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Dualizing the Poisson summation formula

(Fourier transforms/Moebius series/Euler–Maclaurin sum formula)

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ABSTRACT **If $f(x)$ and $g(x)$ are a Fourier cosine transform pair, then the Poisson summation formula can be written as $2\sum_{n=1}^{\infty} g(n) + g(0) = 2\sum_{n=1}^{\infty} f(n) + f(0)$. The concepts of linear transformation theory lead to the following dual of this classical relation. Let $\phi(x)$ and $\gamma(x) = \phi(1/x)/x$ have absolutely convergent integrals over the positive real line. Let $F(x) = \sum_{n=1}^{\infty} \phi(n/x)/x - \int_0^{\infty} \phi(t)dt$ and $G(x) = \sum_{n=1}^{\infty} \gamma(n/x)/x - \int_0^{\infty} \gamma(t)dt$. Then $F(x)$ and $G(x)$ are a Fourier cosine transform pair. We term $F(x)$ the "discrepancy" of ϕ because it is the error in estimating the integral of ϕ by its Riemann sum with the constant mesh spacing $1/x$.**

Section 1. Introduction

Let the Fourier cosine transform of a function $f(x)$ be denoted by $g(x)$ and defined as

$$g(x) = 2\int_0^{\infty} \cos(2\pi xt)f(t)dt. \qquad [1.1]$$

Poisson discovered the important summation formula, which can be written in the form:

PRIMAL POISSON RELATION. *Let* f *be a function and* g *its cosine transform. Then*

$$\sum_{n=1}^{\infty} g(n/x)/x - \int_0^{\infty} g(t)dt = \sum_{n=1}^{\infty} f(nx) - x^{-1}\int_0^{\infty} f(t)dt \text{ for } x > 0. \qquad [1.2]$$

Since Poisson's time there has been a steady stream of literature giving proofs, applications, and generalizations of this formula under various conditions on the function f (see, for example, ref. 1).

Since the cosine transform is linear, the relation **1.2** represents an equation between two linear transformations acting on f. We shall show that the dual of this equation is the following statement:

DUAL POISSON RELATION. *Let* $\phi(x)$ *be a function defined for* $x > 0$ *and set*

$$\gamma(x) = \phi(1/x)/x. \qquad [1.3]$$

Then the functions

$$F(x) = \sum_{n=1}^{\infty} \phi(n/x)/x - \int_0^{\infty} \phi(t)dt \qquad [1.4]$$

and

$$G(x) = \sum_{n=1}^{\infty} \gamma(n/x)/x - \int_0^{\infty} \gamma(t)dt \qquad [1.5]$$

form a pair of Fourier cosine transforms.

In *Section 2* we give a formal analysis of the sense in which the above statements are dual. In *Section 3* we prove that the dual relation is valid for an arbitrary integrable function ϕ that is "small" near zero and infinity.

Section 2. The Concepts of Reversion, Discrepancy, and Duality

We begin by writing the Poisson relation **1.2** in terms of three linear transformations $\mathscr{C}$, $\mathfrak{D}$, and $\mathfrak{R}$ as

$$\mathfrak{D}\mathscr{C}f(x) = \mathfrak{R}\mathfrak{D}f(x). \qquad [2.1]$$

Here $\mathscr{C}$ is the Fourier cosine transform **1.1**. We define the *Reversion* as

$$\mathfrak{R}\phi(x) = \phi(1/x)/x \qquad [2.2]$$

and the *Discrepancy* as

$$\mathfrak{D}f(x) = \sum_{n=1}^{\infty} f(n/x)/x - \int_0^{\infty} f(t)dt. \qquad [2.3]$$

The linear transformations $\mathscr{C}$, $\mathfrak{R}$, and $\mathfrak{D}$ all take classes of functions defined in $(0, \infty)$ into functions defined in $(0, \infty)$. In addition, $\mathfrak{R}$ is an involution in the sense that $\mathfrak{R}^2$ is the identity.

We term $\mathfrak{D}$ the discrepancy because $\mathfrak{D}f$ is the error in estimating the infinite integral of f by its Riemann sum with $\Delta t = 1/x$. Presumably, Riemann would expect the discrepancy to vanish as $x \to \infty$.

