‘Optimal’ FEM approximation for non-linear parabolic problems with non-regular initial value data

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On the occasion of the 20th anniversary year of death of
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Abstract

We provide an ‘optimal’ finite element approximation error estimates for a one-dimensional non-linear parabolic model problem with non-regular initial value data based on the solution concept in [BrK1]. By this we improve a ‘not-optimal’ finite element approximation error estimates for the one-dimensional Stefan problem with non-regular initial value data [NiJ].

For the space dimension $n=3$ ‘not-optimal’ finite element approximation error estimates for the non-stationary, non-linear Navier-Stokes equations have been derived in [HeJ]. As our approach is not depending from the space dimension it can be also applied also to this area.

§ 1 Introduction, the non-linear parabolic Stefan problem

The free boundary Stefan problem with its solution $U(y,\tau)$ can be transformed into the non-linear parabolic equation looking for a solution $u(x,t)=U(y,\tau)$ fulfilling

$$\dot{u}(y,\tau)-u''(x,t)+xu'(1,t)u'=0 \qquad \text{in } Q=\{(x,t)|x\in(0,1),0<t\le T\}$$

$$u'(0,t)=u(1,t)=0 \qquad \text{for } t>0$$
$$u(x,0)=f(x) \quad \text{for } x\in(0,1) \ .$$

Let

$$\dot{H}_1:=\{(w|w\in H_1(0,1),>0,w(0)=0\}=\{(w|w'\in L_2(0,1),>0,w(0)=0\}$$

Then $v:=u_x$ belongs to $\dot{H}_1$ and for any $v\in\dot{H}_1$ the function defined by

$$u(x,t)=-\int_x^1 v(z,t)dz$$

satisfies the boundary condition above. Multiplying the differential equation above with w_x ($w\in\dot{H}_1$) and integration gives the variation equation

$$\int_0^1 u_{xx}w_x+u_{xt}wdx=u_x(1,t)\int_0^1 xu_xw_xdx \ .$$

This leads to a variation representation in the form

Problem P_v: find v such that $v(\cdot,t)\in\dot{H}_1$ and

$$(\dot{v},w)+(v',w')=v(1)(xv,w') \qquad \text{for } w\in\dot{H}_1 \text{ and } t>0$$
$$(v(\cdot,0),w)=(f',w) \qquad \text{for } w\in\dot{H}_1 \text{ and } t=0.$$

This is the weak formulation of

$$v_t(y,\tau)-v_{xx}(x,t)+v(1,t)(xv)_x=0 \text{ in } Q$$
$$v(0,t)=0$$
$$v'(1,t)=v^2(1,t)$$
$$v=f' \quad \text{for } t=0 \ .$$

Obviously the compatibility condition $v'(1)=v^2(1)$ is required in order to ensure the (too high) regularity requirements to the auxiliary function v as indicated by the setting $v:=u_x\in\dot{H}_1$ for $u\in\dot{H}_1$.

For higher regularity assumptions further compatibility assumptions are required in the form $f''(1)=f'^2(1)$. In [NiJ1] for sufficiently high regularity assumptions for the solution v 'optimal' FEM error estimates are derived. The prize to be paid for this imbalanced regularity relationship between the solutions u and v are at least quadratic splines instead of expected linear splines for the FE approximation space.

§ 2 The 'not-regular' Stefan problem case, 'not-optimal' FEM error estimates

In case of reduced regularity assumption of the initial value function $g := f' \in L_2$ and not fulfilled compatibility conditions the problem is called "non-regular". If only $g \in L_2$ is assumed then even $\|v'\|$ and hence $|v|$ is not necessarily bounded for $t \to 0$ ([NiJ2]). In [NiJ] for this case 'not-optimal' FE error estimate of order h^α with $0 < \alpha < 1$ has been derived. The proof of those estimates needs to govern the singularity for $t \to 0$ of the energy norm $\|v'\|$ and hence for the term $v(1)$.

The following a priori estimates are derived and applied

$$\|z\|^2 + \int_0^t \|z'\|^2 d\tau \le 2\|g\|^2 \quad , \quad t\|z'\|^2 + \int_0^t \tau \|\dot z\|^2 d\tau \le c = c(\|g\|)$$

$$\sup_{0\le t\le T}\left\{ t^{2k}\left\|\partial_t^k v_h\right\|^2 + \int_0^t \tau^{2k}\left\|\partial_t^k v_h'\right\|^2 d\tau \right\} \le c_{2k}^2 \quad , \quad \sup_{0\le t\le T}\left\{ t^{2k+1}\left\|\partial_t^k v_h'\right\|^2 + \int_0^t \tau^{2k+1}\left\|\partial_t^{k+1} v_h\right\|^2 d\tau \right\} \le c_{2k+1}^2$$

building on the Young inequality and the Gronwall lemma to tackle inequalities of the following types

$$\lambda := \lambda(t) \le \|g\|^2 + c\int_0^t \lambda^3(\tau)d\tau \quad , \quad \lambda := \lambda(t) \le k_1 + k_2\int_0^t \tau^{-1/}\lambda^{3/2}(\tau)d\tau$$

with $\lambda(t) := \|z\|^2$. From those estimates the specific a priori estimate for the supremum norm of v and v_h can be derived,, i.e.

