The eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|\ldots\|$. We will consider mappings $K: H \to H$. Unless otherwise noticed the standard assumptions on $K$are:

i) K is symmetric, i.e., for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$

ii) K is compact, i.e., any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent

iii) K is injective, i.e., $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} < \infty.$$

Lemma: Let K be bounded, and fulfill condition i) above, but not necessarily the two other conditions ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|}.$$

Theorem: There exists a countable sequence $\{\lambda_i, \phi_i\}$ of eigen-elements and eigenvalues $K\phi_i = \lambda_i \phi_i$ with the properties

i) the eigen-elements are pair-wise orthogonal, i.e. $(\phi_i, \phi_k) = \delta_{i,k}$

ii) the eigenvalues tend to zero

iii) for the generalized Fourier sums it holds

$$S_n := \sum_{i=1}^{n} (x, \phi_i)\, \phi_i \to x \quad \text{with } n \to \infty \text{ for all } x \in H$$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \phi_i)^2$$

holds for all $x \in H$.

Hilbert Scales
J. A. Nitsche

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|..\|$ and let A be a linear operator with the properties

A is self-adjoint, positive definite

A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, one may assume

$$(x, Ax) \geq \|x\| \quad \text{for all } x \in D(A).$$

Any eigen-element of the compact operator $K = A^{-1}$ is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have that there is a countable sequence $\{\lambda_i, \phi_i\}$ with

$$A\phi_i = \lambda_i \phi_i \;,\; (\phi_i, \phi_k) = \delta_{i,k} \text{ and } \lim_{i\to\infty} \lambda_i$$

and any $x \in H$ is represented by

$$x = \sum_{i=1}^{\infty} (x, \phi_i)\, \phi_i \text{ and } \|x\|^2 = \sum_1^{\infty} (x, \phi_i)^2.$$

Lemma 1: Le $x \in D(A)$, then

$$Ax = \sum_{i=1}^{\infty} \lambda_i (x, \phi_i)\, \phi_i \;,\; \|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \phi_i)^2,\; (Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \phi_i)\,(y, \phi_i).$$

Similarly one can define the spaces H_α with scalar product

$$(x, y)_\alpha = \sum_i^{\infty} \lambda_i^\alpha (x, \phi_i)\,(y, \phi_i) = \sum_i^{\infty} \lambda_i^\alpha x_i y_i \text{ and norm } \|x\|_\alpha^2 = (x, x)_\alpha.$$

The relation to $x \in D(A)$ is given by

$$\|x\|_2^2 = (Ax, Ax)_0 \,,\, H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in the context of this section necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $lim\, \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, \ldots)$ with

$$\|\hat{x}\|_2^2 < \infty \,,\, \|\hat{x}\|_0^2 = \infty\,.$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma 2: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma 3: Let $\alpha < \beta < \gamma$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with

$$\mu = \frac{\gamma-\beta}{\gamma-\alpha} \text{ and } \nu = \frac{\beta-\alpha}{\gamma-\alpha}.$$

Lemma 4: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

$$\|x-y\|_\alpha \leq t^{\beta-\alpha}\|x\|_\beta$$
$$\|x-y\|_\beta \leq \|x\|_\beta\ ,\ \|y\|_\beta \leq \|x\|_\beta$$
$$\|y\|_\gamma \leq t^{-(\gamma-\beta)}\|x\|_\beta\ .$$

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x-y\|_\rho \leq t^{\beta-\rho}\|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma-\beta)}\|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \phi_i)^2$$

turned out to be the space H_2, which is densely and compactly embedded into $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that
$D(A) = H_2\ R(A) = H_0$ and $\|x\|_2 = \|Ax\|$.

Extension and generalizations

J. A. Nitsche

For $t > 0$ one may introduce the Hilbert space $H_{(\tau)}$ by an additional inner product resp. norm in the form

$$(x, y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i}t}(x, \phi_i)(y, \phi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 \,.$$

Now the factor has exponential decay $e^{-\sqrt{\lambda_i}t}$ instead of a polynomial decay in case of λ_i^α.

Obviously it holds

$$\|x\|_{(t)} \leq c(\alpha, t)\|x\|_\alpha \text{ for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the (t)-norm is weaker than any α-norm. On the other hand any negative norm index, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (t)-norm.

