$$-\lim_{\epsilon\to 0}\frac{1}{2\pi\epsilon}\int_{-A}^{A}|s(x,u+\epsilon)-s(x,u-\epsilon)|^2\,du.$$

We now appeal to (34.24), (34.21), and (34.34), and obtain in place of the last line of (34.41),

(34.42)
$$\lim_{B\to\infty}\lim_{\epsilon\to 0}\frac{1}{2\pi\epsilon}\int_{-B}^{B}|s(b,u+\epsilon)-s(b,u-\epsilon)|^2 e^{2u(x-b)}\,du$$
$$-\lim_{\epsilon\to 0}\frac{1}{2\pi\epsilon}\int_{-A}^{A}|s(b,u+\epsilon)-s(b,u-\epsilon)|^2 e^{2u(x-b)}\,du,$$

which may be made as small as we please by making A large enough. Thus within $a < x < b$,

$$\lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B} f(x+iy+i\xi)\bar{f}(x+iy)\,dy$$

is continuous in ξ, and $f(x+iy)$ belongs to S' as a function of y.

35. Almost periodic functions. Let us return to (34.16). Let $f(a+iy)$ and $f(b+iy)$ be almost periodic in y: that is, let it be possible, whenever $\epsilon > 0$, to find trigonometrical polynomials

(35.01)
$$P_1(y) = \sum_1^n A_k e^{i\Lambda_k y}$$

and

(35.02)
$$P_2(y) = \sum_1^n B_k e^{iM_k y},$$

such that

(35.03)
$$|f(a+iy)-P_1(y)| < \epsilon \qquad (-\infty < y < \infty),$$

and

(35.04)
$$|f(b+iy)-P_2(y)| < \epsilon \qquad (-\infty < y < \infty).$$

If (34.03) holds uniformly over (a, b), and we may apply (34.16), we obtain

$$\frac{\epsilon}{2\pi}\left[\int_{-\infty}^{\infty}|K_1(x-b+i\lambda)|\,d\lambda+\int_{-\infty}^{\infty}|K_2(x-a+i\lambda)|\,d\lambda\right]$$
$$\geqq\left|f(x+iy)-\frac{1}{2\pi}\int_{-\infty}^{\infty}P_2(\eta)K_1(x+iy-b-i\eta)\,d\eta\right.$$

(35.05)
$$\left.+\frac{1}{2\pi}\int_{-\infty}^{\infty}P_1(\eta)K_2(x+iy-a-i\eta)\,d\eta\right|$$
$$=\left|f(x+iy)-\frac{1}{2\pi}\sum_1^n B_k\int_{-\infty}^{\infty}e^{iM_k\eta}K_1(x+iy-b-i\eta)\,d\eta\right.$$
$$\left.+\frac{1}{2\pi}\sum_1^n A_k\int_{-\infty}^{\infty}e^{i\Lambda_k\eta}K_2(x+iy-a-i\eta)\,d\eta\right|$$

$$=\left|f(x+iy)-\frac{1}{2\pi}\sum_1^n B_k e^{iM_k y}\int_{-\infty}^{\infty}e^{iM_k\eta}K_1(x-b-i\eta)\,d\eta\right.$$
$$\left.+\frac{1}{2\pi}\sum_1^n A_k e^{i\Lambda_k y}\int_{-\infty}^{\infty}e^{i\Lambda_k\eta}K_2(x-a-i\eta)\,d\eta\right|$$

(35.05)
$$=\left|f(x+iy)-\sum_1^n B_k e^{M_k(x-b+iy)}\phi_n(M_k)\right.$$
$$\left.+\sum_1^n e^{\Lambda_k(x-a+iy)}(\phi_n(\Lambda_k)-1)\right|$$
$$=\left|f(x+iy)-\sum_1^{2n}c_k e^{N_k(x+iy)}\right|,$$

where

(35.06)
$$N_{2k-1}=M_k,\qquad N_{2k}=\Lambda_k,\qquad C_{2k-1}=B_k e^{-M_k b}\phi_n(M_k),$$
$$C_{2k}=-A_k e^{-M_k a}(\phi_n(\Lambda_k)-1).$$

Thus $f(x+iy)$ belongs uniformly to the class of almost periodic functions over $a+\epsilon \leqq x \leqq b-\epsilon$ since,

$$\int_{-\infty}^{\infty}|K_1(x-b+i\lambda)|\,d\lambda=\int_{-\infty}^{\infty}d\lambda\left|\int_{-\infty}^{\infty}\phi_n(y)e^{(x-b+i\lambda)y}\,dy\right|$$
$$=\int_{-\infty}^{\infty}d\lambda\left|\int_n^{n+2}\frac{x-n}{2}e^{(x-b+i\lambda)y}\,dy+\int_{n+2}^{\infty}e^{(x-b+i\lambda)y}\,dy\right|$$

(35.07)
$$=\int_{-\infty}^{\infty}\frac{d\lambda}{2|x-b+i\lambda|}\left|\int_n^{n+2}e^{(x-b+i\lambda)y}\,dy\right|$$
$$=\int_{-\infty}^{\infty}\frac{d\lambda}{2((x-b)^2+\lambda^2)}\left|e^{(n+2)(x-b+i\lambda)}-e^{n(x-b+i\lambda)}\right|$$
$$\leqq\frac{\text{const.}}{x-b}\leqq\text{const.},$$

(35.08)
$$\int_{-\infty}^{\infty}|K_2(x-a+i\lambda)|\,d\lambda\leqq\text{const.}$$

Similarly, for a large positive choice of n, we get

(34.35)
$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon\,(2\pi)^{1/2}}\int_{-A}^{A}|\,s(a,\,u+\epsilon)-s(a,\,u-\epsilon)\,|^2\,e^{2u(x-a)}\,du$$
$$=T_2(x,A)-T_2(x,-A).$$

Thus by formula (34.24),

(34.36)
$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-\infty}^{\infty}|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)\,|^2\,du$$

exists for $a < x < b$, and has the value indicated in that formula.

