Supporting data to prove the RH and the Goldbach conjecture

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The Baseline Part

The Riemann Zeta function

For $\psi(x):=\sum_1^\infty e^{-n^2\pi x}$ and $Re(s)>1$ it holds, (EdH) 1.7,

$$\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}=\int_0^\infty \psi(x)x^{\frac{s}{2}}\frac{dx}{x}\,.$$

The entire Riemann function fulfill the duality relation

$$\xi(s):=(s-1)\frac{s}{2}\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}=\xi(1-s)$$

and has the following representation, (EdH) 1.8,

$$\xi(s)=4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx}x^{3/4}cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)logx\right]\ \frac{dx}{x}.$$

or,

$$\Xi(t):=\xi(\frac{1}{2}+it)=4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx}x^{3/4}\cos\left(\frac{t}{2}logx\right)\frac{dx}{x}.$$

If $cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)logx\right]$ is expanded in the usual power series $\cosh(y)=\frac{1}{2}(e^y+e^{-y})=\sum_{n=0}^\infty\frac{y^{2n}}{(2n)!}$ the above formula shows

$$\xi(s)=\sum_{n=0}^\infty a_{2n}(s-\frac{1}{2})^{2n}\,,\quad \Xi(t)=\sum_0^\infty(-1)^n a_{2n}\,t^{2n}$$

where

$$a_{2n}:=4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx}x^{3/4}\,\frac{(\frac{1}{2}logx)^{2n}}{(2n)!}\frac{dx}{x}.$$

The product formula representation is given by

$$\xi(s)=\xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right)\text{ where }\xi(0)=-\xi(0)=\frac{1}{2}.$$

Regarding $\xi(\frac{1}{2}+it)$ resp. $\rho=\frac{1}{2}+i\alpha$ the related representation is given by, (EdH) 1.16,

$$\xi\left(\frac{1}{2}+it\right)=\xi\left(\frac{1}{2}\right)\prod_{Re(\alpha)>0}\left(1-\frac{t^2}{\alpha^2}\right).$$

The mapping $z \to 1 - \frac{1}{z}$ maps the half-plane $Re(z) > \frac{1}{2}$ resp. the critical line onto the unit disk resp. the unit circle $S^1(R) - \{1\}$. The corresponding mappings to the above are given by

$$1 - \frac{s}{\rho} \to 1 - \frac{\left(it-\frac{1}{2}\right)}{\left(it+\frac{1}{2}\right)}\frac{\left(i\alpha+\frac{1}{2}\right)}{\left(i\alpha-\frac{1}{2}\right)} = \frac{\left(it+\frac{1}{2}\right)}{\left(it+\frac{1}{2}\right)}\frac{\left(i\alpha-\frac{1}{2}\right)}{\left(i\alpha-\frac{1}{2}\right)} - \frac{\left(it-\frac{1}{2}\right)}{\left(it+\frac{1}{2}\right)}\frac{\left(i\alpha+\frac{1}{2}\right)}{\left(i\alpha-\frac{1}{2}\right)}$$

$$1 - \frac{s}{\rho} \to \frac{\left(it+\frac{1}{2}\right)\left(i\alpha-\frac{1}{2}\right)-\left(it-\frac{1}{2}\right)\left(i\alpha+\frac{1}{2}\right)}{\left(it+\frac{1}{2}\right)\left(i\alpha-\frac{1}{2}\right)} = \frac{i(\alpha-t)}{\left(it+\frac{1}{2}\right)\left(i\alpha-\frac{1}{2}\right)}.$$

Lemma (Grl) 8.322:

$$\frac{1}{\Gamma(1+z)} = e^{\gamma z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-\frac{z}{n}} = \prod_{n=1}^{\infty}(1+\frac{z}{n})(1+\frac{1}{n})^{-z},\ Re(z) > 0.$$

Lemma (EdH): 1.13, 1.16:

i) $\xi(s) = \xi(0)\prod_{\rho}\left(1-\frac{s}{\rho}\right)$ where $\xi(0) = -\xi(0) = \frac{1}{2}$

ii) $\xi\left(\frac{1}{2}+it\right) = \xi\left(\frac{1}{2}\right)\prod_{Re(\alpha_n)>0}\left(1-\frac{t^2}{\alpha_n^2}\right)$ where $\rho = \frac{1}{2} + i\alpha_n$.

Remark (YoR)p. 124: $\prod_{Re(\alpha_n)>0}\left(1-\frac{t^2}{\alpha_n^2}\right)$ converges to an entire function $f(z)$ with $f(\alpha_0) = 1$, $f(\alpha_n) = 0$, which belongs to the Paley-Wiener space; this is because the sequence α_n is symmetric and the system $\{e^{i\alpha_n x}\}_{n\in Z}$ is exact in $L_2^{\#}(-\pi,\pi)$, (i.e., it is complete, but fails to be complete on the removal of one element).

Putting

$$\Omega(s) := \frac{1}{\pi^{\frac{s}{2}}\Gamma\left(1+\frac{s}{2}\right)} = \pi^{-\frac{s}{2}}e^{\gamma\frac{s}{2}}\prod_{n=1}^{\infty}\left(1-\frac{s}{(-2n)}\right)e^{-\frac{s}{2n}} = \prod_{n=1}^{\infty}(1-\frac{s}{(-2n)})(\pi+\frac{\pi}{n})^{-s/2}$$

one gets

$$(s-1)\zeta(s) = \Omega(s)\xi(s).$$

Lemma:

$$\frac{1}{2\pi}\int_0^{\infty}\left|\Gamma(\frac{1}{2}+it\right|^2 dx = \frac{1}{2}.$$

Lemma (EdH) 9.8: In the sense that the relative error approaches zero as $\omega \to \infty$

$$\frac{1}{2\omega}\int_{-\omega}^{\omega}(\left|\zeta(\frac{1}{2}+it)\right|^2 dt \sim \log\omega.$$

The most general results concerning the average values of $(|\zeta(s)|^2$ are provided in (LaE3b) §228:

Lemma: In the sense that the relative error approaches zero as $\omega \to \infty$

$$\frac{1}{2\omega}\int_{-\omega}^{\omega}(\left|\zeta(\frac{1}{2}+it)\right|^2 dt \sim \log\omega.$$

Lemma (EdH): The entire Zeta function $\xi(s) := (s-1)\frac{s}{2}\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}$ is of order 1 and it holds

i) $\xi(s) = \xi(1-s)$

ii) $\xi(s) = 4\int_1^\infty \frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx} x^{\frac{3}{4}} cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right)logx\right] \frac{dx}{x}$

iii) $\xi\left(\frac{1}{2}+it\right) = 4\int_1^\infty \frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx} x^{\frac{3}{4}} \cos\left(\frac{t}{2}logx\right)\frac{dx}{x}$

iv) $\xi(s) = \sum_{n=0}^\infty a_{2n}(s-\frac{1}{2})^{2n}$, $\xi(\frac{1}{2}+it) = \sum_0^\infty (-1)^n a_{2n}t^{2n}$

where $a_{2n} := 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx} x^{3/4}\frac{(\frac{1}{2}logx)^{2n}}{(2n)!}\frac{dx}{x}$.

v) $\xi(s) = \xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right)$ where $\xi(0) = -\xi(0) = \frac{1}{2}$

vi) $\xi\left(\frac{1}{2}+it\right) = \xi\left(\frac{1}{2}\right)\prod_{Re(\alpha_n)>0}\left(1-\frac{t^2}{\alpha_n^2}\right)$ where $\rho_n = \frac{1}{2}+i\alpha_n$ (EdH) 1.16.

Lemma (TiE)

i) $\lim_{s\to 1}(s-1)\zeta(s) = \gamma$, (TiE) 2.1.16

ii) $\zeta(s) - \frac{1}{s-1} = 1 - \frac{1}{2}s\{\zeta(s+1)-1\} - \frac{s(s+1)}{2\cdot 3}\{\zeta(s+2)-1\} - \cdots$, (TiE) 2.14

iii) $\zeta(s) = \frac{1}{s-1}+\gamma + O(|s-1|)$, $\lim_{n\to\infty}\sum_{m=1}^n(\frac{1}{m+1}+1-logn) = \gamma$, (TiE) 2.1.16

iv) $\zeta(s) = \frac{s}{s-1} + s\int_1^\infty([t]-t)t^{-s-1}dt$ and therefore $log|(1-s)\zeta(s)| \ll log|s|+1$.

Lemma (TiE) 2.4.5:

i) $\zeta'(0) = -\frac{1}{2}\log(2\pi)$, $-\frac{\zeta'(0)}{\zeta(0)} = -\log(2\pi)$, (TiE) 2.4.5

ii) $(s-1)\zeta(s) = e^{bs}\frac{1}{\Gamma\left(1+\frac{s}{2}\right)}\prod_\rho\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}$, where $b := \log(2\pi) - 1 - \frac{\gamma}{2}$ (TiE) 2.12.6.

Lemma (Grl) 8.322 (EdH): For the Gamma resp. Digamma function it holds

$$\frac{1}{\Gamma\left(1+\frac{z}{2}\right)} = e^{\frac{\gamma z}{2}}\prod_{n=1}^\infty\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}} = \prod_{n=1}^\infty\left(1+\frac{z}{2n}\right)\left(1+\frac{1}{n}\right)^{-\frac{z}{2}}.$$

Remark: The Riemann error term $\int_x^\infty \frac{dt}{t(1-t^2)\log t}$, reflects the trivial zeros of the Zeta function; it is derived from the formula, EdH) 1.16,

$$\Gamma\left(1+\frac{s}{2}\right) = \prod\left(1-\frac{s}{-2n}\right)^{-1}\left(1+\frac{1}{n}\right)^{s/2} \text{ resp. } \frac{1}{\Gamma\left(1+\frac{s}{2}\right)} = \prod\left(1+\frac{s}{2n}\right)\left(1+\frac{1}{n}\right)^{-s/2} = e^{\gamma s/2}\prod\left(1+\frac{s}{2n}\right)e^{-s/(2n)}$$

$$\int_x^\infty \frac{dt}{t(1-t^2)\log t} = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} log\Gamma(1+\frac{s}{2})x^s\frac{ds}{s} \to \frac{1}{2\pi i}\left[\int_{1/2-i\infty}^{1/2+i\infty} log\Gamma\left(\frac{s}{2}\right) + \log\left(tan\left(\frac{\pi}{2}s\right)\right)\right]x^s\frac{ds}{s}$$

Lemma (BuH) p.184: For the Gamma resp. Digamma function it holds

$$ {}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right) = e^{\frac{1}{3}z}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}}. $$

Lemma (Grl) 7.612:

$$ \int_0^\infty x^{s-1}\,{}_1F_1(a,c,-x)dx = \frac{\Gamma(c)}{\Gamma(a)}\Gamma(s)\frac{\Gamma(a-s)}{\Gamma(c-s)},\ \ 0 < Re(s) < Re(a). $$

and therefore

$$ \int_0^\infty x^{s/2-1}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2},-x\right)dx = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s},\ 0 < Re(s) < 1. $$

Lemma: Let ω_n denote the imaginary parts of the zeros of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi z\right)$, (whereby none of the zeros lie on the critical line), fulfilling the inequalities, (SeA), $2n-1 < 2\omega_n < 2n < \omega_n+\omega_{n+1} < 2n+1$. For $\omega_0 \in (-\frac{1}{2,},0)$ and $\lambda_n := \frac{3\omega_n+\omega_{n+1}}{4}$ and $\lambda_{-n} := -\lambda_n$ it holds

$$ n-\frac{1}{4} < \lambda_n := \frac{3\omega_n+\omega_{n+1}}{4} < n+\frac{1}{4}\,,\ n = 1,2,\ldots. $$

Therefore, it follows from Kadec's theorem that $\{e^{i\lambda_n t}\}_{n\in Z}$ forms a Riesz basis for $L_2\,(-\pi,\pi)$.

The Laguerre-Polya class LP of functions consists of the entire functions having only real zeros with a Weierstrass factorization of the form

$$ az^q e^{\alpha z-\beta z^2}\prod_k(1-z/\alpha_k)e^{z/\alpha_k}, $$

where a,α,α_k are real, $\beta \geq 0$ and q is a nonnegative integer, and α_k are the nonzero real numbers such that $\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2} < \infty$. The subset LP^* consists of all elements of order < 2. In this case β is neccessarily zero.

RH criterion (CaD): If the function

$$ \Xi(t) := \xi(1/2+it) $$

can be realized as a convolution $\Xi(t) = (K * dF)(t)$ where $K(t) \in LP^*$, i.e., is an entire function from the Laguerre-Polya class of order < 2, i.e.

$$ az^q e^{\alpha z}\prod_k(1-z/\alpha_k)e^{z/\alpha_k}, $$

where c,α,α_k are real, $\beta \geq 0$ and m is a nonnegative integer, this would prove the RH.

