A Bessel function based proof that the Euler-Mascheroni constant γ is irrational

Klaus Braun

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The Euler-Mascheroni constant is defined by $\gamma = \lim_{n\to\infty}(H_n - logn)$, $H_n := \sum_{k=1}^{n}\frac{1}{k}$. In his notebook Ramanujan states the following asymptotics

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{nk}(\frac{x^k}{k!})^n = lnx + \gamma + o(1).$$

The case $n=1$ was proven by Euler and the statement is false for $n>2$, (BeB) p.98. The case $n=2$ has been proven by R. Brent, (BrR), applying the Bessel function of first kind $J_0(2x) = \sum_0^{\infty}(-1)^k(\frac{x^k}{k!})^2$ in combination with the equation $\int_0^{\infty}[e^{-t} - J_0(2t)]\frac{dt}{t} = 0$. This equation is derived from the related Mellin transforms formulae, (WaG) 13-24, (WeH),

$$\int_0^{\infty} t^{\mu-1}J_\nu(2t)dt = \frac{\frac{\mu}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1-\frac{\mu}{2}\right)} \quad \text{for } 0 < Re(\mu) < 3/2,$$

in combination with the asymptotics

$$\Gamma(s) - \frac{\frac{1}{2}\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{1}{s}\left[\Gamma(1+s) - \frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\right] = \frac{1}{s}\left[(1-\gamma s) - \frac{\left(1-\frac{\gamma s}{2}\right)}{\left(1+\frac{\gamma s}{2}\right)} + O(s^2)\right] = O(s) \to_{s\to 0^+} 0 .$$

From the corresponding asymptotics

$$(*) \quad \Gamma(s) - \frac{\Gamma(s)}{\Gamma(1-s)} = \frac{1}{s}\left[\Gamma(1+s) - \frac{\Gamma(1+s)}{\Gamma(1-s)}\right] = \frac{1}{s}\left[(1-\gamma s) - \frac{(1-\gamma s)}{(1+\gamma s)} + O(s^2)\right] = \gamma + O(s) \to_{s\to 0^+} \gamma$$

it follows the representation

$$\gamma = \int_0^{\infty}\left[e^{-t} - J_0(2\sqrt{t})\right]\frac{dt}{t}.$$

We shall consider the generalized Bessel functions

$$\frac{J_\nu(2x)}{x^\nu} := \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{x^{2k}}{\Gamma(\nu+k+1)} = \frac{K_\nu(2x)}{\Gamma(\lambda+1)} \ , \ K_\nu(2x) := \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{x^{2k}}{(\nu+1)(\nu+2)\dots(\nu+k)} .$$

Remark 1, (SiC):

for $\nu \in C$, and α algebraic with $\alpha \neq \pm\frac{1}{2}, -1, \pm\frac{3}{2}, -2 \dots$, $K_\nu(\alpha)$, and $J_\nu(\alpha)$ are transcendental.

For the auxiliary functions

$$B_\nu(x) := \frac{J_\nu(2x)}{x^\nu} - \frac{J_\nu(2\sqrt{x})}{x^\nu/2} = \sum_{k=0}^{\infty}(-1)^{k+1}\frac{x^k}{k!}\frac{(1-x^k)}{\Gamma(\nu+k+1)}$$

we note the following properties (appendix)

Remark 2:

i) $$\int_0^{\infty} x^s B_\nu(x)\frac{dx}{x} = \frac{1}{s}\left[\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(1+\nu-\frac{s}{2}\right)} - \frac{\Gamma(1+s)}{\Gamma(1+\nu-s)}\right]$$

ii) for $0 \le \alpha < \beta$, $0 \le x < 1$: $B_\beta(x) - B_\alpha(x) > 0$ and $B_\alpha(x) = B_{1/\alpha}(x)$ iff $\alpha = 1$.