A proof of exactly the same sort, somewhat more fussy in detail, but not differing at all in principle, will show that if $a < x < b$, and ξ is real,

(34.361)
$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-\infty}^{\infty}|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)\,|^2\,e^{-iu\xi}\,du$$

exists, and equals

(34.362)
$$\lim_{A\to\infty}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-A}^{A}|\,s(b,\,u+\epsilon)-s(b,\,u-\epsilon)\,|^2\,e^{2u(x-b)-iu\xi}\,du.$$

It is clear that

(34.363)
$$\overline{\lim_{A\to\infty}}\ \overline{\lim_{\epsilon\to 0}}\frac{1}{\epsilon}\left|\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]|\,s(b,\,u+\epsilon)-s(b,\,u-\epsilon)\,|^2\,e^{2u(x-b)-iu\xi}\,du\right|$$
$$\leqq \lim_{A\to\infty}\lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]|\,s(b,\,u+\epsilon)-s(b,\,u-\epsilon)\,|^2\,e^{2u(x-b)}\,du,$$

and we have just proved in (34.34) that this is zero. Hence (34.362) exists, and we have only to prove its identity with (34.361). This will readily follow if for a set of values of A increasing to infinity,

(34.364)
$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-A}^{A}e^{-iu\xi}\{\,|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)\,|^2$$
$$-\,e^{2u(x-b)}\,|\,s(b,\,u+\epsilon)-s(b,\,u-\epsilon)\,|^2\}\,du=0.$$

By the use of the Schwarz inequality and the boundedness of

(34.365)
$$\frac{1}{\epsilon}\int_{-A}^{A}|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)\,|^2\,du \quad\text{and}$$
$$\frac{1}{\epsilon}\int_{-A}^{A}e^{2u(x-b)}\,|\,s(b,\,u+\epsilon)-s(b,\,u-\epsilon)\,|^2\,du,$$

(34.364) may be carried back to

(34.366)
$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-A}^{A}|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)$$
$$-\,e^{u(x-b)}\,(s(b,\,u+\epsilon)-s(b,\,u-\epsilon))\,|^2\,du=0,$$

which goes back to (34.21).

Now let us define

(34.37)
$$s_\xi(x,\,u)=\frac{1}{(2\pi)^{1/2}}\left[\int_{1}^{A}+\int_{-A}^{-1}\right]\frac{f(x+iy-i\xi)}{-\,iy}\,e^{-iuy}\,dy$$
$$+\,\frac{1}{(2\pi)^{1/2}}\int_{-1}^{1}f(x+iy-i\xi)\,\frac{e^{-iuy}-1}{-\,iy}\,dy.$$

This will exist for the same reason as (34.17). As on page 158 of Wiener's book,

(34.38)
$$\int_{-\infty}^{\infty}|\,s_\xi(x,u+\epsilon)-s_\xi(x,u-\epsilon)-e^{-iu\xi}(s(x,u+\epsilon)-s(x,u-\epsilon))\,|^2\,du=O(\epsilon^2).$$

As in that argument, this leads to

(34.39)
$$\lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}|\,f(x+iy-i\xi)+w\,f(x+iy)\,|^2\,dy$$
$$=\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty}(2+w\,e^{iu\xi}+\overline{w}\,e^{-iu\xi})\,|\,s(u+\epsilon)-s(u-\epsilon)\,|^2\,du.$$

If we take w successively to equal $\pm\,1$, $\pm\,i$, and add four such expressions as (34.39), with coefficients $\pm\,1$, $\pm\,i$, we get

(34.40)
$$\lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}f(x+iy-i\xi)\,\bar f(x+iy)\,dy$$
$$=\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty}|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)\,|^2\,e^{-iu\xi}\,du,$$

as in (21.17) of Wiener's book. Thus $f(x+iy)$ belongs to S over $a < x < b$.

By (34.40) we see that

(34.41)
$$\lim_{\xi\to 0}\lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}f(x+iy-i\xi)\,\bar f(x+iy)\,dy-\lim_{B\to\infty}\frac{1}{2B}\int_{-B}^{B}|\,f(x+iy)\,|^2\,dy$$
$$=\lim_{\xi\to 0}\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty}(1-e^{-iu\xi})\,|\,s(x,u+\epsilon)-s(x,u-\epsilon)\,|^2\,du$$
$$\leqq \overline{\lim_{\xi\to 0}}\ \overline{\lim_{\epsilon\to 0}}\frac{1}{4\pi\epsilon}\int_{-A}^{A}|\,1-e^{-iu\xi}\,|\,|\,s(x,u+\epsilon)-s(x,u-\epsilon)\,|^2\,du$$
$$+\,\overline{\lim_{\xi\to 0}}\ \overline{\lim_{\epsilon\to 0}}\frac{1}{4\pi\epsilon}\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]|\,1-e^{-iu\xi}\,|\,|\,s(x,u+\epsilon)-s(x,u-\epsilon)\,|^2\,du$$
$$\leqq \overline{\lim_{\xi\to 0}}\,|\,1-e^{-iA\xi}\,|\,\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty}|\,s(x,u+\epsilon)-s(x,u-\epsilon)\,|^2\,du$$
$$+\,\overline{\lim_{\epsilon\to 0}}\frac{1}{2\pi\epsilon}\left[\int_{A}^{\infty}+\int_{-\infty}^{-A}\right]|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)\,|^2\,du$$
$$\leqq \overline{\lim_{\epsilon\to 0}}\frac{1}{2\pi\epsilon}\int_{-\infty}^{\infty}|\,s(x,\,u+\epsilon)-s(x,\,u-\epsilon)\,|^2\,du$$

and

(34.26) $$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_{-A}^{A}|\,s(a, u+\epsilon) - s(a, u-\epsilon)\,|^2 e^{2u(x-a)}\,du$$

exist for all A's of an increasing sequence tending to infinity.

