equal to exponent of the sequence of its zeros.*

t p be the smallest integer such that the series

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}}$$

e $\{a_n\}$ is the sequence of zeros of the canonical product $\Pi(z)$, and nvergence exponent of the sequence $\{a_n\}$. Then $p \le \rho_1 \le p+1$. $< p+1$. We choose $\varepsilon > 0$ such that $\rho_1 + \varepsilon < p+1$. Then

$$n(t) \overset{\text{as}}{<} t^{\rho_1+\varepsilon}\,.$$

he preceding theorem that

$$r) \le K_p r^p \Big\{O(1) + \int_0^r t^{\rho_1+\varepsilon-p-1}\, dt + r\int_r^{\infty} t^{\rho_1+\varepsilon-p-2}\, dt\Big\}$$

$$\le K_p r^p \Big\{O(1) + \frac{r^{\rho_1+\varepsilon-p}}{\rho_1+\varepsilon-p} + \frac{r^{\rho_1+\varepsilon-p}}{p+1-\rho_1-\varepsilon}\Big\} \overset{\text{as}}{<} r^{\rho_1+2\varepsilon}.$$

r the case $\rho_1 = p+1$. We proved in Lemma 1, Section 3.2, that

$$\frac{n(r)}{r^{p+1}}\,, \quad \int_r^{\infty} \frac{n(t)}{t^{p+2}}\, dt$$

$\to \infty$. Hence it follows from Theorem 2 that

$$\log M_\Pi(r) \overset{\text{as}}{<} \varepsilon r^{p+1} = \varepsilon r^{\rho_1}$$

h cases $\rho \le \rho_1$. Comparing this inequality with the corollary to la derived in Section 2.5, we obtain $\rho = \rho_1$. Theorem 3 is proved.

Find a necessary and sufficient conditions for a sequence of com- $\}$ to be such that the infinite product

$$\prod_k \frac{\sin \alpha_k z}{\alpha_k z}$$

tire function. Under what conditions imposed on $\{\alpha_k\}$ this is a ential type?

ncaré theorem.

The Connection between the Growth of Entire Functions and the Distribution of their Zeros

5.1. Functions of noninteger order

THEOREM 1. *The convergence exponent of the zero set of an entire function f of noninteger order is equal to the order of growth of f.*

PROOF. Let f be an entire function of noninteger order ρ, let ρ_1 be the convergence exponent of its zeros , and let $\Pi(z)$ be the canonical product corresponding to the set of zeros of f. According to the Hadamard representation (Theorem 1, Section 4.2), we have

$$(1) \qquad f(z) = z^m e^{P_q(z)}\Pi(z), \quad \deg P_q = q\,.$$

Using the Borel theorem (Theorem 3, Section 4.3), we obtain

$$\log M_f(r) \overset{\text{as}}{<} c_1 r^q + r^{\rho_1+\varepsilon}\,, \quad \varepsilon > 0.$$

Hence

$$\log M_f(r) \overset{\text{as}}{<} r^{\lambda+2\varepsilon}\,, \quad \lambda = \max(\rho_1, q)\,,$$

and $\rho \le \lambda$. The opposite inequality is true, since by virtue of Theorem 2, Section 3.2, we have $\rho_1 \le \rho$, and by virtue of the Hadamard theorem $q \le \rho$. The theorem is proved.

THEOREM 2. *If the order ρ of an entire function $f(z)$ is not an integer, then its type σ_f and the upper density of zeros $\overline{\Delta}_f$ simultaneously are equal either to zero, or to infinity, or to positive numbers.*

PROOF. According to Theorem 3, Section 3.2, we have $\overline{\Delta}_f \le e\rho\sigma_f$. To estimate σ_f from above via $\overline{\Delta}_f$ we shall use the bound of a canonical product of genus p proved in Theorem 2, Section 4.3. The inequality