Remark:

$$ \frac{1}{\pi}arctan\left(\frac{\pi}{x}\right) = \begin{cases} \frac{x}{x^2+0{,}28\pi^2}\,, & |x| \geq \pi \\ \frac{1}{2}-\frac{x}{\pi^2+0{,}28x^2}\,, & x < \pi \\ -\frac{1}{2}-\frac{\pi x}{\pi^2+0{,}28x^2}\,, & x > -\pi \end{cases} $$

Remark: $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$

Remark (GrI) 4.532: $\int_0^\infty t^x\, arctan(t)\frac{dt}{t} = \frac{1}{x}\frac{\pi/2}{cos\,(\frac{\pi}{2}x)}$ for $0 < x < 1$.

Remark: The link of the cot −function to the Bernoulli numbers is given by the formulae, (GrI) 1.411, 9.616

$$cot(x) = \frac{1}{x} - \sum_{k=1}^{\infty}\frac{2^{2k}|B_{2k}|}{(2k)!}x^{2k-1} = \frac{1}{x} - 2\sum_{k=1}^{\infty}\frac{\zeta(2k)}{\pi^{2k}}x^{2k-1},\ |x| \le \pi,$$

$$\pi\, cot(\pi x) = \frac{1}{x} - \sum_{k=1}^{\infty}\frac{(2\pi)^{2k}|B_{2k}|}{(2k)!}x^{2k-1} = \frac{1}{x} - 2\sum_{k=1}^{\infty}\zeta(2k)x^{2k-1},\ |x| \le 1$$

Lemma:

$$\left|\frac{1}{2}\cot\left(\frac{x}{2}\right) - \frac{1}{x}\right| \le \frac{\pi}{6},\ |x| \le \pi \text{ resp. } \left|\pi cot(\pi x) - \frac{1}{x}\right| \le \frac{\pi^2}{3} = \frac{2}{\zeta(2)},\ |x| \le \frac{1}{2}$$

Proof:

$$\left|\frac{1}{2}\cot\left(\frac{x}{2}\right) - \frac{1}{x}\right| = \left|\sum_{-\infty}^{\infty}\frac{x}{2k\pi(x-2k\pi)}\right| \le \frac{\pi}{2\pi^2}\left|\sum_{-\infty}^{\infty}\frac{1}{1-\frac{x}{2k\pi})}\right| \cdot \frac{1}{k^2} \le \frac{\pi}{6},\quad |x| \le \pi.$$

For the sequences $\Psi(1-x_n) = \pi\cot(\pi y_n)$ with the same asymptotics as $log\,(tan\frac{\pi}{2}n) \sim \log(\sin(n)) \sim logn$ it follows from lemma 2.1 iii) and lemma 2.2 the

Corollary: As $0 < y_n < \frac{1}{2}$ it holds

$$\left|\pi cot(\pi y_n) - \frac{1}{y_n}\right| \le \frac{\pi^2}{3} \sim 3{,}2898\ldots\ .$$

Lemma ((LaE3a) §82): it holds $\left|\cot\frac{\pi}{2}s\right| = \left|\frac{e^{i\pi(\sigma+ix)}+1}{e^{i\pi(\sigma+ix)}-1}\right|$, and therefore

i) $\left|\cot\frac{\pi}{2}s\right| \le \frac{e^{-\pi x}+1}{e^{-\pi x}-1} \le \frac{e^{\pi}+1}{e^{\pi}-1}$, for $x \le -1$

ii) $\left|\cot\frac{\pi}{2}s\right| = \left|\frac{-e^{-\pi x}+1}{-e^{-\pi t x}-1}\right| = \frac{e^{-\pi x}-1}{e^{-\pi x}+1} < 1$ for $x \le 0$ and $\sigma = -z, z = 2k+1, k = 1,2,3,\ldots$.

Lemma (EdH): Let B_n denote the Bernoulli numbers; then the values of $\zeta(x)$ for $s = -n, 2n$ can be evaluated by the power series $\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n x^n}{n!}$ leading to

i) $\zeta(-n) = (-1)^n\frac{B_{n+1}}{n+1}$, $B_{2n+1} = 0$,i.e., $\zeta(-2n) = \frac{1}{2}\frac{B_{2n+1}}{n+1} = 0$

ii) $\zeta(2n) = (-1)^{n+1}(2\pi)^{2n}\frac{B_{2n}}{2\cdot(2n)!}$, $\zeta(1-2n) = (-1)^n\frac{B_n}{2n}$, WhE) p. 268

iii) $\zeta(0) = -\frac{1}{2}$, $\zeta(-1) = -\frac{1}{12}$, $\zeta(-3) = -\frac{1}{120}$.

Lemma:

$$\lim_{\sigma\to 1+0}\sum_{n=1}^{\infty}\mu(n)n^{-(\sigma+it)} = \lim_{\sigma\to 1+0}\frac{1}{\zeta(\sigma+it)} = \begin{cases}\zeta^{-1}(1+it) & t \ne 0\\ 0 & t = 0\end{cases}.$$

Remark: We emphasize that for the smallest index $n = 1$ it holds $y_1 \coloneqq 1 + x_1 \sim 0{,}49592$, (SeP). Therefore,

i) the estimate formula of the number of primes $\frac{n}{logn} \sim \int_2^n \frac{dt}{logt}$ can be replaced by the formula ny_n, but starting with the index $n = 1$,

ii) the alternative approximation formula for $\frac{1}{logn}$ also allows to revisit number theoretical formulas, like $\sum_{n=1}^{\infty} \frac{\mu(n)}{s} log^k(n) = (-1)^k (\frac{d}{ds})^k \frac{1}{\zeta(s)}$ for $k \in N$

iii) especially it holds $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} log(n) = -1$ (because of $\frac{1}{\zeta(s)} = (s-1) + a_2(s-1)^2 + \cdots$), which goes deeper than $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$, which is equivalent to the PNT.

Lemma:

i) $\zeta(\sigma + it) = O(logt) = O(t^{\varepsilon})$ for $\sigma \geq 1, x \geq 1$, (LaE3a) § 46

ii) $\zeta(\sigma + it) = O(logt) = O(t^{\frac{1}{2}-\sigma+\varepsilon})$ for $\sigma \leq 0, x \geq 1$, (LaE3a) § 228

iii) $\zeta(\sigma + it) = O\left(t^{\frac{1-\sigma}{2}} logt\right) = O\left(t^{\frac{1-\sigma}{2}+\varepsilon}\right)$ for $0 \leq \sigma \leq 1$, (LaE3b) § 240.

Lemma (LaE3b) §228: let $0 < \beta < 1$, and let $g(s) := \sum_1^{\infty} a_n e^{-s\, log\, n}$ be absolute convergent for $Re(s) = \sigma = \gamma$. Then

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \zeta(\beta + it) g(\gamma - it) dt = \sum_1^{\infty} a_n \frac{logn}{n^{\beta+\gamma}},$$

i.e. it especially holds

$$(*) \lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \zeta(\beta + it) \zeta(\gamma - it) dt = \sum_1^{\infty} \frac{1}{n^{\beta+\gamma}} = \zeta(\beta + \gamma) .$$

The latter formula (*) can be generalized by

Lemma (LaE3b) §228: let $-1 < \beta, \gamma$ with $\beta + \gamma = 1$, $\beta > 1, \gamma > 1$ or $\beta < 1, \gamma < 1$. Then

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \zeta(\beta + it) g(\gamma - it) dt = \zeta(\beta + \gamma).$$

The Hilbert Scale Part

The periodical Hilbert space $L_2^{\#}(-\pi,\pi)$ resp. the corresponding generalized Hilbert scale framework $H_\beta^{\#} \cong l_2^\beta$, $\beta \in R$, is built on the 2π –periodic Hilbert space $L_2^{\#}(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit circle sphere. For $u \in L_2^{\#}(\Gamma)$ and for real $\beta \in R$, $n \in Z$ the Fourier coefficients $u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$ enable the definition of the norms $\|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2$.

For, (see above)

$$\lambda_n := \frac{3\omega_n+\omega_{n+1}}{4} > 0 \ , \ \lambda_{-n} := \frac{3w_n+w_{n+1}}{4} < 0 \ , \ \lambda_0 \in \left(-\frac{1}{4},\frac{1}{4}\right)$$

it holds, (appendix)

Lemma:

i) $\{\lambda_n\}_{n\in Z}$ fulfill the Kadec condition, i.e.,

$$0 < |n-\lambda_n| \le L := \max|\lambda_0| < \frac{1}{4} \text{ for } n = 0, \pm 1, \pm 2, \ldots.$$

ii) The exponential family $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis of $L_2^{\#}(0,1)$.

Remark: putting $\lambda_0 = 0$ provides a Fourier series representation of $u \in L_2^{\#}(0,1)$ in the form

$$u(x) := \sum_1^\infty c_n e^{2\pi i\lambda_n x} \text{ with } \sum_1^\infty |c_n|^2 < \infty.$$

On addition of one exponential the Riesz basis $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis of fractional Sobolev spaces $H_\beta^{\#}(0,1)$ of order β with $0 \le \beta \le 1$ and $\beta \ne 1/2$, (IvS),

Lemma: Let $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis for $L_2^{\#}(0,1)$. Then for each number μ, which do not belong to the spectrum $\{\lambda_n\}_{n\in Z}$, the exponential families

$$E_\mu^{(\beta)} = \left\{\frac{1}{(1+|2\pi\lambda_n|^\beta)}e^{2\pi i\lambda_n x}\right\}_{n\in Z} \cup \{e^{2\pi i\mu x}\}$$

form a Riesz basis for the Sobolev space $H_\beta\ (0,1)$.

Remark: We note that according to the Sobolev embedding theorem any $g \in H_\beta^{\#}(0,1)$ with $\beta < \frac{1}{2}$ is bounded, i.e. $|g(x)| \le c$.

Lemma: Let (YoR) p. 12: Let X denote the vector space of all finite linear combinations of functions of the form $e^{i\lambda x}$, $(-\infty < t < \infty)$, where the parameter λ is real. An inner product in X is defined by

$$((f,g)) := \lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega} f(t)\overline{g(t)}dt.$$

When X is closed by means of the metric generated by this inner product, we obtain a certain Hilbert space B^2 (B is for Besicovitch). The continuum of elements $e^{i\lambda x}$ forms a complete orthogonal subset of B^2. The Hilbert space B^2 contains the important class of Bohr almost periodic functions. Those functions are obtained by adding to X the limits of sequences of function in X that are uniformly convergent on the entire real line.

Lemma: (YoR) p. 183, “the solved Polya problem”: Let $\{\lambda_1, \lambda_2, \lambda_3, \dots.\}$ be an increasing sequence of positive numbers with $\lim\inf \frac{n}{\lambda_n} > 1$, then the conditions

$$\int_{-\pi}^{\pi} f(t) sin\,(\lambda_n t) dt = \int_{-\pi}^{\pi} f(t) cos\,(\lambda_n t) dt = 0$$

imply that $f(t)$ is identically zero.

Under weaker assumptions it holds

Lemma: (YoR) p. 184: Let $\{\lambda_1, \lambda_2, \lambda_3, \dots.\}$ be an increasing sequence of positive numbers with $\lim\sup \frac{n}{\lambda_n} > 1$,then the conditions

$$\int_{-\pi}^{\pi} f(t) e^{\pm i\lambda_n t}\; dt = 0$$

imply that $f(t) = 0$ is almost everywhere on $[-\pi, \pi]$.

Regarding Dirichlet series we recall from (TiE) p. 138:

Lemma: Let $f(s) := \sum_1^{\infty} a_n e^{-s\log n}$, $g(s) := \sum_1^{\infty} b_n e^{-s\log n}$ be absolute convergent for $Re(s) > 1/2$. Then for $\alpha > 1/2$

$$\langle f, g\rangle_{-\alpha} = \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\sigma + it) g(\sigma - it) dt = \sum_1^{\infty} \frac{1}{n^{2\alpha}} a_n b_n,$$

i.e. $f, g \in H_{-\alpha}^{\#} \cong l_2^{-\alpha}$ for $\alpha > 1/2$.

The generalization of lemma 3.6 is provided by the “main theorem” from (LaE3b) §226):

Lemma: Let the series $f(s) := \sum_1^{\infty} a_n e^{-s\log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^{\infty} b_n e^{-s\log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$ and $\frac{\beta-\alpha_1}{\varepsilon_1} + \frac{\gamma-\alpha_2}{\varepsilon_2} > 1$ it holds

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\gamma - it) dt = \sum_1^{\infty} a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

Putting $\alpha = \alpha_1, = \alpha_2$ and $\varepsilon = \varepsilon_1 = \varepsilon_2$ Lemma 3.6 leads to

Lemma: Let the series $f(s) := \sum_1^{\infty} a_n e^{-s\log n}$ and $g(s) := \sum_1^{\infty} b_n e^{-s\log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta, \gamma > \alpha$, $(\beta - \alpha) + (\gamma - \alpha) > \varepsilon$ it holds

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\gamma - it) dt = \sum_1^{\infty} a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

Putting $l := \lim\sup\,(\frac{\log n}{\lambda_n})$ (chosing $\varepsilon_1 = \varepsilon_2 := l$) lemma 3.6 leads to

Lemma: Let the series $f(s) := \sum_1^{\infty} a_n e^{-s\log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^{\infty} b_n e^{-s\log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$, $(\beta - \alpha_1) + (\gamma - \alpha_2) > l$

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\gamma - it) dt = \sum_1^{\infty} a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

chosing $\varepsilon_1 = \varepsilon_2 := \alpha, \beta = \gamma$ leads to

Lemma: Let $l := \limsup \left(\frac{log n}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ and $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\beta - it) dt = \sum_1^\infty a_n b_n \, e^{-2\lambda_n \beta}.$$

Lemma: Let $l := \limsup \left(\frac{log n}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta > \alpha + \varepsilon/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f|(\beta + it)|^2 dt = \sum_1^\infty |a_n|^2 \, e^{-2\lambda_n \beta}$$

Lemma: Let $l := \limsup \left(\frac{log n}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f|(\beta + it)|^2 \, dt = \sum_1^\infty |a_n|^2 \, e^{-2\lambda_n \beta}.$$

Lemma (YoR) p. 12: Let X be the vector space of all finite linear combinations of functions of the form $e^{i\mu t}$, where the parameter μ is real. An inner product in X is defined by

$$\langle f, g \rangle := \lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(t) \overline{g(t)} dt.$$

When X is closed by means of the metric generated by this inner product, we obtain a certain Hilbert space B^2 (B is for Besicovitch).