Let us put

(34.27) $$S(x, u) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{(e^{-iu\xi}-1)\,d\xi}{-i\xi} \times \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+iy+i\xi)\,\overline{f(x+iy)}\,dy\,;$$

(34.28) $$T_1(x, u) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{(e^{-iu\xi}-1)\,d\xi}{-i\xi}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{dy}{4\pi^2} \times \int_{-\infty}^{\infty} f(b+i\eta)\,K_1(x+iy+i\xi-b-i\eta)\,d\eta \times \int_{-\infty}^{\infty}\overline{f(b+i\eta)}\;\overline{K_1(x+iy-b-i\eta)}\,d\eta\,;$$

(34.29) $$T_2(x, u) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\frac{(e^{-iu\xi}-1)\,d\xi}{-i\xi}\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{dy}{4\pi^2} \times \int_{-\infty}^{\infty} f(a+i\eta)\,K_2(x+iy+i\xi-a-i\eta)\,d\eta \times \int_{-\infty}^{\infty}\overline{f(a+i\eta)}\;\overline{K_2(x+iy-a-i\eta)}\,d\eta\,.$$

By Theorem 36 and Lemma 29_6, $T_1(x, u)$ and $T_2(x, u)$ will exist for every u. By (33.09) and (33.13),

(34.30) $$T_1(x, u) = \operatorname*{l.i.m.}_{\epsilon\to 0}\left[\psi(\epsilon) + \frac{1}{2\epsilon\,(2\pi)^{1/2}} \times \int_0^u |\,s(b, u+\epsilon) - s(b, u-\epsilon)\,|^2\,[\phi_n(u)]^2\,e^{2u(x-b)}\,du\right].$$

Now, if we have a sequence of monotone functions $f_n(x)$ converging in the mean to a limit $f(x)$, we shall show that it converges almost everywhere to $f(x)$. We have uniformly

(34.31) $$\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(\xi)\,d\xi = \lim_{n\to\infty}\frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f_n(\xi)\,d\xi\,.$$

From this it follows that independently of $\epsilon > 0$,

(34.32) $$\underline{\lim}_{n\to\infty} f_n(x+\epsilon) \geqq \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(\xi)\,d\xi\,,\qquad \overline{\lim}_{n\to\infty} f_n(x-\epsilon) \leqq \frac{1}{2\epsilon}\int_{x-\epsilon}^{x+\epsilon} f(\xi)\,d\xi\,.$$

Furthermore, by Weyl's lemma to the Riesz-Fischer theorem, there exists a subsequence $\{f_{n_k}(x)\}$ $(k = 1, 2, \cdots)$ of $\{f_n(x)\}$, converging almost everywhere to $f(x)$. Thus $f(x)$ is monotone over a set of points Σ differing from the whole line at most by a null set. This set is of course everywhere dense. Let $F(x)$ stand for the value of $f(x)$ thus defined for values of x in Σ. If y does not lie in Σ, let us put

(34.321) $$f(y) = (1/2)\Big[\operatorname*{l.u.b.}_{x\text{ in }\Sigma,\ x<y} F(x) + \operatorname*{g.l.b.}_{x\text{ in }\Sigma,\ x>y} F(x)\Big]$$

and otherwise let $f(x) = F(x)$. Then $f(x)$ will be everywhere defined and monotone.

Again, by (34.32),

(34.322) $$\underline{\lim}_{n\to\infty} f_n(x+\epsilon) \geqq \overline{\lim}_{\eta\to 0}\frac{1}{2\eta}\int_{x-\eta}^{x+\eta} f(\xi)\,d\xi\,,\qquad \overline{\lim}_{n\to\infty} f_n(x-\epsilon) \leqq \underline{\lim}_{\eta\to 0}\frac{1}{2\eta}\int_{x-\eta}^{x+\eta} f(\xi)\,d\xi\,,$$

and by the fundamental theorem of the calculus, we have almost everywhere

(34.323) $$\underline{\lim}_{n\to\infty} f_n(x+\epsilon) \geqq f(x) \geqq \overline{\lim}_{n\to\infty} f_n(x-\epsilon)\,.$$

Thus almost everywhere

(34.324) $$f(x-\epsilon) \leqq \underline{\lim}_{n\to\infty} f_n(x) \leqq \overline{\lim}_{n\to\infty} f_n(x) \leqq f(x+\epsilon)$$

and hence almost everywhere

(34.325) $$\lim_{\epsilon\to 0} f(x-\epsilon) \leqq \underline{\lim}_{n\to\infty} f_n(x) \leqq \overline{\lim}_{n\to\infty} f_n(x) \leqq \lim_{\epsilon\to 0} f(x+\epsilon)\,.$$

Since a monotone function is almost everywhere continuous, this yields almost everywhere

(34.326) $$f(x) \leqq \underline{\lim}_{n\to\infty} f_n(x) \leqq \overline{\lim}_{n\to\infty} f_n(x) \leqq f(x)\,.$$

Thus it results that for almost all x,

(34.33) $$f(x) = \lim_{n\to\infty} f_n(x)\,.$$

This theorem allows us to use lim in place of l.i.m. in (34.30).

Now let n be taken as an arbitrarily large negative number in (34.30). Then for almost all A,

(34.34) $$\lim_{\epsilon\to 0}\frac{1}{2\epsilon\,(2\pi)^{1/2}}\int_{-A}^{A}|\,s(b, u+\epsilon) - s(b, u-\epsilon)\,|^2\,e^{2u(x-b)}\,du = T_1(x, A) - T_1(x, -A)\,.$$

Now let us consider the function

$$h_1(z) = \frac{1}{z - x - iy} - \frac{1}{z - c} - K_1(x + iy - z) \qquad (c > \Re(z) > x)$$

$$(34.13) \qquad = \int_0^n e^{\xi(x+iy-z)}\, d\xi + \int_n^{n+1} \left(1 - \frac{(\xi - n)^2}{2}\right) e^{\xi(x+iy-z)}\, d\xi + \int_{n+1}^{n+2} \frac{(n + 2 - \xi)^2}{2} e^{\xi(x+iy-z)}\, d\xi + \int_0^\infty e^{\xi(z-c)}\, d\xi ,$$

and

$$h_2(z) = \frac{1}{z - x - iy} - \frac{1}{z - c} - K_2(x + iy - z) \qquad (\Re(z) < x)$$

$$(34.14) \qquad = \int_0^n e^{\xi(x+iy-z)}\, d\xi + \int_n^{n+1} \left(1 - \frac{(\xi - n)^2}{2}\right) e^{\xi(x+iy-z)}\, d\xi + \int_{n+1}^{n+2} \frac{(n + 2 - \xi)^2}{2} e^{\xi(x+iy-z)}\, d\xi + \int_0^\infty e^{\xi(z-c)}\, d\xi .$$

We see that $h_1(z)$ and $h_2(z)$ represent the same analytic function, which is moreover $O(1/\Im(z)^2)$ at infinity. Thus by Cauchy's theorem,

$$(34.15) \qquad 0 = \frac{1}{2\pi} \int_{-\infty}^\infty f(b + i\eta)\, h_1(b + i\eta)\, d\eta - \frac{1}{2\pi} \int_{-\infty}^\infty f(a + i\eta)\, h_2(a + i\eta)\, d\eta ,$$

and by (34.07)

$$(34.16) \qquad f(x + iy) = \frac{1}{2\pi} \int_{-\infty}^\infty f(b + i\eta)\, K_1(x + iy - b - i\eta)\, d\eta - \frac{1}{2\pi} \int_{-\infty}^\infty f(a + i\eta)\, K_2(x + iy - a - i\eta)\, d\eta .$$

By (34.02) these integrals are absolutely convergent.

