Generalized Lemmata of Gronwall

Generalized Lemma of Gronwall (version 1)

Let $\psi(t) \in C^0[0,a]$ be a real valued function and $h(t) \in L_1(0,a)$ be non-negative function with

$$\psi(t) \leq \alpha + \int_0^t h(\tau)\psi(\tau)d\tau \quad , \quad \alpha \in R \, .$$

Then

$$\psi(t) \leq \alpha * e^{\int_0^t h(\tau)d\tau} \, .$$

Generalized Lemma of Gronwall (version 2)

Let $\psi(t) \in C^0[0,a]$ be a real valued function and $h(t) \in L_1(0,a)$ be non-negative function with

$$\psi(t) \leq \alpha(t) + \int_0^t h(\tau)\psi(\tau)d\tau \quad , \quad \alpha \in R \, .$$

Then

$$\psi(t) \leq \alpha(t) + \int_0^t \alpha(\tau)h(\tau)e^{H(t)-H(\tau)}d\tau$$

with $H(\tau) := \int_0^\tau h(s)ds$.

Generalized Lemma of Gronwall (version 3: log type)

Let α, β be non-negative constants. Assume that a non-negative function $a(t,s)$ satisfies $a(*,*) \in C(0 \leq s < t \leq T)$, $a(t,*) \in L_1(0,t)$ for all $t \in ((0,T)]$. Furthermore, we assume that there exists a positive constant ε_0 such that

$$\sup_{0 \leq t \leq T} \int_{t-\varepsilon_0}^t a(t,s)ds \leq 1/2 \, .$$

If a non-negative function $f \in C([0,T])$ satisfies

$$f(t) \leq \alpha + \int_0^t a(t,s)f(s)ds + \beta \int_0^t \{1 + log(1 + f(s))\} f(s)ds$$

for all $t \in [0,T]$. Then we have

$$f(t) \leq e^{\left\{1+\frac{\gamma}{\beta}+log(1+2\alpha)\right\}e^{2\beta t}}$$

for all $t \in [0,T]$. Here we put $\gamma := \sup_{0 \leq t \leq T} \left\{ \sup_{0 \leq s \leq t-\varepsilon_0} a(t,s) \right\}$.

Generalized Lemma of Gronwall (version 4)

Let $a(t)$ and $b(t)$ nonnegative functions in $[0,A)$ and $0 < \delta < 1$. Suppose a nonnegative function $y(t)$ satisfies the differential inequality

$$y'(t) + b(t) \leq \alpha(t)y^\delta(t) \quad \text{on} \quad [0,A)$$

$$y(0) = y_0.$$

Then for $0 \leq t < A$

$$y(t) + \int_0^t b(\tau)d\tau \leq (2^{\delta/(1-\delta)} + 1)y_0 + 2^{\delta/(1-\delta)} \left[\int_0^t \alpha(\tau)d\tau\right]^{\delta/(1-\delta)}$$

Proof: solving $y'(t) \leq \alpha(t)y^\delta(t)$ leads to

$$y(t) \leq y_0 + \left[\int_0^t \alpha(\tau)d\tau\right]^{\delta/(1-\delta)} .$$