Since the Poisson relation **2.1** holds for a large class of f, we can think of it as the identity $\mathfrak{D}\mathscr{C} = \mathfrak{R}\mathfrak{D}$ between linear transformations. The dual relation is obtained by equating the adjoints of the two sides of this equation.

We note that for a large class of functions ϕ and σ the change of variables $s = 1/t$ shows that

$$\int_0^{\infty} \phi(t)\mathfrak{R}\sigma(t)dt = \int_0^{\infty} \mathfrak{R}\phi(s)\sigma(s)ds.$$

Thus $\mathfrak{R}$ is its own formal adjoint.

$\mathfrak{D}$ is also its own formal adjoint in the sense that

$$\int_0^{\infty} \phi(t)\mathfrak{D}f(t)dt = \int_0^{\infty} \mathfrak{D}\phi(s)f(s)ds.$$

To see this make an interchange of summation and integration, a change of the variable of integration in each term of the

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resulting sum, and another interchange of summation and integration.

Parseval's relation shows that the cosine transform $\mathscr{C}$ is also formally self-adjoint. Thus the adjoint of $\mathfrak{D}\mathscr{C}$ is $\mathscr{C}\mathfrak{D}$ and the adjoint of $\mathfrak{R}\mathfrak{D}$ is $\mathfrak{D}\mathfrak{R}$, so that the formal dual of the Poisson relation **2.1** is

$$\mathscr{C}\mathfrak{D}\phi(x) = \mathfrak{D}\mathfrak{R}\phi(x). \quad [2.4]$$

The definition **1.3** of γ can be written as $\gamma = \mathfrak{R}\phi$, so Eq. **2.4** is just what we called the Dual Poisson Relation, $\mathscr{C}F = G$, in *Section 1*.

Now we can state the Poisson summation formula **2.1** in words as follows:

PRIMAL. *The discrepancy of the cosine transform of a function is equal to the reversion of the discrepancy of the function.*

Similarly, the dual relation **2.4** can be stated as follows:

DUAL. *The cosine transform of the discrepancy of a function is equal to the discrepancy of the reversion of the function.*

It should be understood that neither the primal relation nor the dual relation is a universal theorem. Special conditions on the function f or ϕ are needed.

Section 3. A Dual Poisson Theorem

We now make precise the meaning of being "small" near zero and infinity.

THEOREM 1. *In the dual Poisson relation let the function ϕ satisfy*

$$\left(1 + \frac{1}{x}\right)\phi(x) \in L_1(0, \infty). \quad [3.1]$$

Then the series in functions **1.4** *and* **1.5** *converge almost everywhere and also in the* L_1 *norm on finite intervals. The functions* F(x) *and* G(x), *so defined, are a pair of cosine transforms in the sense that*

$$G(x) = \frac{d}{dx}\int_0^\infty \frac{\sin(2\pi xt)}{\pi t} F(t)dt \quad [3.2]$$

and

$$F(x) = \frac{d}{dx}\int_0^\infty \frac{\sin(2\pi xt)}{\pi t} G(t)dt \quad [3.3]$$

almost everywhere.

Proof: We begin by establishing the relation **1.2** for a special family of functions $f_\lambda(x)$. For any $\lambda > 0$ the cosine transform of

$$f_\lambda(x) = \frac{\sin 2\pi\lambda x}{\pi x} \text{ is } g_\lambda(x) = h(x/\lambda). \quad [3.4]$$

Here h is the cut-off function defined as

$$h(x) = 1 \text{ if } x < 1, \qquad h(1) = \frac{1}{2}, \qquad h(x) = 0 \text{ if } x > 1.$$

Then by the definitions of $\mathfrak{R}$ and $\mathfrak{D}$

$$\mathfrak{R}\mathfrak{D}f_\lambda(x) = \frac{1}{x}\left(\sum_{n=1}^{\infty} \frac{\sin 2\pi\lambda nx}{\pi n} - \frac{1}{2}\right) \quad [3.5]$$

and by the definition of $\mathfrak{D}$

$$\mathfrak{D}g_\lambda(x) = \sum_{n=1}^{\infty} h(n/x\lambda) - \lambda = \frac{1}{x}([\lambda x] - \lambda x). \quad [3.6]$$

The symbol $[t]$ is defined as the nearest integer below t when t is not an integer and as $t - ½$ when t is an integer.