Lemma (ApT) p. 188: let $f(s) := \sum_1^\infty a_n e^{-s\lambda_n}$ absolute convergent for $\sigma > \sigma_a$ then

$$\frac{1}{2\omega} \int_{-\omega}^{\omega} e^{\lambda(\sigma + it)} f(\sigma + it) dt = \begin{cases} a_n & if\ \lambda = \lambda(n) \\ 0 & if\ \lambda \neq \lambda(n) \end{cases}.$$

Lemma (ApT) p. 188: let $\mu_n := e^{\lambda_n}$, then $g(s) := \sum_1^\infty a_n e^{-s\mu_n}$ is absolute convergent for $\sigma > 0$; if $\sigma > \sigma_a$ then

$$\Gamma(s) f(s) = \int_0^\infty g(t) t^{s-1} dt$$

which is an extension of the classical formula

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{e^t - 1} t^{s-1} dt.$$

Remark: For the Dirac function is holds

$$\delta(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\omega x} d\omega = \frac{1}{2}\int_0^{2\pi} \cos\,((\omega x) d\omega \in H_{-\frac{1}{2}-\varepsilon}$$

Remark: The family $\{e^{2\pi inx}\}_{n\in Z}$ is the standard tool in the context of the dissection of the circle $x = e^{2\pi i\alpha}$ into „Farey arcs", based on the orthogonal relation

$$\int_0^1 e^{2\pi i(m-n)x} = \begin{cases} 1 & m=n \\ 0 & m\neq n \end{cases}.$$

…. Weyl (trigonometrical) sums $S(x) := \sum_n e^{2\pi inx}$

Let

$$k_1(x) := -\,log\,2\,sin\frac{x}{2},\;\; k_2(x) := cot\frac{x}{2}$$

The convolution operators

$$(Au)(x) := -\oint k(x-y)\,u(y)dy,\; H(u)(x) := \oint cot\frac{x-y}{2}u(y)dy,\;\; D(A) = L_2^*(\Gamma)$$

are linked by the relation

$$(Au')(x) = -\oint log\,2\,sin\frac{x-y}{2}\,d\,u(y) = \oint cot\frac{x-y}{2}u(y)dy = (Hu)(x)\ .$$

The classical derivative can be replaced by a corresponding Calderon-Zygmund singular integral operator in the form

$$S(x) := \frac{1}{\pi}\oint_{0\to 2\pi}\frac{u(y)dy}{4\,sin^2\frac{x-y}{2}} = (Hu')(x).$$

The corresponding Fourier coefficients are given by, (KrR), p. 127, p. 131,

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left(4sin^2\left(\frac{t}{2}\right)\right)e^{int} = \begin{cases} 0 & n=0 \\ -\frac{1}{n} & n=\pm 1,\pm 2,\ldots \end{cases},\; \frac{1}{2\pi}\int_0^{2\pi} cot^2(\frac{t}{2})\,e^{\pm int} = \begin{cases} 0 & n=0 \\ \pm i & n=1,2,\ldots \end{cases}.$$

i.e., $(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|}u_\nu$, $(Hu)_\nu = i\cdot sign(\nu)u_\nu$, $(Su)_\nu = i\nu\cdot i\cdot sign(\nu)u_\nu = -\nu u_\nu$.

The Number Theoretical Part

Remark: The mapping $z \to 1 - \frac{1}{z}$ takes the right half plane $Re(z) > 1/2$ to the interior of the unit circle $D := \{z||z| < 1\}$ in the complex z-plane and maps the critical line $Re(z) = 1/2$ onto the unit circle

Proof:

$$\left|1 - \frac{1}{z}\right|^2 = \left|\frac{z-1}{z}\right|^2 = \frac{z-1}{z} \cdot \frac{\bar{z}-1}{\bar{z}} = \frac{|z|^2 - 2Re(z) + 1}{|z|^2} < \frac{|z|^2 - 1 + 1}{|z|^2} < 1.$$

Remark: The mapping $z \to \frac{z-i}{z+i}$ (with its inverse function $w \to i\frac{1+w}{1-w}$) maps the real line onto the unit circle without 1 and vice versa.

Remark: let $z_n := \frac{1}{2} + 2\pi i \omega_n$ and let x_n denote the negative zeros oft he Digamma function, then the points $1 - \frac{1}{z_n}$ and $\frac{x_n - i}{x_n + i}$ lie on the unit circle.

The number theoretical properties of the zeros of the confluent hypergeometric (Kummer) function and the Digamma function are

i) $Re(z_n) = \sigma_n > 1/2$

ii) $0 < \cot\left(\frac{\pi}{2} y_n\right) - \left|\sum_{n=1}^{m} e^{-2\pi i w_n}\right| < \varepsilon$, $\varepsilon > 0$, $m > m_0$.

Property ii) above is a consequence of the following

Lemma (LaE1): let $m > 1$, β_1 real, $0 < \vartheta \le \beta_2 - \beta_1 \le \beta_3 - \beta_2 \le \cdots \beta_{m-1} - \beta_m \le 1 - \vartheta$ with $0 < \vartheta < 1/2$. Putting $S_m := |\sum_{n=1}^{m} e^{-2\pi i \beta_n}|$ then it holds for every $\varepsilon > 0$:

$$0 < \cot\left(\frac{\pi}{2}\vartheta\right) - S_m < \varepsilon.$$

The following two lemmata are about the building of arithmetical functions based on a certain integrals:

Lemma (Landau, (PoG1)): Let q_n denote a divergent sequence of positive numbers $0 < q_1 \le q_2 \le q_3 \le \cdots$ $\lim_{n\to\infty} q_n = \infty$, $\tau(x)$ the corresponding counting function of the numbers of q_n less than $\le x$ and $w(x)$ a positive, non-decreasing function with $\lim_{x\to\infty} \frac{w(2x)}{w(x)} = \lim_{x\to\infty} \frac{\tau(x)w(x)}{x} = 1$.
Then

$$\lim_{x\to\infty} \frac{1}{\tau(x)} \sum_{q \le x} \rho\left(\frac{x}{q}\right) = 1 - \gamma,$$

where $\rho(x)$ denotes the fractional part function.

In (PoG1) lemma 1 is generalized by

Lemma: Let $w(x)$ a positive, non-decreasing function with $\lim_{x\to\infty} \frac{w(\beta x)}{w(\alpha x)} = 1$ with α, β positive numbers. Then

$$\lim_{x\to\infty} \frac{w(x)}{x} \sum_{n \le x} f\left(\frac{x}{n}\right) = \int_0^1 f(t)dt.$$

Notes

Remark ((PrK) III §5):

$$\lim_{\sigma\to 1+0}\sum_{n=1}^{\infty}\mu(n)n^{-(\sigma+it)}=\lim_{\sigma\to 1+0}\frac{1}{\zeta(\sigma+it)}=\begin{cases}\zeta^{-1}(1+it) & t\neq 0\\ 0 & t=0\end{cases}$$

Especially it holds $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0$, which is equivalent to the PNT.

Remark:

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=1+\sum_{n=2}^{\infty}\frac{\mu(n)}{n}=0$$

$$1+\sum_{n=2}^{\infty}\frac{\mu(n)}{n}logn=0$$

$$1+\sum_{n=2}^{\infty}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}$$

The RH is equivalent to the relation, (LaEb) section 13,

$$M(x)=\sum_{n\le x}\mu(n)=O(x^{\frac{1}{2}+\varepsilon})\text{ for any }\varepsilon>0$$

resp.

$$M(N)=O(N^{\frac{1}{2}+\varepsilon})\text{ for any }\varepsilon>0\text{ and }N\in N$$

(resp. $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}$ is convergent for $\sigma>1/2$).

$$logtan(\frac{\pi}{2}s)=\int_0^1\frac{t^s-t^{1-s}}{(1+t)\log t}\frac{dt}{t}\text{ for }0<Re(s)<1$$

From (GrI) 3.761, 6.246), we recall related Mellin transforms

Lemma 3.1 For $a>0$ it holds

$$\int_0^{\infty}x^s\sin(ax)\frac{dx}{x}=\frac{\Gamma(s)}{a^s}\sin(\frac{\pi}{2}s)\qquad\int_0^{\infty}x^s si(x)\frac{dx}{x}=-\frac{\Gamma(s)}{s}\sin(\frac{\pi}{2}s)\qquad,\quad 0<|Re(s)|<1$$

$$\int_0^{\infty}x^s\cos(ax)\frac{dx}{x}=\frac{\Gamma(s)}{a^s}\cos(\frac{\pi}{2}s)\ ,\int_0^{\infty}x^s ci(x)\frac{dx}{x}=-\frac{\Gamma(s)}{s}\cos(\frac{\pi}{2}s)\qquad,\quad 0<Re(s)<1.$$

The key functions of concern of this paragraph is the $cot(\pi x)-$ function and its related (Ramanujan) divergent Fourier series representations, (BeB),

$$cot(\pi\circ)=2\sum_1^{\infty}sin(2\pi\nu\circ)\ ,$$

which can be formally established by differentiating the equality

$$-\frac{1}{\pi}log\,2\,sin(\pi x)=\sum_1^{\infty}\frac{cos\,2\pi\nu x}{\pi\nu}\ .$$

The following is about ζ –function defining equalities based on the fractional part $\rho(x)$ and the $cot(x)$ functions in the context of appropriate Hilbert scale framework and its relationship to the Riemann duality equation and the Bagchi formulation of the Nyman criterion.

Lemma: Consecutive zeros on the critical line ((IvA)10.3): For any $\varepsilon > 0$ and $n > n_0(\varepsilon)$ it holds

$$\gamma_{n+1} - \gamma_n < \gamma_n^{\vartheta+\varepsilon} \text{ with } \vartheta = 0.1559458\ldots < 0.15625 = \frac{5}{32}.$$

Lemma: If the RH is true, then ((IvA) theorem 12.10):

$$p_{n+1} - p_n \ll p_n^{\frac{1}{2}} \log p_n$$

An only formally valid representation of Riemann's duality equation as transform of an integral operator

Concerning invariant operators, adjoints and their transforms we quote from (EdH), chapter 10:
Let V be the vector space of all complex-valued functions on $R^+$with the inner product $(u,v):=\int_0^\infty u(x)v(x)dx$. By

$$I: v(x) \mapsto \int_0^\infty v(ux)F(u)du$$

an integral operator $I: V \to V$ is defined. An operator is said to be invariant if it commutes with all translation operators $T_u: v(x) \mapsto v(ux)$. The transform of an invariant operator is the function whose domain is the set of complex numbers s such that the function $v(x):=x^{-s}$ lies in the domain of the operator and whose value for such an s is the factor by which the operator multiplies $v(x):=x^{-s}$. Thus e.g. the Zeta function $\zeta(s)$ for $Re(s)>1$ is the transform of the summation operator

$$v(x) \mapsto \sum_1^\infty v(nx) .$$

When defining the adjoint of an invariant operator on V the inner product is defined on a rather small subset of V, whenever both side of $(Lu,v)=(u,L^*v)$ are defined. There is an only formally valid representation of Riemann's duality equation as transform of an integral operator

$$I: v(x) \mapsto \int_0^\infty v(ux)G(u)du$$

in the form

$$\int_0^\infty x^{1-s}G(x)\frac{dx}{x} = \frac{2\xi(s)}{s(s-1)} .$$

But the operator I has no transform at all, as the integral does not converge for any s. The integral would converge at ∞ if the constant term $f(0)=\hat{f}(0)=1$ is absent. If one would find an integral operator in the form I satisfying the same functional equation than G does and if

$$\int_0^\infty x^{1-s}G(x)\frac{dx}{x} \quad \text{converges} \quad \text{and} \quad \int_0^\infty x^{1-s}G(x)\frac{dx}{x} = \int_0^\infty x^{s}G(x)\frac{dx}{x}$$

then this operator would be self-adjoint in the sense of above.