$$|v|, |v_h| \le ct^{-1/4} \ .$$

The key estimate is building on the duality argument according to $-w'' = \Phi$, $w(0) = w'(0) = 0$ and the related Ritz approximation $\varphi := R_h w$ enabling the following inequality

$$\frac{d}{dt}(t\|\Phi\|^2) + t\|\Phi'\|^2 + \frac{1}{2}\frac{d}{dt}\|\varphi'\|^2 \le |a(\varepsilon,\varphi)| + |v(1)||(x\Phi,\varphi')| + |\Phi'(1)||(xv_h,\varphi')| + ct\left\{\|\varepsilon'\|^2 + t^{-1/2}\|\varepsilon\|^2\right\} \ .$$

Corresponding a priori estimates and similar proof techniques are applied to prove similar "not-optimal' FEM error estimates for the non-stationary, non-linear NSE ([HeJ]).

The inadequate high regularity requirements for the auxiliary function $v \in H_1$ is the root cause of the singularity behavior in the form $\approx t^{-1/4}$. Assuming that this singularity behavior still remains when transforming the Stefan problem into a modified variation representation with respect to a $H_{-1/2}$ –framework alternative to current $H_0 = L_2$ –framework the corresponding duality argument would be applied by the equation $-w'' = t^{-1/4}\Phi$, $w(0) = w'(0) = 0$. In combination with the related Ritz approximation $\varphi := R_h w$ this then leads to the following modified inequalities ($e := v - v_h =: \varepsilon - \Phi$)

$$\frac{1}{2}\frac{d}{dt}(t^{1/2}\|\Phi\|^2) + t^{1/2}\|\Phi'\|^2 \le ct^{1/2}\left[\|\varepsilon'\|^2 + t^{-1/2}\|\varepsilon\|^2\right]$$

$$\frac{1}{2}\frac{d}{dt}(t^{1/2}\|\Phi\|_{-1/2}^2) + t^{1/2}\|\Phi'\|_{-1/2}^2 \le ct^{1/2}\left[\|\varepsilon'\|_{-1/2}^2 + t^{-1/2}\|\varepsilon\|_{-1/2}^2\right] \ .$$

From this the 'optimal' error estimation are derived, i.e.

$$\|u - u_h\|_0 \approx \|e\|_{-1/2} \le ch^{1/2}t^{-1/4}\|e\|_0 \le cht^{-1/2} \ .$$

Building on the solution concept as proposed in [BrK1] for the 3D-NSE problem we propose an alternative auxiliary function $v := Au' = Hu$ resp. $u = -Hv$, whereby A and H denotes the one-dimensional Symm resp. Hilbert transform singular integral operator ([BrK], [NiJ6]). The properties of the Hilbert transform operator H ensure same regularity assumption for $u, v \in H_{1/2}$ of the corresponding weak variation representation. From

$$\left|z\right|^2 \leq c(z.z')_{L_2} \leq c\left\|z\right\|_{1/2} \cdot \left\|z'\right\|_{-1/2} \leq c\left\|z\right\|_{1/2}^2$$

it follows that the term $v(1)$ is bounded if $v \in H_{1/2}$. Based on this the technique from the regular case can be applied also for the ‘non-regular’ case [NiJ1]. At the same time the required finite element approximation spaces can be linear splines instead of quadratic splines ([NiJ7]).

§ 3 A non-linear parabolic model problem, 'optimal' FEM/BEM error estimates

In order to enable the full power of approximation theory in a Hilbert scale framework we consider the $2\pi-$periodic continuation of the solution of the Stefan problem. It enables the definition of the problem adequate Hilbert spaces defined per appropriate self-adjoint linear operators with corresponding domains:

Let $H=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u\in H:=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$ and for real β the Fourier coefficients

$$\hat{u}(\nu):=u_\nu:=\frac{1}{\sqrt{2\pi}}\oint u(x)e^{i\nu x}dx=:\oint u(x)\psi_n(x)dx$$

enable the definitions of the norms (see e.g. [BrK], [LiI] Remark 11.1.5)

$$\|u\|_\beta^2:=\sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2$$

defining corresponding Hilbert spaces $H_\beta^\#$. Then H is the space of L_2-periodic function in R. The definition of negative scaled Hilbert scales is enabled by appropriate self-adjoint singular integral operators ([KrR], [LiI]). We build our analysis on the Symm operator A [BrK] and the Hilbert transform (conformal mapping) operator H defined by

$$Au(x)=-\frac{1}{\pi}\oint\log 2\left|\sin\frac{x-y}{2}\right|u(y)dy$$

$$Hu(x)=\frac{1}{2\pi}\oint\cot(\frac{x-y}{2})u(y)dy\ .$$

Both are related in the form $Au'(x)=-Hu(x)$. Some essential properties of the Hilbert transform operator are summaries in