LEMMA 1. *The special Poisson relation*

$$\mathfrak{D}g_\lambda(x) = \mathfrak{R}\mathfrak{D}f_\lambda(x) \quad [3.7]$$

holds for all positive λ *and* x. *Moreover, the partial sums of the series for* $x\mathfrak{R}\mathfrak{D}f_\lambda(x)$ *are uniformly bounded.*

Proof: The Fourier series for the 1-periodic sawtooth function $[t] - t$ is easily found to be

$$[t] - t = \sum_{n=1}^{\infty} \frac{\sin 2\pi nt}{\pi n} - \frac{1}{2}.$$

Because this sawtooth function is piecewise continuously differentiable, and because it is equal to the average of its left and right limits at its jump points, the series converges to the function for all t, and its partial sums are uniformly bounded (see, e.g., section 18 of ref. 2).

Setting $t = \lambda x$, dividing both sides of this equation by x, and using Eqs. **3.5** and **3.6** yields the statements of *Lemma 1*.

To prove *Theorem 1* we shall show that we may multiply both sides of relation **3.7** by ϕ and integrate and that we can justify the interchanges of summation and integration needed to derive the dual relation

$$\int_0^\lambda \mathfrak{D}\phi(t)dt = \int_0^\infty \mathfrak{D}\mathfrak{R}\phi(t)f_\lambda dt. \quad [3.8]$$

Eq. **3.2** will then follow from differentiation with respect to λ.

We see from Eq. **3.6** that the function $|\mathfrak{D}g_\lambda|$ is bounded and that the sequence of partial sums of the series for $\mathfrak{D}g_\lambda$ is nondecreasing. Since ϕ is absolutely integrable, the Lebesgue dominated convergence theorem shows that

$$\int_0^\infty \phi(t)\mathfrak{D}g_\lambda(t)dt = \sum_{n=1}^{\infty}\int_0^\infty \phi(t)h(n/t\lambda)/t\, dt - \lambda\int_0^\infty \phi(t)dt$$

$$= \sum_{n=1}^{\infty}\int_0^\lambda \phi(n/\xi)/\xi\, d\xi - \lambda\int_0^\infty \phi(t)dt. \quad [3.9]$$

Here we have introduced the new variable $\xi = n/t$ and used the cut-off property of h to obtain the second line. The following lemma will permit us to interchange summation and integration in this line.

LEMMA 2. *For every positive* λ *the series in the definition of* $\mathfrak{D}\phi$ *converges,* (i) *in* $L_1(0, \lambda)$ *norm and* (ii) *almost everywhere.*

Proof: First suppose that $\phi \geq 0$. Formula **3.6** gives $\mathfrak{D}g_\lambda \leq 0$. Then Eq. **3.9** shows that

$$\sum_{n=1}^{\infty}\int_0^\lambda \phi(n/\xi)/\xi\, d\xi - \lambda\int_0^\infty \phi(t)dt \leq 0.$$

Thus part *i* follows from the Lebesgue dominated convergence theorem, while part *ii* follows from the Lebesgue monotone convergence theorem.

If ϕ changes sign, applying this argument to $|\phi|$ is seen to finish the proof.

We use part *i* of *Lemma 2* to interchange summation and integration in the second line of Eq. **3.9** to obtain

$$\int_0^\infty \phi(t)\mathfrak{D}g_\lambda(t)dt = \int_0^\lambda \mathfrak{D}\phi(t)dt.$$

By relation **3.7** we can write this equation in the form

$$\int_0^\lambda \mathfrak{D}\phi(t)dt = \int_0^\infty \phi(t)\mathfrak{R}\mathfrak{D}f_\lambda(t)dt. \quad [3.10]$$

Thus we have an expression for the left-hand side of relation **3.8**. We need the following lemma in order to treat the right-hand side.

