The Leray-Hopf and the 3D-curl operators
Extract from (LeN)

1. Introduction

The (Marcel) Riesz operators $(R_j)_{1\leq j\leq n}$ are the following Fourier multipliers (we use the notation $d\hat{u}$ for the Fourier transform of u: our normalization is given in the formula (3.1) of our appendix)

$$(\widehat{R_j u})(\xi) = \xi_j |\xi|^{-1}\hat{u}(\xi), R_j = \frac{D_j}{|D|} = (-\Delta)^{-1/2}\frac{\partial}{i\partial x_j}.$$

The R_j are selfadjoint bounded operators on $L^2(R^n)$ with norm 1. The riesz operators are the natural multidimensional generalization of the Hilbert transform, given by the convolution with $pv\frac{i}{\pi x}$ which is the one-dimensional Fourier multiplier by $sign(\xi)$. These operators are bounded on $L^p(R^n)$ for $1 < p < \infty$ and send L^1 into L^1_ω. However they are not continuous on the Schwartz class, because of the singularity at the origin. The Leray-Hopf projector (that projector is also called the Helmholtz-Weyl projector by some authors) is the following matrix valued Fourier multiplier, given by

$$\boldsymbol{P}(\xi) = Id - \frac{\xi\otimes\xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j\xi_k}{|\xi|^2})_{1\leq j,k\leq n} \quad , \boldsymbol{P} = Id - R\otimes R =: Id - \boldsymbol{Q}.$$

We can also consider the $n\times n$ matrix of operators given by $\boldsymbol{Q} := R\otimes R = (R_jR_k)_{1\leq j,k\leq n}$ sending the vector space of $L^2(R^n)$ vector fields into itself. The operator Q is selfadjoint and is a projection since $\sum_l R_l^2 = Id$ so that $Q^2 = \sum_l (R_jR_lR_lR_k)_{j,k} = Q$. As a result the (Leray-Hopf or Helmholtz-Weyl) operator

$$\boldsymbol{P} = Id - R\otimes R =: Id - \boldsymbol{Q} = Id - \frac{D\otimes D}{D^2} Id - \Delta^{-1}(\nabla\times\nabla)$$

is also an orthogonal projection; the operator is in fact the orthogonal projection onto the closed subspace of L^2 vecor fields with null divergence. We have for a vector field $\sum_l u_j\,\partial_j$, the identity $grad\ div\ u = \nabla(\nabla\cdot u)$, and thus

$$grad\ div = \nabla\otimes\nabla = \Delta\,\mathrm{R}\otimes R, \quad \text{so that}$$

$$\boldsymbol{Q} = R\otimes R = \Delta^{-1}\, grad\ div, \quad div\ R\otimes R = div,$$

which implies $div\ \boldsymbol{P}u = div\ u - div(R\otimes R)u = 0$, and if $div\ u = 0$, we have $\boldsymbol{Q}u = 0$ and $u = \boldsymbol{Q}u + \boldsymbol{P}u = \boldsymbol{P}u$. This operator plays an important role in fluid mechanics since the Navier-Stokes system for incompressible fluids can be written as

$$\begin{aligned} &\partial_t v + \boldsymbol{P}\big((v\cdot\nabla)v\big) - \nu\Delta v = 0 \\ &\boldsymbol{P}((v) = v, \\ &v_{|t=0} = v_0. \end{aligned} \tag{1.6}$$

(*) $\nu = \frac{\eta}{\rho}$ denotes the kinetical viscosity constant, while η denotes the dynamic viscosity constant, and ρ the density of the fluid).

As already said for the Riesz operators, $\boldsymbol{P}$ is not a classical pseudodifferential operator, because of the singularity at the origin; however it is indeed a Fourier multiplier with the same continuity properties as those of , and in particular is bounded on L^p for $p\in(1,\infty)$. In three dimensions the **curl** operator is given by the matrix

$$\boldsymbol{curl} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix} = \boldsymbol{curl}^*$$

so that $\boldsymbol{curl}^2 = -\Delta\, Id + grad\ div$ and (the Bio-Savard law)

$$Id = (-\Delta)^{-1}\boldsymbol{curl}^2 + \Delta^{-1}\, grad\ div = (-\Delta)^{-1}\boldsymbol{curl}^2 + \boldsymbol{Id} - \boldsymbol{P}$$

which gives

$$\boldsymbol{curl}^2 = -\Delta\boldsymbol{P},$$

so that $[\boldsymbol{P}, \boldsymbol{curl}] = 0$ and

$$\boldsymbol{P}\ \boldsymbol{curl} = \boldsymbol{curl}\ \boldsymbol{P} = \boldsymbol{curl}\ (-\Delta)^{-1}\boldsymbol{curl}^2 = \boldsymbol{curl}\ \ (Id - \Delta^{-1}\, grad\ div) = \boldsymbol{curl}$$

since $\boldsymbol{curl}\ grad = 0$ (note also that the transposition of the latter gives $div\ \boldsymbol{curl} = 0$).

The solutions of (1.6) are satisfying

$$v(t) = e^{\nu t\Delta} - \int_0^t e^{(t-s)\nu\Delta}\boldsymbol{P}\nabla(v(s) \otimes v(s))ds.$$

(LeN) Lerner, N. (2009), A Note on the Oseen Kernels. In: Bove, A., Del Santo, D., Murthy, M. (eds) Advances in Phase Space Analysis of Partial Differential Equations. Progress in Nonlinear Differential Equations and Their Applications, vol 78. Birkhäuser Boston

Note: The pressure p of the NSE can be expressed in terms of the velocity u by the formula $p = \sum_{j,k=1}^{3} R_j R_k(u_j u_k)$, where $\boldsymbol{R} \coloneqq (R_1, R_2, R_3)$ is the Riesz transform and $\boldsymbol{u} \otimes \boldsymbol{u} = (u_j u_k)$ is a $3x3$ matrix.

$$H_{1/2} = H_1 \otimes H_1^{\perp}$$