The current physical and mathematical realities with regards to an unified field theory

Klaus Braun
August 25, 2022
riemann-hypothesis.de
unified-field-theory.de

Dedicated to my wife Vibhuta
on the occasion of her 61th birthday, August 25, 2022

A physical world view

The modelling challenges of the SMEP

There are several modelling challenges unifying the so-called „Standard „Model“ of Elementary Particles“ and the general relativity theory, e.g.,

- in the Maxwell theory there has to be a physical „electron“ (fermion) object (a priori) existing outside of the Maxwell equations framework in order to make its motion happen governed by electric and magnetic field operators accompanied by a sophisticated „displacement current“; a similar situation is given by the (a priori) „inflaton“ object in the „big bang theory“
- in plasma physics there is a symmetry between the number of positively and negatively charged interacting particles per considered volume element
- the proportional parts of massive particle matter, plasma particle matter, the „vaccum“ and the unknown rest of the universe
- in quantum theory there is an asymmetry between the kinetics and dynamics of elementary particles and the ground state
- - there is the mathematical "mass gap" problem of the Yang-Mills equations (a generalization of the Maxwell equations, where a chromo-electromagnetic field carries charges), Physically speaking, this means that there is a difference in energy between vacuum energy and the next level lowest kinematical energy level
- the whole particle-antiparticle zoo of the SMEP requires annihilation, however there is no annihilation of the electron-positron pair, but the generation of photons $e^+ + e^- \rightarrow 2\gamma$
- a missing model for the extraordinarily short lifespan in the β decay, (DüH), a missing interpretation of the „fine-structure“ constant and the gyro-magnetic factor (MaW1) p. 75, (MaW) p. 225, in the context of both, wave mechanics and spin dynamics, (YoW)
- it has been questioned whether the NSE really describes general flows. The difficulty with ideal fluids is that in such fluids there are no frictional forces, (BrK9)
- in quantum theory the elements of a single geometric Hilbert space framework provide the model of the „state“ of an elementary particle. This „state“ is described with two attributes (location and momentum); from a mathematical perspective, those attributes relate to two different Hilbert spaces within a Hilbert scale
- in GRT the Einstein space is supposed to provide the framework for a geometric model of the phenomenon „gravity“. However it is a purely metric space accompanied by infinite numbers of locally euclidian space-time „frameworks“; mathematically speaking, the Einstein space has no geometric structure, at all
- a common modelling framework to understand the Hilbert-Polya conjecture and the Berry-Keating conjecture, (BrK4), (*).

(*) (DeJ) p. 292: The Montgomery-Odlyzko law: "The distribution of the spacing between successive non-trivial zeros of the Riemann zeta function (suitable normalized) is statistically identical with the distribution of eigenvalue spacing in a GUE (Gaussian Unitary Ensemble) operator

(DeJ) p. 295: "What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?"

Some crucial modelling requirements for an unified field theory

- to understand the energy factor which determine the distribution of electricity in particles of definite size and charge [(A)] , e.g., to gain insight into
    - the Ritz and Einstein *agreement to disagree* regarding the role of advanced and retarded (Coulomb) potentials (RiW)
    - the Landau damping phenomenon, where electrostatic oscillatory disturbances in plasma are quickly damped out, even without the dissipative help of physical collisions, (ShF)
- to overome the Schrödinger dilemma in case of the harmonic (Planck type) quantum oscillator (e.g. the microscopic model of a molecule with two atoms) with an *observed* zero point energy $\frac{1}{2}h\nu$ in the case of crystal lattices [(B)]
- to understand the discrepancies from simple Dirac theory between an electron and a proton regarding
    - the magnetic momentum in the context of the fine structure constant [(C)] and Ehrenhaft's forgotten discovery, photophoresis by electric vs. electric & magnetic ions, (BrJ), SaG)
    - the Dirac hypothesis [(*)] about the total amount of elementary particles in the universe, the fact, that the proton is the most quantum mechanical particle in the universe, and its astonishing relation to the formula $h \approx cm_p r_p$ [(C)]
- to understand the root cause of the β-decay half life, where on average a neutron decays into a proton, an electron, and an anti-neutrino within 15 minutes
- to understand the relationship between the Hilbert-Polya conjecture and the Berry-Keating conjecture.

(A) (EiA1) p. 28: *„Maxwell's equations determine the electromagnetic field when the distribution of electric charges and currents is known. But we do not know the laws which govern the currents and charges. We do know, indeed, that electricity consists of elementary particles (electrons, positive nuclei), but from a theoretical point of view we cannot comprehend this. We do not know the energy factors which determine the distribution of electricity in particles of definite size and charge, and all attempts to complete the theory in this direction have failed. If then we can build upon Maxwell's equations at all, the energy tensor of the electromagnetic field is known only outside the charged particles*".

(B) (FeE): *„Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field".*

(ScE) p. 62: *"The (second) attitude makes us think of the „n_s particles present in the state α_s" as of a proper vibration (or a „Hohlraum" oscillator to use a customary expression) in its n_sth quantum level. n_s becomes the quantum number and the stipulation that the system of quantum numbers $n_1, n_2, n_3, \ldots n_s, \ldots$ determines only one state of gas, …ceases to be a trange ne adoption, and comes into line with the ordinary view about quantum states and there statistical weight (viz. equal for any two of them). … Bose pointed out, that we could, alternatively to the „Hohlraum" oscillator statistics, speak of photon statistics, but then we had to make it „Bose statistics".*

The wave point of view, … at least in all Bose cases, raises another interesting question. Since in the Bose case we seem to be faced, mathematically, with a simple oscillator of the Planck type, of which the is the quantum number, we may ask whether we ought not adopt for n_s half-odd integers $1/2, 3/2, \ldots (2n+1)/2, \ldots$ rather than integers. <u>One must, I think, call that an open dilemma</u>. From the point of view of analogy one would very much prefer to do so. For, the zero-point energy" $\frac{1}{2}h\nu$ of a Planck oscillator is not only borne out directly observation in the cristal lattices, it is also so intimitely linked up with the Heisenberg uncertainty relation that one hates to dispense with it. On the other hand, if we adopt it striaghtaway, we get in serious trouble, especially on conteplating changes of the volume (e.g. radiabatic compression of a given volume of black-body raditation), because in this process the (infinite) zero-point energy seems to change by infinite amounts! So we do not adopt it, and we continue to take for the n_s the integers, beginning with 0.

(C) (BeH) p. 34: *„If the proton, a $spin$ $(1/2)$ particle with unit charge, obeyed the Dirac equation as precisely as does the electron, it should have a magnetic moment very near* 1 *nuclear magneton. This wide disagreement with the facts is demonstration that the simplified notion of a structureless and well-defined individual particle does not apply to nucleons so well as it does to the electron. Meson theory in its simplest form accounts for the additional moment as the effect of the current of transient mesons near the proton, and of the motion of the proton with the meson emission and reabsorption causes, but Dirac moment for the electron is calculated on the very same basis; the transient presence of photons, rather than mesons, and hence a small additional moment. The agreement is excellent for the electron, where the whole correction effect is only a part in one thousand. But no analogous theory has been found which can account quantitatively for dominating effect in case of the proton*."

[(*)] P.A.M. Dirac, Proc. R. Soc. Lond. A 165 (1938) , 199 ,(UnA1) p. 288, (UnA2) p. 85

Some mathematical modelling requirements
for a quanta action potential model

A quanta action potential model needs to be in line with today‘s physical realities, e.g.

i) Dirac’s proven electron radiation theory based on the mathematical model of a „point charge“ distribution Dirac „function“ $\delta \in H_{-k/2-\varepsilon}$ (where k denotes the space dimension of the underlying domain) is accompanied by indefinite energies of the related Klein-Gordon equation

ii) the stationary energy states of the (thermo-statistics relevant) Schrödinger equation w/o spin accompanied by the set of quantum numbers $E_n = \frac{h}{2\pi}\omega(k + \frac{1}{2})$

iii) the positron and the electron do not annihilate each other; they emit photons in the form $e^+ + e^- \rightarrow \gamma\gamma$

iv) the discrete energy eigenvalues in a hydrogen atom

v) the fine structure constant is approximately equal to the orbital speed of the lowest energy electron „particle“ divided by the speed of light

vi) only about 5 % of the universe is „ordinary matter“, where 99% of it is „plasma matter“; from the remaining 1% there are about 90% hydrogen atoms and about 10% helium atoms, the remaining matter of all known other atoms is 0,1%

vii) the hydrogen atom model is composed of 1 proton (accompanied by $0-2$ neutrons) and 1 elecron; the helium atom model is composed of two „electrons“ to a nucleus containing two protons with either 1 or 2 neutrons, but no closed-form solution to the Schrödinger equation exists.

A quanta action potential model needs to meet the following objectives

i) understand the energy factor which determine the distribution of electricity in particles of definite size and charge

ii) determine the distribution of an assembly of N identical systems over the possible states in which this assembly can find itself, given that the energy of the assembly is a constant E

iii) provide an alternative scalable baseline model to Dirac’s (one-system) model of radiation replacing the small term representing the coupling energy of the atom (system 1) and the radiation field (system 2)

iv) understand the β-decay process (where an instable atomic nucleus characterized by a high number of neutrons and a low number of protons decays into a stable atomic nucleus with high number of protons and a low number of neutrons; $n \rightarrow p + e^- + \nu_e$) from a theoretical perspective

v) understand the difference between nature vs. physical or mathematical (model related) coupling constants, (UnA2).

The Planck oscillator
(ScE)

Notations:

A certain class of states of the assembly will be indicated by saying that $a = a_1, a_2, a_3 \dots a_n, ..$ of the N systems are in state $n = 1,2,3 \dots$ respectively, and all states of the assembly are embraced – without overlapping – by the classes described by all different admissible sets of numbers a_n

State numbers:	$n = 1,2,3 \dots$
Energy:	$E_n = E_1, E_2, E_3 \dots$
Occupation numbers:	$a = a_1, a_2, a_3 \dots a_n, ..$

The number of single states, belonging to this class, is obviously

$$P = \frac{N!}{a_1!a_2!a_3!\dots a_n!..}$$

The set of occupation numbers must comply with the conditions

$$\sum_n a_n = N, \sum_n a_n E_n = E.$$

Those statements really finish our counting. But in this form the result is wholly unsurveyable. …If we regard the set of occupation numbers as obtaining always, we disregard only a very small possible distributions – and this has „a vanishing likelihood of ever being realized“.

The energy levels of the Planck oscillator are, (ScE),

$$E_n = \left(n + \frac{1}{2}\right) h\nu \ , \ n = 0,1,2, \dots$$

The sum-over-states (Zustandsumme) is given by ($\mu = \frac{1}{kT}$, where k denotes the Boltzmann constant)

$$Z = \sum_{n=0}^{\infty} e^{-E_n/kT} = \sum_{n=0}^{\infty} e^{-\mu E_n} = \sum_{n=0}^{\infty} e^{-\mu h\nu\left(n+\frac{1}{2}\right)}.$$

Then

$$\Psi := klogZ = -\frac{1}{T}\sum_{n=0}^{\infty} E_n$$

is the negative free energy divided by T. Putting $x := \mu h\nu = \frac{h\nu}{kT}$ one have

$$Z = e^{-\frac{x}{2}}\sum_{n=0}^{\infty} e^{-nx} = e^{-\frac{x}{2}}\frac{1}{1-e^{-x}} = \frac{1}{2}\frac{1}{\sinh\left(\frac{x}{2}\right)} .$$

Hence $\Psi = klogZ = -klog2(\sinh\left(\frac{x}{2}\right))$,

$$U = \frac{E}{N} = T^2\frac{\partial\Psi}{\partial T} = -kT^2\frac{1}{2}\frac{\cosh\left(\frac{x}{2}\right)}{\sinh\left(\frac{x}{2}\right)}\left(-\frac{h\nu}{kT^2}\right)$$

$$= \frac{h\nu}{2}\frac{e^{x/2}+e^{-x/2}}{e^{x/2}-e^{-x/2}} = \frac{h\nu}{2}\frac{e^{x/2}+1}{e^{x/2}-1}$$

$$= \frac{h\nu}{2} + \frac{h\nu}{e^{\frac{h\nu}{kT}}-1} = \frac{h\nu}{2} + \frac{h\nu}{e^{\mu h\nu}-1} ,$$

which is the well-known expression, in which the „zero-point energy“ is usually dropped.

The Fermi oscillator
(ScE)

This is a particularly simple system (invented, as we shall see later, to formulate „Fermi statistics“). It is a thing capable only of two levels, 0 and $\varepsilon = E_1 = E_2 = \cdots$). Hence

$$Z = 1 + e^{-\varepsilon/kT}$$

$$\Psi = klog(1 + e^{-\frac{\varepsilon}{kT}})$$

$$U = \frac{E}{N} = T^2 \frac{\partial \Psi}{\partial T} = -kT^2 \frac{e^{-\frac{\varepsilon}{kT}}}{1-e^{-\frac{\varepsilon}{kT}}} \frac{\varepsilon}{kT^2} = \frac{\varepsilon}{1+e^{\frac{\varepsilon}{kT}}} .$$

Compare this with the relevant second term of

$$U = \frac{h\nu}{2} + \frac{h\nu}{e^{\mu h\nu}-1}$$

of the Planck oscillator (taking $\varepsilon = h\nu$). There is just one remarkable difference in sign, ± 1 in the denominator. We shall see later that this consitutes the relevant difference between „Einstein-Bose statstics“ and „Fermi-Dirac“ statistics.

The thermodynamical functions of a system composed of L Planck oscillators or L Fermi oscillators would, of course, be obtained on multiplying by L.

Schrödinger's concept of a heat-bath

Schrödinger's thermo-statistical dilemma (*) of the Dirac model (particle – anti-particle – coupling energy) is the fact that the system I (particle – anti-particles) equipped with quantum numbers $n_1, n_2, n_3, \dots n_s, ..$ determines (only) one quantum state, while the system II (coupling energy) is described by the Dirac point charge model equipped with a „quantum state" n_0. This means the overall set of quantum numbers of the Dirac model is $n_s = 0,1,2,3$. However, the zero-point energy" $\frac{1}{2}h\nu$ of a Planck oscillator is borne out directly observation in the cristal lattices inducing a set of quantum numbers of both systems in the form $n_s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots$. We also note that the mathematical potential model to link both „systems" is given by the Coulomb-Newton potential. Its proper mathematical handling and usage require a restriction to the 3D case.

(ScE) p. 1: *"There is, essentially, only one problem in statistical thermodynamics: the distribution of a given amount of energy E over N identical systems. Or perhaps better: to determine the distribution of an assembly of N identical systems over the possible states in which this assembly can find itself, given that the energy of the assembly is a constant E. The idea is that there is weak interaction between them, so weak that the energy of interaction can be disregarded, that one can speak of the "private" energy of every one of them and that the sum of their "private" energies has to equal E. …..*

"To determine the distribution" .. mean in principle to make oneself familiar with any possible distribution-of-the-energy (or state-of-the-assembly) …. is (always the same) the mathematical problem; we shall (soon) present its general solution, from which in the case of every particular kind of system enery particular classification that may be desirable can be found as a special case:

But there are two different attitudes as regards the physical application of the mathematical result. …

The older and more naïve application is to N actually existing physical systems in actual physical interaction with each other, e.g. gas molecules or electrons or Planck oscillators or degrees of freedom ("ether oscillators") of a "hohlraum". The N of them together represent the actual physical system under consideration. This original point of view is associated with the names of Maxwell, Boltzmann and others.

But it suffices only dealing with a very restricted class of physical systems – virtually only with gases. It is not applicable to a system which does not consist of a great number of identical constituents with "private" energies. …

Hence a second point of view … has been developed. It has a particular beauty of its own, is applicable quite generally to every physical system, and has some advantages to be mentioned forthwith. Here the N identical systems are mental copies of the one system under consideration – of the one macroscopic device that is actually erected on our laboratory table. Now what on earth could it mean, physically, to distribute a given amount of energy E over these N mental copies? The idea is, in my view, that you can, of course, imagine that you really had N copies of your system, that they really were in "weak interaction" with each other, but isolated from the rest of the world. Fixing your attention on one of them, you find it in a peculiar kind of "heat-bath" which consists of the $N-1$ others.

Now you have on the one hand, the experience that in thermodynamical equilibrium the behavior of a physical which you place in a heat-bath is always the same whatever be the nature of the heat-bath that keeps it at constant temperature, provided, of course, that the bath is chemically neutral towards your system, i.e., that there is nothing else but heat exchange between them. On the other hand, the statistical calculations do not refer to the mechanism of interaction: they only assume that it is "purely mechanical", that it does not affect the nature of the single systems (e.g., that it never blows them to pieces), but merely transfers energy from one to the other.

These considerations suggest that we may regard the behavior of any one of those N systems as describing the one actually existing system when placed in a heat-bath of given temperature. Moreover, since N systems are a likely and number similar conditions, we can then obviously, from their simultaneous statistics, judge of the probability of finding our system, when placed in a heat-bath of given temperature, in one or other of its private states. Hence all questions concerning the system in a heat-bath can be answered. …

(*) (ScE) p. 62: *The wave point of view, … at least in all Bose cases, raises another interesting question. Since in the Bose case we seem to be faced, mathematically, with a simple oscillator of the Planck type, of which the is the quantum number, we may ask whether we ought not adopt for n_s half-odd integers $1/2,3/2, \dots (2n+1)/2, \dots$ rather than integers.* ***One must, I think, call that an open dilemma****. From the point of view of analogy one would very much prefer to do so. For, the zero-point energy" $\frac{1}{2}h\nu$ of a Planck oscillator is not only borne out directly observation in the cristal lattices, it is also so intimitely linked up with the Heisenberg uncertainty relation that one hates to dispense with it. On the other hand, if we adopt it striaghtaway, we get in serious trouble, especially on conteplating changes of the volume (e.g. radiabatic compression of a given volume of black-body raditation), because in this process the (infinite) zero-point energy seems to change by infinite amounts! So we do not adopt it, and we continue to take for the n_s the integers, beginning with* 0.

The Dirac potential model

(FeE): „Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field".

The Dirac model consists of an „atom" particle equipped with „energy", a „field particle" surrounded by „field energy", and a small „coupling energy term" between both energies. The corresponding model components are particles, anti-particles, and „boson" particles. In the electromagnetism case those are the electrons, the electron neutrinos, and the photons.

Mathematically speaking, the Dirac model is about a mathematical point element $x \in R^k$ and a related (electromagnetical) point charge (distribution) function $c \cdot \delta \in H_{-k/2-\varepsilon}$, carrying a point charge defined the corresponding physical problem related coupling constant c. The simplest related corresponding potential equation is given by the Laplacian operator accompanied by the gravitational constant $c_{grav} \sim 6{,}6 \cdot 10^{-11} \frac{N \cdot m^2}{kg^2}$ in the form

$$-\Delta u = c_{grav}\delta.$$

The Dirac model provides the blueprint for the conceptual design of the SMEP, where each of the three considered „force" phenomena requires correspondingly grouped „affected particles", related „(force specific) energy field particles (anti-particles)", and a small „coupling energy term" between the pair of the two energy types per considered force phenomenon. This conceptual design results into the today's zoo of $17 = 3 + 14$ fermions and bosons of the SMEP „model" representated in the form $U(1) \times SU(2) \times SU(3)$. Each of those symmetry groups represents the physical (conservation of energies) „symmetry" rule, by which the corresponding „zoo member" group are affected by its related „group force".

The mathematical-physical laws, which govern the charges and the related energy factors are represented by variational potential operator equations based on appropriately defined spaces, (ChJ). The Hilbert space for thermo-statistical relevant particles is the standard Hilbert space $L_2 = H_0$ equipped with the Lebesgue integral based inner product $(\cdot,\cdot)_0$.

The standard H_0-based variational energy Hilbert space with respect to the Laplacian potential operator is H_1, equipped with the (Dirichlet integral) inner product $D(\cdot,\cdot) \coloneqq (\cdot,\cdot)_1 \coloneqq (\nabla\cdot, \nabla\cdot)_0$. The elements of the kinematical energy Hilbert space H_1 can be represented as superposition of Fourier waves $\in H_1$. Accordingly, the $H_{-1/2}$-variational energy Hilbert space with respect to the Laplacian potential operator is the Hilbert space $H_{1/2}$, accompanied by the concept of wavelets.

We note that the mathematical link between the Hilbert scales H_α and the Dirac radiation theory is given by the Sobolev embedding theorem in the form $H_{-k/2-\varepsilon} \subset C^0$.

The Dirac (distributional) function provides a density distribution of charged point particles. The mass density of a point mass m at $\vec{r}_0$ is given by $\rho_m(\vec{r}_0) = m\delta(\vec{r} - \vec{r}_0)$. Its relation to the Laplacian operator is given by

$$\Delta\left(\frac{1}{|\vec{r}-\vec{r}_0|}\right) = -4\pi m\delta(\vec{r} - \vec{r}_0).$$

It can be generalized in the form

$$(\Delta + k^2)\left(\frac{e^{\pm ik(\vec{r}-\vec{s})}}{|\vec{r}-\vec{s}|}\right) = -4\pi m\delta(\vec{r} - \vec{s}).$$

The Coulomb potential model

(RiW): „In the special cases in which an electromagnetic process remains restricted to a finite space, the process can be represented in the form

$$f = f_1 = \frac{1}{4\pi}\int \frac{\varphi(x',y',z',t-\frac{r}{c})}{r} dx'dy'dz'$$

as well as in the form

$$f = f_2 = \frac{1}{4\pi}\int \frac{\varphi(x',y',z',t+\frac{r}{c})}{r} dx'dy'dz'$$

and in other forms. … Ritz considers the restriction to the form of retarded potentials as one of the roots of the second law, while Einstein believes that irreveribility is exclusively due to reasons of probability“.

Remark: Matter consists of atomic nuclei and electrons. The Coulomb force acts between atomic nuclei and electrons. It determines the several appearances of matter and most of the observable phenomena ((FIT) p. 37. From a mathematical perspective the Coulomb force corresponds to Newton‘s gravitational force. Both are equipped with the same functional 3D specific k^{-2} „distance“ dependency. the 3D-Coulomb force is proportional to the charge of the considered electrons; the 3D-gravitation force is proportional to the masses of the considered mass particles.

Remark: The Coulomb potential of electromagnetism is an example of a Yukawa potential with factor $e^{-\alpha mr}$ equal to 1, everywhere. This can be interpreted as saying that the photon „mass“ m (i.e. the force-carrier between interacting, charged particles) is equal to 0. The Yukawa potential concept was developed to explain the results of James Chadwick's atomic model, which consisted of positively charged protons and neutrons packed inside of a small nucleus, with a radius on the order of 10^{-14} meters. Finally, the Yukawa potential concept resulted into the concept of meson particles.

Remark: The Coulomb potential resp. the Yukawa potential are accompanied by plane resp. spheric waves.

Remark: The Dirac „function“ $\delta(\vec{r}-\vec{r}_0)$ is a linear continuous functional over the space of continuous functions, which maps a continuous function depending on $\vec{r}$, in this case the constant function q, to the specific function value $q(\vec{r}_0) = q$. The functional cannot be represented as an integral, however, in a distributional sense it holds $f(\vec{r}_0) = (\delta(\vec{r}_0,\cdot), f)$. The corresponding concept in Hilbert space theory are dual Hilbert spaces.

Remark: We note that the mathematical models of both physical concepts, the Dirac potential and the Coulomb potential, very much depend on the 3D-case.

Remark: The Heaviside function

$$\theta(x-x_0) = \begin{cases} 0 \; for \; x<x_0 \\ 1 \; for \; x\geq x_0 \end{cases}$$

is linked to the Dirac function by

$$\frac{d}{dx}\theta(x-x_0) = \delta(x-x_0).$$

Remark: From the identity

$$\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|^3} = -grad\frac{1}{|\vec{r}-\vec{r}_i|} \text{ and } \vec{E}(\vec{r}) \coloneqq \int \rho(\vec{s})\frac{\vec{r}-\vec{s}}{|\vec{r}-\vec{s}|^3}d^3\vec{s} = -grad\int \rho(\vec{s})\frac{1}{|\vec{r}-\vec{r}_i|}d^3\vec{s} = -grad\Phi(\vec{r})$$

it follows that the 3D electric field can be represented by a (scalar) electrostatic potential $\Phi(\vec{r})$ in the form

$$\vec{E}(\vec{r}) = -grad\Phi(\vec{r})$$

where

$$\Phi(\vec{r}) \coloneqq \int \frac{\rho(\vec{s})}{|\vec{r}-\vec{r}_i|}d^3\vec{s}\ .$$

The Coulomb potential model is a generalization of the Coulomb force law, where there are two Coulomb forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ acting in the direction of the connecting line between two charged particles, whereby $\boldsymbol{F}_1 = -\boldsymbol{F}_2$. The two Coulomb forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ are proportional to the product of the two charges q_1 and q_2 and are inverse proportional to the square of the distance of both particles, i.e.

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2 = k q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3}.$$

The constant k determines the interaction force between both particles. It depends from the chosen unit of those charges. If there is a repulsive interaction between those particles this implies $k > 0$.

The charge density of N point charges are expressed by Dirac functions in the form

$$\rho(\vec{r}) = \sum_{i=1}^{N} q_i \delta(\vec{r} - \vec{r}_i).$$

In case one identifies the electrons and atomic nuclei with those charge this corresponds to the charge density of an atom. A continuous charge density

$$\rho(\vec{r}) = \int \rho(\vec{s}) \delta(\vec{r} - \vec{s}) d^3\vec{s}$$

can be approximated by N point charges in related N space partition areas by

$$\rho(\vec{r}) = \int \rho(\vec{s}) \delta(\vec{r} - \vec{s}) d^3\vec{s} = \sum_{i=1}^{N} \int_{\Delta V_i} \rho(\vec{s}) \delta(\vec{r} - \vec{s}) d^3\vec{s} \approx \sum_{i=1}^{N} q_i \delta(\vec{r} - \vec{r}_i)$$ (*).

Then, the electric field (which is also called the electric field-force) $\vec{E}(\vec{r})$ defined by

$$\vec{E}(\vec{r}) \coloneqq \int \rho(\vec{s}) \frac{\vec{r} - \vec{s}}{|\vec{r} - \vec{s}|^3} d^3\vec{s}$$

describes the relationship between an arbitrary charge density ρ and the electric field (**).

A charged particle with charge q in an electric field is governed by N particles with charges q_i experiences a „field force“ in the form

$$\vec{F}_N(\vec{r}) = \sum_{i=1}^{N} q q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} = q \vec{E}_N(\vec{r}) \qquad \text{(resp. } \vec{F}(\vec{r}) = q\vec{E}(\vec{r})\text{)}$$

The physical modelling assumption is that the considered charge q are that small, that its size is not influenced by the charges of the electric field and its underlying charges q_i over time.

The force density of a given distribution of charges are defined by

$$\vec{f}_N(\vec{r}) = \frac{\Delta \vec{F}_N(\vec{r})}{\Delta V} = \frac{\Delta q}{\Delta V} \vec{E}_N(\vec{r}) \qquad \text{(resp. } \vec{f}(\vec{r}) = \rho(\vec{r})\vec{E}(\vec{r})\text{)}$$

where ΔV denotes the volume element at $\vec{r}$ and Δq denotes the related charge of that volume element.