Lemma: The Hilbert transform (conformal mapping) operator H fulfills the following properties:

i) $(Hv,w)=-(v,Hw)$, $\|Hv\|=\|v\|$,

ii) if $v\in H$ then $Hv\in H_0$ and $(Hv,v)=0$

iii) $[xH-Hx]v=\hat{v}(0)$, i.e. for odd functions v it holds $[xH-Hx]v=0$.

Sobolev, $L_\infty-$ and Hölder/Lipschitz space based Galerkin approximation analysis are given in the appendix. The approximation error for finite element approximation spaces $S_h^{k.t}$ of the Symm operator equation is 'optimal' with respect to the complete possible Hilbert scale range, i.e. it holds

$$\|e\|_\kappa\le ch^{\tau-\kappa}\|u\|_\tau\quad,\ -(t+1)\le\kappa\le -1/2\quad,\ \tau\le t\ .$$

The corresponding operators of the one-dimensional Hilbert transform operator H are the Riesz operators R_i ($i=1,...n$) which also play a key role in NSE theory and an alternative definition of the pressure p.

In order to avoid technical difficulties we omit the term x from the Stefan problem. This is without loss of generality as it is always estimates with a constant. With respect to non-parabolic problems for space dimensions $n>1$ this term potential can even support to gain additional regularity (and therefore corresponding convergence orders) in the context of commutator properties which can be interpreted as compact disturbance (appendix and e.g. [BrK]) and finite element super-approximation properties ([NiJ4], [NiJ5], [NiJ6]).

We consider the non-linear parabolic equation

$$\dot{u}-u''=-u'(1)u'.$$

Applying the auxiliary function concept of [NiJ1] in the form $v=Au'=-Hu$ resp. $u=Hv$ leads to

$$A\dot{u}-Au''=-Au'(1)Au' \quad \text{resp.} \quad AH\dot{v}-v'=-v(1)v\ .$$

Multiplying with $Hw\in L_2^{\#}$ leads to the following variation problem:

P_v : find $v\in\dot{H}_{1/2}$ with

$$(\dot{v},w)_{-1/2}+(v',w')_{-1/2}=v(1)(Hv,w)_0 \quad \text{for } w\in\dot{H}_{1/2} \text{ and } t>0$$

$$(v(\cdot,0),w)_{-1/2}=(f',w)_{-1/2}=(f,w)_0 \quad \text{for } w\in\dot{H}_{1/2} \text{ and } t=0\,.$$

The corresponding Galerkin approximation is given by

P_{v_h} : find $v_h\in S_h\subset\dot{H}_{1/2}$ ($v_h(1):=v_h(1,t)$) with

$$(\dot{v}_h,\chi)_{-1/2}+(v_h,\chi)_{1/2}=v_h(1)(Hv_h,\chi)_0 \quad \text{for } \chi\in S_h\subset\dot{H}_{1/2} \text{ and } t>0$$

$$(v_h(\cdot,0),\chi)_{-1/2}=(f,\chi)_0 \quad \text{for } \chi\in S_h\subset\dot{H}_{1/2} \text{ and } t=0\,.$$

For later usage we define the related trilinear form

$$b(\xi,\eta,\zeta):=\frac{1}{2}\xi(1)(\eta,\zeta')_{-1/2}+\frac{1}{2}\eta(1)(\xi,\zeta')_{-1/2}=\frac{1}{2}\xi(1)(\eta,\zeta)_0+\frac{1}{2}\eta(1)(\xi,\zeta)_0\,.$$

Analog to [NiJ1] the corresponding bilinear form for fixed $v\in\dot{H}_{1/2}$ defined by

$$a_v(\xi,\eta):=(\xi,\eta)_{1/2}-2b(v,\xi,\eta)$$

is bounded and coercive in $\dot{H}_{1/2}$, i.e. it holds

i) $\quad |a_v(\xi,\eta)|\le M\cdot\|\xi\|_{1/2}\cdot\|\eta\|_{1/2}$

ii) $\quad a_v(\xi,\xi)\ge m\|\xi\|_{1/2}^2-\Lambda\|\xi\|_{-1/2}^2\,.$

The above coerciveness property ii) is a Garding type inequality which enables the so-called “weak inf-sup condition” (appendix). This results into a positive bilinear form in case the domain is restricted to the approximation spaces $S_h \subset \dot{H}_{1/2}$. Therefore the Galerkin approximation $R_h v$

$$a_v(v - R_h v, \chi) = 0 \quad \text{for } \chi \in S_h \subset \dot{H}_{1/2}$$

is ‘optimal‘, i.e. it holds

$$\|v - R_h v\|_\kappa \le c h^{\tau-\kappa} \|v\|_\tau \quad , \; -(t+1) \le \kappa \le -1/2 \quad , \; \tau \le t \; .$$