Let us now assume that $f(a + iy)$ and $f(b + iy)$ both belong to S, and that for $a \leqq x \leqq b$, (34.03) is uniformly satisfied. Let us put

$$(34.17) \qquad s(x, u) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{1/2}} \left[\int_1^A + \int_{-A}^{-1}\right] \frac{f(x + iy)\, e^{-iuy}\, dy}{-iy} + \frac{1}{(2\pi)^{1/2}} \int_{-1}^1 \frac{f(x + iy)\, (e^{-iuy} - 1)\, dy}{-iy} .$$

Then by (33.13) and (34.16),

$$(34.18) \qquad \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-\infty}^\infty | s(x, u + \epsilon) - s(x, u - \epsilon) - \{s(b, u + \epsilon) - s(b, u - \epsilon)\} \times \phi_n(u)\, e^{u(x-b)} + \{s(a, u + \epsilon) - s(a, u - \epsilon)\}\, (\phi_n(u) - 1)\, e^{u(x-a)} |^2\, du = 0 ,$$

where

$$(34.19) \qquad \phi_n(u)\, e^{u(x-b)} = \frac{1}{2\pi} \int_{-\infty}^\infty K_1(x - b + iy)\, e^{-iuy}\, dy ,$$

and

$$(34.20) \qquad (\phi_n(u) - 1)\, e^{u(x-a)} = \frac{1}{2\pi} \int_{-\infty}^\infty K_2(x - a + iy)\, e^{-iuy}\, dy .$$

It will immediately follow by proper choice of n that over any finite range,

$$(34.21) \qquad \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_A^B | s(x, u + \epsilon) - s(x, u - \epsilon) - \{s(b, u + \epsilon) - s(b, u - \epsilon)\}\, e^{u(x-b)} |^2\, du$$
$$= \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_A^B | s(x, u + \epsilon) - s(x, u - \epsilon) - \{s(a, u + \epsilon) - s(a, u - \epsilon)\}\, e^{u(x-a)} |^2\, du = 0 .$$

Furthermore, if ϵ is smaller than some quantity independent of B and x, using (33.04),

$$(34.22) \qquad \frac{1}{\epsilon} \int_B^\infty | \{s(b, u + \epsilon) - s(b, u - \epsilon)\}\, e^{u(x-b)} |^2\, du \leqq 2e^{2B(x-b)} \frac{1}{2\epsilon} \int_{-\infty}^\infty | s(b, u + \epsilon) - s(b, u - \epsilon) |^2\, du \leqq \text{const.}\, e^{2B(x-b)} .$$

Similarly,

$$(34.23) \qquad \frac{1}{\epsilon} \int_{-\infty}^A | \{s(a, u + \epsilon) - s(a, u - \epsilon)\}\, e^{u(x-a)} |^2\, du \leqq \text{const.}\, e^{2A(x-a)} .$$

Thus

$$(34.24) \qquad \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-\infty}^\infty | s(x, u + \epsilon) - s(x, u - \epsilon) |^2\, du = \lim_{A\to\infty} \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-A}^A | s(b, u + \epsilon) - s(b, u - \epsilon) |^2\, e^{2u(x-b)}\, du = \lim_{A\to\infty} \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-A}^A | s(a, u + \epsilon) - s(a, u - \epsilon) |^2\, e^{2u(x-a)}\, du ,$$

in case

$$(34.25) \qquad \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-A}^A | s(b, u + \epsilon) - s(b, u - \epsilon) |^2\, e^{2u(x-b)}\, du$$

Let $S(u)$ be defined as in (33.06), *and let*

$$T(u) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \frac{(e^{-iux}-1)\,dx}{-ix} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dw \times \int_{-\infty}^{\infty} K(x+w-\xi)\, f(\xi)\, d\xi \int_{-\infty}^{\infty} \overline{K(w-\xi)}\, \overline{f(\xi)}\, d\xi . \tag{33.11}$$

Then

$$T(u) = \int_0^u \left| \int_{-\infty}^{\infty} K(\xi)\, e^{-iu\xi}\, d\xi \right|^2 dS(u) . \tag{33.12}$$

Here we replace the σ of Wiener's formula by S.

Lemma 29_3. *Under the hypothesis of Theorem* 30,

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_{-\infty}^{\infty} \Big| t(u+\epsilon) - t(u-\epsilon) - \{s(u+\epsilon) - s(u-\epsilon)\} \times \int_{-\infty}^{\infty} K(\xi)\, e^{-iu\xi}\, d\xi \Big|^2 du = 0, \tag{33.13}$$

where

$$t(u) = \underset{A\to\infty}{\text{l.i.m.}} \frac{1}{(2\pi)^{1/2}} \left[\int_1^A + \int_{-A}^{-1} \right] \frac{g(x)\, e^{-iux}\, dx}{-ix} + \frac{1}{(2\pi)^{1/2}} \int_{-1}^{1} \frac{g(x)(e^{-iux}-1)\, dx}{-ix} . \tag{33.14}$$