The Laguerre-Polya class

The Laguerre-Polya class LP of functions consists of the entire functions having only real zeros with a Weierstrass factorization of the form

$$az^q e^{\alpha z-\beta z^2}\prod_k (1-z/\alpha_k)e^{z/\alpha_k},$$

where a,α,α_k are real, $\beta\ge 0$ and q is a nonnegative integer, and α_k are the nonzero real numbers such that $\sum_{k=1}^{\infty}\frac{1}{\alpha_k^2}<\infty$. The subset LP^* consists of all elements of order <2. In this case β is neccessarily zero.

RH criterion (CaD): If the function

$$\Xi(t):=\xi(1/2+it)$$

can be realized as a convolution $\Xi(t)=(K*dF)(t)$ where $K(t)\in LP^*$, i.e., is an entire function from the Laguerre-Polya class of order <2, i.e.

$$az^q e^{\alpha z}\prod_k (1-z/\alpha_k)e^{z/\alpha_k},$$

where c,α,α_k are real, $\beta\ge 0$ and m is a nonnegative integer, this would prove the RH.

Holomorphic function in the distributional sense, (PeB), chapter 1, §15

Let $z\to g_z$ be a function defined on a open subset $U\subset C$ with values in the distribution space. Then g_z is called a holomorphic in $U\subset C$ (or $g(z):=g_z$ is called holomorphic in $U\subset C$in the distribution sense), if for each $\phi\in C_c^{\infty}$ the function $z\to(g_s,\phi)$ is holomorphic in $U\subset C$ in the usual sense.

Müntz formula and Bessel function (TiE), (WaG) 13-2, 13-3

For $\omega(x),\omega'(x)$ continuous and bounded in any finite interval with $\omega(x)=o(x^{-\alpha})$ and $\omega(x)=o(x^{-\beta})$ for $x\to\infty$ and $\alpha,\beta>1$ it holds

$$\zeta(s)\int_0^{\infty}x^s\,\omega(x)\frac{dx}{x}=\int_0^{\infty}x^s\left[\sum_1^{\infty}\omega(nx)-\frac{1}{x}\int_0^{\infty}\omega(t)\,dt\right]\frac{dx}{x}\qquad \text{for } 0<Re(s)<1.$$

$$2\pi\int_0^{\infty}x^s\,Y_0(2x)\frac{dx}{x}=\Gamma^2(\tfrac{s}{2})\,cos(\tfrac{\pi}{2}s)=\frac{1}{4}cos(\tfrac{\pi}{2}s)\int_0^{\infty}x^sK_0(2x)\frac{dx}{x}\text{ for } 0<Re(s)<3/2$$

$$\int_0^{\infty}Y_0(x)dx=0\ \ ,\ \ \int_0^{\infty}K_0(x)dx=\frac{\pi}{2}\,.$$

Voronoi Summation Formula, (BeB1)

Theorem 19.5.1, 2.4, 2.5): Let $f \in C^1(0,\infty)$ and

$$\varphi(s):=\sum_1^\infty a(n)\lambda_n^{-s} \quad , \quad \psi(s):=\sum_1^\infty b(n)\mu_n^{-s} \qquad 0<\lambda_n,\mu_n\to\infty$$

be two Dirichlet series with abscissas of absolute convergence $\boldsymbol{\sigma_a}$and $\boldsymbol{\sigma_a^*}$, respectively.
Let $r>0$, and suppose that $\varphi(s)$ and $\psi(s)$ satisfy a functional equation of the type

$$\Gamma(s)\varphi(s)=\Gamma(r-s)\psi(r-s$$

Putting

$$Q(x):=\frac{1}{2\pi i}\int_C \varphi(s)x^s\frac{ds}{s},$$

where $\boldsymbol{C}$ is a simple closed curve(s) containing the integrand's poles in its interior, then if $\boldsymbol{0<a<\lambda_1<x<\infty}$,

$$\sum_{\lambda_n\le x^*} a(n)f(\lambda_n)=\int_a^x Q^{'}(t)\,f(t)dt+\sum_1^\infty b(n)\int_a^x\left(\frac{t}{\mu_n}\right)^{(r-1)/2} J_{r-1}(2\sqrt{\mu_n t})f(t)dt\ .$$

In [(NaC) L_2 – theory of Mellin and Fourier transformation is applied to derive a more symmetrical Voronoi formula (with similar structure as in (DuR), where the fractional part function is applied):
For $\boldsymbol{f,g}\in \boldsymbol{G_1^2(0,\infty)}$ properly defined it holds

$$\lim_{N\to\infty}\left\{\sum_{n=1}^N d(n)f(n)-\int_0^N(\log x+2\gamma)f(x)dx\right\}=\lim_{N\to\infty}\left\{\sum_{n=1}^N d(n)g(n)-\int_0^N(\log x+2\gamma)g(x)dx\right\}$$
.

Dixon & Ferrar ("*On the summation formula of Voronoi and Poisson*") provided a formula with reduced regularity assumptions (bounded variation) to the function $\boldsymbol{g}$.

$$\sum_1^\infty g(nt)=-\frac{1}{2}g(0^+)+\frac{1}{t}\int_0^\infty g(x)dx+\sum_1^\infty 0\int^\infty g(x)\cos(\frac{2\pi n}{t}x)dx.$$

The Hardy Hilbert space

Remark (PaJ) p. 4: The Hardy space $H^2(D)$ is defined as the space of all analytic functions on the disk $D=\{z\in C||z|<1\}$ for which the norm

$$|||f|||_2^2:=sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\varphi}\right|^2 d\varphi\right\}$$

is finite. The radial limit (function) $\tilde{f}(e^{i\varphi})=\lim_{r\to 1} f(e^{i\varphi})$ on $\Gamma=S^1(R^2)$ exist almost everywhere with $|||f|||_2^2=\frac{1}{2\pi}\int_0^{2\pi}\left|f(e^{i\varphi})\right|^2 d\varphi$.

The Hardy space H^2 can also defined as the subspace of those $L_2(\Gamma)$ functions for which the negative Fourier coefficients vanish, that is

$$\hat{f}(n)=f_n=\frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(e^{i\varphi})e^{-in\varphi}d\varphi<0,\ \nu<0.$$

Then a function $\tilde{f}$ with $\tilde{f}\left(e^{i\varphi}\right)\sim\sum_{n=0}^{\infty}f_n e^{i\varphi}$ can be naturally identified with the power series $f(z)=\sum_{n=0}^{\infty}f_n z^n$, defining an analytical function f in D.

In other words, the Hardy space H^2 is a Hilbert space, being a closed subspace of the Hilbert space $H:=L_2^*(\Gamma)$ and the orthogonal projection $P_{H^2}:L_2^*(\Gamma)\ \to\ H^2$ is defined by

$$P_{H^2}:\ \sum_{n=-\infty}^{\infty}a_n e^{i\varphi}\ \to\ \sum_{n=0}^{\infty}a_n e^{i\varphi}.$$

As there is an isometric isomorphism between $L_2^*(\Gamma)$ and $l^2\ (Z)$, and as the sequence space $l^2\ (Z_+)$ maps to the Hardy space H^2, one may regard $l^2\ (Z_+)$ as embedding into $l^2\ (Z)$ as the subspace of all $(a_n)_{n=-\infty}^{\infty}$ with $a_n\ =0$ for $n<0$.

Remark: For $f(z)=\sum_{n=0}^{\infty}a_n z^n\in H^2(D)$ and $g(z)=\sum_{n=0}^{\infty}b_n z^n\in H^2(D)$ it holds $\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\varphi}\right|^2 d\varphi=\sum_{n=0}^{\infty}|a_n|^2 r^{2n}$ for $0\le r<1$ and

$$\big((f,g)\big):=\sum_{n=0}^{\infty}a_n\bar{b}_n$$

defines an inner product of $H^2(D)$ with $|||f|||_2^2=(((f,f))$.

Remark: The mapping $z\to 1-\frac{1}{z}$ takes the right half plane $Re(z)>1/2$ to the interior of the unit circle $D:=\{z||z|<1\}$ in the complex z-plane and maps the critical line $Re(z)=1/2$ onto the unit circle

Proof: $\left|1-\frac{1}{z}\right|^2=\left|\frac{z-1}{z}\right|^2=\frac{z-1}{z}\cdot\frac{\bar{z}-1}{\bar{z}}=\frac{|z|^2-2Re(z)+1}{|z|^2}<\frac{|z|^2-1+1}{|z|^2}<1$.

Remark: The mapping $z\to\frac{1}{z}$ maps the circle $|z-1|=1$ onto the vertical line $Re(z)=1/2$.

Proof: it holds $|x-iy|=|(x-1)+iy|$, $|x-iy|^2=|(x-1)+iy|^2$, and therefore $x^2=(x-1)^2$ resp. $x=1/2$.

Remark: The mappings $w=\frac{z+1}{z-1}=\frac{1+z}{1-z}$ resp. $z=\frac{w-1}{w+1}$ map the unit disk $D:=\{z||z|\le 1\}$ onto the right half-plane $G:=\{z|Re(z)>0\}$ resp. G onto D.

Remark: The mapping $z\to\frac{z-i}{z+i}$ with its inverse function $w\to i\frac{1+w}{1-w}$ (which is also used to define the Caley transform of a Hermitian operator A) maps the real line onto the unit circle without 1 and vice versa.

Remark: The mapping e^z maps the stripe $\left\{z||Im(z)|<\frac{\pi}{2}\right\}$ onto the right half-plane G.

Properties of $\log\left(tan(\frac{\pi}{2}x)\right)$

For $T(x) := \log\left(tan(\frac{\pi}{2}x)\right)$, we summaries a few properties

i) $\frac{\pi}{2}T(x) = -\sum \frac{2h_n}{n} \sin(2\pi n x) \in L_2^{\#}(0,1)$ (ElL)

with $2h_n = \sum_{k=1}^{n} \frac{2}{2k-1} = 2H_{2n} - H_n$ and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ (harmonic numbers) and

$$\int_0^1 T(x)\cos(k\pi x)\,dx = \begin{cases} -1/k & k\ odd \\ 0 & k\ even \end{cases}.$$

ii) the $log\,(tan\,x)$ –integral evaluated by series involving $\zeta(2n+1)$ is provided (ElL1)

iii) for the Hilbert transform evaluation of $T(x)$ we refer to (MaJ)

iv) from (GrI), 1.421,1.518, we recall the series representations

$$\mathrm{T(x)} = logx + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{(2^{2k-1}-1)2^{2k}B_{2k}x^{2k}}{k(2k)!} \quad , x^2 < (\frac{\pi}{2})^2$$

$$T(x) = logx + \frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \cdots \ldots$$

$$log\,(sin\,x) = logx - \frac{1}{6}x^2 - \frac{1}{180}x^4 - \frac{1}{2835}x^6 - \cdots ..$$

Properties of the considered Kummer function $F(z) \coloneqq {}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$

Lemma:

i) $M\left[{}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\right]\left(\frac{s}{2}\right)=\frac{\Gamma\left(\frac{s}{2}\right)}{(1-s)}$, $0<Re(s)<1$, (GrI) 7.612

ii) $M\left[-x\;{}_1F_1{}'\left(\frac{1}{2},\frac{3}{2};-x\right)\right]\left(\frac{s}{2}\right)=\frac{\Gamma\left(1+\frac{s}{2}\right)}{(1-s)}$, following from i.

iii) $\int_0^\infty {}_1F_1\left(\frac{1}{2},\frac{3}{2},-x^2\right)\cos(2xy)dx=\frac{1}{4}\int_0^\infty e^{-(1+t)y^2}\frac{dt}{1+t}=\frac{1}{4}\int_1^\infty e^{-xy^2}\frac{dx}{x}$, (GrI) 7.642, 9.210

iv) $${}_1F_1(a,c,z)=\frac{\Gamma(c)}{\Gamma(c-a)}e^{\pm a\pi i}z^{-a}\left\{\sum_{k=0}^{n}\frac{(-1)^k(a)_k(1+a-c)_k}{k!}z^{-k}+O(|z|^{-n-1}\right\}.$$

$$+\frac{\Gamma(c)}{\Gamma(a)}e^{-(c-a)}z^{-(c-a)}\left\{\sum_{k=0}^{n}\frac{(-1)^k(1-a)_k(c-a)_k}{k!}z^{-k}+O(|z|^{-n-1}\right\}$$ (LeN) 9.12

v) ${}_1F_1(a,a+1;x)\sim a\frac{e^x}{x}\sum_{k=0}^{\infty}(\frac{1}{2})_k x^{-k}$, (SlL) 4.2

vi) $F_a(z)\coloneqq {}_1F_1(a,a+1,z)\sim\frac{1}{\Gamma(a)}\frac{e^z}{z}$, $z\to\infty, ph(z)=0$, (OlF) p.257

vii) $F_a^{(n)}(z)=\frac{a}{a+n}F_{a+n}(z)$, $\frac{1}{2}\frac{d}{dx}F_a^2(x)=\frac{a}{a+1}F_a(x)\cdot F_{a+1}(x)\sim\frac{1}{a+1}\frac{1}{\Gamma^2(a)}(\frac{e^x}{x})^2$.