For our analysis we will apply the analog Galerkin approximation $\tilde{v}_h := R_h v$ for the linear parabolic case, i.e.

$$(\dot{v} - \dot{\tilde{v}}_h, \chi)_{-1/2} + a_v(v - \tilde{v}_h, \chi) = 0 \quad \text{for } \chi \in S_h \subset \dot{H}_{1/2}$$

which is ‚optimal‘ with respect to Hilbert space and L_∞ – norms ([JNi5]), i.e. for $\varepsilon := v - \tilde{v}_h$ it holds e.g.

$$\|\varepsilon\|_{L_2(H_\kappa)} \le c h^{\tau-\kappa} \|v\|_{L_2(H_\tau)} \qquad , \; -(t+1) \le \kappa \le -1/2 \quad , \; \tau \le t \; .$$

$$\|\varepsilon\|_{L_\infty((0,t),L_\infty(0,2\pi))} \le c \cdot \inf_{\chi \in S_h} \|v - \chi\|_{L_\infty((0,t),L_\infty(0,2\pi))} \quad .$$

This approximation behavior will be used to apply the Schauder fix point theorem in order to govern the critical quadratic non-linear term $b(e,e,\chi) = e(1)(e,\chi)$ below.

The defining error variation equation for $e := v - v_h$ is given by

$$(\dot{e}, \chi)_{-1/2} + (e, \chi)_{1/2} - 2b(v, e, \chi) = -b(e, e, \chi) \quad \text{for } \chi \in S_h \subset \dot{H}_{1/2}$$

resp.

$$(\dot{e}, \chi)_{-1/2} + a_v(e, \chi) = -b(e, e, \chi) \quad \text{for } \chi \in S_h \subset \dot{H}_{1/2}$$

whereby $b(e,e,\chi) = e(1)(e,\chi)_0$.

Putting $e := (v - \tilde{v}_h) - (\tilde{v}_h - v_h) =: \varepsilon - \Phi$ with now $\Phi \in S_h \subset \dot{H}_{1/2}$ this leads to

$$(\dot{\Phi}, \chi)_{-1/2} + a_v(\Phi, \chi) = e(1)(e, \chi)_0 \, .$$

In order to show the existence of a finite element solution in the neighborhood of v the quadratic term $e(1) = e(1,t)$ is replaced by $E(1)$ for some function $E = T(e)$. Then

$$(\dot{\Phi}, \chi)_{-1/2} + a_v(\Phi, \chi) + E(1)(\Phi, \chi)_0 = E(1)(\varepsilon, \chi)_0$$

is a linear problem. Therefore, for any $E(1) = E(1,t)$ there exists a solution Φ with $\Phi(0) = 0$. Therefore the same is true for $e = \varepsilon - \Phi$, but now e depends on E.

Following the same arguments as in ([NiJ1] applying the Schauder fix point theorem (appendix) then it follows that there is a E with $E = T(e) = e$. We summaries the above in

Theorem: The problem P_v has an unique bounded solution $v \in H_{1/2}$ and an unique (linear spline) finite element approximation v_h of problem P_{v_h} in the neighborhood of v with optimal order of convergence

$$\|v - v_h\|_{L_\infty(0,t);L_\infty)} \le O(h) .$$

Due to the properties of the Hilbert transform operator the same estimate is true for the original solution $u \in H_{1/2}$, i.e. it holds

$$\|u - u_h\|_{L_\infty(0,t);L_\infty)} \le O(h) .$$

An approximation $u_h \in S_h \subset H_{1/2}$ for the solution $u \in H_{1/2}$ is given by $u_h = Hv_h$ i.e. the spline order of u_h is the same as the spline order of v_h .

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Appendix

Related topics

The Ritz-Galerkin Approximation Theory

A well-defined Ritz-Galerkin approximation method in a Hilbert space H_α and corresponding approximation $u_h := Ru$ of the linear operator equation $Bu = f$ requires certain properties to the linear operator, as well as adequate properties of the operator domain $D(A)$ embedded into a Hilbert space H_α and appropriate related properties of the finite dimensional approximation spaces $S_h \subset H_\alpha$. In the one-dimensional case the parameter h corresponds to space dimension in the form $h \approx n^{-1}$.