It holds:

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Proof: The inequality is a consequence of the following inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1}-\sqrt{\lambda})}, \text{ for any } t, \delta, \alpha > 0 \text{ and } \lambda \geq 1.$$

This holds for the following reasons:

i) if $\lambda^{-1/2} \leq \delta$ then obviously $\lambda^{-\alpha} \leq \delta^{2\alpha}$

ii) in case of $\lambda^{-1/2} \geq \delta$ it holds $e^{t(\delta^{-1}-\sqrt{\lambda})} \geq 1$,

iii) whereas $\lambda^{-\alpha} \leq 1$ is a consequence of $\alpha > 0$ and $\lambda \geq 1$.

The counterpart of the lemma 4 above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

$$\|x - y\| \leq \|x\|$$

$$\|y\|_1 \leq \delta^{-1}\|x\|$$

$$\|x - y\|_{(t)} \leq e^{-t/\delta}\|x\|.$$

Isometric elliptic, parabolic and hyperbolic operators

The proposed mathematical modelling framework is based on appropriately define Hilbert (energy) scales. The baseline model is provided by the potential theory based symmetric mechanical (Laplace) potential energy operator. In classical theoretical physics models this is about a symmetric operator accompanied by the Hilbert scale domain H_2. The Friedrichs extension of the Laplace operator with the H_2 domain provides a self-adjoint potential energy operator defining the inner product of the related potential energy Hilbert space H_1. By construction the Laplacian operator is isometric with respect to the correspondingly defined Hilbert scales, i.e., $\|-\Delta u\|_\alpha^2 \cong \|u\|_{\alpha+2}^2$. A similar property holds for the related parabolic (heat) equation operator $H[u] := \dot{u} - \Delta u$ with respect to the norm $|||u|||_\alpha^2 := \int_0^\infty \|u\|_\alpha^2(t)dt$, i.e.,

$$(*) \qquad |||H[u]|||_\alpha^2 \cong |||u|||_{\alpha+2}^2.$$

In general the above elliptic and parabolic isometries in („*polynomial decay*") Hilbert scales are not valid for the d'Alembert (wave) operator $A[u] := \ddot{u} - \Delta u$ (*). However, in case of („*exponential decay*") Hilbert scales with norm $|||u|||_\alpha^2 := \int_0^\infty \|u\|_{\alpha.\tau}^2 d\tau$, and related inner product in the form

$$(u,v)_{\alpha.(\tau)}^2 := \sum_i \lambda_i^\alpha e^{-\sqrt{\lambda_i}\tau}(u,\phi_i)(v,\phi_i)\ , \tau > 0$$

it holds

$$(**) \qquad |||A[u]|||_\alpha^2 \cong |||u|||_{\alpha+2}^2\ .$$

Proof: Let $w_i := (w,\phi_i)$ resp. $f_i := (f,\phi_i)$ being the generalized Fourier coefficient related to the eigen-pairs $-\Delta v_i = \lambda_i v_i$. Then for $A[w] = f$, it follows $\ddot{w}_i(t) + \lambda_i w_i(t) = f_i(t)$ with the solution

$$w_i(t) = \frac{1}{\sqrt{\lambda_i}}\int_0^t \sin\,(\sqrt{\lambda_i}(t-\tau)\, f_i(\tau)d\tau\ .$$

Then for $\tau \leq t$ one gets

$$\int_0^T \|w\|_{k+2,(t)}^2 dt = \sum \lambda_i^{k+2}\int_0^T e^{-\sqrt{\lambda_i}t} w_i^2(t)dt \leq \sum \lambda_i^{k+2}\int_0^T e^{-\sqrt{\lambda_i}t}\left[\frac{1}{\sqrt{\lambda_i}}\int_0^t \sin\,(\sqrt{\lambda_i}(t-\tau)\, f_i(\tau)d\tau\right]^2 dt$$

$$\leq \sum \lambda_i^{k+1}\int_0^T e^{-\sqrt{\lambda_i}t}(\int_0^t \sin\,(\sqrt{\lambda_i}(t-\tau)d\tau)\left[\int_0^t \sin\,(\sqrt{\lambda_i}(t-\tau)d\tau\, f_i^2(\tau)d\tau\right] dt$$

$$\leq \sum \lambda_i^{k+1/2}\int_0^T e^{-\sqrt{\lambda_i}t}\left[\int_0^t f_i^2(\tau)d\tau\right] dt\ .$$