For the considered Kummer function we summarize the following formulae / properties

i) $M\left[x\;{}_1F_1{}'\left(\frac{1}{2},\frac{3}{2};-x\right)\right](s)=\frac{\Gamma(1+s)}{2s-1}$, $0<Re(s)<1/2$

ii) $F(x)\sim\frac{1}{\sqrt{\pi}}\frac{e^x}{x}\sim\frac{1}{2}\frac{e^x}{x}\sum_{k=0}^{\infty}(\frac{1}{2})_k(\frac{1}{x})^k$, as $x\to\infty$ for x real.

Remark: From (SlL) (3.2.26) we recall the formulae ($Re(z)>0$)

$${}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right)=e^{x^2}\int_0^\infty e^{-y^2}\sin(2xy)\,dy.$$

The proposed enhanced circle method is about fractions series based on the imaginary parts of the zeros of the confluent hypergeometric (Kummer) function

$$F(z) = \ {}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right) = \sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{z^n}{n!},$$

which can be also used to define a Riesz basis $\{e^{i\lambda_n t}\}_{n\in Z}$ for $L_2^{\#}(-\pi,\pi)$. From (LuX) and (YoR) we recall the following properties

Lemma (LuX): For the maximum modulus functions $M(r,F(z))$ of $F(z)$ resp. the order $\sigma(r,F(z))$ of $F(z)$ it holds

$$M\big(r,F(z)\big) = O(e^r) \text{ resp. } \sigma\big(r,F(z)\big) = 1.$$

Theorems 5 & 6 (YoR), p. 55: If $f(z)$ is an entire function of finite order ρ, then for the number of its zeros z_n for which $|z_n| \le r$ it holds $n(r) = O(r^{\rho+\varepsilon})$ and for the z_n other than $z = 0$, the series

$$\sum_{n=0}^{\infty}\frac{1}{|z_n|^{\alpha}}$$

is convergent whenever $\alpha > \rho$.

Remark: We note that if $f(z)$ is an entire function of exponential type, then $\frac{n(r)}{r}$ remains bounded as $r \to \infty$, (YoR) Theorem 3, p. 52.

Lemma (BuH) p.184, (KiA):

$${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right) = e^{\frac{1/2}{3/3}z}\prod\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}} \text{ with } \sum_{n=1}^{\infty}\frac{1}{{z_n}^{1+\varepsilon}} < \infty \text{ for } \varepsilon > 0.$$

Some properties of the Digamma function

Remark (Grl) 8.365:

i) $\Psi(nz) = \frac{1}{n}\sum_{k=0}^{n-1}\Psi\left(z+\frac{k}{n}\right) + logn,\ n = 2,3,4$ ....

ii) $\Psi(x-n) = \Psi(x) - \sum_{k=1}^{n}\frac{1}{x-k}$

iii) $\Psi(x+n) = \Psi(x) + \sum_{k=0}^{n-1}\frac{1}{x+k}$

iv) $\Psi(x+1) = \Psi(x) + \frac{1}{x}$

v) $\Psi(1-x) - \Psi(\mathrm{x}) = \pi\cot(\pi\mathrm{x}) = \frac{\pi}{\tan(\pi x)}$

vi) $\Psi\left(\frac{1}{2}+z\right) - \Psi\left(\frac{1}{2}-z\right) = \pi\tan(\pi z)$

vii) $\Psi\left(\frac{3}{4}-n\right) - \Psi\left(\frac{1}{4}+n\right) = \pi,\ n \in N$

viii) $\pi\cot(\pi\mathrm{x}) = \frac{1}{\mathrm{x}} - \sum_{k=1}^{\infty}\frac{(2\pi)^{2k}|B_{2k}|}{(2k)!}x^{2k-1},\ |x| < 1$

ix) $\pi\cot(\pi\mathrm{x}) = \frac{1}{\mathrm{x}} + 2x\sum_{k=1}^{\infty}\frac{1}{x^2-k^2} = \frac{1}{\mathrm{x}} + x\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{1}{k(x-k)}$

x) $e^{\Psi(x)} = x\prod_{k=0}^{\infty}(1+\frac{1}{x+k})e^{-\frac{1}{x+k}}$

xi) $e^{y\Psi(x)} = \frac{\Gamma(x+y)}{\Gamma(x)}x\prod_{k=0}^{\infty}\left(1+\frac{y}{x+k}\right)e^{-\frac{y}{x+k}}$

xii) $\frac{\Psi(z)}{\Gamma(z)} = -e^{-\gamma z}\prod(1-\frac{z}{x_n})\,e^{\frac{z}{x_n}}$, (LeB)

xiii) $\Psi(1+z) = -\gamma + \sum_{n=2}^{\infty}(-1)^n\zeta(n)z^{n-1},\ |z| < 1$, (AbM) 6.3.14.

Some properties of Dirichlet series

Lemma (PrK) p. 373: Let $\sum_n a_n n^{-s_o}$ be convergent. Then $\sum_n a_n n^{-s}$ converges evenly in every closed interval, which lies in the angular space

$$|ph(s-s_0)| \leq \frac{\pi}{2} - \delta \ , \ \delta > 0,$$

and the function $g(s) = \sum_n a_n n^{-s}$ is regular for $Re(s) > Re(s_0)$.

Regarding Dirichlet series we recall from (TiE) p. 138:

Lemma 3.6: Let $f(s) := \sum_1^\infty a_n e^{-s\log n}$, $g(s) := \sum_1^\infty b_n e^{-s\log n}$ be absolute convergent for $Re(s) > 1/2$. Then for $\alpha > 1/2$

$$\langle f, g\rangle_{-\alpha} = \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\sigma+it)g(\sigma-it)dt = \sum_1^\infty \frac{1}{n^{2\alpha}} a_n b_n,$$

i.e. $f, g \in H_{-\alpha}^{\#} \cong l_2^{-\alpha}$ for $\alpha > 1/2$.

The generalization of lemma 3.6 is provided by the “main theorem” from (LaE3b) §226):

Lemma 3.6: Let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^\infty b_n e^{-s\log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$ and $\frac{\beta-\alpha_1}{\varepsilon_1} + \frac{\gamma-\alpha_2}{\varepsilon_2} > 1$ it holds

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta+it)g(\gamma-it)dt = \sum_1^\infty a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

Putting $\alpha = \alpha_1, = \alpha_2$ and $\varepsilon = \varepsilon_1 = \varepsilon_2$ Lemma 3.6 leads to

Lemma 3.7: Let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ and $g(s) := \sum_1^\infty b_n e^{-s\log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta, \gamma > \alpha$, $(\beta-\alpha) + (\gamma-\alpha) > \varepsilon$ it holds

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta+it)g(\gamma-it)dt = \sum_1^\infty a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

Putting $l := \limsup\left(\frac{\log n}{\lambda_n}\right)$ (chosing $\varepsilon_1 = \varepsilon_2 \coloneqq l$) lemma 3.6 leads to

Lemma 3.8: Let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^\infty b_n e^{-s\log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$, $(\beta-\alpha_1) + (\gamma-\alpha_2) > l$

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta+it)g(\gamma-it)dt = \sum_1^\infty a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

chosing $\varepsilon_1 = \varepsilon_2 \coloneqq \alpha, \beta = \gamma$ leads to

Lemma 3.9: Let $l := \limsup\left(\frac{\log n}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ and $g(s) := \sum_1^\infty b_n e^{-s\log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta+it)g(\beta-it)dt = \sum_1^\infty a_n b_n\, e^{-2\lambda_n\beta}.$$

Lemma 3.10: Let $l := \limsup (\frac{\log n}{\lambda_n})$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta > \alpha + \varepsilon/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f|(\beta + it)|^2 dt = \sum_1^\infty |a_n|^2\, e^{-2\lambda_n \beta}$$

Lemma 3.11: Let $l := \limsup (\frac{\log n}{\lambda_n})$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f|(\beta + it)|^2\, dt = \sum_1^\infty |a_n|^2\, e^{-2\lambda_n \beta}.$$

Lemma 3.12:

$$\frac{1}{2\pi} \int_0^\infty \left|\Gamma(\frac{1}{2} + it\right|^2 dx = \frac{1}{2}.$$

Lemma 3.13 (EdH) 9.8: In the sense that the relative error approaches zero as $\omega \to \infty$

$$\frac{1}{2\omega} \int_{-\omega}^{\omega} (\left|\zeta(\frac{1}{2} + it)\right|^2 dt \sim \log\omega.$$

The most general results concerning the average values of $(|\zeta(s)|^2$ are provided in (LaE3b) §228:

Lemma 3.14: In the sense that the relative error approaches zero as $\omega \to \infty$

$$\frac{1}{2\omega} \int_{-\omega}^{\omega} (\left|\zeta(\frac{1}{2} + it)\right|^2 dt \sim \log\omega.$$

Lemma 3.15 (ApT) p. 188: let $f(s) := \sum_1^\infty a_n e^{-s\lambda_n}$ absolute convergent for $\sigma > \sigma_a$ then

$$\frac{1}{2\omega} \int_{-\omega}^{\omega} e^{\lambda(\sigma+it)} f(\sigma + it) dt = \begin{cases} a_n & if\ \lambda = \lambda(n) \\ 0 & if\ \lambda \neq \lambda(n) \end{cases}.$$

Lemma 3.16 (ApT) p. 188: let $\mu_n := e^{\lambda_n}$, then $g(s) := \sum_1^\infty a_n e^{-s\mu_n}$ is absolute convergent for $\sigma > 0$; if $\sigma > \sigma_a$ then

$$\Gamma(s) f(s) = \int_0^\infty g(t) t^{s-1} dt$$

which is an extension of the classical formula

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{e^t - 1} t^{s-1} dt.$$

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Abel's Identity

(PrK) p. 180: Let q be an integer with $1 \leq q \leq log^u N$, wheßre u is a arebitrary large positive real number independent from N. Let further a an integer with $0 \leq a < q$, $(a,q) = 1$. Then it holds

$$S_N\left(\frac{a}{q}\right) = \sum_{p\leq N} e(p\frac{a}{q}) = \sum_{p\leq N} e^{2\pi i p\frac{a}{q}} = \frac{\mu(q)}{\varphi(q)} li(N) + O(Ne^{c\sqrt{logN}}) \text{ with } li(N) = \int_2^N \frac{dt}{logt} + O(1).$$

Abel's identity (ApT) p. 77: For any arithmetical function $a(n)$ let

$$A(x) = \sum_{n\leq x} a_n$$

where $A(x) = 0$ if $x < 1$. Assume $f(x)$ has continuous derivative on the interval $[y,x]$, where $0 < y < x$. Then we have

$$\sum_{y<n\leq x} a(n)f(n) = A(x)f(x) - A(y)f(y) - \int_y^x A(t)f'(t)dt.$$

Theorem 11 (HaG1): Let $\mu_n = log\lambda_n$. Then

$$\sum_n a_n e^{-\mu_n s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} (\sum a_n e^{-\lambda_n x})\, dx,$$

if $Re(s) > 0$ and the series on the left-hand side is convergent.

A lemma from

On the zeros of Riemann's zeta function
A. Selberg, (SeAt) p. 88

Lemma: If $1 \leq m \leq x$, $1 \leq n \leq x$, $m \neq n$, then

$$\sum \frac{1}{\sqrt{mn}\left|log\frac{m}{n}\right|} = O(xlogx).$$

Proof: We write $\sum = \sum_{m<\frac{n}{2}} + \sum_{\frac{n}{2}\leq m\leq \frac{3}{2}n} + \sum_{m>\frac{3}{2}n} = \sum_{(1)} + \sum_{(2)} + \sum_{(3)}$, then

$$\sum_{(1)} = O(\sum_{m,n<x} \frac{1}{\sqrt{mn}}) = O(\sum_{n<x} \frac{1}{\sqrt{n}}))^2 = O(x), \text{ and so for } \sum_{(3)} .$$

In $\sum_{(2)}$ we have $m = n + r$, where $|r| \leq \frac{1}{2}n$ and $\frac{1}{\left|log\frac{m}{n}\right|} = \frac{1}{\left|\log(1+\frac{r}{n}\right|} = O(\frac{n}{|r|})$, hence

$$\sum_{(2)} = O(\sum_{n=1}^{x} \sum_{r=1}^{n/2} \frac{1}{\sqrt{n\cdot\frac{n}{2}}} \cdot \frac{n}{r}) = O\left(\sum_{n=1}^{x} \sum_{r=1}^{x} \frac{1}{r}\right) = O(xlogx).$$

Additive and analytic number theory

(Translated) extracts from

„Lectures on number theory“, Vol. 1 (LaEa)

E. Landau

(LaEa) p. 188: Let $k \in N, l \in N_0$ and $\eta := e^{\frac{2\pi i}{k}}$. Then

$$\eta^l = e^{2\pi i \frac{l}{k}} \quad 0 \le l \le k-1$$

represent all k-th roots of unity, and $\eta^l = e^{2\pi i \frac{l}{k}}$ $(l,k) = 1$ represent all $\varphi(k)$ primitive k-th unity roots. Obviously, the k^{th} unit roots are identical with the entirety of the primitive d^{th} roots of unity for all $d|k$, i.e. (LaEb) p. 167)

$$\mu(q) = \sum_{\substack{1 \le a \le q \\ (a,q)=1}} e^{2\pi i \frac{a}{q}} .$$

In the whole section VI. it is assumed that $k > 2$; we put $a := \frac{1}{k}$, $K := 2^{k-1}$.

