Proceedings of the Third International Conference on Finite Elements in Water Resources, The University of Mississippi, University, Mississippi, S.Y. Wang, C.V. Alonso, C.A. Brebbia, W.G. Gray and G.F. Pinder eds., 2.270-2.281 (1980)

THE FLOW OF TWO IMMISCIBLE FLUIDS IN A ONE-DIMENSIONAL POROUS MEDIUM AND FINITE ELEMENTS

J.A. Nitsche

3rd International Conference on
Finite Elements in Water Resources
The University of Mississippi
May 19-23, 1980

1. Introduction

The mathematical formulation of the physical phenomenon is

Problem P_U: Find a pair U,s defined in $I \times \tau$, τ with

(1.1) $$I = (-1,1) \; , \quad \tau = (0,T)$$

according to

(1.2) $$U_t = \begin{cases} U_{yy} & \text{in } (-1,s) \times \tau \; , \\ \alpha U_{yy} & \text{in } (s,1) \times \tau \; , \end{cases}$$

(1.3) $$U(\pm 1,.) = f_{\pm} \quad \text{in } \tau \; ,$$

(1.4) $$U(.,0) = g \quad \text{in } I$$

together with

(1.5) $$U(s(t)^-,t) = U(s(t)^+,t) \quad \text{in } \tau \; ,$$

(1.6) $$-\dot{s}(t) = U_y(s(t)^-,t) = U_y(s(t)^+,t) \quad \text{in } \tau \; ,$$

(1.7) $$s(0) = 0 \; .$$

With respect to the physical interpretation as well as the question of existence and unicity we refer to [1] - [4] . The moving of the interface $s(t)$ together with the transient conditions cause special difficulties.

In [8] - [11] we analyzed a finite element method for another class of free boundary problems which covers the three problems discussed in [5] . On the one hand this approach gives a tool for the numerical solution. On the other hand it can be used in order to derive existence and regularity results in an elementary way.

The main features of the method are: (i) The free boundary problem is transformed to a (nonlinear) boundary problem with fixed boundary. (ii) The solution u is characterized by a variational equation of the type (H being an appropriate Hilbert-space)

Problem P: Find $u = u(.,t) \in H$ according to

(1.8) $$a(\dot{u},v) + b(u,v) = N(u,v) + L(v)$$

for all $v \in H$ and $t \in \tau$ with $u(.,0)$ given.

Here a respective b denote positive definite respective semi-definite bilinear forms and the bounded, Frechet differentiable nonlinearity N admits an estimate

(1.9) $$N(v,v) \le \delta\, b(v,v) + c\, a(v,v)^{\gamma}$$

with $\delta < 1$ and some $\gamma > 0$. The finite element method P_h defines for a given approximation space S_h an approximation $u_h \in S_h$ via (1.8) with v restricted to S_h. By more or less standard arguments the existence of a time interval τ can be proved such that problems P and P_h admit unique solutions in τ.

The difference e between the solutions u and u_h, i.e. the error of the method, solves an equation of the form

(1.10) $$a(\dot{e},\chi) + b(e,\chi) = N_u'(e,\chi) + Q(e,\chi) \quad \text{for} \quad \chi \in S_h$$

with Q being at least quadratic in e. The error analysis is obvious provided a Gårding type inequality

(1.11) $$|N_u'(v,v)| \le b(v,v) + \Lambda\, a(v,v)$$

for some $\Lambda \in \mathbb{R}$ holds.

In this paper we develop such a method for problem P_U. The error analysis and the application to an existence proof etc. will be given elsewhere.