Lemma 29_6. *Under the hypothesis of Theorem* 30, *$g(x)$ will belong to S'.*

34. Cauchy's theorem.

Let

$$\int_{-A}^{A} |f(x)|^2\, dx = O(A) . \tag{34.01}$$

Then, as in (33.02),

$$\int_{-A}^{A} \frac{|f(x)|^2\, dx}{1+x^2} = O(1) . \tag{34.02}$$

Thus if $f(x+i\eta)$ is analytic over $a \leqq x \leqq b$ and

$$\int_{-A}^{A} |f(x+iy)|^2\, dy = O(A) \tag{34.03}$$

uniformly in x over $a \leqq x \leqq b$, the function

$$\frac{f(x+iy)}{x+iy-c} \qquad (c > b > a) \tag{34.04}$$

belongs uniformly to L_2 over $a \leqq x \leqq b$. Thus by (3.38), if $a < x < b$,

$$\frac{f(x+iy)}{x+iy-c} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(b+i\eta)\, d\eta}{(b+i\eta-c)(b+i\eta-x-iy)} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{f(a+i\eta)\, d\eta}{(a+i\eta-c)(a+i\eta-x-iy)} . \tag{34.05}$$

Now,

$$\frac{x+iy-c}{(b+i\eta-c)(b+i\eta-x-iy)} = \frac{1}{b+i\eta-x-iy} - \frac{1}{b+i\eta-c}, \tag{34.06}$$

so that

$$f(x+iy) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(b+i\eta)\, d\eta \left(\frac{1}{b+i\eta-x-iy} - \frac{1}{b+i\eta-c} \right) - \frac{1}{2\pi} \int_{-\infty}^{\infty} f(a+i\eta)\, d\eta \left(\frac{1}{a+i\eta-x-iy} - \frac{1}{a+i\eta-c} \right) \tag{34.07}$$

Let us form

$$\phi_n(x) = 0 \quad (x < n); \qquad \phi_n(x) = \frac{(x-n)^2}{2} \quad (n \leqq x < n+1);$$
$$\phi_n(x) = 1 - \frac{(n+2-x)^2}{2} \quad (n+1 \leqq x < n+2); \tag{34.08}$$
$$\phi_n(x) = 1 \quad (n+2 \leqq x);$$

and the transforms

$$K_1(z) = \int_{-\infty}^{\infty} \phi_n(\xi)\, e^{\xi z}\, d\xi \tag{34.09}$$

and

$$K_2(z) = \int_{-\infty}^{\infty} (\phi_n(\xi) - 1)\, e^{\xi z}\, d\xi . \tag{34.10}$$

We shall have at infinity for any $x < 0$,

$$K_1(x+iy) = O\left(\frac{1}{y^3}\right), \tag{34.11}$$

and for any $x > 0$,

$$K_2(x+iy) = O\left(\frac{1}{y^3}\right). \tag{34.12}$$

Chapter VIII

Generalized harmonic analysis in the complex domain

33. **Relevant theorems of generalized harmonic analysis.** Wiener* and others have developed a theory of generalized harmonic analysis. This is a theory of developments in series of trigonometric functions which includes as particular cases the Fourier series and the Fourier integral, but which also includes theories such as that of white light, which do not come under either of the above headings. His theory has so far been confined to functions which may be complex-valued but are of real arguments. Now every theory of the harmonic analysis of functions of arguments in the real domain has an associated theory of functions of arguments in the complex domain. In the case of the Fourier integral, the associated theory is that of the Laplace integral; in the case of the Fourier series, the associated theory is that of the Taylor and Laurent series; and in the case of non-harmonic developments in discrete trigonometric functions such as are found in Bohr's theory of almost periodic functions, the associated theory is that of the Dirichlet theory. It is the purpose of this chapter to extend the Wiener theory in the same sense and to subsume under this general theory certain theorems concerning almost periodic functions. To this end we shall have to carry over certain results from Wiener's book. It does not come under the purpose of this chapter to prove these theorems or, indeed, to give any very detailed discussion of their significance. We shall simply remark that $f(x)$ represents the function subject to harmonic analysis; that $s(u)$ represents the integral of the Fourier transform (which itself does not exist) up to the argument u, and that $S(u)$ represents the total energy in the spectrum of f up to the frequency used. $\phi(x)$, as we shall define it below, is the so-called Faltung of f with itself, and $S(u)$ may be determined in terms of ϕ alone. S is the class of functions $f(x)$ determining a "spectrum" $S(u)$, and S' is the sub-class of S for which in a certain sense the spectrum contains no energy at infinite frequency.

The theorems and definitions which we shall quote come from Chapters III and IV of Wiener's book. They read as follows:

Theorem 20. *Let $f(x)$ be a measurable function for which*

(33.01) $$\frac{1}{2T}\int_{-T}^{T} |\, f(x)\,|^2\, dx$$

is bounded in T. Then

(33.02) $$\int_{-\infty}^{\infty} \frac{|\, f(x)\,|^2}{1+x^2}\, dx < \infty .$$

* *The Fourier Integral and Certain of its Applications*, Cambridge, 1933.

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Definition.

(33.03) $$s(u) = \underset{A\to\infty}{\text{l.i.m.}}\ \frac{1}{(2\pi)^{1/2}}\left[\int_1^A + \int_{-A}^{-1}\right]\frac{f(x)\, e^{-iux}}{-ix}\, dx + \frac{1}{(2\pi)^{1/2}}\int_{-1}^{1} f(x)\,\frac{e^{-iux}-1}{-ix}\, dx .$$

Let it be noted that under the hypothesis of Theorem 20, $s(u)$ will exist for almost every u.

Theorem 22. *Under the hypothesis of Theorem 20,*

(33.04) $$\lim_{\epsilon\to 0}\frac{1}{4\pi\epsilon}\int_{-\infty}^{\infty} |\, s(u+\epsilon) - s(u-\epsilon)\,|^2\, du = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} |\, f(x)\,|^2\, dx$$

in the sense that if either side exists, the other exists and has the same value.

Definition.

(33.05) $$\phi(x) \doteq \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} f(x+\xi)\,\bar{f}(\xi)\, d\xi .$$

Definition. *S is the class of measurable functions $f(x)$ for which $\phi(x)$ exists for every real x. S' is the class of functions of S for which $\phi(x)$ is continuous.*

Theorem 36. *If $f(x)$ belongs to S, then*

(33.06) $$S(u) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}\phi(x)\,\frac{e^{-iux}-1}{-ix}\, dx$$

exists for every u.