The big auxiliary theorem

(LaEa) p. 212

p. 206: let θ denote the smallest number $\theta \ge 1/2$ that in the half plane $\sigma > \theta$ the function $L(s,\chi)$ has no roots. If $L(s,\chi) \ne 0$ for $\sigma > 1/2$, then $\theta = 1/2$; if there exists a root with $L(s,\chi) = 0$, than $\theta > 1/2$. It holds $1/2 \le \theta \le 1$.

For the rest of the section V. we make the assumption, that

$$1/2 \le \theta < 3/4.$$

Theorem 248: For $|x| < 1$ let $f(x)$ denote the regular function $f(x) = \sum_{p>2} logp \cdot x^p$. Let $\rho (= \eta^l)$ denote a primitive k-th root of unity, and

$$y = \eta - i\vartheta,\ 0 < \eta \le \frac{1}{9},\ \vartheta \in R,\ k|y| < 8\sqrt{\eta},\ e^{-y} = X,\ \rho X = x$$

(which means $|x| = |X| = e^{-Re(y)} = e^{-\eta} < 1$)

$$\psi := \psi(y) := \frac{\mu(k)}{h(1-X)}$$

(that ψ only depends from y and ρ). Then it holds for each $\varepsilon > 0$

$$|f(x) - \psi| < A_{20} \eta^{-\theta - \frac{1}{4} - \varepsilon}$$

(where without loss of generality $\varepsilon < 1 - \theta$).

The sense of this desolate theorem will brighten up in the next chapter; first of all we remark, that $f(x)$ for values x lying on an arc within the unit circle is estimate to the circle

$$|x| = |X| = e^{-Re(y)} = e^{-\eta} < 1, \text{ (lying within the unit circle)}$$

to which the point $x = \rho e^{-\eta}$ belongs. I state “arcs“ of $|x| = e^{-\eta} < 1$, because ϑ as a function of η and k is constrained by $k|y| < 8\sqrt{\eta}$. Already k as a function of η will be constrained, because $k|y| < 8\sqrt{\eta}$ demands $k\eta < 8\sqrt{\eta}$, $k < \frac{8}{\sqrt{\eta}}$.

§1 The generating function, die Farey dissection, and a heuristical approach

(LaEa)

In what follows in §1 the considered elements of are presented in different orders as in (LaEa).

p. 237: for every primitive q^{th} unity root ρ (i.e., $\rho = e^{2\pi i\frac{k}{q}}$ $(k,q) = 1$) let

$$S_q \coloneqq S_q(k,\rho) \coloneqq \sum_{h=1}^{q} \rho^{h^k}.$$

(or, which is equivalent, $S_q \coloneqq \sum_{h=1}^{q} \rho^{hk}$ over any complete “rest” system $mod\rho$. S_q depends only from S_q and S_q).

Extended over all $\varphi(q)$ primitive q-th roots of unity let further

$$A(q) := A(q;n,k,s) \coloneqq \frac{1}{q^s} \sum_{\rho(q)} S_q{}^s \rho^{-s}$$

the „singular series“

$$\boldsymbol{S} := \boldsymbol{S}(n,k,s) \coloneqq \sum_{q=1}^{\infty} A(q) \; .$$

Let $r(n) = r(n,k,s)$ $(= r_{k,s}(n))$ denote the solution number of the Waring problem, i.e., there is a $s(k)$ that for every $n > 0$ the equation

$$n = \sum_{\nu=1}^{s} (h_\nu)^k,\ h_\nu \text{ integer and } h_\nu \geq 0$$

has always a solution. Then (p.238)

i) for $s > 2K$ the series $\boldsymbol{S}$ is convergent for all n

ii) for $s > kK$ it holds

(249) $$s > 2K \lim_{n\to\infty} \left\{ \frac{r(n)}{n^{sa-1}} - \frac{\Gamma^s(1+a)}{\Gamma(sa)} \boldsymbol{S} \right\}$$

iii) for $s > c_6$ it holds $\frac{1}{2} < \boldsymbol{S} < 3/2$.

From this it follows with (249) for every $s > kK$ and $n > b_1$

$$\left| \frac{r(n)}{n^{sa-1}} - \frac{\Gamma^s(1+a)}{\Gamma(sa)} \boldsymbol{S} \right| < \frac{1}{4} \frac{\Gamma^s(1+a)}{\Gamma(sa)},$$

$$\frac{\Gamma^s(1+a)}{\Gamma(sa)} \left(\boldsymbol{S} - \frac{1}{4} \right) < \frac{r(n)}{n^{sa-1}} < \frac{\Gamma^s(1+a)}{\Gamma(sa)} \left(\boldsymbol{S} + \frac{1}{4} \right).$$

In combination with iii) for $s > c_7$ and $n > b_1$ this leads to

$$b_2 \coloneqq \frac{1}{4} \frac{\Gamma^s(1+a)}{\Gamma(sa)} < \frac{r(n)}{n^{sa-1}} < \left(1 + \frac{3}{4}\right) \frac{\Gamma^s(1+a)}{\Gamma(sa)} \coloneqq b_3.$$

In other words, for $s > c_7$ the solution number $r(n)$ is of the „truly order“ n^{sa-1}; this means

$$0 < \lim_{n\to\infty} inf \frac{r(n)}{n^{sa-1}} \leq \lim_{n\to\infty} sup \frac{r(n)}{n^{sa-1}} < \infty.$$

Especially it holds the

Hilbert theorem: for $s > c_7$ and $n > b_1$: $r(n) > 0$ (i.e. there exists a solution number).

Remark: (249) already holds for $s > (k-2)K + 4$

p. 243: For $|x| < 1$ let $f(x)$ denote the regular function $f(x) = \sum_{h=0}^{\infty} x^{h^k}$. Then, obviously, for $s \in N$

$$f^s(x) = \sum_{n=0}^{\infty} r(n)x^n \ .$$

For the rest of this section, we consider $n \geq 2$ (except theorems 270-272).

Based on the Cauchy theorem it holds

(258) $$r(n) = \frac{1}{2\pi i}\int \frac{f^s(x)}{x^n}\frac{dx}{x}$$

where the integration is over the circle $|x| = e^{-\frac{1}{n}}$ in positive manner.

Now we consider all fractions l/q $(0 \leq l/q \leq 1)$ of the Farey series with denominator $q \leq n^{1-\alpha}$, with α chosen later properly, but already now demanding $0 < \alpha < a$.

Then the ρ's (i.e., $\rho = e^{2\pi i \frac{l}{q}}$ $(l, q) = 1$) are all $\varphi(q)$ primitive q^{th} unity roots of degrees $q \leq n^{1-\alpha}$.
(whereby $\rho = 1$ occurs two times, for $l = 0, q = 1$, and $l = 1, q = 1$; which is good).

We built the mediants of each of two neighbors of our l/q, we through away the l/q, and we dissect the integration path in the arcs B, defined by $\sum_{q=1}^{[n^{1-\alpha}]} \varphi(q)$, which are marked by the points

$$x = e^{-\frac{1}{n}+2\pi i\cdot mediants}.$$

Then every arc B contains exactly one of our ρ, which we denote now by B_ρ.

According to the theorems 156 and 157 B_ρ has the form,

$$x = \rho X,\ X = e^{-y},\ y = \frac{1}{n} - i\vartheta,\ -\vartheta_1 \leq \vartheta \leq \vartheta_2$$

with

(259) $$\frac{\pi}{q\cdot n^{1-\alpha}} \leq \vartheta_1 < \frac{2\pi}{q\cdot n^{1-\alpha}}\ ,\ \frac{\pi}{q\cdot n^{1-\alpha}} \leq \vartheta_2 < \frac{2\pi}{q\cdot n^{1-\alpha}}.$$

Moreover, it holds $\vartheta_1 \leq \pi$, $\vartheta_2 \leq \pi$, because of

$$2\pi\frac{1}{q\cdot(q+q')} \leq 2\pi\frac{1}{1\cdot 2}\ .$$

Theorem 258: for $-\pi \leq \vartheta \leq \pi$ there is a constant c for which it holds

$$|1 - X| \geq \sqrt{\frac{1}{n^2} + \vartheta^2} \ \text{ and } \ \frac{|1-X|}{1-|X|} \leq 1 + n|\vartheta|.$$

i) on the minor arcs it holds $\frac{|1-X|}{1-|X|} < 8$,

ii) on the major arcs it holds $\frac{|1-X|}{1-|X|} < 8\frac{n^\alpha}{q} = \frac{8n}{qn^{1-\alpha}}$.

p. 261: Putting $\omega = \left[n^{a+\frac{\varepsilon}{2}}\right]$ on the full circle $|x| = e^{-\frac{1}{n}}$ it holds

$$\left|f(x) - \sum_{h=0}^{\omega} x^{h^k}\right| < C_{33}.$$

We now want to estimate the error which occurs, when we replace in (258) the function $f(x)$ by a simpler analytical function; it is allowed different ones of it on each arc B_ρ. Which one we want to choose? The answer is obvious (?!); it will be the function

$$\psi_\rho(x) \coloneqq \Gamma(1+a)\frac{S_q}{q}\frac{1}{(1-X)^a} = \Gamma(1+a)\frac{S_q}{q}\frac{1}{\left(1-\frac{x}{\rho}\right)^a},$$

with $S_q = S_q(k,\rho) \coloneqq \sum_{h=1}^{q} \rho^{hk}$ for every q^{th} unity root ρ, and $\frac{1}{(1-X)^a}$ for $|X| < 1$, denotes the convergent binominal series, (recalling $a = \frac{1}{k}$, $K = 2^{k-1}$)

$$1 + aX + \frac{a(a+1)}{2}X^2 + \cdots,$$

which is the branch with the value 1 for $X = 0$. This is obvious, because of

$$\lim_{X\to 1}(1-X)^a f(\rho X) = \Gamma(1+a)\frac{S_q}{q};$$

therefore, $\psi_\rho(x)$ approximates the function $f(x)$ at the point $x = \rho e^{-\frac{1}{n}}$.

We want to compare $f(x) = \sum_{h=0}^{\infty} x^{h^k}$ on each arc B_ρ with $\psi_\rho(x)$ and we shall estimate the error.

For every $\varepsilon > 0$ we shall find the following estimates

(266) $\qquad \left|f(x) - \psi_\rho(x)\right| < D(k,\varepsilon,\alpha) n^{\frac{aK}{K+1}+\varepsilon} \qquad$ for $q \le n^\alpha$

(267) $\qquad \left|f(x) - \psi_\rho(x)\right| < C_1 n^{a-\frac{\alpha}{K}+\varepsilon} \qquad$ for $n^\alpha < q \le n^{1-\alpha}$.

(266) can be improved, but it would not improve the overall result. (267) cannot be improved based on the current state of science. (from this fact the further development of the Waring problem will depend);

For orientation about the two claims (266), (267) I remark two points.

1. The „joke“ is, that the exponent on the right side for appropriate $\varepsilon = c_9$ is smaller than a. It would be $\left|f(x) - \psi_\rho(x)\right| < c_{10} n^a$ for all arcs B_ρ, which is trivial, and therefore useless; as a matter of fact, already for each of the two functions it holds

$$|f(x)| \le \sum_{h=0}^{\infty} |x|^{h^k} = \sum_{h=0}^{\infty} e^{-\frac{h^k}{n}} < 1 + \int_0^\infty e^{-\frac{u^k}{n}} du = 1 + c_{11} n^a < c_{12} n^a$$

$$\left|\psi_\rho(x)\right| < \left|\frac{\Gamma(1+a)}{q}\sum_{h=1}^{q} \rho^{hk}\frac{1}{(1-X)^a}\right| \le \frac{c_{13}}{q}\cdot q\frac{1}{(1-|X|)^a} = \frac{c_{13}}{\left(1-e^{-\frac{1}{n}}\right)^a} < c_{14} n^a .$$

2. In §2 (theorem 269) we shall prove

$$(*) \quad \left|S_q\right| = \left|S_{q.k}\right| = \left|\sum_{h=1}^{q} \rho^{h^k}\right| \le C_2 q^{\tau+\varepsilon} \text{ for all } \varepsilon > 0, \text{ where } \tau = 1 - \frac{1}{K}.$$

   In chapter 2 we shall even $\left|S_q\right| = \left|S_{q.k}\right| \le c_{15} q^{1-a} = c_{15} q^{1-\frac{1}{k}}$, but that doesn't get us here any further). From this it follows in the claim (267) the function $\psi_\rho(x)$ function can be smoothed out, because according to (*) it holds for $n^\alpha < q \le n^{1-\alpha}$

$$\left|\psi_\rho(x)\right| < \frac{C_3}{q} q^{\tau+\varepsilon}\frac{1}{(1-|X|)^a} < \frac{C_3}{q} q^{1-\frac{1}{K}+\varepsilon} n^a < C_3 q^{-\frac{1}{K}} n^\varepsilon n^a < C_3 n^{a-\frac{\alpha}{K}+\varepsilon} .$$

It is not surprising that we will not replace the function $f(x)$ on the arcs B_ρ with $n^\alpha < q \le n^{1-\alpha}$ by $\psi_\rho(x)$, but by (the function) 0, i.e. by proving just

$$|f(x)| < C_1 n^{a-\frac{\alpha}{K}+\varepsilon} \text{ for } n^\alpha < q \le n^{1-\alpha} \text{ (and not (267).}$$

The arcs with $q \le n^\alpha$ are called major arcs, the ones with $q > n^\alpha$ are called minor arcs. (The names are a bit strange, because we only know about the length of an arc belonging to q according to (259), that they lie between $\frac{2\pi}{q \cdot n^{1-\alpha}} e^{-\frac{1}{n}}$ inclusively, and $\frac{4\pi}{q \cdot n^{1-\alpha}} e^{-\frac{1}{n}}$ exclusively; therefore one can only consider the minor arc as for sure smaller than the major arc, if one would observe those with a doubling magnifying glass.)