(*) this is because of $\sum_{i=1}^{N} \int_{\Delta V_i} \rho(\vec{s}) \delta(\vec{r} - \vec{s}) d^3\vec{s} \approx \sum_{i=1}^{N} \left[\int_{\Delta V_i} \rho(\vec{s}) d^3\vec{s}\right] \delta(\vec{r} - \vec{s}) = \sum_{i=1}^{N} q_i \delta(\vec{r} - \vec{r}_i)$.

(**) this is because of

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} = \sum_{i=1}^{N} \left[\int_{\Delta V_i} \rho(\vec{s}) d^3\vec{s}\right] \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} = \sum_{i=1}^{N} \left[\int_{\Delta V_i} d^3\vec{s}\right] \Delta V_i \rho(\vec{r}) \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} \approx \int \rho(\vec{s}) \frac{\vec{r} - \vec{s}}{|\vec{r} - \vec{s}|^3} d^3\vec{s}).$$

The potential energy of a continuous charge distribution in its own field
(FIT)

The amount of work to be performed to move a point charge in the field $\vec{E}(\vec{r})$ from $\vec{r}_1$ to $\vec{r}_2$ is given by the scalar potential difference

$$W_{1,2} = q(\Phi(\vec{r}_1) - \Phi(\vec{r}_2)).$$

This potential difference is called the „voltage“. The „work“ is the product of „charge“ and „voltage. As the field is assumed to be rotation free the work does not depend from the path between $\vec{r}_1$ and $\vec{r}_2$. The work has the dimension of an energy. The term

$$(*)\quad W(\vec{r}) = q\Phi(\vec{r})$$

is called the potential energy of a charge q in an electric field, whereby the charge q does not contribute to the potential Φ governed by N particles with charges q_i.

As a generalization of this case a charge distribution $\rho(\vec{r})$ within an external field Φ_{ext} can be considered, where there is no contribution of $\rho(\vec{r})$ to that field. Then, every charge element $dq = \rho(\vec{r})d^3\vec{r}$ contributes according to formula (*). The summation over all charged elements then gives

$$W = \int \Phi_{ext}(\vec{r})\rho(\vec{r})d^3\vec{r}.$$

In order to determine the electrostatic energy of N particles with charges q_i one considers $i-1$ point charges q_j, which rest at $\vec{r}_i$ $(j = 1, \dots (i-1))$. Then, the potential energy of another point charge q_i in the field of the given charges q_j is given by

$$W_i(\vec{r}_i) = q_i \sum_{j=1}^{i-1} \frac{q_j}{|\vec{r}_i - \vec{r}_j|}\,.$$

Then, the potential energy of a system with N point charges is given by

$$(*)\quad W = \sum_{i=2}^{N} W_i(\vec{r}_i) = \sum_{i=2}^{N} q_i \sum_{j=1}^{i-1} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}.$$

In order to derive a continuous charge distribution $\rho(\vec{r})$ the distribution is replaced by N discrete charges $\Delta q_i = \rho(\vec{r}_i)\Delta V_i$. In case ρ is continuous, then for $N \to \infty$ and $\Delta V_i \to 0$ this lead to

$$W = \frac{1}{2}\int \frac{\rho(\vec{r})\rho(\vec{s})}{|\vec{r} - \vec{s}|} d^3\vec{s}\, d^3\vec{r}.$$

The limit process is only applicable in case of $\vec{r} \neq \vec{s}$; this corresponds to $i \neq j$

$$(*)\quad W = \sum_{i=2}^{N} W_i(\vec{r}_i) = \sum_{i=2}^{N} q_i \sum_{j=1}^{i-1} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}.$$

In case the charge distribution is continuous the case $\vec{r} = \vec{s}$ contributes only a negligible energy contribution.

Note: The „continuity“ requirement for $\rho(\vec{r})$ cannot be omitted. A charge density of point charges in the form $\rho = \sum q_i \delta(\vec{r} - \vec{r}_i)$ would lead to an infinite energy of the point charges in their own field (*).

(*) The self-energy case (the case $\vec{r} = \vec{s}$) of a continuous charge distribution $\rho(\vec{r})$ can be approximately considered by the finite energy of the (approximation) model of a homogeneously charged ball with radius $r \to 0$:

$$W = \frac{1}{2}\int \Phi(\vec{r})\rho(\vec{r})\, d^3\vec{r}.$$

In this case, the potential $\Phi(\vec{r})$ is generated by $\rho(\vec{r}) = \Delta\Phi(\vec{r})$. It results into the „work“

$$W = -\frac{1}{2}\frac{1}{4\pi}\int \Phi(\vec{r})\Delta\Phi(\vec{r})\; d^3\vec{r} = \frac{1}{8\pi}\int \left|\vec{E}(\vec{r})\right|^2\; d^3\vec{r}.$$

In other words, an *approximating* self-energy density of a continuous charge distribution in its own field is given by

$$w(\vec{r}) \coloneqq \left|\vec{E}(\vec{r})\right|^2.$$

We also note the (Helmholtz) formula $(\Delta + k^2)\left(\frac{e^{\pm ik(\vec{r}-\vec{s})}}{|\vec{r}-\vec{s}|}\right) = -4\pi m\delta(\vec{r} - \vec{s})$.

The Calderón-Zygmund (integrodifferential) operator
(EsG)

The above limit process is only applicable in case of $\vec{r} \neq \vec{s}$; this corresponds to $i \neq j$ in the formula

$$W = \sum_{i=2}^{N} W_i(\vec{r}_i) = \sum_{i=2}^{N} q_i \sum_{j=1}^{i-1} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{2} \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}.$$

The Calderón-Zygmund (integrodifferential) operator Λ is a pseudodifferential operator with symbol $|\xi|^1$. It provides an appropriate alternative tool to adress this kind of modelling requirement. In (BrK8) an alternative Schrödinger momentum differential operator is proposed, which is identical to this pseudodifferential operator. Its integrodifferential representation is given by

$$(\Lambda u)(x) = -(\Delta \Lambda^{-1}) u(x) = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy.$$

The related pseudo-differential operator with symbol $|\xi|^{-1}$ is defined by, (EsG) (3.15‘), (3.17‘),

$$\Lambda^{-1} u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{-\infty}^{\infty} \frac{u(y) dy}{|x-y|^{n-1}}, \quad n \geq 2.$$

An alternative representation of the operator Λ is given by, (EsG) (3.35),

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \sum_{k=1}^{n} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy \ .$$

where describes singular integral (Riesz) operators R_k

$$R_k u := -i \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy.$$

The lifespan of atomic states
(FIT)

In Bohr's atomic model an electron with a certain mass and charge $-e$ circles a proton with a certain mass and charge e, whereby the angular momentum is a multiple of the Planck constant. The balance of power and the angular momentum quantification (where the reduced mass has been approximated by m_e, because of $m_p \gg m_e$) is given by, ($\tilde{h} := \frac{h}{2\pi}$)

$$\frac{m_e v^2}{\mathrm{r}} = \frac{e^2}{r^2} \quad , m_e v r = n,\, n \in N.$$

For $n = 1$ one gets the Bohr radius α_B and the fine structure constant α in the form

$$r := \alpha_B := \frac{\tilde{h}^2}{m_e e^2}\,, \;\; v := v_{at} := \frac{e^2}{\tilde{h}} = \frac{e^2}{\tilde{h}c} c = \alpha c.$$

It follows that the rotation frequency of an atom is given by

$$\omega_{at} := \frac{v_{at}}{\alpha_B} = \frac{m_e e^4}{\tilde{h}^3} \sim 4 \cdot 10^{16}\, \frac{1}{sec}.$$

An excited atomic state ($n \geq 2$) goes into another lower state by radition of an photon. It there has a finite lifespan. This lifespan τ is determined by the time, to release the required energy during the circular motion $P = \frac{2}{3}\frac{e^2}{\alpha_B}\alpha^3 \omega_{at}$, i.e.

$$\tau \sim \frac{E_{at}}{P} \approx \frac{1}{\alpha^3}\frac{1}{\omega_{at}}\,.$$

The paramount role of the proton in fundamental physics
(UnA2) chapter 6

Planck's constant h is approximately equal to the product of the speed of light, the mass m_p of the proton and its radius r_p

$$h \sim \frac{\pi}{2} m_p r_p\,.$$

The formula $h = \frac{\pi}{2} m_p r_p$ is even valid within the current measuring limits of about one percent (!). Of course, this formula displays the definition of the Compton wavelength

$$\lambda_C = \frac{h}{c m_p}\,.$$

However, according to current wisdom, the wavelength λ_C calculated from the mass alone does not reflect the actual size of a particle. … Accordingly, the proton is not given a prominent role among elementary particles. In reality, however, it is the only particle in the universe, that is massive and stable at the same time. The fact that its Compton wavelength approximately matches its real extension measured by experiments is a clear indication of the paramount role of the proton in fundamental physics. …. Since the formula contains fundamental constants of nature only, it would be important to derive it from a theory.

Dirac's observation regarding the size and mass of of particles in the universe is

$$\frac{M_U}{m_p} \sim \frac{R_U^2}{r_p^2}.$$

… without Dirac's conjecture, there cannot be no further progress at all in understanding elementrary particles. A thorought understanding would require a calculation of their masses, which is literally unthinkable in the current paradigm, because the (available nature) constants … cannot be combined in a way that the unit of a mass, kg, emerges. … Dirac's observed large numbers would automatically appear, a consequence of the fact that the very nature of mass can only be understood cosmologically, as E. Mach had suspected.

Atomic nucleus, proton & neutron, carrying charges, and all that
(MaA)

All experiments so far have shown that electric charge in nature comes in units of one magnitude only. The magnitude is denoted by e and is called the electric charge since this magnitude was first measured in connection with the electron. This value is given by

$$e = (1.6021892 \pm 0.0000046) \times 10^{-19} \text{ Coulombs.}$$

All charges in nature occur in interal multiplies of this basic unit and present evidence indicates that all charged elementary particles carry the same magnitude of charge. This occurence of charges in discrete units is called charge quantization. Proton and neutrons which are the constituents of atomic nuclei carry charges $+e$ and 0 respectively. The atomic number Z of an element indicates the number of protons in its nucleus which therefore carries a total charge $+Ze$. This is balanced by the total charge of the Z electrons forming the atomic shells. The atom as a whole is electrically neutral when all its orbital electrons are present. The numerical equality of electron and positron charge has been established to an accuracy of one part in 10^{20}.

It is found tat in all intractions in natue, the total charge of an isolated system remains constant for all times. This is known as the law of conservation of charge. By isolated system, we mean here a system through the boundary of which no charge in allowed to escape or enter. This does not require that the amounts of positive and negative charges are separately conserved; only their algebraic sum is. This means that simultaneous creation or annihilation of pairs of equal and opposite charges is allowed. This means in fact, observed in the following process:

$$\text{Pair production:} \quad 2\gamma \ \ (two\ \gamma\ rays) \ \ \rightarrow \ \ electron\ (e^+) \ + \ positron\ (e^-).$$

The above process can take place only in the presence of matter (e.g. an atom or a nucleus) which absorbs appropriate amounts of linear momentum and energy to satisfy the conservation laws. Electron and positron form a particle-antiparticle pair. They have identical rest masses. Their charges are equal in magnitude but opposite in sing. In addition, we have the inverse reaction.

$$\text{Mutual annihilation:} \ \ electron\ (e^+) \ + \ positron\ (e^-) \ \rightarrow \ 2\gamma \ \ (two\ \gamma\ rays).$$

In addition to the above processes of particle-antiparticle pair creation and annihilation we have radioactive decay processes in which total charge is always conserved.

As an example consider the decay of ${}^{238}_{92}U$

$${}^{238}_{92}U \ \ \rightarrow \ \ {}^{234}_{90}Th + \ {}^{4}_{2}He.$$

The number on the lower left side denotes the atomic number Z of the element. It is also the same as the number of protons in its nucleus. The upper number denotes the number of protons and neutrons (i.e. $Z + N$) in the nucleus. In the above equation to the total charge on each side is identical ($+92e$) as required by the conservation law for electric charge.

The two most important nuclear reactions between U^{235} and neutrons
(WeA) p. 1:

$$U^{235} + neutron \ \rightarrow \ 2\ fission\ fragments + \nu\ \boldsymbol{neutrons} \ + \beta\ and\ \gamma\ rays + energy$$

$$U^{235} + neutron \ \rightarrow \ U^{236} + \gamma\ rays\ (photons)$$

The lifespan of atomic states
(FIT)

In Bohr‘s atomic model an electron with a certain mass and charge $-e$ circles a proton with a certain mass and charge e, whereby the angular momentum is a multiple of the Planck constant. The balance of power and the angular momentum quantification (where the reduced mass has been approximated by m_e, because of $m_p \gg m_e$) is given by, ($\tilde{h} \coloneqq \frac{h}{2\pi}$)

$$\frac{m_e v^2}{\mathrm{r}} = \frac{e^2}{r^2} \quad , m_e v r = n,\, n \in N.$$

For $n = 1$ one gets the Bohr radius α_B and the fine structure constant α in the form

$$r \coloneqq \alpha_B \coloneqq \frac{\tilde{h}^2}{m_e e^2}\,, \;\; v \coloneqq v_{at} \coloneqq \frac{e^2}{\tilde{h}} = \frac{e^2}{\tilde{h}c} c = \alpha c.$$

It follows that the rotation frequency of an atom is given by

$$\omega_{at} \coloneqq \frac{v_{at}}{\alpha_B} = \frac{m_e e^4}{\tilde{h}^3} \sim 4 \cdot 10^{16}\, \frac{1}{sec}.$$

An excited atomic state ($n \geq 2$) goes into another lower state by radition of an photon. It there has a finite lifespan. This lifespan τ is determined by the time, to release the required energy during the circular motion $P = \frac{2}{3}\frac{e^2}{\alpha_B}\alpha^3 \omega_{at}$, i.e.

$$\tau \sim \frac{E_{at}}{P} \approx \frac{1}{\alpha^3}\frac{1}{\omega_{at}}\,.$$

Some facts from nuclear theory
(BeH)

(BeH) I: Each atomic nucleus has a charge Ze, a mass M, and a mass number A. Ze is an integral multiple of the charge e of the proton. M is very close to an integral multiple of the proton mass. The integer A which gives the multiple closest to M is the mass number.

(BeH) VII: The neutron is observed to change spontaneously into a proton, an electron, and a neutrino. This process is so rare that we can often ignore it. Neutron decay takes on the average some thousand seconds for a free neutron, whereas within a nucleus the characteristic time between nucleon-nucleon collisions is only 10^{-21} second. … But since 1947 it has been known that there is another class of particles, called π-mesons, which are capable of strong interaction with nucleons and nuclei. They can emitted, absorbed, scattered. When free, they have an independent existence, long compared to the nuclear collision time, and their intrinsic properties are open to measurement. Their role as transient constituents of nuclei has been theoretically indicated for a long time; it is still by no means clear… Three varieties of π-mesons (with spin zero) are known, classified by their electric charge as positive, negative, and neutral. … The difference in mass between π^+ and π^- is probably not significant.

$$\text{Decay: } \pi^{\pm} \rightarrow \begin{cases} \pi^{\pm} + neutrino & 100\% \\ \beta^{\pm} + neutrino & \leq 1/2 \cdot 10^{-4} \end{cases}$$

$$\text{Decay: } \pi^{0} \rightarrow \begin{cases} 2\gamma & \sim 99\% \\ \gamma + \beta^{+} + \beta^{-} & \sim 0.8 \pm 0.2\% \\ \beta^{+} + \beta^{-} & \sim \quad ? \end{cases} .$$

The total annihilation of nucleons (protons) has been recently observed. This can occur when an „anti-proton“ ($Z = -1, A = +1$) meets a proton ($Z = +1, A = +1$). „Anti-neutrons“ (particles with ($Z = 0, A = +1$), but capable of combining with and annihilating an ordinary neutron, of the same Z and A) presumably also exist, but have not yet been observed.

The word „neutrino“ has been used to represent any assumed product of decay which has half-integer spin, no charge, aand neglegible mass. Whether these particles are all identical with neutrinos of β-decay is so far conjecture.

(BeH) VIII, PROTON, „The Dirac theory of the electron predicts that magnetic moment of an electron is exactly 1 Bohr (electron) magneton, $-|e| \cdot \frac{h}{2\pi} \frac{2}{m \cdot c}$. This quantity can be measured with high precision and turns out not to be exactly unity but actually $1 + \frac{2\pi}{137}$. Even this small discrepancy from simple Dirac theory can be calculated with high accuracy from the quantum theory of radiation, agreeing with the experimental value to within the small error, a few parts per hundred thousands. If the proton, a spin $(1/2)$ particle with unit charge, obeyed the Dirac equation as precisely as does the electron, it should have a magnetic moment very near 1 nuclear magneton. This wide disagreement with the facts is demonstration that the simplified notion of a structureless and well-defined individueal particle does not apply to bucleons so well as it does to the electron. Meson theory in its simplest form accounts for the additional moment as the effect of the current of transient mesons near the proton, and of the motion of the proton with the meson emission and reabsorption causes, but Dirac moment for the electron is calculated on the very same basis; the transient presence of photons, rather than mesons, and hence a small additional moment. The agreement is excellent for the electron, where the whole correction effect is only a part in one thousand. But no analogous theory has been found which can account quantitatively for dominating effect in case of the proton.“

Emmision of electron from nucleus in β-decay
(BoD), p. 508

In the process of β -decay, an electron is emitted from the nucleus with a speed that is, in most cases, close to that of light. The electron leaves in a time of the order r/c, where r is the atomic radius. On the other hand, the periods of the electrons in the atom are of the order of $2\pi \cdot (r/v)$, which is usually at least 100 times as great (v is the speed atomic electrons). This means that for all practical purposes one can say that the nuclear charge is suddenly increased from Z to $Z+1$. At the moment that this charge has occured, the electronic wave function, $u_n(x)$, is that appropriate to a stationary state of an atom of charge Z.

In the new atom of charge $Z+1$, this wave function no longer corresponds to a stationary state, but must be expanded in terms of the stationary-state wave function for the new charge, $Z+1$, as shown in

$$\psi = u_n(x) = \sum_m C_{mn} v_m(x) \ .$$

This means that there will be a certain probability that the atom will be left in an excited state of the new atom as a result of the suddenness of the process of β-decay. This excitation can be detected by the subsequent emission of radiation, which is usually in the x ray region.

The Mie theory

A more general theory of electrodynamics has been proposed by Mie, by which it seems possible to derive the matter from the field.

> (WeH) p. 171: „*On the basis of rather convincing general considerations, G. Mie pointed out a way of modifying the Maxwell equations in such a manner that they might possibly solve the problem of matter, by explaining why the field possesses a granular structure and why the knots of energy remain intact in spite of the back-and-forth flux of energy and momentum*".

The requirement leading to the Mie equations is that the mechanical law must follow from the field equations. Therefore, the Mie equation is fully analogous to that of the fundamental law of mechanics. In the static case that is, the electric force is counterbalanced in the ether by the concept of an „*electric pressure*". It is the essential differentiator to the Lorentz equations, where there is no law that determines how the potentials depend on the phase-quantities of the field and on the electricity; there is only a formula giving the density of the mechanical (*ponderomotorische*) force and the law of mechanics, which governs the motion of electrons under the influence of this force.

Mie's theory resolves the problem of matter into a determination of the expression of the Hamiltonian function in terms of four quantities and the laws for the field may be summarised in a Hamilton's principle.

(WeH1): "*G. Mie in 1912 pointed out a way of modifying the Maxwell equations in such a manner that they might possibly solve the problem of matter, by explaining why the field possesses a granular structure and why the knots of energy remain intact in spite of the back-and-forth flux of energy and momentum. The Maxwell equations will not do because they imply that negative charges compressed in an electron explode; … The preservation of the energy knots must result from the fact that the modified field laws admit only of one state of field equilibrium … The field laws should thus permit us to compute in advance charges and mass of the electron and the atomic weights of the various chemical elements in existence. And the same fact, rather than contrast of substance and field, would be the reason why we may decompose the energy or inert mass of a compound body (approximately) into the non-resolvable energy of its last elementary constituents and the resolvable energy of their mutual bond. …. At a certain stage of the development it did not seem preposterous to hope that all physical phnomena could be reduced to a simple universal field law (in the form of a Hamiltonian principle).*"

In mechanics, a definite function of action corresponds to every given mechanical system and has to be deduced from the constitution of the system. Mie's theory is only concerned with a single system, the world. This is were the real problem of matter takes its beginning: to determine the Mie „world-function of action", belonging to the physical world.

The proposed gravity and quantum field model is basically an enhanced Mie electrodynamic overcoming the above difficulty which is basically caused by a missing truly geometric structure of the underling manifolds w/o any conceptual relationship to all possible mathematical solution of the Mie equations. Therefore, the enhancement is concerned with a replacement of the manifold framework by a Hilbert space, where its inner product induces a corresponding norm and where an existing hermitian operator induces a corresponding energy norm, governing for example the least action or least energy formalisms.

The common baseline with the proposed NSE solution in line with the proposed YME solution is related to the physical notion "*pressure*", which has the same unit of measure than a "*potential difference*". The common additional conceptual new element is the fact that a "*potential difference*" becomes now an intrinsic element of the corresponding PDE systems governed by the closed "potential energy" ("ground state", "internal energy") sub-space.

Global boundedness of the 3D-Navier-Stokes equations in a variational $H_{-1/2}$ based Hilbert space framework
(BrK9)

It turned out that based on the physical modelling assumption of a $H_{-1/2}$ Hilbert space of fluid element the 3D NSE guarantees global solutions, (BrK9).

Putting

$$B(u) := P(u, grad)u)$$

in the NSE and assuming $Pu_0 = u_0$, the NSE initial-boundary equation is given by

$$\frac{du}{dt} + Au + Bu = Pf \ , u(0) = u_0.$$

Multiplying this homogeneous equation with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_\alpha + (Au, u)_\alpha + (Bu, u)_\alpha = 0, \ (u(0), v)_\alpha = (u_0, v)_\alpha \text{ for all } \ v \in H_{-1/2}$$

We note that the the pressure p in the variational representation

$$(Au, v)_{-\frac{1}{2}} \coloneqq (\nabla u, \nabla v)_{-\frac{1}{2}} + (\nabla p, v)_{-\frac{1}{2}} = (u, v)_{\frac{1}{2}} + (p, v)_0 \qquad \text{for all } \ v \in H_{-1/2}$$

$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2}$$

.

can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

with (R_1, R_2, R_3) is the Riesz transform.

In case of $\alpha = -1/2$ one gets from the Sobolevskii-estimates (see (GiY) lemma 3.2), the corresponding generalized “energy” inequality in the form

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\},$$

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

Lemma (GiY): For $0 \le \delta < 1/2 + n \cdot (1 - 1/p)/2$ it holds

$$\left|A^{-\delta}P(u, grad)v\right|_p \le M \cdot \left|A^\theta u\right|_p \cdot |A^\rho u|_p$$

with a constant $M := M(\delta, \theta, \rho, p)$ if $\delta + \theta + \rho \ge n/2p + 1/2$, $\theta, \rho > 0$, $\theta + \rho > 1/2$. Putting $p := 2$, $\delta := 1/4$,$\theta := \rho := 1/2$ fulfilling $\theta + \rho \ge \frac{1}{4}(n + 1) = 1$ it follows

$$\left\|A^{-\delta}P(u, grad)u\right\| \le c\left\|A^\theta u\right\| \cdot \|A^\rho u\| = c\|u\|_{2\theta} \cdot \|u\|_{2\rho} = c\|u\|_1^2$$

resp.

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le \left|(Bu, u)_{-1/2}\right| \le c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

The d'Alembert "paradox"

The d'Alembert "paradox" is not about a real paradox but it is about the failure of the Euler equation (the model of an ideal incompressible fluid) as a model for fluid-solid interaction.

The difficulty with ideal fluids and the source of the d'Alembert paradox is that in incompressible fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by streamline. It is clear, that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The mathematical requirements to define boundary layers and corresponding potentials are very much depending by the definition and regularity requirements of the normal derivative. It is perpendicular to the boundary itself and therefore requires regularity assumptions, affecting "points" outside of the domain w/o any physical meaning. J. Plemelj's alternative "flux" concept. (PlJ), provides the proper tool for the proposed Krein space based mathematical framework, as it does not require "ideal" boundary layer assumptions.

Plasma

Plasma is an inonized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not.

(ChF) 8.8: "*When an electron plasma wave goes nonlinear, the dominant new effect is that the ponderomotive force of the plasma waves causes the background plasma to move away, causing a local depression in density called caviton. Plasma waves trapped in this cavity then form an isolated structure called envelope soliton or envelope solitary wave. Considering the difference in both the physical model and the mathematical form of the governing equations, it is surprising that solitons and evelopes solitons have almost the same shape*".

(CaF) p. 390 ff.: "*The turbulence of plasma differs from the hydrodynamic turbulence by the action of the magnetic field. A more relevant difference is due to the hydrodynamic interaction between the plasma particles, the interaction with the magnetic fields, and the interaction between the electromagnetic waves. ... All of them are the root cause of electromagnetic plasma turbulence. ... The case of interactions between quasi-stationary electromagnetic waves is called weak turbulence. ... The case of non-linear Landau damping (strong plasma turbulence) leads to the generation of virtual waves, which transfer their energy to the affected particles asymptotically with 1/t; the plasma is heated (turbulence heating) faster than this may happen by purely particles collisions*".

Fundamental processes of plasma

Local depression in density

There are two nonlinear equations that have been treated extensively in connection with nonlinear plasma waves: The Korteweg-de Vries equation and the nonlinear Schrödinger equation. Each concerns a different type of nonlinearity.

When an ion acoustic wave gains large amplitude, then main nonlinear effect is wave steeping, …It is mathematically handled by the Korteweg-de Vries equation

(ChF) p. 330 f. :"*When an electron plasma wave goes nonlinear, the dominant new effect is that the ponderomotive force of the plasma waves causes the background plasma to move away, causing a local depression in density called caviton. Plasma waves trapped in this cavity then form an isolated structure called envelope soliton or envelope solitary wave. Considering the difference in both the physical model and the mathematical form of the governing equations, it is surprising that solitons and evelopes solitons have almost the same shape*".

(ChF) p. 336: „*Plane waves of the non-linear **Schrödinger** equation in the form*

$$ih\frac{\partial\psi}{\partial t} + p\frac{\partial^2\psi}{\partial x^2} - q|\psi|^2\psi = 0$$

are modulationally unstable if $pq > 0$*; that is, a ripple on the envelope of the wave will tend to grow. For plasma waves, it is easily to see how the ponderomotive force can cause a modulational instability. The ponderomotive force moves both electrons and ions toward the intensity minima, forming a ripple in the plasma density. Plasma waves are trapped in regions of low density*".

The turbulence of plasma

(TsV) p. 4.: „*Plasma is that state of matter in which the atoms or molecules are found in an ionized state. The interactions of electrons and ions are determined by long-range electrical forces. The many forms of collective motion in a plasma are the result of coupling the charged-particle motion to the electromagnetic field. Therefore, the electromagnetic field which accompanies the particle motion is also a random nonreproducible quantity in a turbulent plasma. Measurements have shown that the fields excited in a plasma during the development of turbulence do in fact have a random nature*."

Charge neutrality and Landau damping
(MiK) p. 1 ff.

One of the fundamental property of plasma is the shielding of the electric potential applied to the plasma. When a probe is inserted into a plasma and positive (negative) potential is applied, the probe attracts (repulses) electrons and the plasma tends to shield the electric disturbance.