2. A Weak Formulation

The linear transformations

$$(2.1)\qquad \begin{aligned} y &= -1 + (1+s)(1+x) \ , \\ y &= 1 - (1-s)(1-x) \end{aligned}$$

map the intervals $I_- = (-1,0)$ resp. $I_+ = (0,1)$ onto $(-1,s)$ resp. $(s,1)$. We introduce the new unknown u by

$$(2.2)\qquad u(x,t) = U(y,t) \ .$$

Then P_U leads to

Problem P_u: Find a pair u,s defined in $I \times \tau$, τ according to

$$(2.3)\qquad u_t = \begin{cases} (1+s)^{-2}u_{xx} + (1+s)^{-1}(1+x)\,\dot{s}\,u_x & \text{in } I_- \times \tau \ , \\ \alpha(1-s)^{-2}u_{xx} + (1-s)^{-1}(1-x)\,\dot{s}\,u_x & \text{in } I_+ \times \tau \ , \end{cases}$$

$$(2.4)\qquad u(\pm 1,t) = f_\pm(t) \quad \text{in } \tau \ ,$$

$$(2.5)\qquad u(x,0) = g(x) \quad \text{in } I \ ,$$

together with

$$(2.6)\qquad u_- = u_+ \qquad \text{in } \tau \ ,$$

$$(2.7)\qquad -\dot{s} = (1+s)^{-1}u_-' = \alpha(1-s)^{-1}u_+' \quad \text{in } \tau \ ,$$

$$(2.8)\qquad s(0) = 0 \ .$$

Here and in the following u_+ etc. means

$$(2.9)\qquad u_+ = u_+(t) = \lim_{x \to 0^+} u(x,t)$$

and $\dot{u}, u'$ will denote the derivatives with respect to time and space.

In the case of a 'classical solution' all the functions entering P_u are continuous. Obviously P_u remains meaningful if u and s have the reduced regularity

(2.10) RR

i_1) $u \in L_\infty(\tau, \dot{H}_2)$ with

$$\dot{H}_2 = H_1(I) \cap \{H_2(I_-) \cup H_2(I_+)\} \ ,$$

i_2) $\dot{u} \in L_\infty(\tau, H_1(I))$,

ii) $s \in W^1_\infty(\tau)$.

Remark: Condition i_2 could be lowered but this is the adequate one for our purposes.

We will reformulate condition (2.7). Let us introduce the functionals

$$(2.11)\qquad \begin{aligned} \lambda(u) &= \lambda(s;u) := \tfrac{1}{2}\{\alpha(1-s)^{-1}u_+' - (1+s)^{-1}u_-'\} \\ \mu(u) &= \mu(s;u) := \tfrac{1}{2}\{\alpha(1-s)^{-1}u_+' + (1+s)^{-1}u_-'\} \ . \end{aligned}$$

Then (2.7) can be splitted into

$$(2.12)\qquad \begin{aligned} \lambda(s;u) &= 0 \quad ,\\ \dot{s} &= -\mu(s;u) \quad . \end{aligned}$$

Multiplication of (2.3) by w'' with $w \in \dot{H}_2$ and integration over $I_\pm$ lead to

$$(2.13)\qquad \begin{aligned} (\dot{u},w'')_- &= (1+s)^{-2}(u'',w'')_- + \dot{s}(1+s)^{-1}((1+x)u',w'')_- \\ &= \dot{u}w'|_{-1}^{0} - (\dot{u}',w')_- \quad , \end{aligned}$$

$$(2.14)\qquad \begin{aligned} (\dot{u},w'')_+ &= \alpha(1-s)^{-2}(u'',w'')_+ + \dot{s}(1-s)^{-1}((1-x)u',w'')_+ \\ &= \dot{u}w'|_{0}^{1} - (\dot{u}',w')_+ \quad . \end{aligned}$$

Here $(.,.)_\pm$ are the L_2-inner products with respect to $I_\pm$. The contribution of u in the terms $\dot{u}w'$ at the points $x = \pm 1$ is given by the data (2.4). The continuity condition (2.6) implies

$$(2.15)\qquad \dot{u}_+ = \dot{u}_- =: \dot{u}_o \quad .$$

Therefore in (2.13) resp. (2.14) the terms $u_o w'_-$ resp. $-u_o w'_+$ enter. The variational formulation will be won by a linear combination of (2.13) and (2.14). For reasons which will become evident below we want to have a contribution $\dot{u}_o\ \lambda(s;w)$. Thus we have to multiply (2.13) by $(1+s)^{-1}$ and (2.14) by $\alpha(1-s)^{-1}$ and to add both equations. Then we get the relations taking into account (2.12_2)

