A one-dimensional turbulent (dynamic fluid) viscosity flow model

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Abstract

An alternative (quantitative model to Kolmogorov's purely qualitative statistical turbulence model is provided. It takes into account the quantitative fluid behavior as its (statistically) described by the Euler or the Navier-Stokes equations. In order to highlight and focus on the new conceptual elements (and to avoid technical difficulties) we restrict ourself to the one-dimensional Constantin-Lax-Majda (CLM) vorticity equation with a viscosity term, ([MaA]. Based on the re-revisited generalized CLM equation with viscosity term we propose a turbulent flow model which allows non-stationary random functions with finite variance and related spectrum ([FrU] (4.54)) with respect to the $H_{1/2}$ *−energy norm. The central modification to the current "revisited CLM proposal" is identical to the alternatively proposed auxiliary function definition of* $v = H[u] = A[u_x]$ *in [BrK1]. The model allows wavelet synthesis according to [FaM], [FaM1].*

Prologs

M. Farge

"Fourier transform would be the appropriate tool to analyze the intrinsic structure of a turbulent flow if and only if the turbulent flow field is a superposition of waves. Only in this case are wave numbers well defined and the Fourier energy spectrum is meaningful for describing and modeling turbulence. If, on the contrary, turbulence were a superposition of point vortices then the Fourier spectrum in this case would be meaningless. The problem we still face in turbulence theory is that we have not yet identified the typical "object" that composes a turbulent field."

[FaM1]: "*The definition of the appropriate "object" that composes a turbulent field is still missing. It would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution. If the appropriate "object" has been defined that composes a turbulent field it would enable the study how turbulent dynamics transports these space-scale "atoms", distorts them, and exchanges their energy during the flow evolution. ...*

...The notion of "local spectrum" is antinomic and paradoxical when we consider the spectrum as decomposition in terms of wave numbers for as they cannot be defined locally. Therefore a "local Fourier spectrum" is nonsensical because, either it is non-Fourier, or it is nonlocal. There is no paradox if instead we think in terms of scales rather than wave numbers. Using wavelet transform then there can be a space-scale energy be defined with a correspondingly defined scale decomposition in the vicinity of location x and a correspondingly defined local wavelet energy spectrum. By integration this defines a local energy density and a global wavelet energy spectrum. The global wavelet spectrum can be expressed in terms of Fourier energy spectrum. It shows that the global wavelet energy spectrum corresponds to the Fourier spectrum smoothed by the wavelet spectrum at each scale. ...

... The concept enables the definition of a space-scale Reynolds number, where the average velocity is being replaced by the characteristics root mean square velocity $Re(l,x)$ *at scale l and location x. At large scale (i.e.* $l \approx L$*)* $Re(L)$ *coincides with the usual large-scale Reynolds number, where* $Re(L)$ *is defined as*

$$Re(L) = \iint_{R^n} Re(L,x)dx. \text{ "}$$

A simple one-dimensional turbulent flow model with viscosity term based on the revisited CLM vorticity equation in [MaA]

extract from original [BrK], 2016

Based on the re-revisited generalized CLM equation ([MaA] 5.2) with viscosity term we propose a turbulent flow model which allows non-stationary random functions with finite variance and related spectrum ([FrU] (4.54)) with respect to the $H_{1/2}$ −energy norm. The modification of current "revisited CLM proposals" is identical to the alternatively proposed auxiliary function definition of $v = H[u] = A[u_x]$ in [BrK1]. The model allows an wavelet synthesis according to [FaM], [FaM1].

Note: In [SaT1] for periodic boundary conditions the Fourier (spectral) representation of the non-linear term $\omega H[\omega] = \omega A[\omega_x]$, whereby ω denotes the vorticity and H the Hilbert transform operator. If the solution of the Euler equation is smooth then the solution to the slightly viscous NSE with same initial data is also smooth. Adding diffusion to the CLM model it makes the solution less regular [MuA]. As a consequence of this the CLM model lost most of the interest in the context of NSE analysis.

Note: In [MuA] a nonlocal diffusion term is proposed removing this drawback. The modification goes along with a reduced regularity of the "dissipation" term resulting in a reduced "energy" Hilbert scale of Hilbert scale factor $-1/2$. As this modification did not modify in same manner the non-linear term this leads to an unbalanced energy equation. As the non-linear term governs the dissipative term in case of turbulence, this is an argument to reject current revisited CLM model with viscosity term [DeS], [MuA], [OkH], [SaT], [SaT1]. At the same time those suggested modifications being applied in same manner to the linear term would fit to the Stieltjes integral based Kolmogorov theory [ShA], as well as to the conceptual idea of this paper (i.e. an $H_{1/2}-$ energy inner product enabling an energy inequality which does not exclude any information from the non-linear term). Combining both conceptual ideas provides a functional analytical common framework ([BrK]) for a statistical fluid mechanics theory [MoA], statistics of gases and highly turbulent fluid flows [HoE].