The other fundamental process of plasma is collective phenomena of charged particles. Waves are associated with coherent motions of charged particles. When the phase velocity v_{ph} of wave or perturbation is much larger than the thermal velocity v_T of charged particles, the wave propagates through the plasma media without damping or amplification. However when the refractive index N of plasma media becomes large and plasma becomes hot, the phase velocity $v_{ph} = c/N$ (c is light velocity) of the wave and the thermal velocity v_T become comparable ($v_{ph} = \frac{c}{N} \sim v_T$), then the exchange of energy between the wave and the thermal energy of plasma is possible. The existence of a damping mechanism of wave was found by L. D. Landau. The process of Landau damping involves a direct wave-particle interaction in collisionless plasma without necessity of randamizing collision. This process is fundamental mechanism in wave heatings of plasma (wave damping) and instabilities (inverse damping of perturbations).

The Landau damping

(DeR) p. 94: „*The Landau damping property is complementary to the properties of electro-magnetic forces, which weaken themselves spontaneously over time w/o increase of entropy or friction. It involves coupling between single-particles and collective aspects of plasma behavior. ..this topic is related to one of the main unsolved questions in physics*." ....

Landau damping involves a flow of energy between single particles on the one hand side, and collective excitations of plasma on the other side".

(BiJ): „*The Landau damping phenomenon is about "wave damping w/o energy dissipation by collisions in plasma", because electrons are faster or slower than the wave and a Maxwellian distribution has a higher number of slower than faster electrons as the wave. As a consequence, there are more particles taking energy from the wave than vice versa, while the wave is damped*."

Nonlinear Landau damping

When the amplitude of an electron or ion wave excited, say, by a grid is followed in space, it is often found that the decay is not expotential, as predicted by linear theory, if the amplitude is large. Instead, one typically finds that the amplitude decays, grows again, and then oscillates before settling down to a steady value. Although other effects may also be operative, these oscillations in amplitude are exactly what would be expected from the nonlinear effect of particle trapping. Trapping of a particle of velocity v occurs when its energy in the wave frame is smaller than the wave potential, (ChF) p. 328.

Analogs to Landau damping
(ShF) p. 402

The capability of stars to organize themselves in a stable arrangement

(ShF) p. 402: "*In its purest form, Landau damping represents a phase-space behavior peculiar to collisionless systems. Analogs to Landau damping exist, for example, in the interactions of stars in a galaxy at the Lindblad resonances of a spiral downsity wave. Such resonances in an inhomogeneous medium can produce wave absorption (in space rather than in time), which does not usually happen in fluid systems in the absence of dissipative forces (an exception in the behavior of corotation resonances for density waves in a gaseous medium)".*

In other words, the Landau damping phenomenon can be interpreted as the capability of stars to organize themselves in a stable arrangement.

Plasma field dynamics & MHD

Nearly all of the matter in the universe consists of "plasma". The key differentiator between plasma to neutral gas or neutral fluid is the fact that its electrically positively and negatively charged kinematical particles are strongly influenced by electric and magnetic fields, while neutral gas is not (*).

The Maxwell (fields) equations determine the electromagnetic field when the distribution of electric charges and currents is known. The laws which govern the currents and charges are unknown. It is known that electricity consists of elementary particles (electrons, positive nuclei), but from a theoretical point of view it is not understood. The energy factors which determine the distribution of electricity in particles of definite size and charge are unkonwn. If one could built upon Maxwell's equations at all, the energy tensor of the electromagnetic field is known only outside the charged particles (**).

MHD is concerned with the motion of electrically conducting fluids in the presence of electric or magnetic fields. In MHD one does not consider velocity distributions. It is about notions like number density, flow velocity and pressure.

The MHD equations are derived from continuum theory of non-polar fluids with three kinds of balance laws, (***):

1. conservation of mass
2. balance of angular momentum (Ampere law and Faraday law, Maxwell equations)
3. balance of linear momentum.

A MHD-based unified field model of „plasma matter“ needs to deal with a two-type kinematical (elementary) quantum element concept.

(*) Similar to the notion „elementary particle“, there is no unique mathematical-physical definition of the notion „plasma particle“. The key differentiator between plasma to neutral gas or neutral fluid is the fact that its electrically positively and negatively charged kinematical particles are strongly influenced by electric and magnetic fields, while neutral gas is not. Conceptually, „plasma particles“ need to fulfill the following two pre-requisites, (CaF) p. 1:

(1) there must be electromagnetic interactions between charged particles
(2) the number of positively and negatively charged particles per considered volume element may be arbitrarily small oder arbitrarily large, but both numbers need to be approximately identical. The number of neutral particles (atomes or molecules) is irrelevant for the definition of a plasma.

(**)
(EiA1) p. 28, *Energy Tensor of the Electromagnetic Field*
We therefore conclude from these considerations that the energy per unit volume has the character of a tensor. This has been proved directly only for an electromagnetic field, although we may claim universal validity for it. Maxwell's equations determine the electromagnetic field when the distribution of electric charges and currents is known. But we do not know the laws which govern the currents and charges. We do know, indeed, that electricity consists of elementary particles (electrons, positive nuclei), but from a theoretical point of view we cannot comprehend this. We do not know the energy factors which determine the distribution of electricity in particles of definite size and charge, and all attempts to complete the theory in this direction have failed. If then we can build upon Maxwell's equations at all, the energy tensor of the electromagnetic field is known only outside the charged particles (+). In these regions, outside of charged particles, the only regions in which we can believe that we have the complete expression for the energy tensor in the form $\frac{\partial T_{\mu,\nu}}{\partial x_\nu} = 0$.

(+) It has been attempted to remedy this lack of knowledge by considering the charged particles as proper singularities. But in my opinion this means giving up a real understanding of the structure of matter. It seems to me much better to admit our present inability rather than to be satisfied by a solution that is only apparent.

(EiA1) p. 29, *Hydrodynamical Equations*
We know that matter is built up of electrically charged particles, but we do not know the laws which govern the constitution of these particles. In treating mechanical problems, we are therefore obliged to make use of an inexact description of matter, which corresponds to that of classical mechanics.

(EiA1) p. 30, *The Eulerian Equations for Perfect Fluids*
In order to get nearer to the behaviour of real matter we must add to the energy tensor a term which corresponds to the pressures. The simplest case is that of a perfect fluid in which the pressure is determined by a scalar p.

(***) Regarding the balance laws 2. and 3. (angular and linear momentum) we quote from A. Einstein and H. A. Lorentz
(EiA): (1): „It is only essential, that next to the observable objects there is another to be viewed as a real but not imperceptible object to accept the acceleration resp. the rotation as something real“, (2): „light speed is caused by the movements of bodies through the ether“.

The Vlasov equation

Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. The continuity equation of ideal magneto-hydrodynamics is given by, (DeR) (4.1)

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

with $\rho = \rho(x,t)$ denoting the mass density of the fluid and $\boldsymbol{v}$ denoting the bulk velocity of the macroscopic motion of the fluid. The corresponding microscopic kinetic description of plasma fluids leads to a continuity equation of a system of (plasma) "particles" in a phase space $(\boldsymbol{x},\boldsymbol{v})$ (where $\rho(\boldsymbol{x},t)$ is replaced by a function $f(\boldsymbol{x},\boldsymbol{v},t)$) given by, (ChF) 7.2, (DeR), (5.1)

$$\frac{\partial}{\partial t}f + v \cdot \nabla_x f + \frac{dv}{dt} \cdot \nabla_v f + f\frac{\partial}{\partial v} \cdot \frac{dv}{dt} = 0\ .$$

In case of a Lorentz force the last term is zero, leading to the so-called collisions-less (kinetic) Vlasov equation, (ShF) (28.1.2).

The Vlasov equation is built under the assumptions, that the plasma is sufficiently hot i.e. „plasma particle" collisions can be neglected.

The mathematical tool to distinguish between unperturbed cold and hot plasma is about the Debye length and Debye sphere (DeR). The corresponding interaction (Coulomb) potential of the non-linear Landau damping model is based on the (Poisson) potential equation with corresponding boundary conditions.

The counterpart of the (collision-free) NSE non-linear critical term in the Vlasov equation is given by the non-linear term $F[f] \cdot \nabla_v f$, whereby

$$F[f](t,x) \coloneqq -\iint \nabla W(x-y) f(t,y,w) dw dy.$$

It is built under the assumptions, that the plasma is sufficiently hot (i.e. „plasma particle" collisions can be neglected) and, that the force F is entirely electromagnetic. The combined system with the related Vlasov-Poisson model

$$F = -\nabla W,\ -\Delta_x W = \rho,\quad W = \frac{1}{4\pi|x|} *_x \rho,\quad \rho(x,t) = \int_{R^n} f(x,v,t) dv$$

is called the Vlasov-Poisson-Boltzmann (VPB) system. The extension of the VPB system, where the Vlasov force F (or self-consistent force, or mean force …) is replaced by the Lorentz force determined by the electro-magnetic field created by the particles themselves, is described in (LiP1).

Landau damping and related norm estimates for the Fourier coefficients of the density of charges of the linearized Vlasov equation

The linearized Vlasov equation is given by ($f = f(t,x,v) = f^0(v) + h(t,x,v)$),

$$\frac{\partial h}{\partial t} + v \cdot \nabla_x h + F[f] \cdot \nabla_v f^0 = 0$$

with

$$F[f](t,x) \coloneqq -\iint \nabla W(x-y) f(t,y,w) dw dy,$$

where $F[f](t,x)$ is the force created at the time t and the position x by the interaction potential W and the mass distribution f. The combination with the Vlasov-Poisson potential equation in the form, ($\rho(x,t) \coloneqq \int_{R^n} f(x,v,t)dv$, $F = -\nabla W$),

$$-\Delta_x W = \rho \quad \text{resp.} \quad W = \frac{1}{4\pi|x|} *_x \rho$$

is called the Vlasov-Poisson-Boltzmann (VPB) system.

We note that the domain of the Laplacian operator $H_2 = D(-\Delta) \subset H_1$ is compactly embedded into the energy Hilbert space H_1 and its inverse operator is the Newton/Coulomb potential. In other words, there is a natural (kinematical operator based) decomposition of the energy Hilbert space H_1 in the form $H_1 = H_2 \otimes H_2^{\perp}$, accompanied by the self-adjoint Friedrichs extension of the symmetric Laplacian, and an inverse operator, which is compact.

In (ViC) the proof of the Landau damping based on the Vlasov equation is provided based on the concepts of glidding analytical regularity and corresponding norms accompanied by a mode-by-mode analysis of the density of charges (*).

The proposed Hilbert scale based enables going beyong the approach in (ViC) by

i) replacing the interaction potential W by its corresponding Riesz transformation $\boldsymbol{W} \coloneqq \underline{R}[W]$, where the framework already guarantees the appreciated assumption of a vanishing constant Fourier term $\widehat{\boldsymbol{W}}_{k=0} = 0$ (Penrose condition)

ii) replacing Villani's "hybrid" and "gliding" analytical norms by the exponential decay Hilbert scale norms providing a problem adequate analysis tool of the to be considered plasma potential function solutions

iii) replacing the concept of velocity distributions by notions like number density, flow velocity and pressure (which is the scope of MHD, which is concerned with the motion of electrically conducting fluids in the presence of electric or magnetic fields.

We mention that in (LiP1) the Vlasov (self-consistent, or mean) force F is replaced by the Lorentz force determined by the electro-magnetic field created by the particles themselves.

(*) For the notations we refer to (ViC). We shall denote different numerical constants with the same symbol c. With the abbreviations $g(v) \coloneqq f^0(v)$, $\hat{g} \coloneqq \hat{f}^0$ and

$$\rho^h(t,x) = \int h(t,x,v)dv,\ K^g(t,k) \coloneqq -4\pi^2 \widehat{W}(k)\hat{g}(kt)|k|^2 t,$$

in the linearized Vlasov equation the Fourier coefficients (i.e. the modes) of $\rho = \rho(x,t)$ (the density of charges) are linked by the following (Volterra integral type) term, (ViC),

$$\rho_k(t) \coloneqq \widehat{\rho^h}(t,k) = \tilde{h}_i(k,kt) + \int_0^t K^g(t-\tau,k)\widehat{\rho^h}(\tau,k)\, d\tau.$$

Putting $\overline{g} \coloneqq \int_0^\infty \hat{g}^2(v)dv$, (and neglecting the term $\tilde{h}_i(k,kt)$) one gets

$$\rho_k^2(t) \le c\widehat{W}_k^2|k|^2\Big[\int_0^t g\big(k(t-\tau)\big)\rho_k(\tau)d\tau\Big]^2$$
$$\le c\widehat{W}_k^2|k|^2 \int_0^t g^2(k(t-\tau)) \int_0^t \rho_k^2(\tau)d\tau\ .$$

Because of $\int_0^t \hat{g}^2\big(k(t-\tau)\big)\,d\tau = \int_0^t \hat{g}^2(ku)\,du = \frac{1}{k}\int_0^{kt} \hat{g}^2(v)\,dv \le \frac{1}{k}\int_0^\infty \hat{g}^2(v)\,dv = \frac{\overline{g}}{k}$ it follows

$$\rho_k^2(t) \le c\widehat{W}_k^2|k| \int_0^t \rho_k^2(\tau)d\tau$$

i.e., the lemma of Gronwall is applicable getting $\rho_k^2(t) \le ce^{c\widehat{W}_k^2|k|t}$.

In case of a Coulomb potential ($\widehat{W}_k \approx |k|^{-1}$) one gets the estimate

$$\rho_k^2(t) \le ce^{c\frac{t}{|k|}}.$$

Einstein relativity principle and the split up of the world into space and time

Einstein relativity principle tells us that the speed of light is independent from the motion state of the light source (it does not tell us, that clocks cannot run differently for observers in other galaxies). In other words, a translation in an ether cannot be distinguished from hibernation. The conclusion of the physicists was, „ether does not exist“ (WeH) §22. The analysis of the Einstein relativity principle in the context of the Lorentz invariance lead to a decomposition of the world into space and time by projection. The related world-points constitute a four-dimensional manifold, i.e., in this world there exists four coordinates that the corresponding space-like zero cones translate into space-like zero cones, and the time-like vectors transform into time-like vectors (WeH) §23.

In terms of Bohm's explicate and implicate order conception, (*), the Einstein-world-model is related to the explicate order with three general transformations considered to be the essential determining features of a geometry in an Euclidean space of three dimensions: displacement operators, rotation operators and dilatation operators, which are the characterizing properties of the Riesz operators.

Gauge symmetry groups and the GRT symmetry groupoid

Regarding the crucial difference between the algebra based gauge theory and the analysis based GRT we quote from (BlC):

„*The correspondence between symmetries and conserved quantities is one of the most important principles of physics. The crucial difference between gauge theories and the GRT is that the symmetries of the GRT act on the space-time itself and not only on the degree of freedoms of the „internal“ fields.*

The vacuum Einstein equations state that the Ricci curvature Ric(g) of a lorentzian metric g is identically zero. Recast as hamiltonian evolution equations, they become a hamiltonian system on the cotangent bundle of the manifold MΣ of smooth riemannian metrics on a manifold Σ which represents the typical Cauchy hypersurface.

As in every lagrangian field theory with symmetries, the initial data must satisfy constraints. But, unlike those of gauge theories, the constraints of general relativity do not arise as momenta of any hamiltonian group action. In this paper, (BlC), we show that the bracket relations among the constraints of general relativity are identical to the bracket relations in the Lie algebroid of a groupoid consisting of diffeomorphisms between space-like hypersurfaces in spacetimes. A direct connection is still missing between the constraints themselves, whose definition is closely related to the Einstein equations, and our groupoid, in which the Einstein equations play no role at all. We discuss some of the difficulties involved in making such a connection.

In contrast to classical mechanics and gauge field theories, the conserved quantities of the GRT do not span a symmetry algebra in the conventional sense. Instead, a so-called Hamiltonian Lie algebroid can be obtained from a naturally constructed symmetry groupoid.“

Mathematical speaking, the ideal classes of generalized quaternion (quadratic form) algebra build a groupoid, (BrH).

(*) (BoD) A.2, p. 200: „What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. … The general transformations are considered to be the essential determining features of a geometry in an Euclidean space of three dimensions; those are displacement operators, rotation operators and the dilation operator.

Propagation phenomena
plane waves arising in hyperbolic case
progressive (free of dispersion or distorted) waves
(CoR) p. 188 ff.

For progressive waves, where ρ is the „frequency and γ denotes the speed of the propagation of the waves, only special (physical relevant, i.e., uniformly bounded in space) waves in the form $e^{i\rho(\alpha x-\gamma t)}$ are possible. For the (limit) speed $\gamma = 1$ there is no progressive wave. For speeds exceeding the limit speed the wave solutions willl no longer be classed as admissible waves since they are not bounded in space. At any rate, the differential equation (for waves in the form $u = f(ax - bt)$)

$$(a^2 - b^2)f''(ax - bt) + cf(ax - bt) = 0$$

represents a phenomenon of dispersion in the following sense: If a solution is a superposition of progressing waves with the same direction, all of the form satisfying the equation above, then the different components are propagated at different speeds; thus the form of a composite wave will change in time. For example, the equation $\ddot{u} - \Delta u - cu = 0$ admits no progressive wave if the prescribed speed is 1 and the prescribed direction arbitrarily.

If for given ρ and α the speed γ (a continuous function $g(\rho, \alpha)$ of α and the frequency ρ) possesses an imaginary part q, then the wave may be written in the form

$$e^{i\rho(\alpha x-pt)}e^{-qt}.$$

We speak of *damed waves* exponentially attenuated in time at a fixed point in space. (The solution with the factor e^{qt} for $q > 0$ is usually discarded, being not bounded for increasing t.). Again we have the phenomenon of distortion or dispersion: An initl harmonic component is propagated at the speed depending on the frequency; thus, an initial shape of u, given by superposition of terms $e^{i\rho(\alpha x}$, is distorted in time (apart from the attenuation of damping), since the different components are propagated at different speeds or „dispersed“ according to their different frequencies.

We summarize:

> The alternative between the case dispersion and undistorted progressing families of plane waves in a given direction is exponential, but the speed can vary continuously with the frequency. In the second case the wave form is arbitrary and the speed is restricted to the discrete roots of the characteristic equation.

Electrodynamics in matter
Maxwell & telegraph equations

In electrodynamics the matter equations (Maxwell equations in discharged areas) are prescribed by the hyperbolic telegraph equations in the form

$$\Delta\vec{E} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} + \sigma\mu_0\mu\frac{\partial\vec{E}}{\partial t},\ \Delta\vec{H} = \frac{\mu\varepsilon}{c^2}\frac{\partial^2\vec{H}}{\partial t^2} + \sigma\mu_0\mu\frac{\partial\vec{H}}{\partial t}.$$

They are wave equations for a lossy dieletric, which correspond to standard wave equations in case of an isolator, i.e. $\sigma = 0$. The dielectric constant governs the resistence and the frequency depending damping phenomena.

The constant ε prescribes an assumed linear relation of the electric displacement field $\vec{D}$ (for homogeneous and isotropic materials) and the electric field by $\vec{D} = \varepsilon\vec{E}$. The constant ε_0 denotes the vacuum (electric) permitivity constant. The magnetic permeability is defined by the ratio μ of the magnetic flux density and the magnetic field strength. The vacuum (magnetic) permeability constant of free space is denoted by μ_0. The two vacuum constants are linked to the speed of light c by $c^2 = \frac{1}{\mu_0\varepsilon_0}$.

A coercive bilinear form for the time-harmonic Maxwell equations

In (CoM) for the time-harmonic Maxwell equations (KiA), there is a coercive bilinear form (for the Sobolev space H_1) provided, containing tangential derivatives of the normal and tangential components of the field on the boundary, vanishing on the subspace H_1. Thus the variational formulations of „electric“ or „magnetic“ boundary value problems with homogeneous bondary conditions are not changed.

The unknown physical parameters of the Maxwell equations

The energy tensor for electromagnetic fields is unknown for elementary particles. The laws by which the currents and charges behave are unknown. Matter is built by electromagnetic particles, but the field laws by which they are constituted are unknown, as well.

(EiA) p. 52: „*However, the laws governing the currents and charges (in the Maxwell equations), are unknown to us. We know, that electricity exists within elementary particles (electrons, positive kernels), but we don't understand it from a theoretical perspective. We do not know the energetical factors, which determine the electricity in particles with given size and charge; and all attempts failed to complete the theory in this directions. Therefore, if at all we can built on the Maxwell equations, we know the energy tensor of electromagnetic fields only outside of the particles*“.

(DiP): "*The Lorentz model of the electron as a small sphere charged with electricity, possessing mass on account of the energy of the electric field around it, has proved very valuable in accounting for the motion and radiation of electrons in a certain domain of problems, in which electromagnetic field does not vary too rapidly and the accelerations of the electrons are not too great. .... The departure from electromagnetic theory of the nature of mass removes the main reason we have for believing in the finite size of the electron. It seems now an unnecessary complication not to have the field equations holding all the way up to the electron's centre, which would then appear as a point of singularity. In this way we are led to consider a point model for the electron*."

The extended complex Maxwell equations, the magnetic scalar and vacuum energy

The Maxwell equations are invariant under Lorentz and gauge transformations. They entail that the photon is massless. In de Broglie hypothesis an elementary particle is accompanied with wave nature. The photon on the other hand exhibits particle nature. Hence, duality is the fundamental nature existing in all elementary particles.

In (ArA1) the extended complex Maxwell equations are considered, where the charge-current densities and the scalar-vector potentials are equally treated. Physically speaking, a complex Lorentz force is introduced and extended to include magnetic scalar.

(ArA1): „***Abstract*** *– The complex Lorentz force is introduced and extended to include magnetic scalar. This scalar is found to be associated with a prevailing magnetic field permeating the whole space. It also introduce an extra force in Lorentz complex force. The magnetic scalar is associated with the vacuum energy. The Proca-Maxwell's massive electrodynamics is derived from the extend current density transformations. Proca-Maxwell's theory is found to be invariant under the extended gauge transformations (current-charge density). The Lorenz gauge condition is shown to express the photon charge conservation. Any violation of Lorenz gauge (photon charge) or electronic charge conservation would lead to spin zero scalar particles. This is manifested in superconductivity. The total charge comprising the electron and photon is always conserved. Owing to superconductivity, the photon charge is related to electron charge by*

$$e_p = e\sqrt{\frac{m_p}{m_e}}.$$

Photons inside superconductors are shown to be massive. It is shown that Maxwell's equations expressed in complex form are more convenient to study duality transformations.

The telegraph equation
(CoR) p. 192 ff.

For the wave equation

$$\frac{1}{c^2}\ddot{u} - \Delta u = 0 \ ,$$

progressing undistorted plane waves with speed c and the arbitrary form

$$\Phi(\sum_{i=1}^{n} \alpha_i x_i - ct), \sum_{i=1}^{n} \alpha_i^2 = 1$$

are possible in every direction. A more general example is given by the telegraph equation

$$\ddot{u} - c^2 u'' + (\alpha + \beta)\dot{u} + \alpha\beta u = 0,$$

satisfied by the voltage or the current u as a function of the time t and the position x along a cable; here x measures the length of the cable from an initial point. Unless this equation represents dispersion. If we introduce $v := e^{\frac{1}{2}(\alpha+\beta)t}u$, we obtain the simpler equation

$$\ddot{v} - c^2 v'' + (\frac{\alpha-\beta}{2})^2 v = 0$$

for the function v. This new equation represents the dispersionless case if and only if $\alpha = \beta$. In this case the *original telegraph equation*, of course, possesses *no absolutely undistorted wave* solutions of arbitrarily prescribed form. However, our result may be stated in the following way:

> *If condition $\alpha = \beta$ holds, the telegraph equation posses damped, yet „relatively" undistorted, progressing wave solutions of the form $u = e^{-\frac{1}{2}(\alpha+\beta)t} f(x \pm ct)$, with arbitrary f, progressing in both directions of the cable.*

The telegraph equation

$$\ddot{u} - c^2 u'' + (\alpha + \beta)\dot{u} + \alpha\beta u = 0,$$

is derived by elimination of one of the unknown functions from the following system of two differential equations of first order for the current $i = i(x,t)$ and the voltage $u = u(x,t)$ as functions of x and t:

$$C\dot{u} + Gu + i' = 0$$

$$L\dot{i_t} + Ri + u' = 0.$$

Here L is the inductance of the cable, R its resistence, C its shunt capacity, and, finally, G, its shunt conductance (loss of current divided by voltage). The constants in the telegraph equation, which arise in the elimination process, have the meaning

$$\frac{1}{c^2} = LC, \ \alpha = \frac{G}{C}, \ \beta = \frac{R}{L}$$

where c is the speed of light and α the capacitive and β the inductive damping factor.

Spin, rotation, half-integer quantum numbers and all that

In quantum theory the underlying physical idea is that the particles are energy quanta without individually. In classical theory but also in quantum theory, symmetry groups are applied to derive conservation laws for energy, translation and angular momenta.

> (FeE): „Dirac‘s theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field“.

Pauli‘s spin($1/2$) concept is about a „rotation“ concept of an elementary particle, which looks the same only after the second „rotation“. This „spin($1/2$)-rotation“ concept is the model for an angular (non-kinematical) momentum. However, in the current quantum theory translation and rotation operators are not interchangeable, which is a consequence of the quantification process of classical partial differential equations (PDE). Therefore, in order to characterize the angular momentum of a system about an axis by a quantum number it is neccessary that the perpendicular translation momentum vanishes or is unknown, (DüH), (HeW).

One basic problem in the quantum interpretation is the „wave function“ to be defined by problem adequate wave equations, based on the imitations of various classical physical processes. Thereby, the Schrödinger and the Pauli equations are non-relativistic equations, while the Klein-Gordon and the Dirac equations are relativistic, (XiK).

In the proposed Krein space model the kinematical (linear) energy momentum and complementary potential energies are intrinsic parts of the modelling framework.

In the most finest „2-fermions-1-boson plasma“ framework $H_1^+ \otimes H_1^- \otimes H_1^\perp$ Pauli’s „half-integer-spin-rotation“ interpretation is replaced by the framework specific „potential“ definition between the two fermion waves.

In the „1-fermion-2-bosons proton/fluid“ framework $H_2 \otimes H_2^\perp \otimes H_1^\perp$, (its solutions can be interpreted as approximation solutions to the most finest quantum world framework) the corresponding „potential“ difference is now between a massive particle and its corresponding additional complementary potential energy.

Physically speaking, there is a single framework specific wave function model, while the modelling of different physical processes / phenomena, like

- relativistic or non-relativistic equations (Schrödinger’s „dilemma“ (*))
- incompressible fluid with or without frictional forces (d’Alembert “paradox”)

become part of appropriately defined PDE accompanied by correspondingly defined domains of the wanted („potential“ function) PDE solutions.

The function domains for non-relativistic hyperbolic (fluid velocity) PDE solutions are defined in that way that a well-posed PDE system is guaranteed (**). Additionally, the domains for relativistic hyperbolic PDE need to enable Lorentz invariances.

In case of a quaternionic Hilbert scale framework there is another operator concept, which is also an “a priori existing“ (as part of the framework) to any physical problem specific PDE. This is the *quaternionic rotation operator*. It is defined independent from an underlying chosen coordinate system.