Let $\underline{\mathrm{m}}$ denote the minor arcs and let $\underline{\mathrm{M}}$ denote the major arcs. Then the auxiliary theorems over $\underline{\mathrm{m}}$ and $\underline{\mathrm{M}}$ are

$$\left|f(x) - \psi_\rho(x)\right| < D(k, \varepsilon, \alpha) n^{\frac{aK}{K+1}+\varepsilon} \quad \text{for } \underline{\mathrm{M}}$$

$$|f(x)| < C_1 n^{a-\frac{\alpha}{K}+\varepsilon} \quad \text{for } \underline{\mathrm{m}}.$$

The proof of the estimates is the real difficulty (taking about 20 pages). After that we will proceed with the following notation

$$r(n) = \frac{1}{2\pi i}\left[\sum_{\underline{\mathrm{m}}} \int_{\underline{\mathrm{m}}} \frac{f^s(x)}{x^n}\frac{dx}{x} + \sum_{\underline{\mathrm{M}}} \int_{\underline{\mathrm{M}}} \frac{f^s(x)}{x^n}\frac{dx}{x}\right].$$

Weyl's theorem (264) enabled Hardy and Littlewood to tame the minor arcs.

Theorem 264, (LaEa) p. 253: let $m > 0$, $r \ge 0$, m, r integers, $\lambda \in R$, and

$$S := \sum_{h=r+1}^{r+m} e^{2\pi\lambda h^k i}\ .$$

Then

$$|S|^K < 4^K \left[m^{K-1} + m^{K-k} \sum_{h_1, h_2, \dots h_k = 1}^{r+m} Min(m, \frac{1}{\{\lambda k! h_1 h_2 \dots h_{k-1}\}}\right].$$

In case $\lambda k!\, h_1 h_2 \dots h_{k-1}$ is entire, i.e. the denominator vanishes, $Min(m, \frac{1}{0})$ means the number m.

Remark (LaEb) p. 178: For the smallest positive primitive root of a prime number p it holds (Winogradoff)

$$g(p) = O\left(p^{\frac{1}{2}+\delta}\right), \delta > 0.$$

Aus der analytischen Zahlentheorie

(Translated) extracts from
„Lectures on number theory", Vol. 2 (LaEb)
E. Landau

Riemann's Hypothesis and Farey fractions
Introduction

p. 167: According to the corresponding Littlewood theorem, the RH is equivalent to the relation

$$M(x) = \sum_{n\le x}\mu(n) = O(x^{\frac{1}{2}+\varepsilon}) \text{ for any } \varepsilon > 0$$

resp.

(645) $$M(N) = O(N^{\frac{1}{2}+\varepsilon}) \text{ for any } \varepsilon > 0 \text{ and } N \in N.$$

(resp. $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}$ is convergent for $\sigma > 1/2$). This is already a property of the Farey fractions, but it is a transcendental one. This is because of (theorem 220) $\mu(q)$ is the sum of all primitive unity roots

$$\mu(q) = \sum_{\substack{1\le a\le q\\(a,q)=1}} e^{2\pi i\frac{a}{q}};$$

Therefore it is

$$M(N) = \sum e^{2\pi i\frac{a}{q}},$$

where $\frac{a}{q} \in]0,1]$ runs through all fractions belonging to Farey series F_n; their total number is

$$A = A(N) = \sum_{q=1}^{N}\varphi(q).$$

Let $r_1 < r_2 < \cdots . < r_v < \cdots r_A$ denote the ordered sequence of those fractions ($r_v = r_v(N), r_v(A) = 1$) then it holds

$$M(N) = \sum_{v=1}^{A} e^{2\pi i r_v},$$

and therefore, because of $M(N)$ is real,

$$M(N) = \sum_{v=1}^{A}\cos(2\pi i r_v).$$

The formula $\sum_{v=1}^{A}\cos(2\pi i r_v) = O(N^{\frac{1}{2}+\varepsilon})$ is therefore equivalent to the RH. As there is the cos function, I call this property a transcendental one.

Franel has discovered the following nice theorem:

He compared the A Farey fractions r_v $(v = 1, \ldots A)$ with the numbers $\frac{v}{A}$ (which are built by dividing the interval $]0,1]$ in A equal parts)

$$\delta_v = \delta_v(N) := r_v - \frac{v}{A}.$$

By analyzing the sum of the errow squares he found that the RH is equivalent to the relation

(647) $$\sum_{v=1}^{A}\delta_v^{\,2}(N) = O(N^{-1+\varepsilon}) \text{ for any } \varepsilon > 0 \text{ and } N \in N.$$

I add, that (*) is equivalent to

(648) $\sum_{v=1}^{A}|\delta_v(N)| = O(N^{\frac{1}{2}+\varepsilon})$ for any $\varepsilon > 0$ and $N \in N$.

The proof that (648) follows from (647) is trivial. This is because of

$$A = A(N) = \sum_{q=1}^{N} \varphi(q) \leq \sum_{q=1}^{N} q \leq \sum_{q=1}^{N} N = N^2$$

and

$$\sum_{v=1}^{A}|\delta_v| \leq \sqrt{A \sum_{v=1}^{A} {\delta_v}^2} \leq N\sqrt{\sum_{v=1}^{A} {\delta_v}^2}$$

and

$$N\sqrt{\frac{1}{N^{1-\varepsilon}}} = N^{\frac{1}{2}+\frac{\varepsilon}{2}} \leq N^{\frac{1}{2}+\varepsilon}.$$

From (648) it follows (645), and therefore from (648) follows the RH (not from the Littlewood theorem, but from its trivial reversal).

The proof that (647) follows from (648) is shown by proving the following statements

i. From (648) it follows the RH
ii. If the RH is true, then (647) it true.

The proof of i., i.e. that the RH follows from (648), is very much easy:

$$M(N) = \sum_{v=1}^{A} e^{2\pi i r_v} = \sum_{v=1}^{A} e^{2\pi i \frac{v}{A}} e^{2\pi i \delta_v} =$$

$$= \sum_{v=1}^{A} e^{2\pi i \frac{v}{A}} + \sum_{v=1}^{A} e^{2\pi i \frac{v}{A}}\left(e^{2\pi i \delta_v} - 1\right) = \sum_{v=1}^{A} e^{2\pi i \frac{v}{A}}(e^{2\pi i \delta_v} - 1) .$$

$$|M(N)| \leq 2\sum_{v=1}^{A}|\sin(\pi\delta_v)| \leq 2\pi\sum_{v=1}^{A}|\delta_v|.$$

The proof of ii. is the results from a list of auxiliary theorems to enable a proof of the

Theorem 491: If the RH is true, then

$$I := I(N) := \int_0^1 G^2(x)dx \, (= \|G(N)\|_{L_2(0,1)}^2) = O(N^{1+\varepsilon})$$

with

$$G(x) := G(x, N) := g(x) - Ax + \frac{1}{2} ,$$

where $g(x)$ denotes the counting function of all numbers r_v, which are $\leq x$, i.e. $g(x) := \#\{r_v | r_v \leq x, x \in [0,1]\}$.

Auxiliary Theorems 485/486/487

i. $g(x) = g(x, N) = \sum_{d=1}^{N} [dx] M(\frac{N}{d})$

ii. $\sum_{d=1}^{N} d \cdot M(\frac{N}{d}) = A$

iii. $G(x) = G(x, N) = -\sum_{d=1}^{N} f(d \cdot x) M(\frac{N}{d})$

with

$$f(x) \coloneqq x - [x] - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{\sin(2\pi n x)}{\pi n} ,$$

fulfilling (auxiliary theorem 483)

$$\sum_{l} f\left(x + \frac{1}{n}\right) = f(nx).$$

Proof of i. (theorem 485): According to theorem 35 it holds for $y \geq 1$

$$(654) \qquad \sum_{n \leq y} M\left(\frac{y}{n}\right) = \sum_{n \leq y} \sum_{m \leq \frac{y}{n}} \mu(m) = \sum_{\substack{m,n \\ mn \leq y}} \mu(m) = \sum_{k \leq y} \sum_{m|k} \mu(m) = 1.$$

(653) is therefore true for $x = 0$. The left side jums only at the entrance into a Farey fraction; the right side as well, only in this case; for $x = \frac{a}{q}$ the jump on the right side is according to (654)

$$\sum_{\dots \leq N/q} M\left(\frac{\frac{N}{q}}{n}\right) = 1 ,$$

because on the right side all terms with $\frac{q}{d}$ come into consideration.

I add two additional little things:

Theorem 492: If the RH is true, then

(655) $$\sum_{v=1}^{A} e^{2\pi i\frac{v}{A}}\delta_v = O(N^{1+\varepsilon})$$

and vice versa.

Remark: (655) is not weaker than (648) (i.e. $\sum_{v=1}^{A}|\delta_v(N)| = O(N^{\frac{1}{2}+\varepsilon})$), because of

$$\left|\sum_{v=1}^{A} e^{2\pi i\frac{v}{A}}\right| \leq |\delta_v|.$$

Proof:

1) The RH is assumed to be true; than it follows (648), and therefore (655).

2) For $-2\pi \leq x \leq 2\pi$ it holds $\left|e^{ix} - 1 - ix\right| \leq \alpha_{200}x^2$.

According to the theorems 488, 489 and 490 it holds

$I \leq (N^2)$ (Th. 488), $\sum_{v=1}^{A} {\delta_v}^2(N) = \frac{1}{A}(I - \frac{1}{12}) < \frac{I}{A}$ (Th. 489),

$\lim\limits_{N\to\infty} \frac{A(N)}{N^2} > 0$ (Th. 490) (because of $\frac{1}{\zeta(2)} = \sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} > 0$)

and therefore

$$\sum_{v=1}^{A} {\delta_v}^2(N) = O(1).$$

From (650), (656) and (657) it therefore follows for $N > 1$ (because of $|2\pi\delta_v| \leq 2\pi$)

$$\left|\sum_{v=1}^{A} e^{2\pi i r_v} - 2\pi i \sum_{v=1}^{A} e^{2\pi i\frac{v}{A}}\,\delta_v\right| = \left|\sum_{v=1}^{A} e^{2\pi i\frac{v}{A}}\,(\sum_{v=1}^{A} e^{2\pi i\delta_v} - 1 - 2\pi i\delta_v)\right|$$

$$\leq \alpha_{200} \cdot 4\pi^2 \sum_{v=1}^{A} {\delta_v}^2 = O(1).$$

So from (655) and (646) it follows (645), and therefore the RH.

Über eine trigonometrische Summe

E. Landau, (LaE1)

Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen,
Mathematisch-Physikalische Klasse, Vol. 1928, p. 21-24, 1928

Theorem (LaE1): let $m > 1$, β_1 real, $0 < \vartheta \leq \beta_2 - \beta_1 \leq \beta_3 - \beta_2 \leq \cdots \beta_{m-1} - \beta_m \leq 1 - \vartheta$, $\sigma_n := e^{-2\pi i\beta_n}$, $S_m := |\sum_{n=1}^{m} \sigma_n|$. Then it holds:

i) $S_m \leq \cot(\frac{\pi}{2}\vartheta)$;

ii) For $\vartheta = 1/2$ and every positive fraction $\vartheta < 1/2$ with odd nominator and odd denominator:

$$S_m = \cot(\frac{\pi}{2}\vartheta);$$

iii) For all other ϑ with $0 < \vartheta < 1/2$: $S_m < \cot(\frac{\pi}{2}\vartheta)$;

iv) For all ϑ with $0 < \vartheta \leq 1$ and every $\varepsilon > 0$: $S_m > \cot\left(\frac{\pi}{2}\vartheta\right) - \varepsilon$.

The Farey dissection of the continuum

Extract from
„An Introduction to the Theory of Number"
G. H. Hardy, E. M. Wright, (HaG)

3.1: The Farey series F_n of order n ist h ascending seires of irreducible fractions between 0 and 1 whose denominators do not exceed n. Thus h/k belongs to F_n if

$$0 \leq h \leq k \leq n, \; (h,k) = 1;$$

The numbers 0 and 1 are included in the forms $\frac{0}{1}$ and $\frac{1}{1}$. For example, F_5 is

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}.$$

The characteristic properties of the Farey series are expressed by the following theorems.

Theorem 28: If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of F_n, then $kh' - hk' = 1$.