The proof of the irrationality of γ is based on an *interval nesting* argument. The considered intervals $[a_n, b_n]$ are defined by the following sequences

$$a_n := \int_0^\infty B_n(x)\,\frac{dx}{x} = \int_0^1 \left[B_n(x) + B_n\left(\frac{1}{x}\right)\right]\frac{dx}{x}\,,$$

$$b_n := \int_0^\infty B_{1/n}(x)\,\frac{dx}{x} = \int_0^1 \left[B_{1/n}(x) + B_{1/n}\left(\frac{1}{x}\right)\right]\frac{dx}{x}.$$

From the above properties it follows

i) $a_n < b_n$ for $n \geq 2$

ii) a_n is strictly monotonously increasing and bounded, and therefore convergent

iii) b_n is strictly monotonously decreasing and bounded, and therefore convergent

iv) $\lim\limits_{n\to\infty} b_n = a_0 = \int_0^\infty B_0(x)\frac{dx}{x} = \int_0^\infty \left[J_0(2x) - J_0(2\sqrt{x})\right]\frac{dx}{x} = \gamma.$

Therefore, $\lim\limits_{n\to\infty} a_n = \gamma$, and the common limit γ of the nested intervals $[a_n, b_n]$ is irrational.

Appendix

(BrR), (WaG) 13-24, (WeH): For $0 < Re(\mu) < Re(\nu) + \frac{3}{2}$ it holds

$$\int_0^\infty t^\mu \frac{J_\nu(t)}{t^\nu}\frac{dt}{t} = \frac{\Gamma\left(\frac{\mu}{2}\right)}{2^{\nu-\mu+1}\Gamma\left(1+\nu-\frac{\mu}{2}\right)}$$

and therefore

i) $$\int_0^\infty t^\mu \frac{J_\nu(2t)}{t^\nu}\frac{dt}{t} = \frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(\nu+1-\frac{\mu}{2}\right)} = \frac{1}{\mu}\frac{\Gamma\left(1+\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)} = \frac{1}{\mu}\frac{\Gamma(1+\mu)}{\Gamma(1+\nu)}\ {}_2F_1\left(\frac{\mu}{2}, \frac{1-\mu+2\nu}{2}; \nu+1; 1\right)$$

ii) $$\frac{1}{2}\int_0^\infty t^{\mu/2}\frac{J_\nu(2\sqrt{t})}{t^{\nu/2}}\frac{dt}{t} = \frac{\frac{1}{2}\Gamma\left(\frac{\mu}{2}\right)}{\Gamma\left(1+\nu-\frac{\mu}{2}\right)}\ ,\ \frac{1}{2}\int_0^\infty t^{\mu}\frac{J_\nu(2\sqrt{t})}{t^{\nu/2}}\frac{dt}{t} = \frac{\frac{1}{2}\Gamma(\mu)}{\Gamma(1+\nu-\mu)}\,.$$

From this it follows

$$\int_0^\infty x^s B_\nu(x)\frac{dx}{x} = \frac{1}{s}\left[\frac{\Gamma\left(1+\frac{s}{2}\right)}{\Gamma\left(1+\nu-\frac{s}{2}\right)} - \frac{\Gamma(1+s)}{\Gamma(1+\nu-s)}\right],$$

which is Remark 2 i).

The estimate of Remark 2 ii) for $0 \leq \alpha < \beta$, $0 \leq x < 1$ follows from the equation

$$B_\beta(x) - B_\alpha(x) = \sum_{k=0}^\infty (-1)^{k+1}\frac{x^k}{k!}\frac{(1-x^k)}{\Gamma(\beta+k+1)} - \sum_{k=0}^\infty (-1)^{k+1}\frac{x^k}{k!}\frac{(1-x^k)}{\Gamma(\alpha+k+1)}$$

$$= \sum_{k=0}^\infty (-1)^k \frac{x^k(1-x^k)}{k!}\left[\frac{1}{\Gamma(\alpha+k+1)} - \frac{1}{\Gamma(\beta+k+1)}\right] > 0\,.$$

References

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