$$(2.16)\qquad a(\dot{u},w) + b(u,w) = \mu(u)\ q(u,w) - 2\dot{u}_o\ \lambda(w) + L(w)$$

with

$$(2.17)\qquad a(v,z) = a(s;v,z) := (1+s)^{-1}(v',z')_- + \alpha(1-s)^{-1}(v',z')_+ \ ,$$

$$(2.18)\qquad b(v,z) = b(s;v,z) := (1+s)^{-3}(v'',z'')_- + \alpha^2(1-s)^{-3}(v'',z'')_+ \ ,$$

$$(2.19)\qquad \begin{aligned} q(v,z) = q(s;v,z) &:= (1+s)^{-2}((1+x)v',z'')_- + \\ &+ \alpha(1-s)^{-2}((1+x)v',z'')_+ \quad , \end{aligned}$$

$$(2.20)\qquad L(z) = L(s;z) := \alpha(1-s)^{-1}\ \dot{f}_+\ w'_{+1} - (1+s)^{-1}\ \dot{f}_-\ w'_{-1} \quad .$$

In order to get the needed coerciveness properties we have to replace $\dot{u}_o$. For the sake of symmetry we take the arithmetic mean of the right hand sides of (2.3). Using the fact $\lambda = 0$ and $\dot{s} = -\mu$ we come to

$$(2.21)\qquad \begin{aligned} 2\dot{u}_o = M(u) = M(s;u) &:= (1+s)^{-2}u''_- + \alpha(1-s)^{-2}u''_+ - \\ &- \alpha^{-1}(1+\alpha)\ \mu^2(u) \quad . \end{aligned}$$

Finally we may add a term

$$(2.22)\qquad \nu\ \lambda(u)\ \lambda(w)$$

since this vanishes anyway. Now we are ready to define

A pair u,s having the reduced regularity <u>RR</u> is a weak solution if for all $w \in \dot{H}_2$ and almost all $t \in \tau$

$$a(\dot{u},w) + b(u,w) + \gamma\,\lambda(u)\,\lambda(w) = \mu(u)\,q(u,w) - M(u)\,\lambda(w) + L(w) \qquad (2.23)$$

and in addition for almost all $t \in \tau$

$$\dot{s} = -\mu(u) \qquad (2.24)$$

together with the initial conditions

$$u(.,0) = g \quad \text{in } I \;, \qquad (2.25)$$

$$s(0) = 0 \;. \qquad (2.26)$$

A straight forward analysis shows for $\gamma \neq 0$:

Lemma 1: Let u,s be a weak solution and both functions sufficiently smooth then they solve P_u .

3. The Finite Element Method

Let for $h < 1$

$$\pi_h := \left\{x_i \,\middle|\, -1 = x_o < x_1 < \ldots < x_n = 1\right\} \qquad (3.1)$$

be a subdivision of I such that $x = 0$ is one of the knots and with some $\varkappa > 0$ fixed

$$\varkappa\, h \leq \inf\,(x_{i+1}-x_i) \leq \sup\,(x_{i+1}-x_i) \leq h \;. \qquad (3.2)$$

The approximation spaces we will work with consist of all splines of a fixed degree r continuous in I , continuously differentiable in I_+ and I_- and with a possible jump of the first derivative at $x = 0$. The finite element approximation is defined by

Problem P_{u_h}: Find a pair u_h, s_h with $u_h(.,t) \in S_h$ according to

$$a(s_h;\dot{u}_h,\chi) + b(s_h;u_h,\chi) + \gamma_h\,\lambda(s_h;u_h)\,\lambda(s_h;\chi) = \mu(s_h;u_h)\,q(s_h;u_h,\chi) - M(s_h;u_h)\,\lambda(s_h;\chi) + L(s_h;\chi) \qquad (3.3)$$

for $\chi \in S_h$ and $t \in \tau$,

$$\dot{s}_h = -\mu(s_h;u_h) \quad \text{for } t \in \tau \;, \qquad (3.4)$$

$$u_h(.,0) = P_h g \;, \qquad (3.5)$$

$$s_h(0) = 0 \qquad (3.6)$$

with P_h being the L_2-projection onto S_h .

