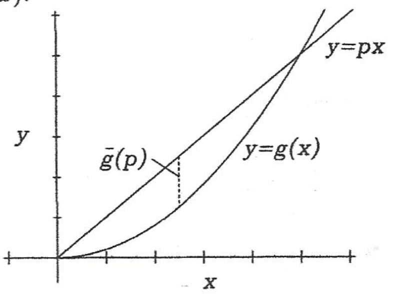
Box 1.2: Legendre transforms

Let $g(x)$ be a **convex function**, that is, a function such that $g''(x) > 0$. Then the **Legendre transform** $\overline{g}(p)$ of g is defined by

$$\overline{g}(p) \equiv xp - g(x) \quad \text{where } x(p) \text{ is implicitly defined as the root for given } p \text{ of} \quad p = \frac{\partial g}{\partial x}. \qquad (1)$$

The convexity of g guarantees that the equation defining $x(p)$ can be solved for any p that lies between the maximum and minimum gradients of g. Thus $\overline{g}(p)$ is well defined. It is straightforward to show that Legendre transforms are invertible. In fact a Legendre transform is its own inverse: $\overline{\overline{g}}(x) = g(x)$ (see Problem 1.6).

It is often helpful to consider the function $\mathcal{G}(x,p) \equiv xp - g(x)$ of two independent variables (x,p). Graphically, $\mathcal{G}(x,p)$ is the vertical displacement at ordinate x between the straight line $y = px$ and the upward curving graph of $g(x)$:



The Legendre transform $\overline{g}(p)$ is the value of $\mathcal{G}$ at the point $x(p)$ at which the curve runs parallel to the line. Since

$$\frac{\partial \mathcal{G}}{\partial x} = p - \frac{\partial g}{\partial x}, \qquad (2)$$

$x(p)$ is the value of x which extremizes $\mathcal{G}$ for given p, as is already evident from the figure.