Theorem 29: If $\frac{h}{k}$, $\frac{h''}{k''}$ and $\frac{h'}{k'}$ are three successive terms of F_n, then $\frac{h''}{k''} = \frac{h+h'}{k+k'}$.

We shall prove that the two theorem are equivalent in the next section. … We conclude this section by proving two still simpler properties of F_n.

Theorem 30: If $\frac{h}{k}$ and $\frac{h'}{k'}$ are two successive terms of F_n, then $k + k' > n$. The „mediant"

$$\frac{h+h'}{k+k'}$$

of $\frac{h}{k}$ and $\frac{h'}{k'}$ falls into the interval

$$\left(\frac{h}{k}, \frac{h'}{k'}\right).$$

Theorem 31: If $n > 1$ then no two successive terms of F_n have the same denominator.

3.8: It is often convenient to represent the real numbers on a circle instead of, as usual, on a straight line, the object of the circular representation being to eliminate integral parts. We take a circle C of unit circumference, and an arbitrary point O of the circumference as the representive of 0, and represent x by the point P_x whose distance from O, measured round the circumference in the counter-clockwise direction, is x. Plainly integers are representated by the same point O, and numbers which differ by an integer have the same representative point.

It is sometime useful to divide up the circumference of C in the following manner. We take the Farey series F_n, and form all mediants

$$\mu = \frac{h+h'}{k+k'}$$

of successive pairs $\frac{h}{k}$, $\frac{h'}{k'}$. The first and last mediants are

$$\frac{0+1}{1+n} = \frac{1}{n+1}, \frac{n-1+1}{n+1} = \frac{n}{n+1}.$$

The mediants naturally do not belong themselves to F_n.

We now represent each mediant μ by the point P_x. The circle is thus divided up into arcs which we call Farey arcs, each bounded by two points P_μ and containing one Farey point, the representative of a term of F_n. Thus

$$\left(\frac{n}{n+1}, \frac{1}{n+1}\right)$$

Is a Farey arc containing the one Farey point O. The aggregate of Farey arcs we call the Farey dissection of the circle. In what follows we suppose that $n > 1$.

> **Theorem 35**: In the Farey dissection of order n, where $n > 1$, each part of the arc which contains the representative of $\frac{h}{k}$ has a length between
>
> $$\frac{1}{k}\frac{1}{(2n+1)} \text{ and } \frac{1}{k}\frac{1}{n+1}.$$

The dissection, in fact, has a certain „uniformity“ which explains its importance.

We use the Farey dissection here to prove a simple theorem concerning the approximation of arbitrary real numbers by rationals, a topics tow hich wes hall reture in Ch. XI.

> **Theorem 36**: If ξ is any real number, and n a positive integer, then there is an irreducible fraction $\frac{h}{k}$ such that
>
> $$\left|\xi - \frac{h}{k}\right| \leq \frac{1}{k}\frac{1}{n+1}.$$

Orthogonal polynomials on the unit circle

Extract from

„Orthogonal Polynomials“

G. Szegö

11.1 Definition; preliminaries

Let $f(\theta)$ be a non-negative function of period 2π, integrable on $[-\pi,\pi]$ in Lebesgue's sense, and assume $\int_{-\pi}^{\pi} f(\theta)d\theta > 0$. We introduce the Fourier constants

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cdot e^{-in\theta}d\theta \quad n = 0, \pm 1, \pm 2, \ldots.$$

Obviously, $c_{-n} = \bar{c}_n$so that the matrix of "Toeplitz type" $T_n = (c_{\nu-\mu}),\ \nu,\mu = 0,1,2,\ldots n$ is Hermitian. The corresponding Hermitian form

$$H_n = \sum_{\nu=0}^{n}\sum_{\mu=0}^{n} c_{\nu-\mu}u_\mu\bar{u}_\nu = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cdot|u_0 + u_1 z + u_2 z^2 + \ldots + u_n z^n|^2 d\theta$$

where $z = e^{i\theta}$, is positive definite and has the positive determinant $D_n = [c_{\nu-\mu}],\ \nu,\mu = 0,1,2,\ldots n$.

Definition: If we orthogonalize the system

$$\{z^n\cdot\sqrt{f(\theta)}\}\ ,\ z = e^{i\theta}\quad n = 0,1,2,\ldots n$$

we obtain a system of polynomials $\varphi_0(z), \varphi_1(z), \varphi_2(z), \ldots.\varphi_n(z), \ldots$ with the following properties:

i) $\varphi_n(z)$ is a polynomial of precise degree n in which the coefficients of z^n is real and positive;

ii) the system $\{\varphi_n(z)\}$ is orthogonal; that is,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\varphi_n(z)\bar{\varphi}_m(z)d\theta = \delta_{nm}\ ,\ z = e^{i\theta}\ ,\ n,m = 0,1,2,\ldots$$

Moreover, the system $\{\varphi_n(z)\}$ is uniquely determined by the conditions i) and ii). If $f(\theta)$ is an even function the coefficients of $\varphi_n(\mathrm{z})$ are real. The coefficient of z^n in $\varphi_n(z)$ is $\sigma_n = \sqrt{D_{n-1}D_n^{-1}}$.

Theorem 11.1.1: Let $F(e^{-i\theta})$ be a given measurable function for which $\int_{-\pi}^{\pi} f(\theta)\left|F(e^{-i\theta})\right|^2 d\theta$ exists. The weighted quadratic deviation

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)|F(z) - \rho(z)|^2 d\theta$$

where $\rho(z)$ ranges over the set of all (polynomials of degree n) π_n, is a minimum, if $\rho(z)$ is the n^{th} partial sum of the Fourier expansion

$$F(z) \sim F_0\varphi_0(z) + F_1\varphi_1(z) + \cdots. + F_n\varphi_n(z) + ..$$

$$F_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)F(z)\overline{\varphi_n(z)}d\theta\ ,\ ,\ z = e^{i\theta}\ ,\ n = 0,1,2,\ldots$$

As a ready consequence, there follows Bessel's inequality

$$|F_0|^2 + |F_1|^2 + \cdots + |F_n|^2 + \cdots. \leq \int_{-\pi}^{\pi} f(\theta)\left|F\left(e^{-i\theta}\right)\right|^2 d\theta.$$

In addition, the equality sign holds (i.e. the Parseval's formula) if one of the following sets of conditions is satisfied:

i) $\mathrm{F(z)}$ is regular and bounded for $|z| < 1$

ii) $f(\theta)$ is bounded and $\mathrm{F(z)}$ is of the class H_2 (see §10.1; Fatou's theorem)

A consequence of theorem 11.1.1 is the following

Theorem 11.1.2: The polynomial $\sigma_n^{-1}\varphi_n(z)$ minimizes the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta) \cdot |z^n + a_1 z^{n-1} + a_2 z^{\mathrm{n}-2} + \ldots + a_n|^2 d\theta,\ z = e^{i\theta}$$

if $z^n + a_1 z^{n-1} + a_2 z^{\mathrm{n}-2} + \ldots + a_n$ ranges over the set of all π_n with the highest term z^n. The minimum is σ_n^{-2}.

References

(AbM) Abramowitz M., Stegen I. A., Handbook of Mathematical Functions, Dover Publications, Inc., New York, 1965

(ApT) Apostel T. M., Modular Functions and Dirichlet Series in Number Theory, Springer-Verlag, New York, Berlin, Heidelberg, 2000

(BeB) Berndt B. C., Andrews G. E., Ramanujan´s Notebooks Part I, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1989

(BeB1) Berndt B. C., "Ramanujan´s Notebooks Part IV", Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1994

(BrK) Braun K., Looking back, part A, (A1) - (A3)

(BuH) Buchholtz H., The Confluent Hypergeometric Function, Springer-Verlag, Berlin, Heidelberg, New York, 1969

(CaD) Cardon D. A., Convolution operators and zeros of entire functions, Proc. Amer. Math. Soc., 130, 6 (2002) 1725-1734

(DuR) Duffin R. J., Weinberger H. F., Dualizing the Poisson summation formula, Proc. Natl. Acad. Sci. 88 (1991) pp. 7348-7350

(EdH) Edwards, Riemann's Zeta Function, Dover Publications, Inc., Mineola, New York, 2001

(ElL) Elaissaoui L., Guennoun Z. El-Abidine, Relating log-tangent integrals with the Riemann zeta function, arXiv, May 2018

(ElL1) Elaissaoui L., Guennoun Z. El-Abidine, Evaluation of log-tangent integrals by series involving zeta(2n+1), arXiv, May 2017

(GrI) Gradshteyn I. S., Ryzhik I. M., Table of Integrals Series and Products, Fourth Edition, Academic Press, New York, San Francisco, London, 1965

(HaG) Hardy G. H., Wright E. M., An Introduction to the Theory of Numbers, Oxford University Press, Oxford, 2008

(HaG1) Hardy G. H., The General Theory Of Dirichlets Series, Cambridge University Press, Cambridge, 1915

(IvS) Ivanov S. A., Nonharmonic Fourier series in the Sobolev spaces of positive fractional orders, New Zealand Journal of Mathematics, Vol. 25, p. 39-46, 1996

(KiA) Kienast A., Untersuchungen über die Lösungen der Differentialgleichung $xy'' + (\gamma - x)y' - \beta y = 0$, Mitt. Naturforsch. Ges. Bern, 57, 247-325, (1921)

(KrR) Kress R., Linear Integral Equations, Springer-Verlag, Berlin, Heidelberg, New York, 1989

(LaEa) Landau E., Vorlesungen über Zahlentheorie, Aus der additiven Zahlentheorie, 5. & 6. Teil, Chelsea Publishing Company, New York, 1927

(LaEb) Landau E., Vorlesungen über Zahlentheorie, Aus der analytischen und geometrischen Zahlentheorie, 7. & 8. Teil, Chelsea Publishing Company, New York, 1927

(LaEc) Landau E., Vorlesungen über Zahlentheorie, Aus der algebraischen Zahlentheorie und über die Fermatsche Vermutung, 9.-13. Teil, Chelsea Publishing Company, New York, 1927

(LaE1) Landau E., Über eine trigonometrische Summe, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, Vol. 1928, p. 21-24, 1928

(LaE2) Landau E., Die Goldbachsche Vermutung und der Schnirelmannsche Satz, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse, 255-276, 1930

(LaE3a) Landau E., Handbuch der Lehre von der Verteilung der Primzahlen I, Teubner Verlag, Leipzig, Berlin, 1090

(LaE3b) Landau E., Handbuch der Lehre von der Verteilung der Primzahlen II, Teubner Verlag, Leipzig, Berlin, 1090

(LeB) Levin B. Y., Lectures on Entire Functions, American Mathematical Society, 1996

(LuX) Luo X.-D., Lin W.-C., Some new properties of Confluent Hypergeometric Functions, arXiv:1509.06465v1

(MaJ) Mashreghi, J., Hilbert transform of log(abs(f)), Proc. Amer. Math. Soc., Vol 130, No 3, p. 683-688, 2001

(McC) McMullen C., Complex Analysis, Course Notes, Harward University Fall 2000, 2006, 2010

(NaC) Nasim C., On the summation formula of Voronoi, Trans. Amer. Math. Soc., 163 (1972) 35-45

(NiN) Nielsen N., Handbuch der Theorie der Gammafunktion, Chelsea Publishing Company, New York, 1965

(OlF) Olver F. W. J., Asymptotics and Special Functions, Academic Press, Inc., Boston, San Diego, New York, London, Sydney, Tokyo, Toronto, 1974

(PaJ) Partington J., R., Linear Operators and Linear Systems, Cambridge University Press, Cambridge, 2004

(PeB) Petersen B. E., Introduction to the Fourier Transform & Pseudo-differential Operators, Pitman Publishing Limited, Boston, London, Melbourne

(PoG1) Polya G., Über eine neue Weise bestimmte Integrale in der analytischen Zahlentheorie zu gebrauchen, Göttinger Nachr. (1917) 149-159

(PrK) Prachar K., Primzahlverteilung, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1957

(RaH) Rademacher H., Topics in Analytic Number Theory, Springer-Verlag, Berlin, Heidelberg, New York, 1973

(SeA) Sedletskii A. M., Asymptotics of the Zeros of Degenerated Hypergeometric Functions, Mathematical Notes, Vol. 82, No. 2, 229-237, 2007

(SeAt) Selberg A., Collected Papers I, Springer, Heidelberg, New York, Dortrecht, London, 2014

(SeP) Sebah P., Gourdon X., Introduction to the Gamma function, GammaFunction.dvi (ntu.edu.tw)

(SlL) Slater L. J., Confluent Hypergeometric Functions, Cambridge University Press, Cambridge, 1960

(TiE) Titchmarsh E. C., The Theory of the Riemann Zeta-Function, Oxford Science Publications, Clarendon Press, Oxford, 1986

(WhE) Whittaker E. T., Watson G. N., A Course of Modern Analysis, Cambridge University Press, Cambridge, 1984

(WaG) Watson G. N., A Treatise on the Theory of Bessel Functions, Cambridge University Press, Cambridge, 2nd edition first published 1944, reprinted 1996, 2003, 2004, 2006

(YoR) Young R. M., An Introduction to Nonharmonic Fourier Series, Pure and Applied Mathematics, Academic Press Inc., New York, 1980