(*) (ScE) p. 50: *In the Bose case we seem to be faced, mathematically, with a simple oscillator of Planck type, of which the quantum numbers n_s, we may ask whether we ought not to adopt for n_s half-odd integers $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots n + \frac{1}{2}, \dots$ rather than integers. One must, I think, call that an open dilemma. Form this point of view of analogy one would very much prefer to do so. For, the „zero-point energy“ of a Planck oscillator is not only borne out by direct observation in the case of crystal lattices, it is also so intimately linked up with the Heisenberg uncertainty relation that one hates to dispense it. On the other hand, if we adopt it straightaway, we get into serious trouble, especially on contemplating changes of the volume, because in this process the (infinite) zero-point energy seems to change by infinite amounts!*

(**) The Riesz transform (R_1, R_2, R_3) is the generalization of the Hilbert transform for space dimensions $n > 1$. The „pressure“ force in the NSE can be expressed in terms of the velocity by the formul $p = -\sum_{i,j=1}^{3} R_i R_j (u_i, u_j)$a . The Riesz commutes with translations & homotheties, and have nice properties relative to rotations, (PeB) p. 113. For the well posedness of hypermaximal accretive multivalued operators A in a Banach space accompanied by continuous semi-groups of non-linear contractions on $D(A)$ we refer to (BrH2).

Renormalization groups
(DeP) p. 551

The behavior of a physical system depends on a scale (of energies, distances, momenta, etc.) at which the behavior is studied. The change of a behavior when the scale is changed, is described by the renormalization group equation. In quantum field theory, the dependence of the behavior on the scale is often expressed mathematically by the fact that in order to regularize (i.e. render finite) Feynman diagram integrals one must introduce auxiliary scales, cutoffs, etc. The effect of these choices on the physics is encoded into the renormalization group equation. This equation then becomes an important tool for the study of physical theories.

Very generally speaking, the method of renormalization group is a method designed how to describe how the dynamics of some system changes when we change the scale (distance, energies) at which we probe it.

Two basic facts about physics:

Scale dependence

Physics is scale dependent. For example, consider a fluid. At each scale of distances, we need a different theory to describe ist behavior:

- classical continuum mechanics (NSE)
- theory of granular structure
- nucleus + electronic cloud
- nuclear physics
- quantum chromodynamics
- string theory .

At each scale, we have different degrees of freedom and different dynamics.

Decoupling

Physics at large scale (largely) decouples from the physics at a smaller level. For example, to describe the behavior of fluid at the scale ~ 1cm, we don't know about the granular structure, nor about the atoms and nucleons. The only things we need to know are th viscosity and the density of the fluid. ….

Similarly, if we want to describe atoms, we don't need to know anything about the nucleus except its mass and electric charge.

When we pass fom a smaller scale to a larger scale, we average over irrelevant degrees of freedom. Mathematically this means that they become integration variables and thus disappear in the answer. This decoupling is the reason why we are able to do physics. If there was no decoupling, it would be necessary for Newton to know string theory to describe the motion of a viscous fluid.

In *classical mechanics* we deal with three scales according to its 3 basic measurements (distance D, time T, mass M)

In *non-relativistic quantum theory* (M can be expressed through T & D using the Planck constant) and classical relativity (T can be expressed via D using the speed of light) we have 2 scales

In *relativistic quantum theory* we only have 1 scale – the scale of distance (or equivalently – the (inverse) scale of momenta).

Solutions of the string vibration equation

The wave equation

$$u_{tt} - k^2 u_{xx} = 0$$

has a solution $u(x,t) = f(x - kt)$ for any function of one variable f; it has the physical interpretation of a „traveling wave“, described by a „shape“ $f(x)$ moving at velocity k.

There is no physical reason for the “shape” to be differentiable, but if it is not, the differential equation is not satisfied at some points. In order to not through away physically meaningful solutions because of technicalities, the concept of distributions can be applied.

If the equation above is also meaningful, if u is a distribution, then u is called a weak solution of it. If u is twice continuously differentiable and the equation holds, one calls u a strong or classical solution.

Each classical solution is a weak solution. In case of the equation above it’s also the other way around. The same is NOT TRUE for the elliptic Laplace equation (counter example is the classical solution $u(x.y) := log(x^2 + y^2)$ with $\Delta\, log(x^2 + y^2) = 4\pi\delta$) and the function $u(x.y) := f(x - kt) \in L^1_{loc}(R^2)$ with

$$(u_{tt} - k^2 u_{xx}, \phi) = 0\ .$$

We note that the Schrödinger equation for a harmonic oscillator may be obtained by using the classical string potential function

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

where $\omega = \sqrt{\frac{k}{m}}$ denotes the angular frequency.

Quaternionic multiplication and rotation operators

In classical theory but also in quantum theory, symmetry groups are applied to derive conservation laws for energy, translation and angular momenta.

In the current quantum theory framework in order to characterize the angular momentum of a system about an axis by a quantum number it is neccessary that the perpendicular translation momentum vanishes or is unknown, (DüH). Mathematically speaking, translation and rotation operators are not interchangeable, as a consequence of the quantification process of classical partial differential equations.

Vector analysis is the standard tool in Maxwell’s electrodynamics for expressing the electrical and magnetic fields. There is a close relation between vector analysis and quaternion algebra, (UnA2) p. 152:

> *The quaternionic multiplication of a spatiotemporal derivative vector with electromagnetic potential is given by*
>
> $$\left(\frac{\partial}{\partial t}, \vec{\nabla}\right) \times \left(\varphi, \vec{A}\right) = \frac{\partial \varphi}{\partial t} - \vec{\nabla} \cdot \vec{A}\frac{\partial \vec{A}}{\partial t} + \vec{\nabla}\varphi + \vec{\nabla} \times \vec{A}\ ,$$
>
> *where the last two terms precisely match the known expressions for the electric and magnetic fields $\vec{E}$ and $\vec{B}$.*

The GRT is based on the mathematical concept of „differentiable manifolds“: Physically speaking, this results into pure locally relevant metrics, which are depending from the chosen coordinate system. In other words, the physical world as described as local euclidian space-time structure for all space-time points in the universe.

The quaternions provide an appropriate field addressing the „translation-rotation“ (linear and angular rotation) „permutation“ requirement by the concept of a quaternion rotation operator. This is a special quaternion triple-product (unit quaternions and rotating imaginary vector) competing with the conventional (Euler) matrix rotation operator.

Heisenberg's lost key

Indefinite metrics in a Hilbert space
and the degeneracy of the ground state

In (HeW) a mathematical formalism for an unified field theory is proposed. Its cornerstones are an indefinite metric in a Hilbert space and the degeneracy of the ground state (accompanied with the action variable J).

The corresponding classical and variational kinematical energy model world is governed by the (hyperbolic) wave equation accompanied by the physical (cosmic or objective) time variable (*). It is connected to the ground state variable (which is defined as an differential) by, (HeW1),

$$t := \frac{1}{h\frac{\partial \nu}{\partial J}} \ .$$

The wave equation is accompanied by the Green function, which is connected to the quantum world by the Green function integral, underlying particle interaction terms, and related Weyl spinors, (HeW).

In (HeW) the deviation from iso-spin-symmetry in electrodynamics is taken as indication for an asymmetry of the ground state, (DüH):

> (HeW) 7-1: „*The deviation from iso-spin-symmetry in electrodynamics shall be taken as indication for an asymmetry of the ground state, (DüH). In fact the number of protons in the world seems to be very different from the number of neutrons, the number of electrons is very different from the number of neutrinos. Even the matter and antimatter should be distributed in the universe with equal average density – many glaxies might be consist of matter, equally many of antimatter – and if total isospin should be small in this way, the big asymmetry would remain, since in matter the total isospin would point in one direction, in antimatter in the opposite direction. Hence there would be a macroscopic deviation from symmetry in isospace.*
>
> *An asymmetry of the ground state and therefore a degeneracy of this state is a well-known phenomenon in many systems discussed in conventional quantum mechanics. Ferromagnetism, superfluidity, superconductivity, crystal structure are obvious examples. In such cases two important new phenomena appear …..: The degeneracy of the ground state enforces the existence of bosons of rest mass zero, as has been pointed out in a mathematical form by Goldstone (the Goldstone theorem). Some property of the ground state can be attached to the particles thereby changing normal particles into strange particles*."
>
> (HeW) 8-1: „*The asymmetry of the ground state with respect to the isospin group has been used in chapter 7 as explanation for the strange particle poles in the Green's functions and as basis for the spurion formalism. … It has been emphasized already in earlier papers on this subject, that empirically the asymmetry of the ground state seems to be closely connected with the existence of long-range forces, i.e. of particles with rest mass zero, (DuH). The asymmetry with respect to the isospin group comes in through the long-range forces of electrodynamics, the asymmetry of with respect to the the space reflection parity appears in the weak interactions, and this is the first interaction which affects neutrinos. It can be well understood that short-range forces allow a clear separation of the particles from the rest of the world, while long-range forces may lead to a dependence of the properties of the particles on the state of the world in large dimensions. This connexion has been found a mathematical expression in the theorem of Goldstone. .. In the present theory the goldstone theorem is the basis for an understanding of quantum electrodymamics*.

(*) Husserl differentiates between the objective time of appearing objects, the subjective or preempirical time of acts and experiences and the prephenominal absolute flow of the internal time consciousness, ((ZaD) chapter 3.

Ehrenhaft's forgotten discovery

Photophoresis by electric vs. electric & magnetic ions
and the fine structure constant

(EhF): Key words: light negative and positive motion, light positive and light negative longitudinal photophoresis, trembling effect, difficulties of radiometer forces

III. THE INTERPRETATION OF THE EXPERIMENTS
The Interpretation of the Photophoresis

In order to explain the phenomena of photophoresis one conclusion is drawn from the movement of the illuminated particles in the homogeneous electric and magnetic fields.

The light induces electric and magnetic charges (poles) upon the particles if they are illuminated by concentrated light pre-ponderantly shorter wave lengths. …

For the magnetic charges this conclusion is new, but is justified because of the complete analogy of this phenomenon with the electric phenomenon.

It should be noted here that there are no empirical facts which contradict this conclusion as was shown above. The difficulties such an explanation encounters are not due to experimental facts but only to theoretical considerations which go back as far as Ampere who introduced the hypothesis of molecular currents and reduce the phenomena of magnetism to purely electrical phenomena.

V. Conclusion

Particles of matter irradiated by light between electrodes behave as if they carry positive or negative electric charges. Therefore we can say that through the action of the light uncharged particles obtain unipolar charges, either negative or positive.

Particles of matter, sufficiently irradiated by light between magnetrodes behave as if they carry single south or north magnetic poles (charges). Light therefore produces unipolar magnets (magnetic ions, charges). Unipolar particles flow in homogeneous fields E or H in the direction of the field and reverse their movement with the field. Such a flow of particles simultaneously in both directions can be observed directly by means of a microscope (dark field). One can actually see the flow of an electric current in the above mentioned arrangement. It is very remarkable that the same picture appears if the magnetic field is applied as if the electric field is applied. From the visual appearance it is impossible to determine when an electric and when a magnetic field is applied. The generality of this effect is not diminished by the necessity of using light to produce magnetic ions. It will be a question of further investigation to find out if magnetic ions exist also without light. It should be remembered that when electric currents were discovered, dissociation in the voltaic cell was considered all important, but nobody could explain it. No model to picture what happens in a voltaic cell was known to aid ones imagination and in the same way the author does not attempt to use a model to explain the mechanism of the production of the magnetic ions ....

It is evident that a great number of problems are suggested by the conclusions described above. Thus, for instance, one may think of the existence of conductors of magnetic ions, of the heat created by the flow of magnetic ions, etc.

In this paper, the attempt has been made to show that a beam of light causes or induces not only heat and electricity but also magnetism at the same time.

Although an attempt has been made to separate the thermal and mechanical forces from the electric and magnetic ones one cannot be certain, from a general point of view, whether this is entirely possible in the observation of physical phenomena.

Photophoresis, Nature's cycloidal space curve movement and all that

F. Ehrenhaft, V. Schauberger, E. Schrödinger

Ehrenhaft's comment on his discovered phenomenon of photophoresis by electric and magnetic ions was, (EhF):

> *„Particles of matter irradiated by light between electrodes behave as if they carry positive or negative electric charges. Therefore we can say that through the action of the light uncharged particles obtain unipolar charges, either negative or positive*".
>
> *It is unlikely, that all those movement phenomena in light with or without the action of a field can be explained with the helf of today's hypothesis; we may be forced to reach for something new*, (*).

In (AlO) p. 222, Ehrenhaft is further quoted (referring to (*)) with the following words

> *„Completely new and amazing is the fact, that the movements of the particles in the field do not run in straight lines, but run in paths in extremly regular forms, sizes and orbital frequencies …*
>
> *It was also interesting too, that a centripetal force occured, which acted on the particles 130 times stronger than the gravity force".*

Ehrenhaft's experimental obserbed photophoresis phenomenon is in line with the proposed Krein space based quantum dynamics model, (EhF). Concerning the related potential operators of the proposed quantum dynamics model we may reformulate Ehrenhalt's discovery in the following form:

> „*The occured centripetal force generated by an angular momentum (governed by the potential of* $H_2 \otimes H_2^{\perp}$*) acts on the massive particle 130 (!) times stronger than the gravity force."*

If this interpretation is accepted, then the Ehrenhaft's experiments also confirm Schauberger's theory of physical and biological spiral movement phenomena (including spiral galaxies) governed by two related physical resp. biological force types, centrifugal (linear movements) and centripetal forces. In other words, it provides two physical and biological forms of motions with related „negative" and „positive" energies.

It also puts the spot on Schrödinger's concept of a „biological potential" of living cells (**):

the conjecture is that Schauberger's biological form of motion (which is a centripetal motion corresponding to the negative entropy) is caused by the same biological potential, which raises consciousness in the sense of Schrödinger.

(*) Acta Physicia Austriaca, Band 4, 1950 and Band 5, 1951

(**) (ScE1) „Mind and Matter", p. 95 ff: „A tentative answer, *You see that all the attempts at extending the domain of consciousness, asking oneself whether anything of the sort might be reasonably associated with other than nervous processes, needs must run into unproved and unprovable speculation. But we tread on firmer ground when we start in the opposite way. Not every nervous process, nay by no means every cerebral process, is accompanied by consciousness. Many of them are not, even though physiologically and biologially they are much like the „conscious" ones, both in frequently consisting of afferent impulses (centripetal course) followed by efferent (centrifugal course) ones, and in their biological significance of regulating and timing reactions partly inside the system, partly towards a changing environment. ….Even so, at first anyhow, only these modifications or „differentials" intrude into the conscious sphere that distinguish the new incidence from previous ones and thereby usually call the „new considerations". … But whenever the situation exhibits a relevant differential this differential and our response to it introde into consciousness, …"*

Edward Leedskalnin‘s view of the magnetic current world

> *Original german text: Magnetstrom ist dasselbe wie elektrischer Strom, es sind zwei Ströme; eine Strömung setzt sich zusammen aus konzentrierten Strömen von individuellen Nordpol-Magneten und individuellen Südpol-Magneten. Ein Strom verläuft mit hoher Geschwindigkeit gegen den anderen gerichtet in wirbelnder, schraubenartiger Form, (LeE) p. 31.*
>
> *Um einen Strom fließen zu lassen, muss es notwendigerweise gegen den anderen gerichtet zu laufen.*
>
> *translation:*
>
> *A magnetic current is the same as an electric current, those are two currents, which are made up of individual North Pole particles and South Pole particles. One current runs against one another with high velocity in a whirling helical form.*
>
> *In order to let a current flow, it must must be necessarily run against the other current.*

Leedskalnin concept of „a *magnetic current running against electric current with high velocity in a whirling helical form In order to let a current flow“* is in line with the proposed Krein space based quantum dynamics model, (LeE).

Khun Dee’s Story about the universe creation

Khun Dee’s „Implosion Theory of Universe Formation“ is governed by two principles

- steps toward thermodynamic initiation
- steps toward gravitational initiation.

Those two kinds of movements are in line with Schauberger‘s cycloidal movement theory based on two kinds of kinematical energies resp. their related biological force types, centrifugal (linear movements) and centripetal forces.

The two principles of Khun Dee’s „Implosion Theory“ are in line with the proposed Krein space based quantum dynamics model.

The conclusion with respect to the „inflaton“ cause „big bang“ „theory“ may be summarised

> (BlS) p. 3, (see also (DrW): „*Consider the Big Bang Theory. A mass the size of our Universe appeared out of nowhere. Even though it was the biggest black whole ever, it then exploded. Our universe is a real thing but the Big Bang Theory is a fairy tale.*“

From R. Penrose we quote (PeR1) p. 444:

> *In order to produce an universe resembling the one in which we live, the Creator would have to aim for an absurdly tiny volume of the phase space of possible universes*
>
> *- about* $1/10^{10^{123}}$
>
> *of the entire volume, for the situation under consideration.*

The big-bang theory assumes, that god had succesfully outsourced this phantastic task to an „inflaton“. The theory provides *verbally formulated* „model“ starting a very short after the inflation has finished ist job during the „first“ 3 minutes period. Its objective is to describe the emergence of the zoo of SMEP-particles until the first hydrogen atomic nucleus appears.

The first *mathematical-physical formulated* models appear after this three minutes period, when the GRT and background radiation phenomena are connected. Those differential equations are basically simple ordinary differential equations with the time parameter t. Similar kind of models are used to explain for instance the butterfly effect, i.e., a wing flapping of a butterfly in europe *can* cause a tsunami near by Japan (~ 8500 km distance).

A mathematical world view

Hilbert Scales

(NiJ)

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|..\|$ and let A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A).$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{\lambda_i, \phi_i\}$ with

$$A\phi_i = \lambda_i \phi_i \ , \ (\phi_i, \phi_k) = \delta_{i,k} \text{and } \lim_{i\to\infty} \lambda_i$$

ii) any $x \in H$ is represented by

$$(*) \quad x = \sum_{i=1}^{\infty} (x, \phi_i)\, \phi_i \ \text{ and } \ \|x\|^2 = \sum_1^{\infty} (x, \phi_i)^2.$$

Lemma 1: Let $x \in D(A)$, then

$$(**) \ Ax = \sum_{i=1}^{\infty} \lambda_i (x, \phi_i)\, \phi_i \ , \ \|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \phi_i)^2,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \phi_i)\, (y, \phi_i).$$

Because of (*) there is a one-to-one mapping I of H to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, \dots)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \phi_i) \ .$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_1^{\infty} (x, \phi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x}, \hat{y})_\alpha = \sum_i^{\infty} \lambda_i^{\alpha} (x, \phi_i)\, (y, \phi_i) = \sum_i^{\infty} \lambda_i^{\alpha} x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x}, \hat{x})_\alpha.$$

Let $\hat{H}_\alpha$ denote the set of all sequences with finite α –norm. then $\hat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space l_2.

Similarly one can define the spaces H_α: they consist of those elements $x \in H$ such that $Ix \in \widehat{H}_\alpha$ with scalar product

$$(x,y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x,\phi_i)\,(y,\phi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x,x)_\alpha .$$

Because of the Parseval identity we have especially

$$(x,y)_0 = (x,y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax,Ax)_0 \ , \ H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $lim\, \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, \dots)$ with

$$\|\hat{x}\|_2^2 < \infty \ , \ \|\hat{x}\|_0^2 = \infty \ .$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma 2: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma 3: Let $\alpha < \beta < \gamma$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with

$$\mu = \frac{\gamma-\beta}{\gamma-\alpha} \text{ and } \nu = \frac{\beta-\alpha}{\gamma-\alpha}.$$

Lemma 4: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x-y\|_\alpha \leq t^{\beta-\alpha}\|x\|_\beta$

ii) $\|x-y\|_\beta \leq \|x\|_\beta \ , \ \|y\|_\beta \leq \|x\|_\beta$

iii) $\|y\|_\gamma \leq t^{-(\gamma-\beta)}\|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x-y\|_\rho \leq t^{\beta-\rho}\|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma-\beta)}\|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x,\phi_i)^2$$

turned out to be the space H_2, which is densely and compactly embedded into $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \ R(A) = H_0 \text{ and } \|x\|_2 = \|Ax\|.$$

Extension and generalizations
(NiJ1)

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)^2_{(t)} = \sum_{i=1} e^{-\sqrt{\lambda_i}t}(x, \phi_i)(y, \phi_i)$$

$$\|x\|^2_{(t)} = (x, x)^2_{(t)} \ .$$

Now the factor has exponential decay $e^{-\sqrt{\lambda_i}t}$ instead of a polynomial decay in case of λ_i^{α}.

Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t)\|x\|_{\alpha} \text{ for } x \in H_{\alpha}$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the (t)-norm is weaker than any α-norm. On the other hand any negative norm index, i.e. $\|x\|_{\alpha}$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (t)-norm.

It holds:

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|^2_{-\alpha} \leq \delta^{2\alpha}\|x\|^2_0 + e^{t/\delta}\|x\|^2_{(t)}$$

with $\delta > 0$ being arbitrary.

Proof: The inequality is a consequence of the following inequality

$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1}-\sqrt{\lambda})}$, for any $t, \delta, \alpha > 0$ and $\lambda \geq 1$.

This holds for the following reasons:

- if $\lambda^{-1/2} \leq \delta$ then obviously $\lambda^{-\alpha} \leq \delta^{2\alpha}$
- in case of $\lambda^{-1/2} \geq \delta$ it holds $e^{t(\delta^{-1}-\sqrt{\lambda})} \geq 1$, whereas
- $\lambda^{-\alpha} \leq 1$ is a consequence of $\alpha > 0$ and $\lambda \geq 1$.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the α-norm, which can be reformulated in the form ($\mu, \nu > 0, \mu + \nu > 1$)

$$\|x\|^2_{\beta} \leq \nu\varepsilon\|x\|^2_{\gamma} + \mu e^{-\nu/\mu}\|x\|^2_{\alpha}$$

applying Young's inequality to

$$\|x\|^2_{\beta} \leq (\|x\|^2_{\alpha})^{\mu}(\|x\|^2_{\gamma})^{\nu} \ .$$

The counterpart of the lemma 4 above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\|x - y\| \leq \|x\|$

ii) $\|y\|_1 \leq \delta^{-1}\|x\|$

iii) $\|x - y\|_{(t)} \leq e^{-t/\delta}\|x\|$.

In this paper we are especially concerned with the $H_{-1/2}-$ Hilbert space, as the proposed alternative framework to model quantum elements in the form

$$\psi = \psi_0 + \psi_0^{\perp} \in H_0 \otimes H_0^{\perp} \ .$$

From the above lemma this means that for any bounded $\psi_0 \in H_0$ it holds $\|\psi\|^2_{-1/2} \leq \delta\|\psi_0\|^2_0 + e^{t/\delta}\|\psi\|^2_{(t)}$.

Eigen-functions and Eigen-differentials

Let H *be* a (infinite dimensional) Hilbert space with inner product $(.,.)$, the norm $\|\ldots\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1}compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$

$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \phi_\mu(\phi_\mu,*)d\mu \quad , \quad \mu \in [\lambda_0,\infty) \ ,$$

i.e.

$$dE_\lambda = \phi_\lambda(\phi_\lambda,*)d\lambda \ .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ. The set E_λ fulfills the following properties:

i) E_λ is a projection operator for all $\lambda \in R$

ii) for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

iii) $\lim\limits_{\lambda\to-\infty} E_\lambda = 0$ and $\lim\limits_{\lambda\to\infty} E_\lambda = Id$

iv) $\lim\limits_{\substack{\mu\to\lambda \\ \mu>\lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f: [a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f: H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda)d(E_\lambda x, x) \ .$$

Symbolically one writes $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$. Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \ , \ d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax,y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x,y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \quad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2x,y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x,y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ , \ \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function. It holds the following eigen-pair relations

$$A\phi_i = \lambda_i\phi_i \quad A\phi_\lambda = \lambda\phi_\lambda \quad \|\phi_\lambda\|^2 = \infty \ , (\phi_\lambda,\phi_\mu) = \delta(\phi_\lambda - \phi_\mu).$$

The ϕ_λ are not elements of the Hilbert space. The so-called eigen-differentials, which play a key role in quantum mechanics, are built as superposition of such eigen-functions.

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\phi_\lambda(x) = \lambda\phi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\phi_\lambda(x)$ it results into elements of H. The practical way to find eigen-differentials is looking for solutions of a distribution equation.

Krein spaces and potential operators
(AzT), (BoJ)

A Krein space is a Hilbert space H with inner product (x,y), which can be written in the form $H = H^+ \otimes H^-$. There are two equivalent approaches defining Krein spaces based on

- the concept of an indefinite metric (also called a Q-metric) $Q(x,y) \coloneqq [x,y]$, $\forall x,y \in H$
- a self-adjoint operator B defined on all of the Hilbert space H inducing the decomposition of H.

A canonical decomposition of $H = H^+ + H^-$ enables the (positive definite) inner product of H according to

$$(*)\quad (x,y) = [x^+,y^+] - [x^-,y^-],\ x = x^+ + x^-,\ y = y^+ + y^-.$$

For vectors $u,v \in H^+$ we have $(u,v) = [u,v]$; for vectors $u,v \in H^-$ we have $(u,v) = -[u,v]$. If $u \in H^+$ and $v \in H^-$, then it follows from (*) that $(u,v) = [u,,\theta] - [,\theta,v]$.

The formula (*) can be inverted in the following way

$$[x,y] = (x^+,y^+) - (x^-,y^-) \text{ resp. } [x,x] = (x^+,x^+) - (x^-,x^-)$$

from which it follows

„Positivity, negativity, and neutrality of a vector $x \in H$ are equivalent to the relations

$$\|x^+\| > \|x^-\| \quad , \quad \|x^+\| < \|x^-\| \quad , \text{ or } \quad \|x^+\| > \|x^-\| \quad \textit{respectively.“}$$

In short, a Krein space can be looked on as an arbitrary Hilbert space decomposed into usual orthogonal sums of two subspaces, equipped in addition to the original Hilbert metric (i.e., the inner product (x,y)) with an additional indefinite metric $[x,y]$.

The decomposition of a Krein space generates two mutually complementary projectors P^+ and P^- mapping H on to H^+ and H^- respectively. Those orthogonal projection operators P^+ and P^- are linked to the indefinite metric by, (VaM) chapter IV,

$$\varphi(x) \coloneqq [x,x] = \|P^+x\|^2 - \|P^-x\|.$$

The indefinite metric $\varphi(x)$ can be interpreted as a „potential“. The related „potential operator“ (in mathematics it is called „the canonical symmetry“ J, (AzT) §3, (BoJ) p. 52) is then given by, (VaM) (10.7), (12.6)

$$\boldsymbol{W}(x) \coloneqq \tfrac{1}{2} grad\varphi(x) = P^+x - P^-x = x^+ - x^- \quad .$$

The fundamental properties of the potential operator $\boldsymbol{W}(x)$ are completeness, invertibility, ($\boldsymbol{W} = \boldsymbol{W}^{-1}$) isometry, and symmetry. Thus, the bilinear form $(x,y)_W \coloneqq (\boldsymbol{W}(x),y)$ defines an inner product, (BoJ) p. 52.

The sub-space H^+ is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = 1$.

The sub-space H^- is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = -1$.

The whole spectrum of $\boldsymbol{W}$ lies on the join of the points $\lambda = \pm 1$.

The definition of the potential (canonical symmetry) operator enables a treatment of the results of its action as the „mirror reflection“ of the space H in the subspace H^+ .