Exchanging the order of integration gives

$$\int_0^T\int_0^t e^{-\sqrt{\lambda_i}t} f_i^2(\tau)d\tau dt = \int_0^T\int_t^T e^{-\sqrt{\lambda_i}t} f_i^2(\tau)dt d\tau = \int_0^T f_i^2(\tau)dt\left[\int_t^T e^{-\sqrt{\lambda_i}t}\, d\tau\right]$$

$$\leq \frac{1}{\sqrt{\lambda_i}}\int_0^T f_i^2(\tau)dt$$

from which it follows $\int_0^T \|w\|_{k+2,(t)}^2 dt \leq c\int_0^T \|f\|_{k,(t)}^2 dt$.

Note: The (exponential decay type) Hilbert scales $H_{\alpha.(\tau)}$ provide the baseline framework to define Krein space based potential energy Hilbert scales accompanied by related self-adjoint potential energy operators.

(*) the counter example is given by the function $\Phi(x,t) := e^{-(\frac{1}{2}-(x-t))^2}$, $u(x,t) := t^2\Phi(x,t)$, $f(x,t) := 2\Phi(x,t) - 4t\Phi'(x,t)$ fulfilling the relationships $\dot{\Phi}(x,t) = -\Phi'(x,t)$, $\ddot{\Phi}(x,t) = \Phi''(x,t)$, $\ddot{u}(x,t) - u''(x,t) = f(x,t)$ and $\|u''\|_{L_2(L_2)} \sim \|\Phi''\|_{L_2(L_2)}$ but $\|f\|_{L_2(L_2)} \sim \|\Phi'\|_{L_2(L_2)}$.

Coercive operators with compact disturbances

A variational representation of an operator in the form $B = A + K$, where A is a H_α - coercive operator with a compact disturbance K fullfills a coerciveness (Garding type inequality) condition in the form, (AzA), (see also (KaY), (BrK10)),

$$(Bu, v) \geq c \cdot \|u\|_\alpha \|v\|_\alpha - (Ku, v) \text{ or } (Bu, v) \geq c_1 \cdot \|u\|_\alpha^2 - c_2 \cdot \|u\|_\beta^2$$

with $H_\beta \subset H_\alpha$ compactly embedded.

The Riesz and the Calderón-Zygmund operators

The Riesz transformations are the n-dimensional generalizations of the 1-dimensional Hilbert transformation. They arise when study the Neumann problem in upper half-plane. The Riesz transforms

$$R_k u = -ic_n p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy\ , \quad c_n := \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}}$$

commutes with translations and homothesis, having nice properties relative to rotation, (PeB), (StE). The "rotation property" plays a key role in the context of the rotation group $SO(n)$:

> let $m := m(x) := (m_1(x), \dots m_n(x))$ be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then it holds $m(\rho(x)) = \rho(m(x))$,
> i.e. $m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$.

The Calderón-Zygmund operators Λ with symbol $|\nu|$ and its inverse operator Λ^{-1} may be represented in the following forms, (EsG) 3.15, 3.17, 3.35, (LiI) p. 58 ff.,

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{n} p.v. \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy$$

$$= -\frac{\Gamma\left(\frac{n-1}{2}\right)}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta \Lambda^{-1}) u(x)$$

$$(\Lambda^{-1} u)(x) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{u(y)}{|x-y|^{n-1}} dy\ , n \geq 2\ .$$

Note: For space dimension $n = 1$ this is about $\Lambda = DH = PH$, where H denotes the Hilbert transformation and $D = P$ the Schrödinger momentum operator $P = -i\frac{d}{dx}$, (MeY) p. 5. In (BrK6) the Calderón-Zygmund operators Λ is proposed as alternative Schrödinger$^{2.0}$ momentum operator.