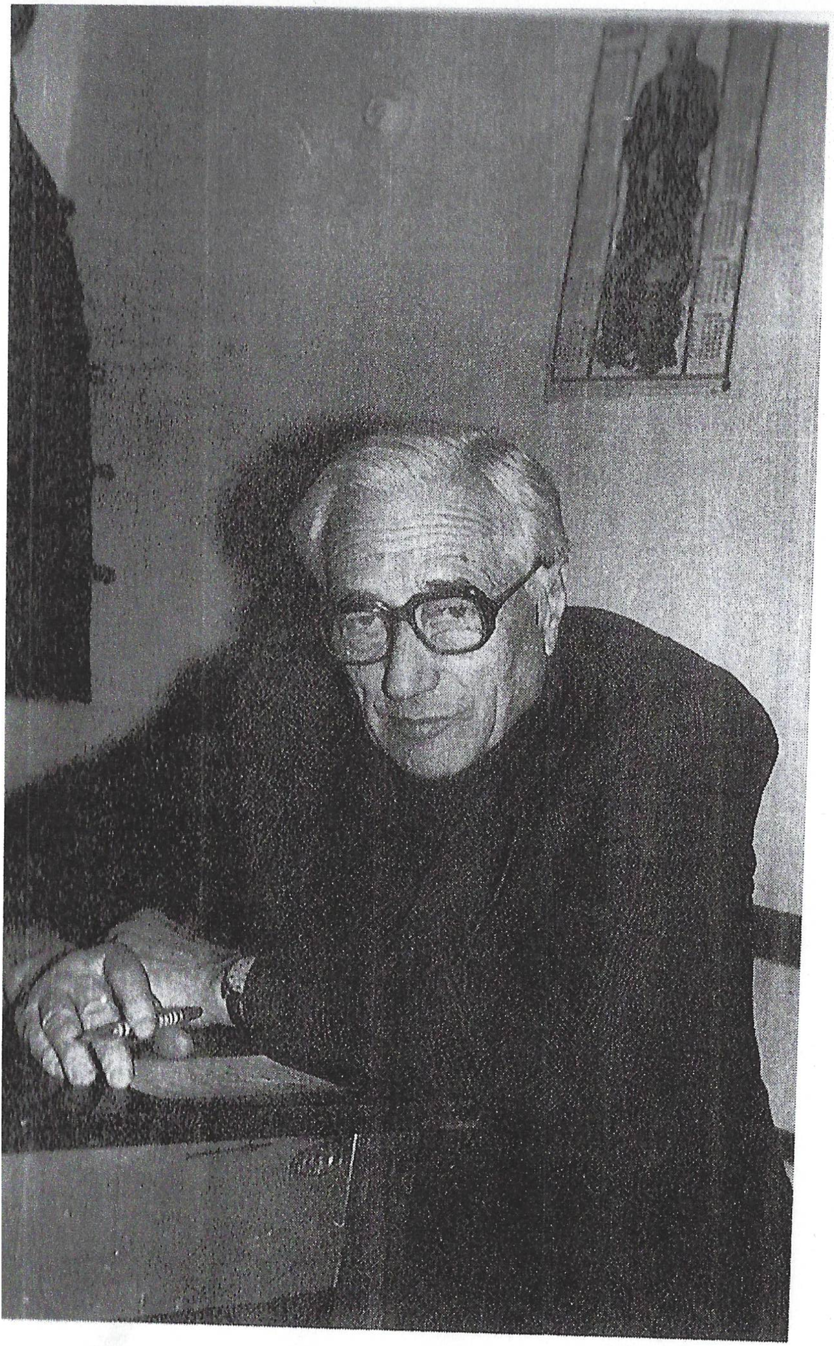
$$n(t) \overset{\text{as}}{<} (\overline{\Delta}_f + \varepsilon)t^{\rho}\,, \quad \varepsilon > 0\,,$$

yields

$$\log M_\Pi(r) \le K_p r^p \Big\{O(1) + (\overline{\Delta}_f + \varepsilon)\int_0^r t^{\rho-p-1}\, dt + (\overline{\Delta}_f + \varepsilon) r \int_r^{\infty} t^{\rho-p-2}\, dt\Big\}\,.$$

Since $p < \rho < p+1$, we have

$$\log M_\Pi(r) \overset{\text{as}}{<} C_\rho(\overline{\Delta}_f + \varepsilon) r^{\rho}\,,$$



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1906–1993

MATHEMATICAL
MONOGRAPHS

Volume 150

Lectures on Entire Functions

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lecture 5

American Mathematical Society