In order to apply the generalized Lax-Milgram lemma the $H_\beta -$coerciveness can be weakened to the Garding type inequality which is given by one of the following forms

$$(Bu,v) \geq c_1\|u\|_\alpha^2 - c_2\|v\|_\beta^2 \quad \text{or} \quad (Bu,v) \geq c\|u\|_\alpha \cdot \|v\|_\alpha - (Ku,v)$$

whereby H_α is compactly embedded in H_β resp. K describes a corresponding compact operator. This enables the so-called “weak inf-sup condition” ([AzA], [BrK], [NiJ7], [NiJ8]) which we summaries in the following

Lemma: Given three Hilbert spaces H, H_1, H_2 with $H_1 \subset H$ compactly embedded and the bilinear form $b(u,v) := (Bu,v) : H_1 \times H_2 \to R$ with

i) $|b(u,v)| \leq c \cdot \|u\|_{H_1} \|v\|_{H_2}$ for all $u \in H_1, v \in H_2$
ii) For $u \in H_1$ with $b(u,v)$ for all $v \in H_2$ it follows $u = 0$
iii) For all $n \in N$ the approximation spaces fulfill $S_n \subset H_1, T_n \subset H_2$ and $\dim S_n = \dim T_n$
iv) $T_n \subset T_{n+1}$ and $\bigcup_{n \in N} T_n$ is dense in H_2
v) $\forall \varphi \in S_n \exists \psi \in T_n : |b(\varphi,\psi)| \geq (c_1\|\varphi\|_{H_1} - c_2\|\psi\|_H) - \|\psi\|_{H_2}$ (the weak inf-sup condition).

Then there is a $N > 0$ and a constant $c_3 > 0$ that for all $n \geq N$ it holds

$$\forall \varphi \in S_n \exists \psi \in T_n \text{ with } |b(\varphi,\psi)| \geq c_3\|\varphi\|_{H_1} \cdot \|\psi\|_{H_2}.$$

The required properties the approximation spaces are e.g. given by finite elements. The standard notation of finite element spaces is given by $S_h^{k.t} \subset H_k$ with $k < t$ which is about $(k-1)-$time continuously differentiable functions $\chi \in S_h^{k.t}$ with the property that the restriction of χ to any triangle Δ of the triangulation $\Delta \in \Gamma_h$ is a polynomial of degree less than t.

The corresponding approximation properties are usual described in the following way:

i) $S_h^{k.t} \subset H_k$

ii) $\inf_{\chi \in S_h} \|v-\chi\|_k \le ch^{l-k}\|v\|_l$ for $v \in H_l$

iii) $\|\chi\|_k \le ch^{-(k-l)}\|v\|_l$ for $\chi \in S_h$.

The Hölder resp. Lipschitz spaces are the adequate ones to derive approximation estimates for non-linear problems. A proof of the boundedness of the Ritz operator in Hölder resp. Lipschitz spaces is a consequence of the below AI-condition and lemma (by the choice $X_1 = C^k$ and $X_2 = C^{k.\lambda}$) in combination with the boundedness of the Ritz operator in in $L_\infty -$ norm ([JNi4]).

based on the boundedness of the Ritz operator in in the C^0 Banach space as the Hölder spaces $C^{k.\lambda}$ are compactly embedded into C^0. The boundedness of the Ritz operator as mapping of C^0 into itself is a consequence of the boundedness with respect to the $L_\infty -$ norm

The so-called approximation (A) and inverse (I)-condition for the collection of approximation spaces $\{S_h | 0 < h \le 1\}$ are given by

AI-condition: let $\{S_h | 0 < h \le 1\}$ a collection of subspaces of X_2 with **a**pproximation and **i**nverse-quantities σ_h and τ_h according to

i) $\forall y \in X_2 \exists \eta \in S_h \quad \|y-\eta\|_1 \le \sigma_h \|y\|_2$, $\|\eta\|_2 \le c_1 \|y\|_2$ with c_1 independently of h .

ii) $\forall \eta \in S_h$ a Bernstein type inequality holds $\|\chi\|_2 \le \tau_h \|\chi\|_1$.

The collection $\{S_h | 0 < h \le 1\}$ fulfills the AI condition if $K := \sup \sigma \ \tau_h < \infty$.

Lemma: let $\{P_h | X_1 \to S_h\}$ be a collection of linear projection operators of X_1 onto S_h which is uniformly bounded as mappings of X_1 into itself, i.e.

$$\|P_h\|_1 = \sup \frac{\|P_h y\|_1}{\|y\|_1} \le p_1 \text{ with } p_1 \text{ independently of } h.$$

If $\{S_h\}$ fulfills the AI-condition then $\{P_h\}$ as mapping of X_2 into itself is uniformly bounded with

$$\|P_h\|_2 = \sup \frac{\|P_h y\|_2}{\|y\|_2} \le p_2 := (c_1 + 3K) p_1 \ .$$

The boundedness of the Ritz operator in in $L_\infty -$ norm can be proven by the weighted norm technique of J. A. Nitsche ([JNi4], next section). In [BrK] interior 'optimal' error estimates of the Ritz method for Pseudo-differential operators are derived.