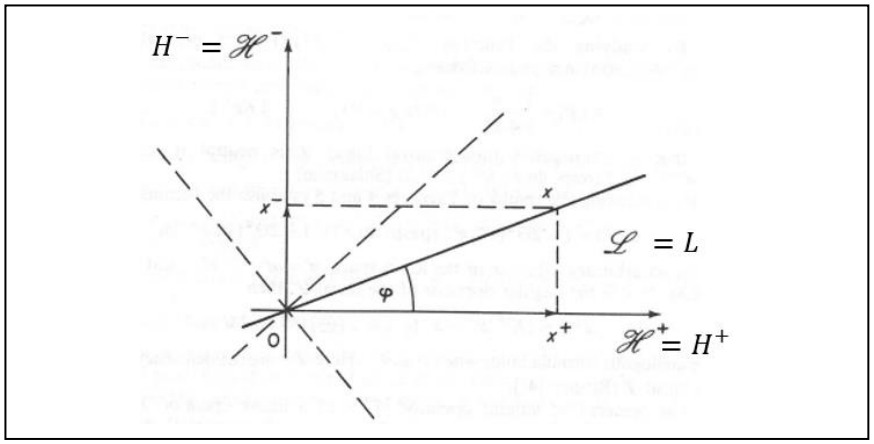
Krein spaces and angular operators
(AzT), (BoJ)

By the aid of J-norms a description of semi-definite subspaces L can be given enabling the definition of an *angular operator* $K^+: H^+ \to H^-$ with domain $D(K^+) = P^+(L)$ and range $R(K^+) = P^-(L)$, (BoJ) p. 54. For the following we refer to (AzT) p. 48 ff. and (BoJ) p. 54.

Let $L \subset H$ in a Krein space $H = H^+ \otimes H^-$ and $P^\pm$ the canonical projectors. Then the bounded linear operator

$$K^+ := P^-(P^+|L)^{-1} : P^+|L \to H^-$$

is called the angular operator for L with respect to H^+. The meaning of this nomenclature is explained by the following picture, (AzT) p. 61:



In the figure above a non-negative (even positive) subspace $L \subset H$ is shown. For any $x \in L$ we have $x = x^+ - x^-$, and $x^- = Kx^+$, where K is the operator of rotating the vector x^+ through an angle $\pi/2$ (in the positive direction), and then multiplying by a scalar $k = tan\varphi$ - the angular coefficient of the „line" L ...

If φ is always understood to be the *minimal* angle between L and „the axis" H^+, then $\tan(\varphi) = \|K\|$. In the general case too ($dimH \leq \infty$) for the angular operator K of a non-negative subspace L we have $tan(\varphi(L, H^+) = \|K\|$, if the (minimal) angle φ is defined by the equality $sin(\varphi(L, H^+) = sup_{e \in S(L)} \|e - ZP^+e\|$, where $S(L)$ is the unit sphere of the lineal L ($\|e\| = 1$).

Theorem 8.2 (AzT) p. 49 (see also Theorem 11.6, (BoJ) p. 54): The set of vectors

$$L = \{x^+ + Kx^+\}_{x^+ \in L^+}$$

in which L^+ is an arbitrary lineal from H^+, and $K: L^+ \to H^-$ is an arbitrary compression ($\|K\| \leq 1$), gives the general form of all $L^+ \subset H^+$ of the Krein space $H = H^+ \otimes H^-$, and $L^+ = P^+(L)$ and K is the angular operator for L with respect to H^+.

Let $\|x\|_W^2 = \|x\|_J^2 = \|x^+\|^2 - \|x^-\|^2$ denote the $J = W$-inner product related (potential) norm.

Theorem 11.7 ((BoJ) p. 54): A subspace $L \subset H$ is positive if and only if the angular operator K^+ of L with respect to H^+ exists and satisfies the condition

$$\|K^+x^+\|_W^2 \leq \|x^+\|_W^2 \,,\, x^+ \in D(K^+).$$

In particular, positive definite subspaces are characterized by the property

$$\|K^+x^+\|_W^2 < \|x^+\|_W^2 \,,\, x^+ \in D(K^+),\, x^+ \neq 0,$$

and neutral subspaces by

$$\|K^+x^+\|_W^2 = \|x^+\|_W^2 \,,\, x^+ \in D(K^+).$$

Remark: For negative subspaces similar statements, involving K^- instead of K^+, are valid.An ordered pair of subspaces $\{L_1, L_2\}$ of the Krein space H will be called an alternating pair provided L_1 is positive, L_2 is negative, and $L_1 \perp L_2$. If, in addition, L_1 is maximal positive and L_2 is maximal negative, the pair$\{L_1, L_2\}$ is called alternating maximal pair. For related (maximal) dissipative operators and the spectra of unitary and self-adjoint operators we refer to we refer to (BoJ) V, VI.

Hyperboloids generated by operators
(VaM) p. 92

Let B be self-adjoint operator defined on all of the Hilbert space H. Since it follows that B is bounded, then

$$inf\{\,(Bx,x) = a \mid \|x\| = 1\} > \infty\ ,\ sup\{\,(Bx,x) = b \mid \|x\| = 1\} < \infty.$$

We shall assume that $a < 0$, $b > 0$. Further, let E_t be the resolution of the odentity corresponding to B; then $E_b - E_0 = P_1$ is a projection operator onto subspace $H_1 \subset H$ which reduces B. Thus, the operator induces a decomposition of into the direct sum of subspaces H_1 and H_2 ($H = H_1 \otimes H_2$) and thereby generated a hyperboloid

$$\varphi(x) = \varphi(x^+ + x^-) = \sqrt{\|P_1\|^2 - \|P_2\|^2} = c > 0,$$

where P_2 is the projection onto H_2.

In the case where the positive part of the spectrum of B lies in an interval $[m,b]$, where $m > 0$, then the inequality

$$\|Bx\| \geq \frac{m}{\sqrt{2}}\sqrt{\varphi^2(x) + \|x\|^2} \geq \frac{m}{\sqrt{2}}\sqrt{c^2 + \|x\|^2}$$

holds for every x in the hyperbolic region V_c defined by

$$\varphi(x) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq c > 0,$$

as well as in the conical region V_0 defined by

$$\varphi(x) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq 0\ .$$

Remark: It should be remarked that in some cases the operator B leaves invariant the hyperbolic regions V_c, which it generates. This is the case, for example, when the positive part of the spectrum of B lies in the interval $[1,b]$ and the negative part lies in $[-1,0]$. In fact, we then have

$$\big((Bx)\big) = \|P^+Bx\|^2 - \|P^-Bx\|^2 = \|BP^+x\|^2 - \|BP^-x\|^2$$

$$= \int_1^b t^2 d(E_t\,P^+x, P^+x) - \int_{-1}^0 t^2 d(E_t\,P^-x, P^-x)$$

$$\geq \|P^+x\|^2 - \|P^-x\|^2 \geq c^2.$$

Alternating pairs and dissipative operators in Hilbert space

(BoJ) p. 39: Let H_0 denote a Hilbert space with with inner product $(x,y)_0$, $x,y \in H_0$ and norm $\|x\|$ and let W be an arbitrary bounded self-adjoint operator ($W = W^*$) given on H_0. Then the Hermitian sesquilinear form $[x,y] = (Wx,y)_0 = Q(x,y)$ defines in H_0 an indefinite metric which we shall call the W-metric, and we shall call the space H_0 itself with the W-metric a W-space. W is called the Gram operator of the space H_0.

(BoJ) p. 91: A linear operator A with an arbitrary domain of definition $D(A)$, operating in a W-space H_0, is said to be W-dissipative if $Im[Ax,x] \geq 0$ for all $x \in D(A)$, and to be maximal W-dissipative if it is W-dissipative and coincides with any W -dissipative extension of it.

An ordered pair of subspaces $\{L_1, L_2\}$ of the Krein space H will be called an alternating pair provided L_1 is positive, L_2 is negative, and $L_1 \perp L_2$. If, in addition, L_1 is maximal positive and L_2 is maximal negative, the pair $\{L_1, L_2\}$ is called alternating maximal pair.

By an alternating extension of the alternating pair $\{L_1, L_2\}$ we mean an alternating pair $\{L_1' L_2'\}$ such that $L_1 \subset L_1'$, $L_2 \subset L_2'$.

> **Theorem 9.1** (BoJ) p. 115: Every alternating pair in the Krein space H can be extended to an alternating maximal pair.

The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1)$, $\mathrm{D}(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1 x_1, x_1] + [x_1, T_1 x_1] \leq 0,\ x_1 \in D(T_1)$$

$$[T_2 x_2, x_2] + [x_2, T_2 x_2] \leq 0,\ x_2 \in D(T_2).$$

> **Theorem** (BoJ) p. 118: If $\left\{L_1^{(0)}, L_1^{(0)}\right\}$ is an alternating maximal pair extending $\{D(-T_1), D(-T_2)\}$, then the operators $T_1^{(0)}, T_2^{(0)}$ defined by the relations $L_1^{(0)} = D(-T_1^{(0)})$, $L_2^{(0)} = D(-T_2^{(0)})$ are maximal dissipative operators of the dissipative operators T_1, T_2, and every solution can be obtained in this way.

Krein space based ellipsoids & hyperboloids accompanied by hyperbolic and conical regions

Putting $x^+ \coloneqq P^+x$, $x^- \coloneqq P^-x$ the self-adjoint operator B defined on all of the Hilbert space H generates a hyperboloid and a related ellipsoid

i) Hyperboloid: $\varphi(x^+ + x^-) = \|x^+\|^2 - \|x^-\|^2 = c > 0$

ii) Ellipsoid: $\frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} = 1$; elliptical region: $E_c \coloneqq \left\{x \in H | \frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} \leq c, c > 0\right\}$.

A hyperbolic region V_c is defined by

$$\big((x)\big) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq c > 0$$

A conical region V_0 is defined by

$$\big((x)\big) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq 0 \ .$$

Evidently V_c is a subspace of V_0. If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0,\infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0,\infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\|P^+x\| = \|P^-x\|$ belong to K, and those points of S^1 for which $\|P^+x\| > \|P^-x\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by

$$t = \mathrm{c}(\|P^+x\|^2 - \|P^-x\|^2)^{-1/2}.$$

From this it is seen that $t \to \infty$ if $\|P^+x\|^2 - \|P^-x\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$.

Theorem (ZaC) p. 291: Let H denote a Hilbert space with inner product $(\cdot,\cdot)$ and $K \subset H$ be a closed convex cone. For every $x \in H$ let $P^K x$ (which is uniquly defined) denote the projection of x on K. Putting

$$K^- \coloneqq -K^+ \coloneqq \{y \in H] \ (x,y) \leq 0, \forall x \in H\}$$

it holds

$$\forall x \in H \ x = P^K x + P^{K^-}x \text{ and } (P^K x, P^{K^-}x) = 0.$$

Conversely, if $x = x_1 + x_2$ with $x_1 \in K$, $x_2 \in K^-$ and $(x_1, x_2) = 0$ then $x_1 = P^K x$ and $x_2 = P^{K^-}x$.

The Gauss-Weierstrass and Abel-Poisson kernels
(PeB)

Let $\boldsymbol{F}$ denote the Fourier transform operator, and $\varepsilon > 0$. The Gauss-Weierstrass and Abel-Poisson kernels are defined by

$$W\xi, \varepsilon) \coloneqq (2\pi)^{-n}\boldsymbol{F}\left(e^{-\varepsilon|x|^2}\right) = (4\pi\varepsilon)^{-n/2}e^{-|\xi|^2/4\varepsilon}$$

$$P\xi, \varepsilon) \coloneqq (2\pi)^{-n}\boldsymbol{F}\left(e^{-\varepsilon|x|}\right) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{\varepsilon}{(|\varepsilon|^2+|\xi|^2)^{(n+1)/2}}$$

The two kernels play a key role in the theory of Fourier transforms. For $f \in L^1(R^n)$ it holds

$$(2\pi)^{-n}f(x) = \lim_{\varepsilon\to 0}\int e^{i\langle x,\xi\rangle}\hat{f}(\xi)e^{-\varepsilon|\xi|^2}d\xi$$

$$(2\pi)^{-n}f(x) = \lim_{\varepsilon\to 0}\int e^{i\langle x,\xi\rangle}\hat{f}(\xi)e^{-\varepsilon|\xi|}d\xi$$

If $f \in L^2(R^n)$, then

$$\hat{f}(\xi) = \lim_{\varepsilon\to 0}\int e^{-i\langle x,\xi\rangle}f(x)e^{-\varepsilon|x|^2}dx$$

$$\hat{f}(\xi) = \lim_{\varepsilon\to 0}\int e^{-i\langle x,\xi\rangle}f(x)e^{-\varepsilon|x|}dx.$$

The Dirichlet (energy inner product) integral

The mathematical-physical laws, which govern the charges and the related energy factors are represented by variational potential operator equations based on appropriately defined spaces, (ChJ). The Hilbert space for thermo-statistical relevant particles is the standard Hilbert space $L_2 = H_0$ equipped with the Lebesgue integral based inner product $(\cdot,\cdot)_0$.

The standard H_0-based variational energy Hilbert space with respect to the Laplacian potential operator is H_1, equipped with the (Dirichlet integral) inner product $D(\cdot,\cdot) \coloneqq (\cdot,\cdot)_1 \coloneqq (\nabla\cdot,\nabla\cdot)_0$. The elements of the kinematical energy Hilbert space H_1 can be represented as superposition of Fourier waves $\in H_1$. Accordingly, the $H_{-1/2}$-variational energy Hilbert space with respect to the Laplacian potential operator is the Hilbert space $H_{1/2}$, accompanied by the concept of wavelets.

We note that the mathematical link between the Hilbert scales H_α and the Dirac radiation theory is given by the Sobolev embedding theorem in the form $H_{-k/2-\varepsilon} \subset C^0$.

Hilbert scales H_α are built resp. equipped with discrete eigenpair based orthogonal systems $\{\varphi_n\}_{n\in N}$.

The model problem is the Laplacian operator accompanied by a countable (eigenpair) sequence $\{\lambda_i, \phi_i\}$ with

$$A\phi_n = \lambda_n\phi_n\ ,\quad (\phi_i,\phi_k) = \delta_{i,k} \text{ and } \lim_{n\to\infty}\lambda_n.$$

The extended Hilbert space H_α is defined by the following inner product resp. norm

$$(x,y)_\alpha = \sum_{i=1}\lambda_i^\alpha (x,\phi_i)(y,\phi_i),\quad \|x\|_\alpha^2 = (x,x)_\alpha\,.$$

Lemma: for any triple α,β,γ with $\alpha<\beta<\gamma$ it holds $\|x\|_\alpha \le \|x\|_\beta^2\ \forall x\in H_\beta$ and

$$\|x\|_\beta \le \|x\|_\alpha^\mu \|x\|_\gamma^\nu\ \forall x\in H_\gamma$$

with $\mu = \frac{\gamma-\beta}{\gamma-\alpha}$, $\nu = \frac{\beta-\alpha}{\gamma-\alpha}$.

Hyperbolic shift theorems

We consider the following hyperbolic (wave equation) model problem

$$\begin{aligned} \ddot{w} - w'' &= f && \text{in } (0,1)\times[0,T] \\ w(0,t) = w(1,t) &= 0 && \text{for } t \in (0,T] \\ \dot{w}(0,t) = \dot{w}(1,t) &= 0 && \text{for } t \in (0,T] \\ w(x,o) &= 0 && \text{for } x \in (0,1). \end{aligned}$$

There is no corresponding shift theorem possible in Hilbert scales characterized by a polynomial decay in case of λ_i^{α} (*). It turned out that the Hilbert space characterized by an exponential decay $e^{-\sqrt{\lambda_i}t}$ with the inner product resp. norm in the form

$$(x,y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i) \quad t>0$$
$$\|x\|_{(t)}^2 = (x,x)_{(t)}^2 \ .$$

governs „optimal“ shift theorems for hyperbolic equations.

Let $w_i := (w,\phi_i)$ resp. $f_i := (f,\phi_i)$ being the generalized Fourier coefficient related to the eigen-pairs $-v_i'' = \lambda_i v_i$. Then it holds

$$\ddot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \text{ and } w_i(0) = \dot{w}_i(0) = 0 \ .$$

with the solution

$$w_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\, f_i(\tau)d\tau \right. .$$

Lemma:

$$\int_0^T \|w\|_{k+2,(t)}^2 dt \le c \int_0^T \|f\|_{k,(t)}^2 dt \ .$$

Proof: It holds for $\tau \le t$

$$\begin{aligned} \int_0^T \|w\|_{k+2,(t)}^2 dt &= \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} w_i^2(t)dt \le \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\, f_i(\tau)d\tau\right.\right]^2 dt \\ &\le \sum \lambda_i^{k+1} \int_0^T e^{-\sqrt{\lambda_i}t} \left(\int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)d\tau\right.\right) \left[\int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)d\tau\, f_i^2(\tau)d\tau\right.\right] dt \\ &\le \sum \lambda_i^{k+1/2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\int_0^t f_i^2(\tau)d\tau\right] dt \ . \end{aligned}$$

Exchanging the order of integration gives

$$\begin{aligned} \int_0^T \int_0^t e^{-\sqrt{\lambda_i}t} f_i^2(\tau)d\tau dt = \int_0^T \int_t^T e^{-\sqrt{\lambda_i}t} f_i^2(\tau)dtd\tau &= \int_0^T f_i^2(\tau)dt \left[\int_t^T e^{-\sqrt{\lambda_i}t}\, d\tau\right] \\ &\le \frac{1}{\sqrt{\lambda_i}} \int_0^T f_i^2(\tau)dt \ . \end{aligned}$$

(*) the counter example is given by the function

$$\Phi(x,t) := e^{-\left(\frac{1}{2}-(x-t)\right)^2},\ u(x,t) := t^2\Phi(x,t),\ f(x,t) := 2\Phi(x,t) - 4t\Phi'(x,t)$$

fulfilling the relationships

$$\dot{\Phi}(x,t) = -\Phi'(x,t),\ \ddot{\Phi}(x,t) = \Phi''(x,t),\ \ddot{u}(x,t) - u''(x,t) = f(x,t)$$

and

$$\|u''\|_{L_2(L_2)} \sim \|\Phi''\|_{L_2(L_2)} \quad \text{but} \quad \|f\|_{L_2(L_2)} \sim \|\Phi'\|_{L_2(L_2)}.$$

Dirac's point charge model $\delta \in H_{-n/2-\varepsilon}$ and …

(FeE): „Dirac's theory of radiation is based on a very simple idea; instead of considering an atom and the radiation field with which it interacts as two distinct systems, he treats them as a single system whose energy is the sum of three terms: one representing the energy of the atom, a second representating the electromagnetic energy of the radiation field, and a small term representing the coupling energy of the atom and the radiation field".

(RiW): „In the special cases in which an electromagnetic process remains restricted to a finite space, the process can be represented in the form, (RiW)

$$f = f_1 = \frac{1}{4\pi}\int \frac{\varphi(x',y',z',t-\frac{r}{c})}{r} dx'dy'dz'$$

as well as in the form

$$f = f_2 = \frac{1}{4\pi}\int \frac{\varphi(x',y',z',t+\frac{r}{c})}{r} dx'dy'dz'$$

and in other forms. … Ritz considers the resstriction to the form of retarded potentials as one of the roots of the second law, while Einstein believes that irreversibility is exclusively due to reasons of probability.

The mathematical link between the Hilbert scales H_α and the Dirac radiation theory is given by

- the Dirac point charge "function" $\delta \in H_{-k/2-\varepsilon}$ (where k denotes the space dimension)
- the Sobolev embedding theorem in the form $H_{-k/2-\varepsilon} \subset C^0$.

The proposed „quanta potential Krein scale framework" deals with an „Dirac approximation function" $\delta^* \in H_{-1/2}$.

<u>… Sobolev functions</u>

The mathematical link between the Hilbert scales H_α and the Dirac radiation theory is given by

- the Dirac point charge "function" $\delta \in H_{-n/2-\varepsilon}$ (where n denotes the space dimension)
- the Sobolev embedding theorem in the form $H_{-n/2-\varepsilon} \subset C^0$.

For the proposed Dirac approximation function δ^* it holds $\delta^* \in H_{-1/2}$.

<u>… piecewise defined functions</u>

The regularity of piecewise defined functions C_0' with respect to scales of Sobolev spaces is given by, (NiJ2) $C_0' \subset H_\vartheta$, $\vartheta \in (0,1/2)$.

<u>… wavelets</u>

The elements of the „approximating Dirac" energy Hilbert space $H_{1/2}$ (related to $\delta_* \in H_{-1/2}$) are mathematically related to wavelets. A wavelet tool analysis can be interpreted as a „mathematical microscope" analysis.:A mathematical microscope is to look at the details that are added if one goes from a scale a to a scale $a - da$, where da is infinitesimally small. The mathematical microscope tool 'unfolds' a function f over the one-dimensional space R into a function over the two-dimensional half-plane of "positions" and "details/enlargements" by "optics (wavelet function g)". This two-dimensional parameter space may also be called the position-scale half-plane, (HoM), (LoA), (MeY).

Dirac's proven electron radiation theory based on the mathematical model of a „point charge" distributional function Dirac function $\delta \in H_{-n/2-\varepsilon}$ (where n denotes the space dimension of the underlying domain) accompanied by indefinite energies of the related Klein-Gordon equation → $\delta^* \in H_{-1/2}$.

<u>… thermostatistics</u>

The „Dirac function approximation" $\delta^* = \delta_0 + \delta_0^\perp$ is related to a „thermo-statistical" relevant particles $\delta_0 \in L_2 = H_0$ by the inequality

$$\|\delta^*\|_{-1/2}^2 \leq \delta\|\delta_0\|_0^2 + e^{t/\delta}\|\delta^*\|_{(t)}^2 \text{ with } \delta > 0 \text{ being arbitrary.}$$

The Coulomb potential model

(RiW): „In the special cases in which an electromagnetic process remains restricted to a finite space, the process can be represented in the form, (RiW)

$$f = f_1 = \frac{1}{4\pi}\int \frac{\varphi(x',y',z',t-\frac{r}{c})}{r} dx'dy'dz'$$

as well as in the form

$$f = f_2 = \frac{1}{4\pi}\int \frac{\varphi(x',y',z',t+\frac{r}{c})}{r} dx'dy'dz'$$

and in other forms. … Ritz considers the resstriction to the form of retarded potentials as one of the roots of the second law, while Einstein believes that irreversibility is exclusively due to reasons of probability.

Mathematically speaking in the 3D case any vector field can be decomposed into a rotation free field and a source free component:

Theorem (Helmholtz decomposition): Let $\boldsymbol{V}: R^3 \rightarrow R^3$ a continuous differentiable vector field with $\boldsymbol{V}(\boldsymbol{x}) = O(\frac{1}{|\boldsymbol{x}|^{2+\varepsilon}})$ and $\boldsymbol{V}'(\boldsymbol{x}) = O(\frac{1}{|\boldsymbol{x}|^{3+\varepsilon}})$ for $|\boldsymbol{x}| \rightarrow \infty$ and

$$\varphi(\boldsymbol{x}) := -\frac{1}{4\pi}\iiint_{R^3} \frac{div\boldsymbol{V}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} d\tau_{\boldsymbol{y}} \text{ , } \boldsymbol{A}(\boldsymbol{x}) := \frac{1}{4\pi}\iiint_{R^3} \frac{rot\boldsymbol{V}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|} d\tau_{\boldsymbol{y}}.$$

Then it holds

i) $\boldsymbol{V} = grad(\varphi) + rot\boldsymbol{A}$

ii) $\Delta\varphi = div(grad(\varphi)) = div\boldsymbol{V}$.

For the (scalar) electrostatic potential $\Phi(\vec{r})$ of the electric field $\boldsymbol{V}(\vec{r}) = \vec{E}(\vec{r})$ it follows

$$div\vec{E}(\vec{r}) = -divgrad\Phi(\vec{r}) = -\Delta\Phi(\vec{r}) = 4\pi\rho$$
$$rot\vec{E}(\vec{r}) = divgrad\Phi(\vec{r}) = 0.$$

In summary, in the 3D case the (scalar) electrostatic potential $\Phi(\vec{r})$ in an electric field $\vec{E}(\vec{r})$ and the Dirac point charge density $\rho(\vec{r})$ are related to each other by the equations

$$\Delta\Phi(\vec{r}) = -4\pi\rho(\vec{r}) \text{ and } \vec{E}(\vec{r}) = -grad\Phi(\vec{r}).$$

The three-dimensional Coulomb potential (function) is defined by $f(\vec{r}) = \frac{1}{r} = \frac{1}{|(\vec{r})|}$. The related regularized „screened“ three-dimensional Yukawa-Coulomb potential (function) is given by

$$f(\vec{r}) = \frac{1}{r}e^{-\varepsilon r}.$$

The related (scalar) electrostatic potential function $u = \Phi$ is given by the solution of the Poisson equation $\Delta u(\vec{r}) = f(\vec{r})$ resp. by the three-dimensional Fourier transform equation in the form

$$-k^2\hat{u}(\vec{k}) = \hat{f}(\vec{k})$$

in which f is proportional to the charge „density $\rho(\vec{r}) = q\delta(\vec{r} - \vec{r}_0)$, (i.e. the density is a point charge density of a 3-D space element) in the form $f(\vec{r}) = -4\pi\rho(\vec{r})$. The *distributional* relationship between the 3D-Coulomb potential function u and its Fourier transform $\hat{u}$ is given by

$$\frac{1}{r} \leftrightarrow \frac{4\pi}{k^2},$$

from which it follows that

$$u(\vec{r}) = -\int \frac{f(\vec{s})}{|\vec{r}-\vec{s}|} d^3\vec{s} \text{ .}$$

The k^{-2} dependency of the Coulomb force is confirmed by experiments in the range of $10^{-12} \ldots 10^{-11}$. For experiments and related calculations in a smaller range the Coulomb law is only restrictedly applicable. In case of the Coulomb interaction within a hydrogen atom the related correction term is given by the fine structure constant $\sim 1/137$, i.e., the error is in the percentage range, (FIT) p. 39.

Complementary variational principles and the method of Noble

The method of Noble ((VeW) 6.2.4), (ArA) 4.2), is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a "Hamiltonian" function W(·,·) which combines the pair of underlying operator equations (based on the "Gateaux derivative" concept)

Let $(E,\langle,\rangle)$ and $(E',(,))$ be Hilbert spaces and $T: E \to E'$, $T^*: E' \to E$ linear operators fulfilling $(u', Tu) = \langle T^*u', u\rangle$ and let $W: E'xE \to R$ a functional fulfilling

$$T = \frac{\partial W(u',)}{\partial u'} \quad \text{and} \quad T^* = \frac{\partial W(.,u)}{\partial u}$$

i.e., the operators T and T^* are deviations from $W(.,.)$ in the sense of Gateaux, i.e.

$$lim \frac{F(u+tv)-F(v)}{t} = F_u(v) \text{ for all } v \in E \ .$$

Putting $W(u',u) := \frac{1}{2}(u',u') - F(u)$ the minimization problem

$$(*) \qquad J(u) := (Tu, Tu) + 2F(u) \to min \quad , \quad u \in U \subset E$$

leads to $Tu = u'$ and $(T^*u', .) = -F_u(.)$ and therefore to

Lemma A.2 (method of Noble): If $F(.)$ is a convex functional it follows that $W(u',u)$ is convex concerning u' and concave concerning u. The minimization problem (*) is equivalent to the variational equation

$$(v', T\phi) + F_u(\phi) = 0 \text{ for all } \phi \in U \quad \text{resp.} \quad (T^*v', \phi) = -F_u(\phi) \text{ for all } \phi \in U.$$

i.e., there is a characterization of the solution of (*) as a saddle point.