On page 162 of Wiener's book there is a formula which reads

(33.07) $$\sigma(u) = \text{const.} + \underset{\epsilon\to 0}{\text{l.i.m.}}\ \frac{1}{2\epsilon(2\pi)^{1/2}}\int_0^u |\, s(u+\epsilon) - s(u-\epsilon)\,|^2\, du ,$$

and is numbered (21.257). This formula states more than has actually been established at the point in question, and should read

(33.08) $$\sigma(u) = \underset{\epsilon\to 0}{\text{l.i.m.}}\left[\psi(\epsilon) + \frac{1}{2\epsilon(2\pi)^{1/2}}\int_0^u |\, s(u+\epsilon) - s(u-\epsilon)\,|^2\, du\right].$$

In Theorem 36, Wiener shows that if $f(x)$ belongs to S, $\sigma(u) - S(u)$ is a constant, except at most over a null set. Thus except at most over a null set, we have

(33.09) $$S(u) = \underset{\epsilon\to 0}{\text{l.i.m.}}\left[\psi(\epsilon) + \frac{1}{2\epsilon(2\pi)^{1/2}}\int_0^u |\, s(u+\epsilon) - s(u-\epsilon)\,|^2\, du\right].$$

Theorem 30. *Let $f(x)$ belong to S. Let $xK(x)$ belong to L_1, and $(1+|\,x\,|)K(x)$ to L_2. Let*

(33.10) $$g(x) = \int_{-\infty}^{\infty} K(x-\xi)\, f(\xi)\, d\xi .$$

whence

(23.19) $$\log \left| \frac{1 - x^2/z_\nu^2}{1 - x^2/\lambda_\nu^2} \right| \geqq 0.$$

Furthermore

(23.20) $$0 \leqq I_\nu = \int_{-\infty}^{\infty} \log \left| \frac{1 - x^2/z_\nu^2}{1 - x^2/\lambda_\nu^2} \right| x^{-2}\, dx$$
$$\leqq \frac{1}{\lambda_n} \int_{-\infty}^{\infty} \log \left| \frac{1 - t^2 e^{iO(1/\lambda_n)}}{1 - t^2} \right| t^{-2}\, dt = O\left(\frac{1}{\lambda_n^2}\right).$$

If we integrate term-wise, we obtain

(23.21) $$0 < \int_{-\infty}^{\infty} \log \left| \frac{\Xi(x)}{H(x)} \right| x^{-2}\, dx = \sum_{\nu=1}^{\infty} I_\nu < \infty.$$

Then by (23.17)

(23.22) $$(\log y)^{-1} \int_{-y}^{y} \log | c^{-1}\, \Xi(x) |\, x^{-2}\, dx \to -\frac{\pi}{2}.$$

If we now apply (23.10), and express everything in terms of the zeta function, we obtain

THEOREM XXV.*

(23.23) $$\int_1^y \log \frac{| \zeta(\frac{1}{2} + ix) |}{x^2}\, dx = o(\log y).$$

24. Some theorems of Titchmarsh. Titchmarsh† has discussed asymptotic properties of entire functions with real negative zeros. In this paragraph we indicate some results which overlap those of Titchmarsh. The method used in deriving these results is closely analogous to that used in proving Theorem XXII; therefore we shall give here only a brief outline of the proof, leaving the details to the reader.

Let

(24.01) $$f(z) = \prod_{\nu=1}^{\infty} \left(1 + \frac{z}{a_\nu}\right)$$

be an entire function all of whose zeros $\{-a_\nu\}$ are negative. It will be assumed that

(24.02) $$0 < a_1 \leqq a_2 \leqq \cdots, \qquad \sum_{\nu=1}^{\infty} a_\nu^{-1} < \infty.$$

* This result is less powerful than one obtained by Titchmarsh in his Cambridge tract on the zeta function. He establishes that

$$\int_0^T \log | \zeta(\tfrac{1}{2} + it) |\, dt = O(T \log\log T).$$

† Loc. cit.

We shall use the symbol $n(r)$ for the number of a_n's not exceeding r. The letter x will designate a real positive variable which tends to infinity.

THEOREM XXVI. *Let λ, ρ, θ be fixed numbers such that*

(24.03) $$\lambda > 0, \qquad 0 < \rho < 1, \qquad |\theta| < \pi.$$

Then the statements

(i) $$n(x) \sim \lambda\, x^\rho;$$

(ii) $$\log f(x) \sim \pi\, \lambda \operatorname{cosec} \pi\rho\, x^\rho;$$

(iii) $$\log | f(xe^{i\theta}) | \sim \pi\, \lambda \operatorname{cosec} \pi\rho \cos \theta\rho\, x^\rho \qquad (|\theta| < \pi);$$

(iv) $$\int_0^x r^{-1-\pi/(2|\theta|)} \log | f(re^{i\theta}) |\, dr \sim \frac{\pi\lambda \operatorname{cosec} \pi\rho \cos \theta\rho}{\rho - \frac{\pi}{2|\theta|}}\, x^{\rho - \pi/(2|\theta|)} \qquad \left(\frac{\pi}{2\rho} < |\theta| < \pi\right);$$

are all equivalent. In the last statement (iv) *the right-hand member in the case $\rho = \pi/(2\theta)$ should be replaced by its limiting value as $\rho \to \pi/(2\theta)$.*

We first observe that the convergence of the series implies

(24.04) $$n(x) = o(x).$$

Next, let us put

(24.05) $$\omega(x) = x^{-\rho}\, n(x).$$

In view of the fact that $n(x)$ is monotone increasing, it is readily seen that the statements (i), which can be written as

(24.06) $$\omega(x) \to \lambda,$$

and

(24.07) $$\int_0^x \omega(r)\, dr \sim \lambda x,$$

are equivalent.*

Our next step is to transform the left-hand members of (ii–iv) in such a way as to allow an immediate application of Wiener's Tauberian theorems. We have

$$x^{-\rho} \log f(x) = x^{-\rho} \int_0^\infty \log\left(1 + \frac{x}{t}\right) dn(t)$$
$$= x^{-\rho} \int_0^\infty n(t) \frac{x}{t(t+x)}\, dt = \frac{1}{x} \int_0^\infty \omega(t) \frac{\left(\frac{t}{x}\right)^{\rho-1}}{1 + \frac{t}{x}}\, dt,$$

* This is readily proved directly or derived from a theorem of Wiener (loc. cit., Theorem XIII, pp. 34–35); it also follows from a well known theorem of Landau, *Beitrage zur analytischen Zahlentheorie*, Rendiconti del Circolo Matematico di Palermo, vol. 26 (1908), pp. 169–302 (p. 218).

then the statements

(23.01) $$\log \phi(iy) \sim \pi A\, |y| \log |y| \quad \textit{as } y \to \infty$$

and

(23.02) $$\int_{-y}^{y} \log |\phi(x)|\, x^{-2}\, dx \sim -\pi^2 A \log |y|$$

are completely equivalent.