$L_\infty -$ **Boundedness of the Ritz Operator for Singular Integral Equation Problems**

The (Hilbert transform) singular integral equation problem is given by

$$Hu = f$$

with

$$Hu(s) = \frac{1}{2\pi} \oint \cot \frac{s-t}{2} u(st)dt = f(s) \cdot$$

In [JNi6] the $L_\infty -$ boundedness of the Ritz-Galerkin approximation operator R_h onto finite element subspaces S_h is defined by

$$(\chi, Hu) = (\chi, H(R_h u)) =: (\chi, H\varphi) \text{ for } \chi \in S_h .$$

As a consequence the boundedness holds true also in the norm of $C^{0,\lambda}$.

Analogue to the analysis in case of boundary value problems the proof deals with weighted norms in the form

$$\mu := \rho^2 + \sin^2(s - s_0) \text{ , with } s_0 \in [0,2\pi) \text{ appropriately chosen.}$$

For $\alpha \in R$ the weighted scalar products resp. norms are defined by

$$((v,w))_\alpha := \oint \mu^{-\alpha} v \cdot w ds \quad , \quad |||v|||_\alpha := ((v,v))_\alpha .$$

The estimates require for space dimension case $n=1$ a value e.g. $\alpha = 1 > n/2$, for the space dimension case $n=2$ a value e.g. $\alpha = 2 > n/2$. The connection of weighted norms and the $L_\infty -$ norm is given by the inequality (for $\alpha = 1$)

$$|||v|||_\alpha \le ch^{-1/2} |||v|||_\infty \text{ for } v \in C^0$$

and

$$\|\chi\|_{L_\infty} \le ch^{1/2} \sup\{|||\chi|||_\alpha | s_0 \in [0,2\pi)\} \text{ for } \chi \in \dot{S}_h .$$

An analogue proof shows the $L_\infty -$ boundedness of the Ritz-Galerkin approximation operator R_h of the singular integral equation problem $Au = f$ with

$$Au(x) = -\frac{1}{\pi}\oint \log 2\left|\sin\frac{x-y}{2}\right| u(y)dy \quad .$$

The BEM is given by

$$(\chi, Au) = (\chi, A(R_h u)) \text{ for } \chi \in S_h .$$

For the error $e := u - R_h u$ we use the split $e := u - R_h u = (u-\eta) - (R_h u - \eta) =: \varepsilon - \Phi$, $\eta, \Phi \in S_h$.
Then the error is defined by

$$(\chi, A\varepsilon) = (\chi, A\Phi) \text{ for } \chi \in S_h .$$

The central estimates of this proof follow same concept as [JNi6]. We recall those by the following inequalities.
The starting equation is based on weighted norm with respect to $\Phi \in S_h$

$$\|\Phi\|_\alpha^2 = (\Phi, \mu^{-\alpha}\Phi) = (\Phi, \mu^{-\alpha}\Phi - A\lambda) + (\Phi, \varepsilon)_\alpha - (\varepsilon, \mu^{-\alpha}\Phi - A\lambda).$$

The objective is to derive a final estimate in the form $\|\Phi\|_\alpha^2 \le c\|\varepsilon\|_\alpha^2$ which then gives

$$\|e\|_\alpha^2 \le c\|\varepsilon\|_\alpha^2$$

from which the boundedness follows (by appropriately chosen $\eta \in S_h$) according to the relationship of weighted and $L\infty -$norm of finite elements. With the definition

$$\chi := R_h(\mu^{-\alpha}\lambda) \quad : \quad (\mu^{-\alpha}\lambda, \psi) = (A, \psi) \text{ for } \psi \in \dot{S}_h .$$

it follows from the above (for $n = 1,2$)

$$\begin{aligned} &\le c_\delta \|\varepsilon\|_\alpha^2 + \delta\|\mu^{-\alpha}\Phi - A\lambda\|_{-\alpha}^2 \\ &\le c_\delta \|\varepsilon\|_\alpha^2 + \delta \cdot c \cdot \inf_{\psi \in S_h} \|\mu^\alpha A\lambda - \psi\|_\alpha^2 \\ &\le c_\delta \|\varepsilon\|_\alpha^2 + \delta \cdot c \cdot h^{2n} \|\nabla^n(\mu^\alpha A\lambda)\|_\alpha^2 . \end{aligned}$$