Non-linear minimization problems

Non-linear minimization problems can be analyzed as saddle point problems on convex manifolds in the following form (VeW):

$$(*) \quad J(u): a(u,u) - F(u) \to min \ , \quad u - u_0 \in U.$$

Let $a(\cdot,\cdot) : V \times V \to R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V$ and $F(\cdot): V \to R$ a functional with the following properties:

i) $F(\cdot): V \to R$ is convex on the linear manifold $u_0 + U$, i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \le (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

ii) $F(u) \ge \alpha$ for every $u \in u_0 + U$

iii) $F(\cdot): V \to R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \to R$ with

$$\lim_{t \to 0} \frac{F(u+tv)-F(v)}{t} = F_u(v).$$

Then the minimum problem (*) is equivalent to the variational equation

$$a(u, \phi) + F_u(\phi) = 0 \text{ for every } \phi \in U$$

and admits only an unique solution.

In case the sub-space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \to R$ is continuous with respect to convergence in the energy norm, then there exists a solution. We note that the energy functional is even strongly convex in whole V.

The Hilbert transform operator

Some key properties of the Hilbert transform

$$(Hu)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x-y} d\, y = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy$$

are given in

Lemma:

i) The constant Fourier term vanishes, i.e., $(Hu)_0 = 0$

ii) $H\big(xu(x)\big) = xH\big(u(x)\big) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy$

iii) For odd functions it holds $H(xu(x)) = x(Hu)(x)$

iv) If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e., $\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = 0$

v) $\|H\| = 1$, $H^* = -H$, $H^2 = -I$, $H^{-1} = H^3$

vi) $H(f \cdot g) = f \cdot Hg = Hf \cdot g$, $f \cdot g = -Hf \cdot Hg$,

vii) $H(fHg + gHf) = Hf \cdot Hg - f \cdot g$, $H(fHf) = \frac{1}{2}((Hf)^2 - f^2)$

viii) If $(\phi_n)_{n \in N}$ is an orthogonal system, so it is for the system $(H(\phi_n))_{n \in N}$, i.e.,

$$(H\phi_n, H\phi_n) = -(\phi_n, H^2 \phi_n) = (\phi_n, \phi_n)$$

ix) $\|Hu\|^2 = \|u\|^2$, i.e. if $u \in L_2$, then $Hu \in L_2$.

Proof:

i) i) and v)-viii): (PeB), 2.9

ii) ii) The insertion of a new variable $z = x - y$ into the Hilbert transform of $xu(x)$ gives

$$H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z} dz$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xu(x-z)}{z} dz - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x-z) dz = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y) dy.$$

iii) follows from i) and ii)

iv) $\int_{-\infty}^{\infty} u(y)(Hu)(y) dy = \frac{i}{2\pi} \int_{-\infty}^{\infty} sign((\omega)|\hat{u}(\omega)|^2 d\omega$ whereby $|\hat{u}(\omega)|^2$ is even.

The Hilbert transform operator & the mean ergotic theorem

Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. Then Hilbert spaces $\{H_\alpha | \alpha \in R\}$ and H_τ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \,, \|x\|_{(\tau)}^2 = \sum_{n=1}^\infty e^{-\sqrt{\lambda_n}\tau} x_n^2, x_n := (x, \varphi_n).$$

The Hilbert transform of the orthogonal system$\Phi_n := \varphi_n^H := H[\varphi_n]$, where $(\Phi_n, \varphi_n) = 0$ provides an unitary operator U on those Hilbert spaces and theor Hilbert sub-space.

Mean ergotic theorem (HaP), HoE): Let U be an isometry on a Hilbert space H; let P be the projection on the space of all x invariant under U, then

$\frac{1}{n} \sum_{j=0}^{n-1} U^j x \to Px$ in a weak L_2 sense for all $x \in H$.

Note: If $x = y - Uy$ for some y, then $\frac{1}{n} \sum_{j=0}^{n-1} U^j x$ is a telescoping sum equal to $y - U^n y$ and $\left\| \frac{1}{n} \sum_{j=0}^{n-1} U^j x \right\| \leq \frac{2}{n} \|y\| \to 0$.

The Riesz transform operator
(PeB), (StE)

The Riesz transforms are the generalization of the one-dimension Hilbert transform. The properties of the Riesz transforms have the following converse (StE) p. 58,

Proposition 2: Let $T = (T_1, T_2, \dots T_n)$ be an n-tuple of bounded transformations on $L_2(R^n)$. Suppose

- each T_j commutes with the translation of R^n
- each T_j commutes with the dillations of R^n
- for every rotation of $\rho = (\rho_{jk})$ of R^n, $\rho T_j \rho^{-1} f = \sum_k \rho_{jk} T_k f$.

Then the T_j are a constant multiple of the Riesz transforms, i.e. there exists a constant c, so that $T_j = cR_j$, $j = 1, \dots, n$.

In other words, the Riesz operators fulfill certain properties with respect to commutation with translations homothesis and rotation (PeB), (StE). Let $SO(n)$ denote the rotation group. If $j \neq j$ then $R_j R_k$ is a singular convolution operator. On the other hand, it holds $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. The following identities are valid

$$\|R_j\| = 1 \ , R_j^* = -R_j \ , \ \sum R_j^2 = -I \ , \ \sum \|R_j u\|^2 = \|u\|^2 \ , u \in L_2 \ .$$

Let

$$m := m(x) := (m_1(x), \dots m_n(x))$$

be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then

$$m(\rho(x)) = \rho(m(x)), \text{ whereby } m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$$

and

$$m(\rho(x)) = c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} sign(x\rho^{-1}(y)) + log \left|\frac{1}{x\rho^{-1}(y)}\right|\right) \frac{y}{|y|} d\sigma(y)$$

$$= c_n \int_{S^{n-1}} \left(\frac{\pi i}{2} sign(xy) + log \left|\frac{1}{xy}\right|\right) \frac{y}{|y|} d\sigma(y) \ .$$

The Leray-Hopf (projection) operator

The Leray-Hopf projector is the matrix valued Fourier multiplier given by

$$P(\xi) = Id - \frac{\xi \otimes \xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2})_{1 \le j,k \le n} \quad , \; P = Id - R \otimes R =: Id - Q$$

resp.

$$P = Id - R \otimes R =: Id - Q = Id - \frac{D \otimes D}{D^2} Id - \Delta^{-1}(\nabla \times \nabla) \; .$$

As the operator $Q := R \otimes R = (R_j R_k)_{1 \le j,k \le 1} = Q^2$ ($\mathrm{R_i}$ denote the Riesz operators) is an orthogonal projector, the Leray-Hopf operator is also an orthogonal projection, where the domain can be defined on each Hilbert scale. In (LeN1) an explicit expression for the kernels of the Fourier multipliers of the corresponding Oseen operators are provided, which involves the incomplete gamma function and the confluent hypergeometric function of first kind.

The Boltzmann equation

The Boltzmann equation is a (non-linear) integro-differential equation which forms the basis for the kinetic theory of gases. This not only covers classical gases, but also electron /neutron /photon transport in solids & plasmas / in nuclear reactors / in super-fluids and radiative transfer in planetary and stellar atmospheres. The Boltzmann equation is derived from the Liouville equation for a gas of rigid spheres, without the assumption of “molecular chaos”; the basic properties of the Boltzmann equation are then expounded and the idea of model equations introduced. Related equations are e.g. the Boltzmann equations for polyatomic gases, mixtures, neutrons, radiative transfer as well as the Fokker-Planck (or Landau) and Vlasov equations. The treatment of corresponding boundary conditions leads to the discussion of the phenomena of gas-surface interactions and the related role played by proof of the Boltzmann H-theorem.

The Landau collision operator

The Landau equation (a model describing time evolution of the distribution function of plasma consisting of charged particles with long-range interaction) is about the Boltzmann equation with a corresponding Boltzmann collision operator where almost all collisions are grazing.

Its solutions enjoy a rather striking compactness property, which is main result of (LiP1).

The collision operator of the Landau equation is given by

$$Q(f,f) = \frac{\partial}{\partial v_i} \left\{ \int_{R^N} a_{ij}(v-w) \left[f(w) \frac{\partial f(v)}{\partial v_j} - f(v) \frac{\partial f(w)}{\partial w_j} \right] dw \right\}$$

with

$$a_{ij}(z) = \frac{a(z)}{|z|} \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} = \frac{a(z)}{|z|} P(z) := \frac{1-[1-a(z)]}{|z|} [Id - Q](z) \;\; Q(z) := (R_i R_j)_{1 \le i,j \le N}$$

and $\mathrm{a(z)}$ symmetric, non-negative and even in z and with an unknown function f corresponding at each time t to the density of particle at the point x with velocity v. It can be approximated by a linear Pseudo Differential Operator (PDO) of order zero with symbol

$$b_{ij}(z) = z \cdot a_{ij}(z) = \frac{z}{|z|} \left\{ \delta_{ij} - \frac{z_i z_j}{|z|^2} \right\} = \frac{z}{|z|} P(z) := \frac{z}{|z|} [Id - Q](z)$$

whereby $a_{ij}(z)$ denotes the symbol of the Oseen kernel (LeN).

The Calderón-Zygmund (integrodifferential) operator
(EsG)

The pseudodifferential operator Λ with symbol $|\xi|^1$ is called the Calderón-Zygmund integrodifferential operator. Its integrodifferential representation is given by

$$(\Lambda u)(x) = -(\Delta\Lambda^{-1})u(x) = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy.$$

The related pseudo-differential operator with symbol $|\xi|^{-1}$ is defined by, (EsG) (3.15‘), (3.17‘),

$$\Lambda^{-1}u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} \int_{-\infty}^{\infty} \frac{u(y)dy}{|x-y|^{n-1}}, \quad n \geq 2.$$

An alternative representation of the operator Λ is given by, (EsG) (3.35),

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \sum_{k=1}^{n} \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy \ .$$

where describes singular integral (Riesz) operators R_k

$$R_k u := -i \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy.$$

We note that the Calderón-Zygmund integrodifferential operator is identical to the proposed alternative Schrödinger momentum operator (BrK8).

The Prandtl operator

The Prandtl operator $\overline{P}$, which is the double layer (hyper-singular integral) potential operator of the Neumann problem, fulfills the following properties, (LiI) Theorems 4.2.1, 4.2.2, 4.3.2:

i) the Prandtl operator $\overline{P}: H_r \to \widehat{H}_{r-1}$ is bounded for $0 \leq r \leq 1$

ii) the Prandtl operator $\overline{P}: H_r \to \widehat{H}_{r-1}$ is Noetherian for $0 < r < 1$

iii) for $1/2 \leq r < 1$, the exterior Neumann problem admits one and only one generalized solution.

The two momentum laws systems are modelled by corresponding momentum operator equations with corresponding domains according to $H_{1/2} = H_1 \times H_1^{\perp}$. The Prandtl operator is proposed to balance between conservation of mass & (linear & angular) momentum integral equations system.

The D'Alembert operator and the time-harmonic Maxwell equation

The operator concerned with the time-harmonic Maxwell equation and the radiation problem is the D'Alembert operator related to the wave equation:

$$\square u := \ddot{u} - \Delta u \ .$$

The electrodynamic in the special relativity theory is described by the four-vector formalism of the space-time given by the equation $\square \vec{A} = \frac{4\pi}{c}\vec{j}$, with the four-vector potential $\vec{A}$, where its curvature determines the electric and magnetic field forces, and $\vec{j}$ denotes the four-current-density.

The solution of time-harmonic Maxwell equations in a vacuum leads to the Helmholtz equation. The fundamental solution of the Helmholtz equation at the origin is given by spherical wave fronts. The time-dependent magnetic field has the form of the Hertz dipole centered at the origin, (KiA) p. 14.

"Optimal" shift theorems of the potential and wave equation

Obviously, the Hilbert scales characterized by a polynomial decay in case of λ_i^{α} enables optimal shift theorem for the Laplacian operator.

The operator concerned with the time-harmonic Maxwell equation and the radiation problem is the D'Alembert operator related to the wave equation:

$$\square w := \ddot{w} - \Delta w \ .$$

The Hilbert space characterized by an exponential decay $e^{-\sqrt{\lambda_i}t}$ enables corresponding optimal shift theorem for the the D'Alembert operator in the following form

$$\int_0^T \|w\|_{k+2,(t)}^2 dt \le c \int_0^T \|f\|_{k,(t)}^2 dt \ .$$

Proof: Let $w_i := (w, \phi_i)$ resp. $f_i := (f, \phi_i)$ being the generalized Fourier coefficient related to the eigen-pairs $-w_i'' = \lambda_i w_i$ of the Laplacian operator. Th corresponding solution of $(\square w = f)$,

$$\ddot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \ \text{ and } w_i(0) = \dot{w}_i(0) = 0 \ .$$

is given by

$$w_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\right) f_i(\tau) d\tau .$$

It holds for $\tau \le t$

$$\int_0^T \|w\|_{k+2}^2 dt = \sum \lambda_i^{k+2} \int_0^T w_i^2(t) dt \le \sum \lambda_i^{k+2} \int_0^T \left[\int_0^t e^{-\lambda_i(t-\tau)} d\tau\right] \left[\int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau\right] dt$$
$$\le \sum \lambda_i^{k+2} \int_0^T \lambda_i^{-1} \left[\int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau\right] dt .$$

Exchanging the order of integration gives

$$\int_0^T \int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau dt = \int_0^T \int_t^T e^{-\lambda_i(t-\tau)} f_i^2(\tau) dt d\tau$$
$$= \int_0^T f_i^2(\tau) dt \left[\int_t^T e^{-\lambda_i(t-\tau)} d\tau\right] \le \lambda_i^{-1} \int_0^T f_i^2(\tau) dt \ .$$

Parabolic shift theorems

We consider the following two parabolic (heat equations) model problems

$\dot{w} - w'' = f$	$\dot{z} - z'' = 0$	in $(0,1) \times [0,T]$
$w(0,t) = w(1,t) = 0$	$z(0,t) = z(1,t) = 0$	for $t \in (0,T]$
$w(x,o) = 0$	$z(x,o) = g(x)$	for $x \in (0,1)$.

The following compatibility relations for the initial value function need to be fulfilled to ensure corresponding regularity of the solution z:

$$g(1) = 0 \text{ , } g'(0) = 0 \text{ , } g''(1) = g'^2(1) \text{ , etc.}$$

Let $w_i := (w, \phi_i)$ resp. $f_i := (f, \phi_i)$ being the generalized Fourier coefficient related to the eigen-pairs $-v_i'' = \lambda_i v_i$. Then it holds

$$\dot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \text{ and } w_i(0) = 0 \text{ .}$$

with the solution

$$w_i(t) = \int_0^t e^{-\lambda_i(t-\tau)} f_i(\tau) d\tau \text{ .}$$

The following shift theorem holds true:

Lemma:

i) $\int_0^T \|w\|_{k+2}^2 dt \le c \int_0^T \|f\|_k^2 dt$

ii) $\|z(t)\|_k^2 \le c t^{-(k-l)} \|g\|_l^2$, $\int_0^T t^{-1/2} \|z'\|_{-1/2}^2 dt \le c\|g\|$.

Proof: i) It holds for $\tau \le t$

$$\int_0^T \|w\|_{k+2}^2 dt = \sum \lambda_i^{k+2} \int_0^T w_i^2(t) dt \le \sum \lambda_i^{k+2} \int_0^T \left[\int_0^t e^{-\lambda_i(t-\tau)} d\tau\right] \left[\int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau\right] dt$$
$$\le \sum \lambda_i^{k+2} \int_0^T \lambda_i^{-1} \left[\int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau\right] dt.$$

Exchanging the order of integration gives

$$\int_0^T \int_0^t e^{-\lambda_i(t-\tau)} f_i^2(\tau) d\tau dt = \int_0^T \int_t^T e^{-\lambda_i(t-\tau)} f_i^2(\tau) dt d\tau$$
$$= \int_0^T f_i^2(\tau) dt \left[\int_t^T e^{-\lambda_i(t-\tau)} d\tau\right] \le \lambda_i^{-1} \int_0^T f_i^2(\tau) dt \text{ .}$$

ii) From $z(x,t) = \sum z_\nu(t) \phi_\nu(x)$ it follows

$$\dot{z} - z'' = \sum (\dot{z}_\nu(t) + \lambda_\nu z_\nu(t)) \phi_\nu(x) = 0 \text{ .}$$

Therefore

$$z_\nu(t) = z_\nu(0) e^{-\lambda_\nu t} \text{ and } z_\nu(0) = g_\nu = (g, \phi_\nu) \text{ .}$$

Putting

$$z_{k,l}(t) = sup_{\lambda_\nu \ge m > 0} \lambda_\nu^{k-l} e^{-2\lambda_\nu t}$$

it follows

$$\|z(t)\|_k^2 = \sum \lambda_\nu^k z_\nu^2(t) = \sum \lambda_\nu^k e^{-2\lambda_\nu t} g_\nu \le C_{k.l}(t) \sum \lambda_\nu^l e^{-2\lambda_\nu t}$$

The conditions

$$(k-l)\lambda^{k-l-1} e^{-2\lambda_\nu t} + \lambda^{k-l}(-2t) e^{-2\lambda_\nu t} = 0 \quad \text{resp.} \quad (k-l)\lambda^{k-l-1} e^{-2\lambda_\nu t} = 2t\lambda^{k-l} e^{-2\lambda_\nu t}$$

leads to (for the critical case $k > l$) $\lambda \approx t^{-1}$.

Solutions of the string vibration equation

The wave equation

$$u_{tt} - k^2 u_{xx} = 0$$

has a solution $u(x,t) = f(x-kt)$ for any function of one variable f; it has the physical interpretation of a „traveling wave“, described by a „shape“ $f(x)$ moving at velocity k.

There is no physical reason for the “shape” to be differentiable, but if it is not, the differential equation is not satisfied at some points. In order to not through away physically meaningful solutions because of technicalities, the concept of distributions can be applied.

If the equation above is also meaningful, if u is a distribution, then u is called a weak solution of it. If u is twice continuously differentiable and the equation holds, one calls u a strong or classical solution.

Each classical solution is a weak solution. In case of the equation above it’s also the other way around. The same is NOT TRUE for the elliptic Laplace equation (counter example is the classical solution $u(x.y) := log(x^2+y^2)$ with $\Delta\, log(x^2+y^2) = 4\pi\delta$) and the function $u(x.y) := f(x-kt) \in L^1_{loc}(R^2)$ with

$$(*) \quad (u_{tt} - k^2 u_{xx}, \phi) = 0 \ .$$

Proof of (*): From the following identities

i) $\quad (u_{tt}, \phi) = (u, \phi_{tt}) = \int\int f(x-tk)\, \phi_{tt} dxdt$

ii) $\quad (u_{xx}, \phi) = (u, \phi_{xx}) = \int\int f(x-tk)\, \phi_{xx} dxdt$

it follows

$$(u_{tt} - k^2 u_{xx}, \phi) = \int\int f(x-tk)\, [\phi_{tt} - k^2 \phi_{xx}] dxdt \ .$$

Substituting the variable in the form $y = x - kt$ and $z = x + kt$ means

$$\frac{\partial(y,z)}{\partial(x,t)} = \begin{pmatrix} 1 & -k \\ 1 & k \end{pmatrix} \text{ and } 2kdxdt = dydz \ .$$

From this it follows

$$(u_{tt} - k^2 u_{xx}, \phi) = -2k \int\int f(y)\, \phi_{yz} dzdy = -2k \int_{-\infty} f(y) (\int_{-\infty}^{\infty} \phi_{yz} dz) dy.$$

As $\int_{-\infty}^{\infty} \phi_{yz} dz = \phi_y \big|_{z=-\infty}^{z=\infty} = 0$ it follows (*).

The Divergence

The concept of divergence is especially important in those areas concerned with the behavior of vector fields. The divergence of the energy-stress tensor is zero, which plays a key role, when defining Einstein's gravitation equation. The divergence theorem (the flux of a vector field through a closed surface S is equal to the integral of the divergence of that field over a volume V for which S is a boundary) plays a key role within the Maxwell equations.

Positive divergence is associated with the "flow" of electric field lines away from positive charges. Flux is defined over an area, while divergence applies to individual points. In case of a fluid, the divergence at any point is a measure of the tendency of the flow vectors to diverge from that point, i.e., to carry out more "material" away from it than is brought towards to it. Those points of positive divergence are sources, while points of negative divergence are sinks.

In case of a point charge at the origin, the flux through an infinitesimally small surface is nonzero only if that surface contains the point charge. Everything else, the flux into and out of that tiny surface must be the same, and the divergence of the electric field must be zero.

The mathematical definition of divergence may be understood by considering the flux through an infinitesimal surface surrounding the point of interest. If you were to form the ratio of the flux of a vector field $\vec{E}$ through a surface S to the volume enclosed by that surface as the volume shrinks towards zero, one would have the divergence of $\vec{E}$:

$$div\vec{E} = lim\,\frac{1}{\Delta V}\oint_S \vec{E} \circ \vec{n} da \ .$$

This definition also states the relation between flux and divergence. Vector fields with zero divergence are called "solenoidal" fields.

The Maxwell equations produce the wave equation, which is the basis for the electromagnetic theory of light. A phenomenon of the Maxwell equations is the electromotive force (emf), involving movement of a charged particle through a magnetic field, defined by

$$emf := -\frac{d}{dt}\oint_S \vec{B} \circ \vec{n} da \ .$$

The negative right hand side plays a key role in Lenz's law, which is about the direction of the current induced by changing magnetic flux (flow always in the direction so as to oppose the change in flux).

The flux (i.e., the "number of field lines") of a vector field $\vec{E}$, which represents a "fluid flow/stream model to "which places" the fluid flow transports a particle along an integral curve γ (characterized by $\gamma'(t) = \vec{E}(\gamma(t))$ during a certain time span $t \in (0, a)$). For a point $P = \gamma(0)$ this is mathematically described as bundle of functions

$$\Phi_t(P) = \gamma(t) \ .$$

In differential geometry, the Ricci flow is an intrinsic geometric flow - a process which deforms the metric of a Riemannian manifold - in this case in a manner formally analogous to the diffusion of heat, thereby smoothing out irregularities in the metric. It plays an important role in the proof of the Poincaré conjecture.

The Stokes operator

The Stokes operator is a projector from $A: L_2 \to L_\sigma^2 := \{v | v \in L_2 \wedge div(v) = 0\}$. The Hilbert scale is built on the Stokes operator on $\Omega \subseteq R^n$ $(n \geq 2)$ in the form $A = \int_0^\infty \lambda dE_\lambda$. The Stokes operator enables the definition of a related Hilbert scale ($\alpha \in R$) with a corresponding norm $\|u\|_\alpha := \|A^{\alpha/2}u\|$, enabled by the corresponding positive selfadjoint fractional powers ((SoH), IV15)

$$A^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda \ , -1 \leq \alpha \leq 1$$

The corresponding Stokes semi-group family $\{S(t)\}$ is built on the everywhere bounded, positive selfadjoint operator

$$S(t) := e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda \, | \lambda \geq 0, t \geq 0.$$

An unusual proof of the shift theorem for the Stokes problem

The proof is restricted to $n = 2$, as the argument is based on the Cauchy-Riemann differential equations in order to de-coule the solenoid condition from the Stokes equations.

Stationary Stokes problem: let $n = 2$; consider the boundary value problem

$$\begin{aligned} -\Delta \underline{v} - \nabla p &= \underline{f} \quad \text{in } \Omega \\ div \underline{u} &= h \quad \text{in } \Omega \\ \underline{u} &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

Let $\dot{L}_2$ denote the factor space L_2/R equipped with the corresponding factor norm and let the right hand sides $\underline{f}$ with a reduced regularity assumptions in the form

$$\underline{f} = -div(\underline{\sigma}) \text{ i.e. } \quad f_i = \sum_{j=1}^n \sigma_{ij|j}$$

In this case the weak solution of the Stokes boundary value problem is characterized by

$$\begin{aligned} (\nabla \underline{v}, \nabla \underline{w}) - (p, div \underline{w}) &= (\underline{\sigma}, \nabla \underline{w}) \quad \text{for all } \underline{w} \in \underline{D}(\Omega) \\ (q, \nabla \underline{v},) &= (q, h) \quad \text{for all } q \in \dot{L}_2 \ . \end{aligned}$$

The following two shift theorems hold true (NiJ), (SoH) p. 107)

Theorem:

1. Assume the regularity $\underline{f} = -div(\underline{\sigma})$ with $\underline{\sigma} \in \underline{H}_0$ and $h \in \dot{L}_2$. Then the unique (weak) solution $\{\underline{v}, p\}$ of the boundary value problem has the regularity $\underline{v} \in \underline{H}_1$ and $p \in \dot{L}_2$ and the a priori estimate holds true:

$$\|v\|_{\underline{H}_1} + \|p\|_{\dot{L}_2} \leq c\|\sigma\|_{\underline{H}_0} + \|h\|_{\dot{L}_2}$$

2. Assume the regularity $\underline{f} = -div(\underline{\sigma})$ with $\underline{\sigma} \in \underline{C}_{0.\lambda}$ and $h \in \dot{C}_{0.\lambda} := C_{0.\lambda} \cap \dot{L}_2$. Then the unique (weak) solution $\{\underline{v}, p\}$ of the boundary value problem has the regularity $\underline{v} \in \underline{C}_{1.\lambda}$ and $p \in \dot{C}_{0.\lambda}$ and the a priori estimate holds true:

$$\|v\|_{\underline{C}_{1.\lambda}} + \|p\|_{\dot{C}_{0.\lambda}} \leq c\|\sigma\|_{\underline{C}_{0.\lambda}} + \|h\|_{\dot{C}_{0.\lambda}}.$$

Four versions of the "lemma of Gronwall"

The lemma of Gronwall is a well-established tool for instance to derive evolution equation based classical or variational inequalities. However, applying this tool to Hilbert norm base estimates jeopardizes the balance of any problem adequate norm, e.g. energy / conservation law equations or related inequality estimate (e.g. Garding type inequalities). In this sense, when applying the Gronwall lemma the balance of the „conservation law" norms is jeopardized. For discrete analogues of some generalizations of Gronwall's inequality we refer to (WiD).

Generalized Lemma of Gronwall (version 1): Let $\psi(t) \in C^0[0,a]$ be a real valued function and $h(t) \in L_1(0,a)$ be non-negative function with

$$\psi(t) \leq \alpha + \int_0^t h(\tau)\psi(\tau)d\tau,\ \alpha \in R\ .$$

Then

$$\psi(t) \leq \alpha * e^{\int_0^t h(\tau)d\tau}\ .$$

Generalized Lemma of Gronwall (version 2): Let $\psi(t) \in C^0[0,a]$ be a real valued function and $h(t) \in L_1(0,a)$ be non-negative function with

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau$$

with

$$H(\tau) := \int_0^\tau h(s)ds\ .$$

Generalized Lemma of Gronwall (version 3: log type): Let a, β be non-negative constants. Assume that a non-negative function $a(t,s)$ satisfies $a(\cdot,\cdot) \in C(0 \leq s < t \leq T)$, $a(t,\cdot) \in L_1(0,t)$ for all $t \in ((0,T)]$. Furthermore, we assume that there exists a positive constant ε_0 such that

$$\sup_{0\leq t\leq T} \int_{t-\varepsilon_0}^t a(t,s)ds \leq 1/2 \quad .$$

If a non-negative function $f \in C([0,T])$ satisfies

$$f(t) \leq \alpha + \int_0^t a(t,s)f(s)ds + \beta\int_0^t \{1 + log(1+f(s))\}f(s)ds$$

for all$t \in [0,T]$. Then we have

$$f(t) \leq e^{\left\{1+\frac{\gamma}{\beta}+log(1+2\alpha)\right\}e^{2\beta t}}$$

for all$t \in [0,T]$. Here we put

$$\gamma := \sup_{0\leq t\leq T}\left\{\sup_{0\leq s\leq t-\varepsilon_0} a(t,s)\right\}.$$

Lemma of Gronwall (version 4): Let $a(t)$ and $b(t)$ nonnegative functions in $[0,A)$ and $0 < \delta < 1$. Suppose a nonnegative function $y(t)$ satisfies the differential inequality

$$y'(t) + b(t) \leq \alpha(t)y^\delta(t) \quad \text{on} \quad [0,A)$$

$$y(0) = y_0.$$

Then for $0 \leq t < A$

$$y(t) + \int_0^t b(\tau)d\tau \leq (2^{\delta/(1-\delta)} + 1)y_0 + 2^{\delta/(1-\delta)}\left[\int_0^t \alpha(\tau)d\tau\right]^{\delta/(1-\delta)}$$

Proof: solving $y'(t) \leq \alpha(t)y^\delta(t)$ leads to $y(t) \leq y_0 + \left[\int_0^t \alpha(\tau)d\tau\right]^{\delta/(1-\delta)}$.