Let $y > 0$, and let us use the kernels $N_1(\lambda)$, $N_2(\lambda)$ of a previous paragraph. Then (23.01) and (23.02) may be replaced respectively by

(23.03) $$(y \log y)^{-1} \int_0^\infty N\left(\frac{t}{y}\right) d\lambda(t) \to A\,; \qquad N = N_1(\lambda),\quad N_2(\lambda)\,.$$

We now observe that either of the statements (23.03) implies

(23.04) $$\lambda(y) = O(y \log y)\,.$$

Indeed if (23.03) is satisfied then

(23.05) $$\begin{aligned} O(1) &\geq (y \log y)^{-1} \int_0^y N\left(\frac{t}{y}\right) d\lambda(t) \\ &= N(1)\,(y \log y)^{-1} \lambda(y) - (y \log y)^{-1} \int_0^y \lambda(t)\, d_t N\left(\frac{t}{y}\right) \\ &> N(1)\, \lambda(y)\, (y \log y)^{-1}, \end{aligned}$$

since $N(\lambda)$ is positive and decreasing. Next we prove that under condition (23.04), (23.03) is equivalent to

(23.06) $$\frac{1}{y} \int_0^\infty N\left(\frac{t}{y}\right) d\Lambda^*(t) \to A\,; \qquad N(\lambda) = N_1(\lambda)\,,\quad N_2(\lambda)\,;$$

where as a Cauchy principal value,

(23.07) $$\Lambda^*(y) = \int_0^y (\log t)^{-1}\, d\lambda(t)\,.$$

We can assume without essential restriction that $\lambda(t)$ is continuous at 1. It is readily seen from (23.04) and (23.07) that $\Lambda^*(y)$ vanishes for sufficiently small y, while

(23.08) $$\Lambda^*(y) = O(y) \text{ as } y \to \infty\,.$$

Now the difference between the left-hand members of (23.06) and (23.03) is equal to

(23.09) $$\begin{aligned} I(y) &= \frac{1}{y} \int_0^\infty N\left(\frac{t}{y}\right) \left(\frac{1}{\log t} - \frac{1}{\log y}\right) d\lambda(t) \\ &= (y \log y)^{-1} \int_0^\infty N\left(\frac{t}{y}\right) \log \frac{y}{t}\, d\Lambda^*(t) \\ &= -(y \log y)^{-1} \int_0^\infty \Lambda^*(t)\, d_t\left[N\left(\frac{t}{y}\right) \log \frac{y}{t}\right] \\ &= O\left\{(\log y)^{-1} \int_0^\infty t \frac{d}{dt}\left[N(t) \log \frac{1}{t}\right] dt\right\} = O\left(\frac{1}{\log y}\right), \end{aligned}$$

which tends to zero as $y \to \infty$ or $y \to 0$. The same theorem of Wiener which was applied in the proof of Theorem XXII shows immediately the equivalence of the two statements (23.06): hence the two statements (23.03) are equivalent, as are (23.01) and (23.02).

In order to apply Theorem XXIV to the theory of the Riemann zeta function, we introduce

(23.10) $$\begin{aligned} \Xi(z) &= \xi\left(\frac{1}{2} + iz\right) \\ &= \frac{1}{2}\left(\frac{1}{2} + iz\right)\left(\frac{1}{2} - iz\right) \pi^{-1/4-iz/2}\, \Gamma\left(\frac{1}{4} + \frac{iz}{2}\right) \zeta\left(\frac{1}{2} + iz\right). \end{aligned}$$

It is known that $\Xi(z)$ is an entire function, is even, and has all its zeros in the strip $|\Im(z)| < \frac{1}{2}$.* Moreover

(23.11) $$\log \Xi(iy) = O(y) + \log \Gamma(y/2) \sim \tfrac{1}{2}\, y \log y\,;$$

(23.12) $$\Xi(z) = c \prod_{\nu=1}^{\infty} \left(1 - \frac{z^2}{z_\nu^2}\right), \qquad \sum_1^\infty |z_\nu|^{-2} < \infty, \qquad c = \Xi(0).$$

We set

(23.13) $$z_\nu = z_\nu' + i z_\nu''; \qquad z_\nu' > 0; \qquad |z_\nu''| < \tfrac{1}{2}; \qquad |z_\nu| = \lambda_\nu.$$

Let us put

(23.14) $$H(z) = c \prod_{\nu=1}^{\infty} \left(1 - \frac{z^2}{\lambda_\nu^2}\right).$$

Along the imaginary axis we have

(23.15) $$\begin{aligned} \log \frac{|H(iy)|}{|\Xi(iy)|} &= -\sum_{\nu=1}^{\infty} \log \left|\frac{z_\nu^2 + y^2}{\lambda_\nu^2 + y^2}\right| = -\sum_{\nu=1}^{\infty} \log \left|1 + \frac{z_\nu^2 - \lambda_\nu^2}{\lambda_\nu^2 + y^2}\right| \\ &= O(1). \end{aligned}$$

Thus if $y > 0$, by (23.11)

(23.16) $$\log H(iy) \sim \tfrac{1}{2}\, y \log y,$$

and by Theorem XXIV,

(23.17) $$(\log y)^{-1} \int_{-y}^{y} \log |c^{-1} H(x)|\, x^{-2}\, dx \to -\frac{\pi}{2} \text{ as } y \to \infty\,.$$

Again, on the real axis,

(23.18) $$\left|1 - \frac{x^2}{z_\nu^2}\right| \geq \left|1 - \frac{x^2}{\lambda_\nu^2}\right|,$$

* Cf. A. E. Ingham, *The Distribution of Primes*, Cambridge Tract in Mathematics and Mathematical Physics, No 30, chapter III, §7.