Remark: The formulae are somewhat lengthy. But besides the point evaluations λ, μ and M only L_2-scalar-products are to be computed. Also a time stepping procedure does not cause difficulties.

In general, i.e. in the space $\dot{H}_2$, no coerciveness relation of the type (1.9) will hold. This is due to the fact that $M(u)$ depends on $u''_{\pm}$ whereas in $b(.,.)$ only the L_2-norm of u'' is covered. But still we can show

Lemma 2: Assume s_h is bounded away from ± 1 . There exists a constant γ_o such that for any choice

$$\gamma_h \geq \gamma_o h^{-1} \tag{3.7}$$

the term

$$\mu(s_h;\varphi)\, q(s_h;\varphi,\varphi) - M(s_h;\varphi)\, \lambda(s_h;\varphi) \tag{3.8}$$

appearing on th right hand side of (3.3) is bounded by

$$\delta\left\{b(s_h;\varphi,\varphi) + \gamma_h \lambda^2(s_h;\varphi)\right\} + c\left\{a(s_h;\varphi,\varphi)^2 + a(s_h;\varphi,\varphi)^3\right\} . \tag{3.9}$$

Proof: In the following $c_1, c_2, \dots$ denote numerical constants. If they are strictly positive we write $\underline{c}_\nu$. We will need the two a priori estimates:

Proposition 1: Let J be a bounded interval. For $v \in H_1(J)$ the L_∞-norm may be bounded by

$$\|v\|_{L_\infty(J)} \leq c_1\left\{\|v\|_{L_2(J)} + \|v\|^{1/2}_{L_2(J)} \|v'\|^{1/2}_{L_2(J)}\right\} . \tag{3.10}$$

Proposition 2: Let J be I_- resp. I_+ . Under the above assumptions on S_h inverse relations of the type

$$\|\varphi''\|_{L_\infty(J)} \leq c_2\, h^{-1/2} \|\varphi''\|_{L_2(J)} ,$$
$$\|\varphi''\|_{L_2(J)} \leq c_2\, h^{-1} \|\varphi'\|_{L_2(J)} \tag{3.11}$$

hold for $\varphi \in S_h$.

Remark: (3.11) is proved in [6] .

Because of the assumption there is a $c_3 < 1$ with

$$|s_h| \leq c_3 . \tag{3.12}$$

This gives at once

$$a(z,z) \geq \underline{c}_4 \|z'\|^2 ,$$
$$b(z,z) \geq \underline{c}_5 \|z''\|^2 . \tag{3.13}$$

$\|.\|$ denotes the $L_2(I)$-norm. For functions in the space $\dot{H}_2$ the second derivatives are defined in $I_{\pm}$ and are in $L_2(I_{\pm})$. In this way $\|z''\|$ has to be understood.

Further we find the bounds

$$|\mu(z)| \leq c_6\left\{\|z'\| + \|z'\|^{1/2} \|z''\|^{1/2}\right\} , \tag{3.14}$$

$$|q(z,z)| \leq c_7 \|z'\|\, \|z''\| . \tag{3.15}$$

For functions $\varphi \in S_h$ we get because of Proposition 2

$$(3.16) \quad |M(\varphi)| \le c_8\, h^{-1/2}\|\varphi''\| + c_9\{\|\varphi'\|^2 + \|\varphi'\|\,\|\varphi''\|\} \ .$$

In this way we come to the estimate of the first term in (3.8):

$$(3.17) \quad |\mu(\varphi)\, q(\varphi,\varphi)| \le c_{10}\{\|\varphi'\|^2\,\|\varphi''\| + \|\varphi'\|^{3/2}\,\|\varphi''\|^{3/2}\}$$

which by means of Young's inequality may be estimated by

$$(3.18) \quad |\mu(\varphi)\, q(\varphi,\varphi)| \le \delta\|\varphi''\|^2 + c_{11}(\delta)\{\|\varphi'\|^4 + \|\varphi'\|^6\}$$

with $\delta \in (0,1)$ arbitrary and $c_{11}(\delta)$ depending on the constants $c_1,\dots,c_{10}$ and on $1+\delta^{-1}$.