Wavelets

(HoM), (LoA), (MeY)

The decomposition of the quantum state space $H_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. the quantum energy space $H_{1/2} = H_1 \otimes H_1^{\perp}$ goes along with the Fourier wave resp. the Calderón wavelet tool (*). While the Fourier waves enable an analysis of the test space H_0, wavelets enable an alternative analysis tool for a specific densely embedded subspace of H_0, as the (wavelet) admissibility condition for a $\psi \in H_0$ is a weak one, as for each $\psi, \hat{\psi} \in H_0$: it holds $\|\psi_\varepsilon - \psi\|_{L_2}^2 \to 0$ for

$$\hat{\psi}_\varepsilon := \begin{cases} \hat{\psi}(\omega), & |\omega| \geq \varepsilon \\ 0, & else \end{cases}$$

There are at least two approaches to wavelet analysis, both are addressing the somehow contradiction by itself, that a function over the one-dimensional space R can be unfolded into a function over the two-dimensional half-plane. The Fourier transform of a wavelet transformed function f is given by, (LoA), (MeY),

$$\widehat{W_\vartheta[f]}(a,\omega) := (2\pi|a|)^{\frac{1}{2}} c_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-a\omega)\hat{f}(\omega)\ .$$

For $\varphi, \vartheta \in L_2(R)$, $f_1, f_2 \in L_2(R)$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\overline{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \leq c_\vartheta c_\varphi$ one gets the duality relationship, (LoA)

$$(W_\vartheta\ f_1, W_\varphi^* f_2)_{L_2(R^2, \frac{dadb}{a^2})} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$$

i.e.

$$W_\varphi^* W_\vartheta\ [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{ –sense.}$$

For $\varphi, \vartheta \in L_2(R)$, $f_1, f_2 \in L_2(R)$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\overline{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \leq c_\vartheta c_\varphi$ one gets the duality relationship (LoA)

$$(W_\vartheta\ f_1, W_\varphi^* f_2)_{L_2(R^2, \frac{dadb}{a^2})} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$$

i.e.

$$W_\varphi^* W_\vartheta\ [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{ –sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula.

(*) (HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify

b ↔ position; $(a\Delta)^{-1}$ ↔ enlargement; g ↔ optics. “

Hermite Polynomials

The weighted Hermite polynomials, $n = 0,1,2,\ldots$,

$$\varphi_n(x) := \frac{e^{-\frac{x^2}{2}} H_n(x)}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{with} \quad H_n(x) := (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}\ ,\ H_0(x) = 1,\ H_1(x) = x$$

form a set of orthonormal functions in $L_2(-\infty,\infty)$, i.e., it holds $\varphi_n \in L_2$ and $\hat{\varphi}_n = (-i)^n \varphi_n$, and the following recursion formula is valid:

$$H_{n+1} = 2xH_n - 2nH_{n-1} = 2xH_n - H_n'.$$

Let H denote the Hilbert transform, then it follows $H\varphi_n \in L_2$ and $(\phi_n, H\phi_n) = 0$, because $|\hat{u}(\omega)|^2$ is even and $\int_{-\infty}^{\infty} u(y)(Hu)(y)dy = \frac{i}{2\pi}\int_{-\infty}^{\infty} sign((\omega)|\hat{u}(\omega)|^2 d\omega$. The expansion of the $sign((\omega)$ function in a series of Hermite polynomials is given by, (LeN) 4.16,

$$sign(x) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(2n+1)n!} H_{2n+1}(x).$$

Kummer function related representations of $H_n(x)$ are given by, (LeN) 9.13,

$$H_{2m}(z) = (-1)^m \frac{(2m)!}{m!}\ {}_1F_1\left(-m, \frac{1}{2}; z^2\right)$$

$$H_{2m+1}(z) = (-1)^m \frac{(2m+1)!}{m!} 2z\ {}_1F_1\left(-m, \frac{3}{2}; z^2\right).$$

The following integral equations with real terms are valid ($m = 0,1,2,\ldots$), (LeN) 4.11, 4.12,

$$e^{-x^2} H_{2m}(x) = \frac{1}{\sqrt{\pi}} (-1)^m 2^{2m+1} \int_0^{\infty} e^{-t^2} t^{2m} \cos(2xt)\, dt$$

$$e^{-x^2} H_{2m+1}(x) = \frac{1}{\sqrt{\pi}} (-1)^m 2^{2m+2} \int_0^{\infty} e^{-t^2} t^{2m+1} \sin(2xt)\, dt$$

$$e^{-\frac{x^2}{2}} H_{2m}(x) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{y^2}{2}} H_{2m}(y) \cos(xy)\, dy$$

$$e^{-\frac{x^2}{2}} H_{2m+1}(x) = (-1)^m \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-\frac{y^2}{2}} H_{2m+1}(y) \sin(xy)\, dy\ .$$

Hermite polynomials have only real zeros all of which are simple. The Hermite polynomials in the form

$$f_n(x) := 2^{-\frac{n}{2}} H_n(\sqrt{2\pi}x) e^{-\pi x^2}$$

are the eigenfunctions of the Hamiltonian

$$x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}$$

of the quantum mechanical harmonic oscillator. That is, they satisfy the Schrödinger equation

$$\left(x^2 - \frac{1}{4\pi^2}\frac{d^2}{dx^2}\right) f_n = \frac{2n+1}{2\pi} f_n.$$

The Mellin transforms of the Hermite functions have their zeros on the critical line $Re(s) = 1/2$, and the related polynomials p_n fulfill (in common with the Riemann zeta function) the following functional equation, (BuD),

$$p_n(1 - \mathrm{s}) = \begin{cases} p_n(\mathrm{s}) & n=0,1\ mod\ 4 \\ -p_n(\mathrm{s}) & n=2,3\ mod\ 4 \end{cases}.$$

The Dawson function $F(x) := e^{-x^2} \int_0^x e^{t^2} dt$ resp. the Hilbert transform of the Gaussian function $I(x) := \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{x-t} dt$, (GaW), are linked to the zeros and the weight factors of the Hermite polynomials by, (AbM) 7.1.15, 7.1.23,

$$I(x) = \frac{1}{\sqrt{\pi}} \lim_{n\to\infty} \sum_{k=1}^{n} \frac{H_k^{(n)}(x)}{x - x_k^{(n)}} = 2F(x) \cong \frac{1}{x}\ .$$

The quaternion rotation operator

In classical theory but also in quantum theory, symmetry groups are applied to derive conservation laws for energy, translation and angular momenta. However, in the current quantum theory translation and rotation operators are not interchangeable, which is a consequence of the quantification process of classical partial differential equations. Therefore, in the current quantum theory framework in order to characterize the angular momentum of a system about an axis by a quantum number it is neccessary that the perpendicular translation momentum vanishes or is unknown, (DüH). This handicap can be seen as the „root of evil" of all challenges of the standard model of (elementary) particles & cosmology, loop quantum & gravity, but also of Heisenberg's mathematical formalism for an unified field theory (based on an asymmetry of the ground state and nonlinear spinors, (DüH), (HeW)).

The quaternions provide an appropriate field to address the „translation-rotation" (linear and angular rotation) „permutation" requirement. The perhaps primary application of quaternions is the quaternion rotation operator. This is a special quaternion triple-product (unit quaternions and rotating imaginary vector) competing with the conventional (Euler) matrix rotation operator.

The quaternion rotation operator can be interpreted as a frame or a point-set rotation, (KuJ). Its outstanding advantages compared to the Euler geometry are

- the axes of rotation and angles of rotation are independent from the underlying coordinate system and directly readable (*)
- there is no need to to take care about the sequencing of the rotary axes.

In other words, a fractional quaternionic Hilbert scale framework accompanied by indefinite inner products and corresponding variational theory applied for Pseudo Differential Operators provides an appropriate toolset, which overomes current challenges in quantum and gravity theory (including „the problem of „*time*"), (*).

In (SaM) a generalized quaternionic quantum wave equation formulation is used to construct general plane waves enabling corresponding generalized Klein Gordon and Helmholtz equations.

MacFarlane (MaF) introduced the set of hyperbolic quaternions. The hyperbolic quaternions are not commutative like real quaternions. But the set of hyperbolic quaternions contains zero divisors.

In (LeS) the isomorphism between unitary quaternions and space time rotations is extended to Lorentz boosts. From the transformation properties of two-component spinors a quaternionic representation for the space-time algebra is derived. Additionally, a quaternionic bi-dimensional version of the Dirac equation is derived.

In (KrR) variable orthonormal sets, so-called moving frames, are provided to generalize the notion „differentiability" and „complex-analyticity" by a hypercomplex (quaternionic) differential form calculus.

In (BoI) a continuous wavelet transform is built on the upper sheet of the 2-hyperboloid H_+^2.

(*) Husserl differentiates between the objective time of appearing objects, the subjective or preempirical time of acts and experiences and the prephenominal absolute flow of the internal time consciousness, ((ZaD) chapter 3.

Quaternion groups and real & complex Lorentz groups

The Lorentz transformation in special relativity is a simple type of rotation in hyperbolic space.

The Lorentz group L has four components, each of which is connected in the sense that any point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component. One of this components is the restricted Lorentz group, which is the group of $2x2$ complex matrices of determinant one, $SL(2,C)$. It is isomophic to the symmetry group $SU(2) \cong SL(2,C)$, containing as elements the complex-valued rotations, which can be written as a complex-valued matrix of type

$$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$$ with determinant one.

It is important in describing the transformation properties of spinors. In SMEP the group $SU(2) \cong SL(2,C)$ describes the weak force interaction with 3 bosons W^+, W^-, Z.

Another group associated with the Lorentz group L is the complex Lorentz group $L(C)$. It has just two connected components, $L_+(C)$ and $L_-(C)$. The transformations 1 and -1, which are disconnected in L are connected in $L(C)$. In other words, the complex Lorentz transformation connects

- the two components containing the 1-transformation and space-time inversion
- the two components containing the space inversion and the time inversion.

Just as the restricted Lorentz group is associated with $SL(2,C)$, the complex Lorentz group is associated with $SL(2,C) \otimes SL(2,C) \cong SU(2) \otimes SU(2)$. There is also a two-to-one homomorphism from $SL(2,C) \times SL(2,C)$ onto $L(+,C)$.

For the relations of the Lorentz group in the context of the SRT and the GRT to Minkowskian quaternions, belonging to the wider class of complex quaternions (biquaternions), we refer to (GiP).

In (LeS1) new real linear quaternions are introduced to obtain a quaternionic version of the Lorentz group without the use of complexified quaternions) and a quaternionic metric tensor is defined, overcoming difficulties concerning the appropriate transformations on the $3+1$ space-time.

For quaternionic analysis and elliptic boundary value problems we refer to (GuK).

The spin of an elementary particle is its eigen-rotation with exactly two rotation axes, one parallel and one anti-parallel axis to a magnetic field. This is the 2×2 complex number scheme, where every „normal“ rotation is contained twice. Consequently, an electron has a charge only half of the Planck's quantum of action. For a quaternionic equation representation of the motion of a particle with an electric charge in a electromagnetic field manifesting the relativistic covariance of classical electromagnetism we refer to (GiP). In (ArI) a quaternionic unification of electromagnetism and hydrodynamics is provided.

In (HuM) the concept of the unit quaternion is applied to enable statistical analysis for rotations in 3D electron cryo-microscopy. Concepts of distance and geodesic between spatial rotations is introduced and developed to enable comparisons and interpolations between rotations. Statistical methods for performing sampling and numerical analysis in the rotational space are introduced and developed. A description for the molecular symmetry and the corresponding method of space division for asymmetric units are developed based on the unit quaternion.

Rotations in hyperbolic 4-spaces, hyperbolic function theory and quaternionic quantum waves

The Lorentz transformation in special relativity is a simple type of rotation in hyperbolic space.

(CoA): „*In flat* 4*-space a general rotation can be made up of two planar rotations in orthogonal planes, and these component rotations are commutative. In hyperbolic space a general rotation is made up of two planar rotations in planes which are reciprocal with respect to the hypercone,*

$$t^2 = x^2 + y^2 + z^2.$$

Of the two planes one cuts the hypercone in real 4*-vectors and the rotation in this plane is hyperbolic, the rotation in the other plane is circular. The rotations are commutative.“*

In (ErS) a study of hyperbolic function theory in the total skew field of quaternions is provided, where the considered function depend on all four coordinates of quaternions. The considered (so-called α-hyperbolic harmonic) functions are harmonic with respect to the Riemannian metric

$$ds_\alpha^2 := \frac{ds_0^2+ds_1^2+ds_2^2+ds_3^2}{x_3^\alpha}$$

in the upper half space $R_+^4 := \{(x_0, x_1, x_2, x_3) \in R^4 : x_3 > 0\}$. If $\alpha = 2$, the metric is the hyperbolic metric of the Poincare´s upper half-space.

In (SaM) a generalized quaternionic quantum wave equation formulation is used to construct general plane waves enabling corresponding generalized Klein Gordon and Helmholtz equations.

Electrons cannot rotate in a classical way but manifest in a 3-dimensional S^3 reality

The spin of an elementary particle is its eigen-rotation with exactly two rotation axes, one parallel and one anti-parallel axis to a magnetic field. This is the 2×2 complex number scheme, where every „normal“ rotation is contained twice. Consequently, an electron has a charge only half of the Planck's quantum of action, (UnA1) S. 188. On average an neutron decays into a proton, an electron, and an anti-neutrino in 15 minutes. The root cause of this so-called „beta decay“ is unknown, (UnA1) S. 189.

> (UnA2) chapter 12: „*The Stern-Gerlach experiment showed that electrons cannot rotate in a classical way. … The quantification of spin, as one might call Stern and Gerlach's result, is an additional, very irritating property of nature. The result can only be interpreted in such a way that elementary particles basically appear in two different states as soon as they are somewhere located in space. … Nature thus surprises us with a mysterious doubling of states as soon as we consider orientations in space. … Apparently, however, the* 3*-dimensional* S^3 *unit sphere shows exactly this structure. … The difference from* $SO(3)$ *is the double cover, i.e.,* S^3 *provides exactly two states for each point in* $SO(3)$*. …. It is clear that the double cover must be the deeper mathematical cause of the spin phenomenon, for which there is no explanation in the conventional paradigm of space and time.“*

Statistics of spatial rotations in 3D electron cryo-microscopy by unit quaternion description

In (HuM) the concept of the unit quaternion is applied to enable statistical analysis for rotations in 3D electron cryo-microscopy. Concepts of distance and geodesic between spatial rotations is introduced and developed to enable comparisons and interpolations between rotations. Statistical methods for performing sampling and numerical analysis in the rotational space are introduced and developed. A description for the molecular symmetry and the corresponding method of space division for asymmetric units are developed based on the unit quaternion.

The real and complex Lorentz groups

The real indefinite orthogonal Lorentz group $O(1,3)$, preserving $t^2 - x^2 - y^2 - z^2$, has to do with the symmetry of vectors (tensors). The restricted Lorentz group $O^+(1,3)$ preserves orientation of space and time. The special linear group $SL(2,C)$ is a double covering of $O^+(1,3)$, where $O^+(1,3)$ is a specific representation of it. The biquaternions are also isomorphic to $SL(2,C)$. The Lorentz group that preserves the direction of time is called *orthochronous*; the restricted Lorentz group is also called *proper orthochronous* Lorentz group. The restricted Lorentz group is generated by ordinary space rotation and Lorentz boosts (which are rotations in a hyperbolic space that include a time-like direction); in other words, the restricted Lorentz group is generated by 6 parameters.

The complex, homogeneous Lorentz group, $L(C)$, (the set o, (4×4)-complex matrices) has two connected components $L(+,C)$ and $L(-,C)$, where $L(+,C)$ is the only sub-group of $L(C)$. It leaves the complex Minkowski scalar product z^*z invariant, without complex conjugation of one of the complex four-vectors z. Just as the restricted Lorentz it is associated with $SL(2,C) \times SL(2,C)$; there is a two-to-one homomorphism from $SL(2,C) \times SL(2,C)$ onto the complex Lorentz sub-group $L(+,C)$, which is a proper Lorentz transform, (WiD).

For quaternionic Lorentz group representations in relation to the Dirac equation we refer to (LeS).

For quaternionic analysis and elliptic boundary value problems we refer to (GuK).

The real Lorentz transform

(StR): A Lorentz transformation is a linear transformation Λ mapping space-time onto space-time which preserves the scalar product $(\Lambda\vec{\mathrm{x}}, \Lambda\vec{\mathrm{y}}) = (\vec{\mathrm{x}}, \vec{\mathrm{y}})$, where

$$(\vec{\mathrm{x}}, \vec{\mathrm{y}}) \coloneqq (x^0, y^0) - [(x^1, y^1) + (x^2, y^2) + (x^3, y^3)] = x^\mu g_{\mu\nu} x^\nu = x^\mu y_\mu.$$

If $(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu$, the (real) matrix $\Lambda^\mu{}_\nu$ of the transformation must satisfy

$$\Lambda^\kappa{}_\mu \Lambda_{\kappa\nu} = g_{\mu\nu} \text{ or } \Lambda^T G \Lambda = G, \qquad (1\text{-}5)$$

where the transpose Λ^T of Λ is defined by $(\Lambda^T)^\mu{}_\nu = \Lambda^\mu{}_\nu$ and indices on Λ are lowered according to

$$\Lambda_{\kappa\nu} = g_{\kappa\sigma} \Lambda^\sigma{}_\nu = (G\Lambda)_{\kappa\nu}.$$

If Λ and M satisfy (1-5), so do $\Lambda\mathrm{M}$ and Λ^{-1}. Here

$$(\Lambda\mathrm{M})^\mu{}_\nu = \Lambda^\mu{}_\nu \, \mathrm{M}^\kappa{}_\nu \quad (\Lambda^{-1})^\mu{}_\nu \Lambda^\kappa{}_\nu = \mathrm{g}^\mu{}_\nu = \begin{cases} 0 & \mu \neq \nu \\ 1 & \mu = \nu \end{cases},$$

so the (real) Lorentz transformations form a group, the Lorentz group L.

Two Lorentz transformations Λ and M are defined to be close to one another if the numbers $\Lambda^\mu{}_\nu$ and $\mathrm{M}^\mu{}_\nu$ are close for all $\mu, \nu = 0,1,2,3$. Clearly, with this definition, Λ^{-1} and $\Lambda\mathrm{M}$ are continuous functions of Λ and M, respectively. Furthermore, it make sense to say that two (real) Lorentz transformations can be connected to one another by a continuous curve of Lorentz transformations.

The Lorentz group L has four components, each of which is connected in the sense that any point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component.

One of this components is the restricted Lorentz group, which is the group of $2x2$ complex matrices of determinant one, $SL(2, C)$. It is isomophic to the symmetry group $SU(2)$, containing as elements the complex-valued rotations, which can be written as a complex-valued matrix of type

$$\begin{pmatrix} a + ib & c + id \\ -c + id & a - ib \end{pmatrix} \text{ with determinant one.}$$

It is important in describing the transformation properties of spinors. In SMEP the group $SU(2)$ describes the weak force interaction with 3 bosons W^+, W^-, Z.

This restricted Lorentz group contains the 1-transformation. Its connected component contains the space-time inversion. The other pair of connected components of the Lorentz group contains the space inversion resp. the time inversion.

The Lorentz group has also three important subgroups, which are

- the orthochronous Lorentz group
- the proper Lorentz group
- the orthochorous Lorentz group.

The complex Lorentz transform

Another group associated with the Lorentz group L is the complex Lorentz group, which we shall denote by $L(C)$. It is essential in the proof of the PCT theorem as we shall see. It is composed of all complex matrices satisfying

$$\Lambda^{\kappa}{}_{\mu}\Lambda_{\kappa\nu} = g_{\mu\nu} \text{ or } \Lambda^{T}G\Lambda = G, \qquad (1\text{-}5).$$

It has just two connected components, $L_{+}(C)$ and $L_{-}(C)$ according to the sign of det(Λ). The transformations 1 and -1, which are disconnected in L are connected in $L(C)$. In other words, the complex Lorentz transformation connects

- the two components containing the 1-transformation and space-time inversion, i.e. the pair

$$\{\det(\Lambda) = +1\,, \det(\Lambda^{0}{}_{0} = +1)\}, \{\det(\Lambda) = +1\,, \det(\Lambda^{0}{}_{0} = -1)\},$$

- the two components containing the space inversion and the time inversion, i.e. the pair

$$\{\det(\Lambda) = -1\,, \det(\Lambda^{0}{}_{0} = +1)\}, \{\det(\Lambda) = -1\,, \det(\Lambda^{0}{}_{0} = -1)\}.$$

Summary:

While two (real) Lorentz transformations need to be connected to one another by an appropriately defined continuous curve of Lorentz transformations, there are two pairs of components of the complex Lorentz transform, which are both already connected by definition.

Just as the restricted Lorentz group is associated with $SL(2,C)$, the complex Lorentz group is associated with $SL(2,C) \otimes SL(2,C)$. The latter group is the set of all pairs of $2x2$ matrices of determinants one with the multiplication law

$$\{A_1, B_1\} \cdot \{A_2, B_2\} = \{A_1 A_2, B_1 B_2\}.$$

Is is easy to see that only matrix pairs which yield a given $\Lambda(\mathrm{A},\mathrm{B})$ are $(\pm\mathrm{A}, \pm\mathrm{B})$. In particular,

$$\Lambda(-1,1) = \Lambda(1,-1) = -1.$$

The corresponding complex Poincare group admits complex translation but also the multiplication law

$$\{a_1, \Lambda_1\} \cdot \{a_2, \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1 \Lambda_2\}.$$

It has two components $\mathrm{P}_{\pm}(C)$, which are distinguished by $\det(\Lambda)$ and a corresponding inhomogeneous group to $SL(2,C)$.

The Lorentz group and Minkowskian quaternions

(GiP): „*The Lorentz group is the group of homogeneous transformations which conserve the quantity*

$$c^2t^2 - x^2 - y^2 - z^2.$$

If there is no time reversal and no space reflection, the transformations are said to be orthochronous proper. Any orthochronous proper Lorentz transformation can be expressed by a quaternion transformation of the type

$$q' = aqa_c^*$$

where a *is a complex quaternion such that* $aa_c = 1$*, and where* $q = (ct, ix, iy, iz)$*,* $q' = (ct', ix', iy', iz')$*. I shall refer to quaternions such that* $q_c^* = q$ *as minquats (Minkowskian quaternions).*"

Minkowskian quaternions belong to the wider class of complex quaternions (biquaternions). The norm and the scalar product of Minkowskian quaternions are real, (SyJ) p.2,10.

The special theory of relativity and the pure Lorentz transform

(GiP): „*Consider the pure Lorentz transformation*

$$q' = bqb_c^*$$

with $b = (cosh\Phi, isinh\Phi \boldsymbol{e})$*,* $q = (ct, ix, iy, iz)$*,* $q' = (ct', ix', iy', iz')$ *and where* $\boldsymbol{e}$ *is a real unit vector.*

The four-veclocity transforms according to the relation $u' = bub_c^*$ *where* $u = \frac{dq}{ds}$*,* $u' = \frac{dq'}{ds}$*, and* $ds^2 = dqdq_c$*.*

The above transformation transforms a point at rest into a point moving with a speed u *in the direction* $-e$*; it corresponds to a standard Lorentz transformation (from a passive point of view) in the* e *direction.*

The four-angular momentum of a material point can be defined by

$$\boldsymbol{L} = \boldsymbol{V}(qp_c) = \left[0, \boldsymbol{r} \times \boldsymbol{p} + i(\frac{E\boldsymbol{r}}{c} - ct\boldsymbol{p})\right]$$

The transformation rule of $\boldsymbol{L}$ *yields the relativistic invariant*

$$\boldsymbol{LL_c} = (\boldsymbol{r} \times \boldsymbol{p})^2 - (\frac{E\boldsymbol{r}}{c} - ct\boldsymbol{p})^2.$$"

The general theory of relativity group the generalised Lorentz group

(GiP): „*The GTR group is simply the generalised Lorentz group*

$$u' = aua_c^*$$

where a *is an arbitrary function such that* $aa_c = 1$ *and where* u *is a Minkowskian quaternions. At any given space-time point, the GTR group reduces to the Lorentz group*."

Special Relativity by a quaternionic algebra on real linear quaternions

In (LeS1) new real linear quaternions are introduced to obtain a quaternionic version of the Lorentz group without the use of complexified quaternions) and a quaternionic metric tensor is defined, overcoming difficulties concerning the appropriate transformations on the $3 + 1$ space-time.

Hyperbolic quaternions

(KöA): „*The multiplication of real quaternions is not commutative. The roots of a real quaternions were given by Niven (Nil), and Brand [8] proved De Moivre's theorem and used it to find* n^{th} *roots of a real quaternion.*

In (CoE) Euler's formula and De Moivre's formula for real quaternions are generalized. It is also shown that there are uncountably many unit quaternions satisfying $x^n = 1$ *for* $n \geq 3$, (*)

Using De Moivre's formula to find roots of real quaternion is a more useful way. After the discovery of real quaternions by Hamilton, MacFarlane (MaF) introduced the set of hyperbolic quaternions. The hyperbolic quaternions are not commutative like real quaternions. But the set of hyperbolic quaternions contains zero divisors. In this work, we express Euler and De Moivre's formulas for hyperbolic quaternions after we give some algebraic properties of hyperbolic quaternions."

The hyperbolic quaternions are not commutative like real quaternions. But the set of hyperbolic quaternions contains zero divisors. In (KöA) some algebraic properties of hyperbolic quaternions are given, including Euler and De Moivre's formulas for hyperbolic quaternions.

The S^1, S^2, S^3 unit spheres

The 1-dimensional unit sphere in R^2 corresponds to the Lie group $U(1)$. The related number grid is built by the Eisenstein numbers.

The 3-dimensional S^3 unit sphere is isomorphic to $SU(2)$. The S^1 and S^3 are the only spheres with a "continuous" group structure, (EbH) 7.2. We note that S^0, S^1, S^3, S^7 the only parallelizable sheres.

The compactification of the field of complex numbers C, the Riemann sphere, is homeomorphic to S^2. It plays a key role in the Teichmueller theory. In the context of the proposed Hilbert space framework we note the relationship of the Teichmüller space with the fractional Hilbert space $H_{1/2}$.