Thus by another theorem of Wiener,*

(21.28) $$\frac{1}{y}\log f_1(iy) \sim A\pi$$

and

(21.29) $$\lambda(u) \sim Au$$

are completely equivalent. This is also a result of Titchmarsh.† Thus all that we have to establish is (21.28), or by Theorem XXII,

(21.30) $$\int_{-\infty}^{\infty} \log |f_1(x)|\, x^{-2}\, dx = -\pi^2 A .$$

We already know that

(21.31) $$\log^+ |f_1(w)| = O(|w|)$$

and

(21.32) $$\int_{-\infty}^{\infty} \log^+ |f_1(u)|\, u^{-2}\, du < \infty ,$$

as follows by a direct computation. It follows from (21.31) that the ratio $\lambda(t)/t$ is bounded. Hence by (21.05) and (21.09),

$$\int_{-y}^{y} \log |f_1(u)|\, u^{-2}\, du = -\frac{\pi^2}{y}\int_0^{\infty} N_2\left(\frac{t}{y}\right) d\lambda(t)$$

(21.33) $$= \frac{\pi^2}{y}\int_0^{\infty} \lambda(t)\, d_t N_2\left(\frac{t}{y}\right) = -2\int_0^{\infty} t^{-2}\lambda(t)\log\left|\frac{y+t}{y-t}\right| dt$$

$$= O\left\{\int_0^{y}\frac{dt}{t}\log\left(\frac{1+t/y}{1-t/y}\right) + \int_y^{\infty}\frac{dt}{t}\log\left(\frac{1+y/t}{1-y/t}\right)\right\} = O(1)$$

Combining this with (21.32), we obtain

(21.34) $$\int_{-y}^{y} \log^- |f_1(u)|\, u^{-2}\, du = O(1) ,$$

the integral being taken along the real axis. Thus we have

(21.35) $$\int_{-\infty}^{\infty} \log^- |f_1(u)|\, u^{-2}\, du < \infty .$$

This and (21.32) together yield (21.30). We shall clearly have

(21.36) $$\lim_{u\to\infty}\frac{\lambda(u)}{u} = A ,$$

from which (20.16) follows at once.

* N. Wiener, *A new method in Tauberian theorems*, Journal of Mathematics and Physics, Massachusetts Institute of Technology, vol. 7 (1928), pp. 161–184. The theorem is not there stated in the form applicable to Stieltjes integrals, but the proof is substantially independent of that form.

† E. C. Titchmarsh, *On integral functions with real negative zeros*, Proceedings of the

Let us return to $f(x)$, in the general case discussed in the hypothesis of Theorem XXI, where not all the roots are real. By (20.13)

(21.37) $$\lambda(u) = O(u) .$$

On the other hand,

(21.38) $$\int_0^{\infty}\frac{\log^+|f(x)|}{x^2}dx = \int_0^{\infty}\frac{\log^+\left|\prod_1^{\infty}\left(1-\frac{x^2}{z_\nu^2}\right)\right| dx}{x^2}$$

$$\geq \int_0^{\infty}\frac{\log^+\left|\prod_1^{\infty}\left(1-\frac{x^2}{|z_\nu^2|}\right)\right| dx}{x^2} = \int_0^{\infty}\frac{\log^+|f_1(x)|\, dx}{x^2} .$$

It follows that

(21.39) $$A = \lim_{u\to\infty}\frac{\lambda(u)}{u} \leq \frac{1}{\pi^2}\int_0^{\infty}\frac{|\log|f_1(x)||\, dx}{x^2}$$

exists.

22. **A condition that the roots of an entire function be real.** We wish to prove

THEOREM XXIII. *Let $f(z)$ be an even entire function of order not exceeding 1, and let the number of its roots $\pm z_n$ within a circle of radius r about the origin be $2\lambda(r)$. Let*

(22.01) $$\lambda(r) \sim Br \qquad (r\to\infty).$$

Then all the roots of $f(z)$ will be real, when and only when

(22.02) $$B = -\frac{1}{\pi^2}\int_{-\infty}^{\infty}\frac{\log|f(x)|\, dx}{x^2} .$$

By Theorem XXII and (21.28–9), all that we have to prove is that

(22.03) $$\int_{-\infty}^{\infty}\left[\log\left|\prod_1^{\infty}\left(1-\frac{u^2}{z_\nu^2}\right)\right| - \log\left|\prod_1^{\infty}\left(1-\frac{u^2}{|z_\nu^2|}\right)\right|\right] u^{-2}\, du = 0$$

when and only when every z_ν is real. However, if any z_ν is not real, we shall have

(22.04) $$\log\left|\prod_1^{\infty}\left(1-\frac{u^2}{z_\nu^2}\right)\right| - \log\left|\prod_1^{\infty}\left(1-\frac{u^2}{|z_\nu^2|}\right)\right| = \sum_{\nu=1}^{\infty}\log\frac{|z_\nu^2-u^2|}{||z_\nu^2|-u^2|} > 0$$

for real u, which is incompatible with (22.03).

23. **A theorem on the Riemann zeta function.** We proceed to prove

THEOREM XXIV. *If the λ_n's are real and positive, if the series $\Sigma\, 1/\lambda_n^2$ converges, and if*

(23.001) $$\phi(w) = \prod_1^{\infty}\left(1-\frac{w^2}{\lambda_n^2}\right),$$

23. A theorem on the Riemann zeta function

33. Relevant theorems of generalized harmonic analysis

AMERICAN MATHEMATICAL SOCIETY
COLLOQUIUM PUBLICATIONS
VOLUME XIX

!! S. 374 $H(\omega) \sim \frac{1}{2} \gamma \log$, $H(\omega) = \prod (1 - \frac{z^2}{\lambda^2})$

FOURIER TRANSFORMS
IN THE
COMPLEX DOMAIN

BY

The late RAYMOND E. A. C. PALEY
Late Fellow of Trinity College, Cambridge

AND

NORBERT WIENER
Professor of Mathematics at the Massachusetts Institute of Technology

S. 178

$\Rightarrow \frac{1}{2T} \int_{-T}^{T} |f(x)| dx$ bounded

$\Rightarrow \int_{-\infty}^{\infty} \frac{|f(x)|}{1+x^2} dx < \infty$

S. 77 $\log \mathfrak{X}(\omega) \sim \frac{1}{2} \gamma \log \gamma$

PUBLISHED BY THE
AMERICAN MATHEMATICAL SOCIETY
501 WEST 116TH STREET, NEW YORK