The building concept of the revisited generalized CLM model is therefore as follows: we consider periodic boundary condition and assume that the initial condition of ω is symmetric with respect to the origin ([SaT1]). We propose a weak $H_{-1/2}-$ variation representation of the extended Schochet-CLM model ([ScS]) in the form

$$(\dot{\omega}, v)_{-1/2} - \varepsilon(\omega_{xx}, v)_{-1/2} = (\omega H[\omega], v)_{-1/2}\,, \forall v \in H_{-1/2}.$$

With the notation of [BrK] this representation is equivalent to

$$(A\dot{\omega}, v)_0 + \varepsilon(H[\omega], v)_{1/2} = (A\big[\omega H[\omega]\big], v)_0\,, \forall v = H[w] \in H_{-1/2}.$$

Taken into account that the Hilbert transform is an isometry on all Hilbert scales and that $H^2[v] = -v$ and putting $\omega_H := H[\omega]$ this can be reformulation in the form

$$(\dot{\omega}_H, w)_{-1/2} + \varepsilon(\omega, w)_{1/2} = (H\big[\omega H[\omega]\big], w)_{-1/2}\,, \forall w \in H_{-1/2}.$$

From [MaA] we recall the identity

$$2H\big[\omega H[\omega]\big] = \omega_H^2 - \omega^2$$

leading to

$$(\dot{\omega}_H - \varepsilon\omega', w)_{-1/2} = \frac{1}{2}(\omega_H^2 - \omega^2, w)_{-1/2}\,, \forall w \in H_{-1/2}.$$

The left hand side of the variation representations above is reflecting to current revisited proposals of the CLM model, while now the right hand side of the variation equation shows a modified non-linear CLM model operator (as the domain has changed).

The spectral method analysis of the equation follows the same way as in ([SaT1]) leading to:

$$\dot{\omega}_n \approx n(\varepsilon\omega_n + \sum_{k=1}^{n-1} \omega_k \omega_{n-k}) \quad , \quad \omega_n(0) = \frac{A_n}{2}$$

whereby

$$\omega(x,0) = \sum_1^\infty A_n \, sin(nx).$$

The spectral analysis above is also linked to the solution framework of [BrK3]. The Hilbert transform of the Gaussian is the Dawson function, which is norm equivalent to the Gaussian due to the related property of the Hilbert transform. Therefore a Dawson basis function based Hilbert space framework enables an alternative statistical hydromechanics.

[FaM]: "*The turbulent regime develops when the non-linear term of the NSE strongly dominates the linear term. Superposition principle holds no more for non-linear phenomena. Therefore turbulent flows cannot be decomposed as a sum of independent subsystems that can be separately studied. A wavelet representation allows analyzing the dynamics in both space and scale, retaining those degrees of freedom which are essential to compute the flow evolution*".

[MeM] "*Methods based on wavelet (Galerkin) expansions in L_2 framework face the issue that in Galerkin methods the degrees of freedom are the expansion coefficients of a set of basis functions and these expansion coefficients are not in physical space (means in wavelet space). First map wavelet space to physical space, compute non-linear term in physical space and then back to wavelet space, is not very practical*".

The $cot(\circ)$ (with its distributional Fourier series representation) and the first derivative of the Dawson function are proposed candidates for a wavelet as element of the complementary subspace $H_{-1/2} - H_0$ of L_2. The Hilbert transform is an isomorphism on any Hilbert scale H_β. Therefore the Hilbert transformed $cot(\circ)$ is a wavelet, as well ([WeJ]).

Following the concept of [FaM] the turbulent $H_{-1/2}$ –signal can be split into two contributions: coherent bursts, corresponding to that part of the signal which can be compressed in a H_0 –wavelet basis, plus incoherent noise, corresponding to that part of the signal which cannot be compressed in a H_0 –wavelet basis, but in the $H_{-1/2}$ –wavelet basis. For the $n = 1$ periodic case the later one corresponds to the alternative zero-state energy model of the harmonic quantum oscillator.

We note from [ShA] that a homogenous random field is a stationary process $X(s)$ with the correlation function $R(t) = EX(s+t)X(s)$ of the process X. Ist spectral (Stieltjes integral) representation based on a finite spectral measure F (spectral distribution function) with "spectral density" dF , where $dF(k)$ is the contribution to the "energy" of the harmonics whose frequencies are within the interval $(k, k + dk)$. $F(k)$ is characterized by the properties "symmetry" ($dF(-k) = dF(k)$, "monotonicity" ($F(k) \leq F(l)$ for $k \leq l$), $R(0)$ "boundedness"). If X is a real-valued process, then the spectral "function" is symmetric with respect to the point $k = 0$. As a consequence for $G(k) = F(k) - F(-k)$ the correlation function $R(t)$ is given by

$$R(t) = \int_0^\infty cos(\lambda t) G(d\lambda).$$

This corresponds to a purely cos-Fourier series representation which is given by the Hilbert transformed $cot(\circ)$.

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