In the following we consider only estimate relevant estimates for the case $n = 1 = \alpha$. Then the second term gives

$$\nabla(\mu A\lambda) \le \rho^2 |\nabla(A\lambda)| + \left|\sin^2\frac{s-s_0}{2}\right| |A\lambda| + \mu|\nabla(A\lambda)|$$

and therefore (because of $\sin^2((s-s_0)/2)/\mu \le 1$)

$$\|\nabla(\mu^\alpha A\lambda)\|_\alpha^2 \le \rho^4 \|\nabla(A\lambda)\|_\alpha^2 + \|(A\lambda)\|^2 + \|\nabla(A\lambda)\|_{-\alpha}^2$$

$$\|\nabla(\mu^\alpha A\lambda)\|_\alpha^2 \le \rho^2\|\lambda\|^2 + \|(A\lambda)\|^2 + \left\|\sin\frac{s-s_0}{2}\nabla(A\lambda)\right\|^2 .$$

The following estimates enable to the proof:

$$\left\|\sin\frac{s-s_0}{2}\nabla A\chi\right\|_0^2 \le \left\|\nabla\sin\frac{s-s_0}{2}A\chi\right\|_0^2 + c\|A\chi\|_0^2$$

$$\left\|\nabla\sin\frac{s-s_0}{2}A\chi\right\|_0^2 \le \|\nabla(sA-As)\chi\|_0^2 + \|\nabla A(s\chi)\|_0^2$$

$$\|s\chi\|_0^2 \le \|s\chi-\psi\|_0^2 + \|\psi\|_0^2 \le ch^2\|\chi\|_0^2 + \sup_{\xi\in S_h}\frac{(\psi, A\xi)}{\|A\xi\|_0^2} \le c\|A\chi\|_0^2 + \sup_{\xi\in S_h}\frac{(s\chi, A\xi)-(s\chi-\psi, A\xi)}{\|A\xi\|_0^2}$$

$$(s\chi, A\xi) = (\chi, (sA-As)\xi) + (\chi, A(s\xi))$$

$$(\chi, (sA-As)\xi) \le \|\chi\|_{-1}\|(sA-As)\xi\|_1 \le \|A\chi\|\cdot\|\xi\|_{-1} \le \|A\chi\|\cdot\|A\xi\|$$

$$(s\chi-\psi, A\xi) \le ch\|\chi\|\cdot\|A\xi\| \le c\|\chi\|_{-1}\cdot\|A\xi\| \le c\|A\chi\|\cdot\|A\xi\|$$

$L_\infty -$ **Boundedness of the Ritz-Galerkin Operator for Linear and Non-linear Elliptic and Parabolic Problems**

In [JNi5] $L_\infty -$ boundedness of the finite element method Ritz-Galerkin operator for parabolic problems and related optimal FEM approximation estimates are derived. In In [NiJ3]) $L_\infty -$ error estimates are derived for a nonlinear boundary value problem. $L_\infty -$ boundedness in Hölder resp. Lipschitz norms of the Ritz-Galerkin operator for the Laplace equation are analyzed in [JNi4].
Hölder/Litschitz spaces are the adequate ones in treating nonlinear elliptic and parabolic problems. The boundedness of the Ritz operator in the corresponding norms at least simplifies the analysis of finite element procedures in some cases it is essential.

The two central elements of [JNi5] are

i) an 'optimal' "parabolic type" shift theorem for the solution of the heat equation in the form

$$|||w|||_{H_{k+2}}^2 \le c|||Aw|||_{H_k}^2 \qquad \text{with} \qquad |||w|||_{H_k}^2 := \int_0^T \|w\|_{H_k}^2 \, dt$$

ii) Nitsche's weighted norms

$$\|\nabla v\|_{\alpha.\Omega'}^2 := \sum_{|\xi|=k} \iint_{\Omega'} \mu^{-\alpha} \left| D^\xi v \right|^2 dx$$

with $\mu(x,t) := |x - x_0|^2 + |t - t_0|$, and x_0, t_0 chosen that $\|u\|_{L_\infty(L_\infty)} = u(x_0, t_0)$.

The proof of the shift theorem is based on appropriate estimates of the generalized Fourier coefficients $w_i\ (t)$ of the heat equation

$$\dot{u} - \Delta u = f \ , \ u(0) = u_0 \, , \ u|_{\partial\Omega} = 0$$

with

$$w_i(t) = e^{-\lambda_i t} u_0 + \int_0^t e^{-\lambda_i (t-\tau)} f_i(\tau) d\tau \ .$$

The "trick" to go there is about changing the order of integration in the following form:

$$\int_0^T w_i^2(t)dt \le \int_0^T \left[\int_0^t e^{-\lambda_i (t-\tau)} d\tau \right] \left[\int_0^t e^{-\lambda_i (t-\tau)} f_i^2(\tau) d\tau \right] dt \, .$$

$$\le \lambda_i^{-1} \int_0^T f_i^2(\tau) \left[\int_\tau^T e^{-\lambda_i (t-\tau)} dt \right] d\tau \le \lambda_i^{-2} \int_0^T f_i^2(\tau) d\tau \ .$$

Approximation Theory for Non-linear Problems

Lecture Notes

J. A. Nitsche

In [NiJ3]) $L_\infty -$ error estimates are derived for the nonlinear boundary value problem

$$div(a(\circ,u)\nabla u)=f$$

As for $u\in C^{1.1}$

$$Tu(x):=a(x,u)\Delta u+\frac{\partial a}{\partial u}\sum(\frac{\partial u}{\partial x_i})^2+\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_i}\in C^{2.1}$$

the Schauder fix point theorem can be applied ($M:=C^{1.1}$)to prove the existence of a solution u.