The groups S^1 and S^3 have parameter representations, (EbH) 3.5.4(2'), 7.3.2(3). There are epimorphisms between S^3 and $SO(3)$, resp. between $S^3 \times S^3$ and $SO(4)$. The group $SO(4)$ contains isomorphic normal subgroups to the group S^3, i.e. it is a not „simple" Lie group. The groups $SO(n)$, $n > 4$, are all „simple", i.e. they have not trivial coherent normal subgroups. The groups $SO(2n+1)$ have no normal subgroup unequal (e). The groups $SO(4)$ have exactly the not trivial normal subgroup $\{e, -e\}$, (EbH) 7.3.4.

From the fundamental theorem of algebra for quaternions it follows that there are exactly n roots of any quaternion with not vanishing imaginary part, (EbH) 7.1.8. For each quaternion of S^3 there is a quaternion represention as a sum of two product terms in the form $e \cdot cos\left(\frac{\omega}{2}\right) + q \cdot sin\left(\frac{\omega}{2}\right)$, where e denotes the „real" quaternion unit, q denotes a purely imaginary quaternion with norm equal one, and ω denotes an angle between zero and 2π, (EbH) 7.3

The two-semicircle method to prove the Kummer conjecture
(BrK4), (BrK10)

The Kummer conjecture deals with cubic characters in the form $p = 3k + 1$. This set can be decomposed into

- all odd squares of $3k + 1$
- all even squares of $3k + 1$
- all remaining odd numbers
- all remaining even numbers.

The link to the related $\{4n - 3{,}4n - 1{,}2n\}$ decomposition of the set of integers is given by the fact, that the „distance“ between the consecutive odd squares of $n = 3k + 1$ is $\{4l - 1\}$, and that the „distance“ between the consecutive even squares of of $n = 3k + 1$ is $\{2l\}$. This property provides the conceptual data for an appropriate framework set-up of the proposed two-semicircle method with the following key differentation to the Hardy-Littlewood circle method:

Hardy-Littlewood circle method	**Two-semicircle method**
winding number n	A pair of semicircle numbers $\left(\mathrm{n} - \frac{1}{2}, n\right)$
a single distribution function	two distinct distribution functions (**)
zeros of the orthonormal system $\{e^{2\pi i n z}\}$ of $L_2(S^1)$	complex-valued zeros $\{z_n\}_{n\in N}$ of the Kummer function ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right)$ and absolute values of their imaginary parts $\lvert Im(z_n)\rvert = 2\pi\omega_n$ with $n - \frac{1}{2} < \omega_n < n$ and related retarded/condensed sequences ω_n^*, (*), (BrK)
$\{n\}$	$\{2n - 1{,}2n\} = \{4n - 3{,}4n - 1{,}2n\}$ (**)
Gaussian numbers, (HaG)	Hurwitz numbers, (HuA)
norm: sum of two squares, (Moc)	norm: sum of four squares, (MoC)
Euclidian rotations with fixed winding axis governed by the winding number n	quaternion rotation with dynamic winding axes governed by the odd and even squares of integers resp. their corresponding indices of the retarded/condensed sequence ω_n^* enjoying the Kadec condition

(*) In the context of non-harmonic Fourier series governed by Kadec's theorem and Avdonin's (generalized) theorem of ¼-in the mean we note that the "retarded" sequences of $(2k - 1)$ resp. $((4k - 3), (4k - 1))$ resp. $((8k - 7), (8k - 5), (8k - 3), (8k - 1))$ are condensed by the factor ¼. „*It is interesting to note that Euclid's procedure to prove that the sequences of primes is infinite also works starting with* $n = 0$ *, i.e., without any knowledge about primes*“, (HaH) S. 4.

(**) and related distribution functions built according to (PoG1). We note that the set of even integers is an ideal in the ring of Z. In case the Goldbach conjecture is valid this means that each even integer $2n = p + q$ is the norm of a quaternion if $p, q = 1 mod 4$.

References

(AbM) Abramowitz M., Stegun A., Handbook of mathematical functions, Dover Publications Inc., New York, 1970

(AdS) Adler S. L., Quaternionic quantum mechanics and quantum fields, Volume 88 of International Series of Monographs on Physics. The Clarendon Press Oxford University Press, New York, 1995

(AlD) Alpay D., Colombo F., Sabaini I., Inner Product Spaces and Krein Spaces in the Quarternionic Setting, in Springer, Operator Theory: Advances and Applications, pp. 33-65, 2015

(AlO) Alexandersson O., Lebendes Wasser, Viktor Schauberger und das Geheimnis natürlicher Energie, Ennsthaler Verlag Steyr, 1998

(AnE) Anderson E., The Problem of Time, Quantum Mechanics Versus Generality, Springer, 2017

(AnJ) Annett J. F., Superconductivity, Superfluids and condensates, Oxford University Press, Oxford, 2004

(ArA) Arbab A. I., The analogy between electromagnetism and hydrodynamics, Physics Assays Vol. 24, No. 2, 2011

(ArA1) Arbab A. I., The consequences of complex Lorentz force and violation of Lorentz gauge condition,2013, arXiv:1302.0695v1

(ArA2) Arbab A. I., The generalized Newton's law of gravitation, Astrophys. Space Sci., Vol. 325, 2010 pp. 37-40

(AzT) Azizov T. Ya., Iokhvidov I. S., Linear Operators in Spaces with an Indefinite Metric, John Wiley & Sons, Chichester, New York, Brisbane, Toronto, Singapore, 1989

(BaB) Bagchi B., On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann Hypothesis, Proceedings of the Indian Academy of Sciences - Mathematical Sciences, Vol. 116, No. 22, 2006, pp. 137-146

(BaJ) Barbour J. B., Time and complex numbers in canonical quatum gravity, Physical Review D, Vol. 47, No. 12, 1993, pp. 5422-5429

(BaJ1) Barbour J. B., The End of Time, Oxford University Press, Oxford, 2000

(BeH) Bethe H. A., Morrison P., Elementary Nuclear Theory, Dover Publications, Inc., Mineola, New York, 2006

(BeH1) Bethe H. A., The electromagnetic shift of energy levels, Phys. Rev. 72, 1947, pp. 339-343

(BiJ) Benney J. J., Dowrick N. J., Fisher A. J., Newman M. E. J., The Theory of Critical Phenomena, Oxford Science Publications, Clarendon Press, Oxford, 1992

(BiI) Biswas I., Nag S., Jacobians of Riemann surfaces and the Sobolev space $H^{1/2}$ on the circle, Mathematical Research Letters, Vol. 5, (1998) pp. 281-292

(BlC) Blohmann, Fernandes M. C., Weinstein, A., Groupoid symmetry and constraints in general relativity, Commun. Contemp. Math., 15, 25 pp. (2013), 1003.2857.pdf (arxiv.org)

(BlS) Blue Star, The Implosion Theory of Universe Creation, Khun Dee's Story, Blue Star, 2018

(BlS1) Blundell S., Magnetism in Condensed Matter, Oxford University Press, Oxford, 2001

(BoA) Boboc A., Zeit und Zeitbewusstsein bei Edmund Husserl, www.cercetari.institutuldefilosofie.ro/

(BoD) Bohm D., Quantum Theory, Dover Publications, Inc. New York, 1979

(BoD1) Bohm D., Wholeness and the Implicate Order, Routledge Classics, New York, 1980

(BoI) Bogdanova I., Vandergheynst P., Gazeau J.-P., Continuous wavelet transform on the hyperboloid, Appl. Comput. Harmon. Anal. 23 (2007) 285–306

(BoJ) Bognar J., Indefinite Inner Product Spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1974

(BoJ1) Bourgain J., Kozma G., One cannot hear the winding number, J. Eur. Math. Soc., Vol. 9, 2007, pp. 637-658

(BrH) Brandt H., Über eine Verallgemeinerung des Gruppenbegriffes, Math. Ann. 97 1927, S. 360-366

(BrH) Brandt H., Idealtheorie in Quaternionenalgebren, Math. Ann. 99, 1928, S. 1–29

(BrH) Brandt H., Über die Komponierbarkeit quaternärer quadratischer Formen, Math. Ann. 94, 1925, S. 179-197

(BrJ) Braunbeck J., Der andere Physiker, Das Leben von Felix Ehrenhaft, Leykam, Technisches Museum München, Wien, 2003

(BrK) Braun K., Interior Error Estimates of the Ritz Method for Pseudo-Differential Operators, Japan Journal of Applied Mathematics, Vol. 3, No. 1, 1986, pp. 59-72

(BrK1) Braun, K., The mathematical and physical realities of an unified field model, August 25, 2022

(BrK2) Braun K., A mathematical tool box for an unified field model, 2022

(BrK3) Braun K., A Hilbert scale and MHD based unified plasma, quantum and gravity field theory, 2022

(BrK4) Braun K., A toolbox to solve the RH and to build a non-harmonic Fourier series based two-semicircle method, 2022

(BrK5) Braun K., A geometric Hilbert scale based gravity and quantum field model, the modelling landscape, 2021

(BrK6) Braun K., A Hilbert space based Mie electricity field theory accompanied by a complementary 0-point energy space, 2021

(BrK7) Braun K., 3D-NSE, YME, GUT solutions, 2019

(BrK8) Braun K., An alternative Schroedinger (Calderon) momentum operator enabling a quantum gravity model, 2017

(BrK9) Braun K., Global existence and uniqueness of 3D Navier-Stokes equations, 2016, navier-stokes-equations.com

(BrK10) Braun, K., The Kummer conjecture and the two-semicircle method, 2022

(BrH1) Brézis H., Asymptotic behavior of some evolution systems, Nonlinear evolution equations, Academic Press, New York, (1978), pp. 141-154

(BrH2) Brézis H., On a problem of T. Kato, Communication on Pure and Applied Mathematics, Vol. 24, 1971, pp. 1-6

(BrL) Brand L., The Roots of a Quaternion, Amer. Math. Monthly 49 (8) (1942) 519–520

(BuD) Bump D., Choi K.-K., Kurlberg P., Vaaler J., A Local Riemann Hypothesis I, Math. Zeit. 233 (2000) pp. 1-19

(CaC) Callender C., The Oxford Handbook of Philosophy of Time, Oxford University Press, Oxford, 2013

(CaD) Cardon D. A, Convolution Operators and Zeros of Entire Functions, Proc. Amer. Math. Soc., Vol. 130, No. 6, (2002) pp. 1725-1734

(CaD1) Cardon D. A, de Gaston S. A., An equivalence for the Riemann hypothesis in terms of orthogonal polynomials, Journal of Approximation Theory, Vol. 138, No. 1, (2006) pp. 54-64

(CaF) Cap F., Lehrbuch der Plasmaphysik und Magnetohydrodynamik, Springer-Verlag, Wien, New York, 1994

(ChF) Chen F. F., Introduction to Plasma Physics and Controlled Fusion, Volume I: Plasma Physics, Plenum Press, New York, 1984

(ChJ) Chabrowski J., Variational Methods for Potential Theory, Walter de Gruyter, Berlin, New York, 1997

(CoA) Conway A. W., Application of quaternions to rotations in hyperbolic space in four dimensions, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, Vol. 191, No. 1025, 1947, pp. 137-145

(CoE) Cho, E., De Moivre's Formula for Quaternions, Appl. Math. Lett. 6 (1998), 33-35

(CoM) Costabel M., A Coercive Bilinear Form for Maxwell's Equations, J. Math. Anal. Appl. 157, 2 (1991) 527-541

(CoR) Courant R., Hilbert D., Methods of Mathematical Physics, Volume II, John Wiley & Sons, New York, Chichester, Brisbane, Toronto, Singapore, 1989

(DeH) Dehnen H., Hönl H., Westphal K., Ein heuristischer Zugang zur allgemeinen Relativitätstheorie, Annalen der Physik, 7. Folge, Band 6, 1960

(DeJ) Derbyshire J., Prime Obsession, Joseph Henry Press, Washington D.C., 2003

(DeP) Deligne P. et. al., Quantum Fields and Strings: A Course for Mathematicians, Volume 1, AMS, Institute of Advanced Study, 2000

(DeP1) Deligne P. et. al., Quantum Fields and Strings: A Course for Mathematicians, Volume 2, AMS, Institute of Advanced Study, 2000

(DeR) Dendy R. O., Plasma Dynamics, Oxford Science Publications, Oxford, 1990

(DiP) Dirac P., Classical theory of radiating electrons, Proc. Roy. Soc. London, 167, (1938) pp. 148-169

(DiR) Dicke R. H., Gravitation without a Principle of Equivalence, Rev. Mod. Phys. 29, 1957, pp. 363-376

(DoC) Dolph C. L., Recent developments in some non-self-adjoint problems of mathematical physics, Bull. Amer. Math. Soc., Vol.67 1961, pp.1-69

(DrW) Drees W. B., Interpretation of „The Wave Function of the Universe", International Journal of Theoretical Physics, Vol. 26, No. 10, 1987, pp. 939-942

(DüH) Dürr, H.P., Heisenberg, W., Mitter, H., Schlieder, S., Yamazaki, K., Zur Theorie der Elementarteilchen, Zeitschrift für Naturforschung, 14a, 1959, S. 441-485

(EbH) Ebbinghaus H.-D., Numbers, Springer Science + Business Media, New York, 1991

(EdH) Edwards H. M., Riemann's Zeta Function, Dover Publications, Inc., Mineola, New York, 2001

(EhF) Ehrenhaft, F., Photophoresis and Its Interpretation by Electric and Magnetic Ions, Journal of the Franklin Institute, 233, (1942), 235-256

(EiA) Einstein A., Grundzüge der Relativitätstheorie, Vieweg & Sohn, Braunschweig, Wiesbaden, 1992

(EiA1) Einstein A., The meaning of relativity, Princeton University Press, New Jersey, 1921

(EiR) Eisberg R., Resnick R., Quantum Physics of Atoms, Molecules, Solids, Nuclei and Particles, Wiley, New York, 1985

(ErS) Eriksson S.-L., Orelma H., Hyperbolic Function Theory in the Skew-Field of Quaternions, Adv. Appl. Clifford Algebras, Vol. 29, 2019

(EsG) Eskin G. I., Boundary Value Problems for Elliptic Pseudodifferential Equations, Amer. Math. Soc., Rhode Island, 1981

(EyG) Eyink G. L., Stochastic Line-Motion and Stochastic Conservation Laws for Non-Ideal Hydrodynamic Models. I. Incompressible Fluids and Isotropic Transport Coefficients, arXiv:0812.0153v1, 30 Nov 2008

(FaM1) Farge M., Schneider K., Wavelet transforms and their applications to MHD and plasma turbulence: a review, arXiv:1508.05650v1, 2015

(FeC) Fefferman C., Phong D. H., Symplectic geometry and positivity of pseudo-differential operators, Proc. Natl. Acad. Sci. USA, Vol. 79, (1982) pp. 710-713

(FeE) Fermi E., Quantum Theory for Radiation, Reviews of Modern Physics, Vol. 4, 1932

(FiD) Finkelstein D., Jauch J. M., Schiminovich S., and D. Speiser D., Foundations of quaternion quantum mechanics. J. Mathematical Phys., 3:207–220, 1962

(FlT) Fließbach T., Elektrodynamik, Springer Spektrum, Berlin, 2021

(GaA) Ganchev A. H., Greenberg W., van der Mee C. V. M., A class of linear kinetic equations in Krein space setting, Integral Equations and Operator Theory, Vol. 11, 518-535, 1988

(GaW) Gautschi W., Waldvogel J., Computing the Hilbert Transform of the Generalized Laguerre and Hermit Weight Functions, BIT Numerical Mathematics, Vol. 41, pp. 490–503 (2001)

(GiP) Girard P. R., The quaternion group and modern physics, Eur. J. Phys. S (1984) 25-32

(GiY) Giga Y., Weak and strong solutions of the Navier-Stokes initial value problem, Publ. RIMS, Kyoto Univ. 19 (1983) 887-910

(GlJ) Gleick J., Genius, the life and science of Richard Feynman, First Vintage Books Edition, New York, 1992

(GöK) Gödel K., An Example of a New Type of Cosmological Solutions of Einstein's Field Equations of Gravitation, Reviews of Modern Physics, Vol. 21, No. 3, 1949, pp. 447-449

(HaP) Halmos P. R., Lectures on Ergotic Theory, Martino Publishing, Mansfield Centre, CT, 2013

(HeM) Heidegger M., The Age of the World Picture, in W. Lovitt (trans. and ed.), The Question Concerning Technology and Other Essays, New York, Harper & Row, 1977

(HeW) Heisenberg W., Introduction to the Unified Field Theory of Elementary Particles, John Wiley & Sons, London, New York, Sydney, 1966

(HeW1) Heisenberg W., Physikalische Prinzipien der Quantentheorie, Wissenschaftsverlag, Mannheim, Wien, Zürich, 1930

(HoE) Hopf E., Ergodentheorie, Springer-Verlag, Berlin, Heidelberg, New York, 1970

(HoM) Holschneider M., Wavelets, An Analysis Tool, Clarendon Press, Oxford, 1995

(HuM) Hu M., Zhang Q., Yang J., Li X., Statistics of spatial rotations in 3D electron cryo-microscopy by unit quaternion description, Journal of Structural Biology, Vol. 212, No. 3, 2020, 44 pages

(KaM) Kaku M., Introduction to Superstring Theory, Lecture Notes, Springer-Verlag, New York, 1984

(KaY) Kato Y., The coerciveness for integro-differential quadratic forms and Korn's inequality, Nagoya Math. J. 73, 7-28, 1979

(KiA) Kirsch A., Hettlich F., The mathematical theory of time-harmonic Maxwell´s equations, expansion-, integral-, and variational methods, Springer-Verlag, Heidelberg, New York, Dordrecht, London, 2015

(KiA1) Kiselev A. P., Relatively Undistorted Cylindrical Waves Depending on Three Spacial Variables, Mathematical Notes, Vol. 79, No. 4, 587-588, 2006

(NiI) Niven I., The Roots of a Quaternion, Amer. Math. Monthly 449 (6) (1942) 386-388

(KöA) Kösal I., A note on hyperbolic quaternions, Universal Journal of Mathematics and Applications, Vol. 1, No 3, 2018, pp. 155-159

(KrA) Krall A. M., Spectral analysis of the generalized Hermite polynomials, Transaction of the American Mathematical Society, Vol. 344, (1994) pp. 155-172

(KrA1) Krall A. M., On the generalized Hermite Polynomials $\{H_n^k\}_{n=0}^{\infty}$, $k < -1/2$, Indiana J. Math. Vol. 30, 1981, pp. 73-78

(KrR) Kraußhar, R. S., Malonek H. R., A characterization of conformal mappings in R^4 by a formal differentiability condition, Bulletin de la Société Royale des Sciences de Liège, Vol. 70, No. 1, 2001, pp. 35-49

(KuJ) Kuipers J. B., Quaternions & Rotation Sequences, A Primer with Applications to Orbits, Aerospace and Virtual Reality, Princeton University Press, Princeton, New Jersey 1999

(LeE) Leedskalnin E., Magnetischer Strom, Hesper Verlag, 2018, Magnetic Current, Wilder Publications Inc., Blacksburg, 2011

(LeN) Lebedev N. N., Special Functions and their Applications, translated by R. A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, New York, 1965

(LeN1) Lerner N., A note on the Oseen kernels, in Advances in phase space analysis of partial differential equations, Siena, pp. 161-170, 2007

(LeS) Leo De Stefano, Quaternionic Lorentz group and Dirac equation, Foundations of Physics Letters, Vol. 14, No 1, 2001, 37-50

(LeS1) Leo De S., Quaternions and special relativity, American Institute of Physics, 1996

(LiI) Lifanov I. K., Poltavskii L. N., Vainikko G. M., Hypersingular Integral Equations and their Applications, Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D. C., 2004

(LiP) Lions P. L., On Boltzmann and Landau equations, Phil. Trans. R. Soc. Lond. A, 346, 191-204, 1994

(LiP1) Lions P. L., Compactness in Boltzmann's equation via Fourier integral operators and applications. III, J. Math. Kyoto Univ., 34-3, 539-584, 1994

(LoA) Louis A. K., Maass P., Rieder A., Wavelets, Theorie und Anwendungen, B. G. Teubner Stuttgart, 1998

(MaA)Mahajan A. S., Rangwala A. A., Electricity and Magnetism, McGraw Hill Education, Chennai, (India) Private Limited, 1988

(MaA1) Majda A. J., Bertozzi A. L., Vorticity and Incompressible Flow, Cambridge University Press, Cambridge, 2002

(MaA1) Mahajan A. S., Rangwala A. A., Electricity and Magnetism, McGraw-Hill, 1989

(MaF) MacFarlane, A., Hyperbolic Quaternions, Proc. Roy. Soc. Edinburgh, 1900, pp. 169-181

(MaW) Macke W., Quanten, Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1959

(MaW1) Macke W., Quanten und Relativität, Akademische Verlagsgesellschaft, Geest & Portig K.-G., Leipzig, 1965

(MeY) Meyer Y., Coifman R., Wavelets, Calderón-Zygmund and multilinear operators, Cambridge studies in advanced mathematics 48, Cambridge University Press, 1996

(MiK) Miyamoto K., Fundamentals of Plasma Physics and Controlled Fusion, NIFS-PROC-48, Oct. 2000

(MoC) Mouhot C., Villani C., On Landau damping, Acta Mathematica, Vol. 207, No. 1, p. 29-201, 2011

(NiJ) Nitsche J. A., Direct Proofs of Some Unusual Shift-Theorems, Anal. Math. Appl., Gauthier-Villars, Montrouge, 1988, pp. 383-400

(NiJ) Nitsche J. A., lecture notes, Approximation Theory in Hilbert Scales

(NiJ1) Nitsche J. A., lecture notes, Extensions and Generalizations

(NiJ2) Nitsche J. A., The regularity of piecewise defined functions with respect to scales of Sobolev spaces

(OrE) Orth E. W., Zeit und Zeitlichkeit bei Husserl und Heidegger, Verlag Karl Alber, Freiburg, München, 1983

(PeB) Petersen B. E., Introduction to the Fourier Transform & Pseudo-Differential Operators, Pitman Publishing Limited, Boston, London, Melbourne

(PeM) Perel M., Gorodnitskiy E., Representations of solutions of the wave equation based on relativistic wavelet, arXiv:1205.3461v1, 2012

(PeR) Penrose R., Cycles of Time – An Extraordinary New View of the Universe, The Bodley Head, 2010

(PeR1) Penrose R., The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics, Oxford University Press, Oxford, 1989

(PhR) Phillips R. S., Dissipative Operators and Hyperbolic Systems of Partial Differential Equations, Trans. Amer. Math. Soc., Vol. 90, No. 2, 1959, pp. 193-254

(PlJ) Plemelj J., Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911

(RiW) Ritz W., Einstein A., On the present status of the radiation problem, Physikalische Zeitschrift 10, S. 323-324, 1909

(RoH) Rollnick H., Quantentheorie 2, Springer, Berlin, Heidelberg, New York, 2003

(SaG) Santos G. M., A debate on magnetic current, the troubled Einstein-Ehrenhaft correspondence, The British Journal for the History of Science , Volume 44 , Issue 3 , September 2011 , pp. 371 - 400

(SaM) Salih M., Quaternionic quantum wave equation and applications, International Journals of Sciences and High Technologies, Vol. 13 No. 2, 2019, pp. 233-232

(ScD) Sciama D. W., On the origin of inertia, Monthly Notice of the Royal Astronomical Society, Vol. 113, No. 1, 1953, pp. 34-42

(ScE) Schrödinger E., Statistical Thermodynamics, Dover Publications, Inc., New York, 1989

(ScE1) Schrödinger E., What is Life? The Physical Aspects of the Living Cell *with* Mind and Matter, Cambridge University Press, Cambridge, 1967

(SeW) Sears W. R., Resler E. L., Theory of thin airfoils in fluids of high electrical conductivity, Journal of Fluid Mechanics, Vol. 5, No. 2 (1959) 257-273

(ShF) Shu F. H., Gas Dynamics, Vol II, University Science Books, Sausalito, California, 1992

(SoH) Sohr H., The Navier-Stokes Equations. An Elementary Functional Analytical Approach. Birkhäuser Verlag, Basel, Boston, Berlin, 2001

(SmL) Smolin L., Einstein's Unfinished Revolution, The search for what lies beyond the quantum, Penguin Random House, UK, 2019

(StE) Stein E. M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970

(StR) Streater R. F., Wightman A. S., PCT, Spin & Statistics, and all that, W. A. Benjamin, Inc., New York, Amsterdam, 1964

(SuL) Susskind L., Friedman A., Special relativity and classical field theory, Basic Books, New York, 2017

(SyJ) Synge J. L., Quaternions, Lorentz transformations, and the Conway-Dirac-Eddington matrices, Communications on the Dublin Institute for Advanced Studies, 1972

(TsV) Tsytovich V., Theory of Turbulent Plasma, Consultants Bureau, New York, 1977

(UnA1) Unzicker A., Vom Urknall zum Durchknall, Springer Spektrum, Berlin, Heidelberg, 2010

(UnA2) Unzicker A., The Mathematical Reality, Why Space and Time are an Illusion, 2020

(UnA3) Unzicker A., Bankrupting Physics, Alexander Unzicker and Sheilla Jones, 2013

(UnA4) Unzicker A., Einstein's Lost Key: How We Overlooked the Best Idea of the 20th Century, 2015

(VaM) Vainberg M. M., Variational Methods for the Study of Nonlinear Operators, Holden-Day Inc., San Francisco, London, Amsterdam, 1964

(VeW) Velte W., Direkte Methoden der Variationsrechnung, Teubner Studienbücher, Stuttgart, 1976

(ViC) Villani C., Landau damping, 2010

(WeA) Weinberg A. M., Wigner E. P., The physical theory of neutron chain reactors, The University of Chicago Press, Chicago, 1958

(WeH) Weyl H., Space, Time, Matter, Cosimo Classics, New York, 2010

(WeH1) Weyl H., Philosophy of Mathematics and Natural Science, Princeton University Press, Princeton and Oxford, 2009

(WeP) Werner P., Self-Adjoint Extensions of the Laplace Operator with Respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. Appl., Vol 70, 1979, pp. 131-160

(WeP1) Werner P., Spectral Properties of the Laplace Operator with respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. Appl., Vol 92, 1983, pp. 1-65

(WiD) Willett D., Wong J. S. W., On the Discrete Analogues of Some Generalizations of Gronwall's Inequality, Monatshefte für Mathematik, Tome, Vol. 69, 1965, pp. 362-367

(WiE) Wigner E., The unreasonable effectiveness of mathematics in the natural sciences, Comm. Pure and Appl. Math., Vol. 13, No. 1 (1960), p. 1-14

(XiK) Xiao K., The Fine Structure Constant and Interpretations of Quantum Mechanics, EJTP,Vol. 9, No. 26 (2012) pp. 135-146

(XuY) Xu Y., Yan D., The Bredrosian identity for the Hilbert transform of product functions, Proc. Amer. Math. Soc., Vol. 134, No. 9, 2006, pp. 2719-2728

(YoW) Yourgrau W., Mandelstam S., Variational Principles in Dynamics and Quantum Theory, Sir Isaac Pitman & Sons Ltd., London, 1955

(ZaD) Zahave D., Husserl's Phenomenology, Stanford University Press, 2002

(ZaC) Zalinescu C., Convex Analysis in General Vector Spaces, World Scientific, New Jersey, London, Singapore, Hong Kong, 1952