The second term in (3.8) may be estimated by

$$(3.19) \quad |M(\varphi)\, \lambda(\varphi)| \le c_{12}|\lambda(\varphi)|\,\{h^{-1/2}\|\varphi''\| + \|\varphi'\|^2 + \|\varphi'\|\,\|\varphi''\|\}$$

The first term on the right hand side is bounded by

$$(3.20) \quad c_{12}\, h^{-1/2}|\lambda(\varphi)|\,\|\varphi''\| \le \delta\|\varphi''\|^2 + \frac{1}{4\delta}\, c_{12}^2\, h^{-1}\,\lambda(\varphi)^2 \ .$$

The second causes no difficulties. The third can be estimated by

$$(3.21) \quad c_{12}|\lambda(\varphi)|\,\|\varphi'\|\,\|\varphi''\| \le h^{-1}\,\lambda(\varphi)^2 + c_{13}\, h\|\varphi'\|^2\|\varphi''\|^2 \ .$$

Because of Proposition 2 we get

$$(3.22) \quad h\|\varphi'\|^2\,\|\varphi''\|^2 \le c_2\|\varphi'\|^3\|\varphi''\| \le \delta\|\varphi''\|^2 + \frac{1}{4\delta}\, c_2^2\,\|\varphi'\|^6 \ .$$

This completes the proof.

<u>Remark:</u> The method discussed has its predecessor in [7] . In order to approximate the solution of the Dirichlet problem of a second order elliptic differential equation we constructed a bilinear-form defined for functions not necessarily fulfilling any boundary conditions. Whereas this form is in general indefinite it is positive definite in the approximation spaces by the aid of certain inverse properties.

Bibliography:

[1] CANNON, J.R. and A. FASANO: A nonlinear parabolic free boundary value problem. Ann. Mat. p. Appl. 112, 119-149 (1977).

[2] EVANS, L.C.: A free boundary problem: The flow of two immiscible fluids in a one-dimensional porous medium. I. Indiana Univ. Math. J. 26, 915-952 (1977).

[3] EVANS, L.C.: A free boundary problem: The flow of two immiscible fluids in a one-dimensional porous medium. II. Indiana Univ. Math. J. 27, 93-111 (1978).

[4] EVANS, L.C. and A. FRIEDMAN: Regularity and asymptotic behavior of two immiscible fluids in a one-dimensional porous medium. J. Differential Equations 31, 366-391 (1979).

[5] MAGENES, E.: Topics in parabolic equations: Some typical free boundary problems. Boundary value problems for linear partial differential equations, Proceedings of the NATO Advanced Study Institute held at Liège, Belgium, Sept. 6-17, 1976; H.G. Garnier ed., D. Reidel Publ. Comp., Dortrecht, 239-312 (1976).

[6] NITSCHE, J.A.: Umkehrsätze für Spline-Approximationen. Comp. Math. 21, 400-416 (1969).

[7] NITSCHE, J.A.: Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. Abh. d. Hamburger Math. Sem. 36, 9-15 (1971).

[8] NITSCHE, J.A.: Finite element approximation to the one-dimensional Stefan-problem. Proceedings on Recent Advances in Numerical Analysis, C. de Boor and G. Golub eds., Academic Press, New York, 119-142 (1978).

[9] NITSCHE, J.A.: Approximation des eindimensionalen Stefan-Problems durch finite Elemente. International Congress of Mathematicians, Helsinki, August 15-23 (1978).

[10] NITSCHE, J.A.: Finite element approximations for free boundary problems. 'TICOM Second International Conference on Computational Methods in Nonlinear Mechanics', Austin, Mar. 26-29 (1979).

[11] NITSCHE, J.A.: A finite element method for parabolic free boundary problems. Intensive Seminar on Free Boundary Problems, Pavia, Sept. 4-21 (1979).