Note: If $j \neq j$ then $R_j R_k$ is a singular convolution operator. On the other hand, it holds $R_j^2 = -(1/n)I + A_j$ where A_j is a convolution operator. The following identities are valid

$$\|R_j\| = 1\ , R_j^* = -R_j\ \ , \sum R_j^2 = -I\ \ , \ \sum \|R_j u\|^2 = \|u\|^2\ , u \in L_2\ .$$

Let $m := m(x) := (m_1(x), \dots m_n(x))$ be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then

$$m(\rho(x)) = \rho(m(x)), \text{ whereby } m_j\big(\rho(x)\big) = \sum \rho_{jk} m_k(x)$$

and

$$m(\rho(x)) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(x\rho^{-1}(y)) + log\left|\frac{1}{x\rho^{-1}(y)}\right|) \frac{y}{|y|} d\sigma(y)$$

$$= c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(xy) + log\left|\frac{1}{xy}\right|) \frac{y}{|y|} d\sigma(y)\ .$$

Note: The Riesz operator is a special Calderón-Zygmund (Pseudo Differential-, convolution-) operator $T(f) = S * F$ with a distribution S defined by symbols $m(\omega) \in C^\infty(R^n - \{0\})$ with the following properties, (MeY),

- i) $m(\mu\omega) = m(\omega), \mu > 0$
- ii) the mean of $m(\omega)$ on the unit sphere is zero
- iii) it holds $m(\omega) = \frac{\omega_j}{|\omega|}$.

The Leray-Hopf and the Riesz operators

The Leray-Hopf (Helmholtz-Weyl) operator $P(z)$ is the matrix valued Fourier multiplier given by

$$P(\xi) = Id - \frac{\xi \otimes \xi}{|\xi|^2} = (\delta_{jk} - \frac{\xi_j \xi_k}{|\xi|^2})_{1 \le j,k \le n} \quad , P = Id - R \otimes R =: Id - Q.$$

It is like the Riesz operator, not a classical Pseudo Differential Operator (PDO) of order zero with the symbol

$$b_{ij}(z) = \delta_{ij} - \frac{z_i z_j}{|z|^2}$$

because of the singularity at the origin, (LeN).

The operator Q is an orthogonal projector, as it holds $Q := R \otimes R = (R_j R_k)_{1 \le j,k \le 1} = Q^2$. As a result the Leray-Hopf operator

$$P = Id - R \otimes R =: Id - Q = Id - \frac{D \otimes D}{D^2} Id - \Delta^{-1}(\nabla \times \nabla)$$

is also an orthogonal projection operator from $(L^3(R^3))^3$ onto the closed subspace of divergence free vector fields. It can be computed through the following identity

$$P = (-\Delta)^{-1} curlcurl.$$

(LeN) Lerner, N. (2009). A Note on the Oseen Kernels. In: Bove, A., Del Santo, D., Murthy, M. (eds) Advances in Phase Space Analysis of Partial Differential Equations. Progress in Nonlinear Differential Equations and Their Applications, vol 78. Birkhäuser Boston

In der Fluiddynamik, speziell der Lösbarkeitstheorie der NSE , spielt die Helmholtz-Projektion eine wichtige Rolle. Wird die Helmholtz-Projektion auf die linearisierte inkompressiblen Navier-Stokes-Gleichungen angewandt, erhält man die Stokes-Gleichung. Diese ist nur noch von der Geschwindigkeit der Teilchen in der Strömung abhängig, jedoch nicht mehr vom statischen Druck, wodurch die Gleichung auf eine Unbekannte reduziert werden konnte.

Note: We note that under rotation in R^n, the Riesz operators transform in the same manner as the components of a vector ([SteE1] III, 1.2).

Note: The one-component plasma model of the non-linear collision operator of the Landau equation is given by

$$Q(f,f) = \frac{\partial}{\partial v_i}\left\{\int_{R^N} a_{ij}(v-w)\left[f(w)\frac{\partial f(v)}{\partial v_j} - f(v)\frac{\partial f(w)}{\partial w_j}\right] dw\right\}$$

with

$$a_{ij}(z) := \frac{1}{|z|} P(z) := \frac{b_{ij}(z)}{|z|} = \frac{1}{|z|}[Id - \bar{Q}](z) \text{ and } \bar{Q}(z) := (R_i R_j)_{1 \le i,j \le N}.$$

The symbol function $a(z)$ is symmetric, non-negative and even in z; f denotes an unknown function corresponding at each time t to the density of a particle at the point x with velocity v.