Schauder fix point theorem

> Let X be a Banach space and $M\subseteq X$ closed, bounded, convex and $T:M\to M$ compact. Then there exists a $\bar{x}\in M$ with $\bar{x}=T\bar{x}$.

In the following we give a FEM 'optimal' error estimate in $L_\infty -$ norm for non-linear problems.

Let the problem be given by

$$F(x,u)=0$$

with the (roughly) regularity assumptions:

i) there is a unique solution

ii) F,F_u are Lipschitz continuous.

The approximation problem is given by:

find $\varphi\in S_h$ $\quad (F(\cdot,\varphi),\chi)=0$ for $\chi\in S_h$.

Theorem: The FEM admits the error estimate

$$\|u-\varphi\|_{L_\infty}\le c\inf_{\chi\in S_h}\|u-\chi\|_{L_\infty} .$$

Error analysis

Put

$$f(x) = F_u(x, u(x)) \quad \text{and} \quad \varphi = u - e$$

then

$$(fe, \chi) = (R, \chi)$$

with a remainder term

$$R := R(e) := F(\cdot, u - e) + fe$$

resp.

$$(fe, \chi) = (fu - R(e), \chi)\,.$$

Let P_h denote the L_2 – projection related to $(f\cdot,\cdot) = (R, \chi)$, then

$$\varphi = P_h(u - \frac{1}{f}R(e))$$

resp.

$$e = (I - P_h)u + P_h \frac{1}{f}R(e)) =: T(e)\ \cdot$$

Therefore the difference $e = u - e$ is a fix point of T.

Let

$$B_{\kappa\bar{\varepsilon}} := \left\{ e \middle| \|e\|_{L_\infty} \le \kappa\bar{\varepsilon} \right\} \quad \text{and} \quad \bar{\varepsilon} := \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty}\,.$$

The application of the Schauder fix point theorem is enabled by the following properties of T:

Lemma:

i) There is a $\kappa > 0$ such that for $\bar{\varepsilon}$ sufficiently small, then T maps the ball $B_{\kappa\bar{\varepsilon}}$ into itself.

ii) for $\bar{\varepsilon}$ sufficiently small, T is a contradiction in $B_{\kappa\bar{\varepsilon}}$.

Proof: i) Because of P_h and f^{-1} are being bounded it holds

$$\|I - P_h\|_{L_\infty} \le c_1 \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty} = \bar{\varepsilon}$$

and

$$\left\| P_h(\frac{1}{f}R(e)) \right\|_{L_\infty} \le c_2 \|R(e)\|_{L_\infty}\ .$$

It is

$$\|F(\cdot, u - e) + fe\|_{L_\infty} \le c_3 \|e\|_{L_\infty}^2 = c_3 \kappa^2 \bar{\varepsilon}^2$$

with c_3 being the Lipschitz constant of F_u and therefore

$$\|T(e)\|_{L_\infty} \leq c_1\bar{\varepsilon} + c_3 c_2 \kappa^2 \bar{\varepsilon}^2 \quad .$$

Now fixing $\kappa > c_1$ and choosing $\bar{\varepsilon}_0$ according to $\kappa = c_1 + c_3 c_2 \kappa^2 \bar{\varepsilon}_0$ gives i)

ii) it holds
$$\|T(e_1) - T(e_2)\|_{L_\infty} = \left\|P_h(\frac{1}{f}(R(e_1) - R(e_2))\right\|_{L_\infty} \leq c_2 \|R(e_1) - R(e_2)\|_{L_\infty}$$

and
$$R(e_1) - R(e_2) = F(\cdot, u - e_1) - F(\cdot, u - e_2) = (F_u(\cdot, \vartheta) - F_u(u)(e_1 - e_2) \quad .$$

With
$$F_u(\cdot, \vartheta) = F_u(\cdot, u - \vartheta e_1 - (1 - \vartheta)e_2)$$

one gets
$$\|F_u(\cdot, \vartheta) - F_u(\cdot, u)\| \leq \kappa\bar{\varepsilon}c_3 \quad .$$

Choosing $\bar{\varepsilon} < Min(\varepsilon_0, (c_2 c_3 \kappa)^{-1})$ then proves ii).

Consequence: The operator T has a unique fix point in the ball $B_{\kappa\bar{\varepsilon}}$.

From this it follows the theorem above.