[CoP] Constantin P., Gallavotti G., Kazhikhov A. V., Meyer Y., Ukai S., Mathematical Foundation of turbulent Viscous Flows, Lectures given at the C.I.M.E. Summer School held in Martina Franca, Italy, September 1-5, 2003

[StE1] Stein E. M., Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, new Jersey, 1970

The Stokes operator

Sohr: The Stokes operator A is basic for our functional analytic approach to the Navier-Stokes system, see the discussion in Section 2, I. We need some elementary Hilbert space methods, see Section 3.2, II. We develop only the L_2-theory for A. The advantage of this approach is that we can admit arbitrary domains. In particular we can include the interesting case of unbounded boundaries.

Fujiwara, D. and Morimoto, H., An L_r-theorem of the Helmholtz decomposition of vector fields, J. Fac. Sc. Univ. Tokyo, vol.24 (1977), 685-700

- precise definition of P(f):= f – grad q

Hieber, M., Saal, J., The Stokes equation in the Lp-setting: well-posedness and regularity properties. Handbook of math. analysis in mech. of viscous fluids, 117-206. Springer, 2018.

Abstract
This article discusses the Stokes equation in various classes of domains $\Omega \subset R^n$ within the L^p-setting for $1 \le p \le \infty$ from the point of view of evolution equations. Classical as well as modern approaches to well-posedness results for strong solutions to the Stokes equation, to the Helmholtz decomposition, to the Stokes semigroup, and to mixed maximal L^q -L^p-regularity results for 1 < p; q < ∞ are presented via the theory of R-sectorial operators. Of concern are domains having compact or noncompact, smooth or nonsmooth boundaries, as well as various classes of boundary conditions including energy preserving boundary conditions. In addition, the endpoints of the L^p-scale, i.e., p=1 and p=∞ are considered and recent well-posedness results for the case p =∞ are described. Results on L^q -L^p-smoothing properties of the associated Stokes semigroups and on variants of the Stokes equation (e.g., nonconstant viscosity, Lorentz spaces, Stokes-Oseen system, flow past rotating obstacles, hydrostatic Stokes equation) complete this survey article.

Giga, Y., Sohr, H., On the Stokes operator in exterior domains. J. Fac.Sci. Univ. Tokyo Sect. IA Math. 36, 103–130 (1989)

The content of this section is basically taken from [GiY]. With c we denote numeric constants which may have different values at different places.

Let P be the orthogonal projection operator of $(L_2(\Omega))^n$ onto the divergence free vector field H_ω consisting of all solenoidal vector functions u, i.e. the operator is an orthogonal projection onto the kernel of the divergence operator. It is a Pseudo-Differential operator (PDO) of degree zero [(EsG]). The Stokes operator A is a selfadjoint operator in H_σ, being the Friedrichs extension of the non-negative symmetric operator $-P\Delta$ in H_ω defined for all $u \in C^2$ with $divu = 0$ and $u_{n|\partial\Omega} = 0$. T

The Stokes operator enables the definition of a related Hilbert scale ($\alpha \in R$) with corresponding norm

$$\|u\|_\alpha := \left\|A^{\alpha/2}u\right\| \ .$$

Throughout this paper, if not explicitly mentioned, we assume $p = 2$and $n = 3$ for $(L_p(\Omega))^n$.

Using the Stokes operator and its related Hilbert scale framework the Navier-Stokes equations can be represented as an evolution equation in H_0 . Since $P(gradp) = 0$ one gets

$$Au = Pf \quad \text{in } H_0$$

Putting $B(u) := P(u, grad)u)$ and assuming $Pu_0 = u_0$ the NSE initial-boundary equation is given by

(*) $$\frac{du}{dt} + Au + Bu = Pf\ , u(0) = u_0.$$

As u is divergence free and $u \cdot v$ identically vanishes on $\partial\Omega$ one gets

$$b(u, v, w) := ((u, grad)v, w) = \iint_{\Omega} (u, grad)v \cdot w dx = -b(u, w, v)$$

and especially $b(u, v, v) = 0$.