LEMMA 3. *If* $(1 + 1/x)\phi(x) \in L_1(0, \infty)$, *the same is true if* ϕ *is replaced by* $\mathfrak{R}\phi$, *and*

$$\int_0^\infty \left(1 + \frac{1}{t}\right)|\mathfrak{R}\phi(t)|dt = \int_0^\infty \left(1 + \frac{1}{s}\right)|\phi(s)|ds.$$

Proof: This result follows from definition **2.2** of $\mathfrak{R}\phi$ when the variable of integration s is replaced by $t = 1/s$.

By combining *Lemmas 2* and *3* we see that the series for $\mathfrak{D}\mathfrak{R}\phi$ converges in the $L_1(0, \mu)$ norm for any $\mu > 0$. The function f_λ defined by Eq. **3.4** is bounded, and hence

$$\int_0^\mu \mathfrak{D}\mathfrak{R}\phi(x)f_\lambda(x)dx = \sum_{n=1}^\infty \int_0^\mu \phi(x/n)/n\, f_\lambda(x)dx - \int_0^\infty \phi(t)/t\, dt \int_0^\mu f_\lambda(x)dx$$

$$= \sum_{n=1}^\infty \int_0^\infty \phi(x/n)/n\, f_\lambda(x)h(x/\mu)dx - \int_0^\infty \phi(t)/t\, dt \int_0^\mu f_\lambda(x)dx$$

$$= \sum_{n=1}^\infty \int_0^\infty \phi(t)f_\lambda(nt)h(nt/\mu)dt - \int_0^\infty \phi(t)/t\, dt \int_0^\mu f_\lambda(x)dx. \quad [3.11]$$

In order to interchange integration and summation in the last line we first consider a partial sum. Because ϕ is absolutely integrable and $f_\lambda(nt)h(nt/\mu)$ is bounded, we have

$$\sum_{n=1}^N \int_0^\infty \phi(t)f_\lambda(nt)h(nt/\mu)dt = \int_0^\infty \phi(t)/t \sum_{n=1}^N \frac{\sin 2\pi\lambda nt}{\pi n} h(nt/\mu)dt. \quad [3.12]$$

We see from Eq. **3.5** that for each fixed t, μ, and N the sum on the right is a partial sum of the series for $t\mathfrak{R}\mathfrak{D}f_\lambda(t)$. By *Lemma 1* this sum is bounded, uniformly in t, N, and μ.

Since $\phi(t)/t$ is absolutely integrable, the Lebesgue dominated convergence theorem permits us to let N approach infinity in Eq. **3.12**, so that Eq. **3.11** becomes

$$\int_0^\mu \mathfrak{D}\mathfrak{R}\phi(x)f_\lambda(x)dx = \int_0^\infty \phi(t) \sum_{n=0}^\infty f_\lambda(nt)h(nt/\mu)dt - \int_0^\infty \phi(t)/t\, dt \int_0^\mu f_\lambda(x)dx. \quad [3.13]$$

Next we let μ approach infinity. The partial sums of the series are still bounded, and $h(nt/\mu)$ converges to 1. Therefore the dominated convergence theorem shows that this limit exists and that

$$\int_0^\infty \mathfrak{D}\mathfrak{R}\phi(x)f_\lambda(x)dx = \int_0^\infty \phi(t)\left\{\sum_{n=0}^\infty f_\lambda(nt) - t^{-1}\int_0^\infty f_\lambda(x)dx\right\}dt.$$

By the definition of $\mathfrak{R}\mathfrak{D}$, this is the duality relation

$$\int_0^\infty \mathfrak{D}\mathfrak{R}\phi(x)f_\lambda(x)dx = \int_0^\infty \phi(t)\mathfrak{R}\mathfrak{D}f_\lambda(t)dt. \quad [3.14]$$

We substitute this into Eq. **3.10** to get

$$\int_0^\lambda \mathfrak{D}\phi(t)dt = \int_0^\infty \mathfrak{D}\mathfrak{R}\phi(x)f_\lambda(x)dx \quad [3.15]$$

for all $\lambda > 0$.

Because $\mathfrak{D}\phi$ is integrable on every bounded interval, the left side has the derivative $\mathfrak{D}\phi(\lambda)$ almost everywhere. We differentiate both sides of Eq. **3.14**, replace λ by x, and recall that by definition $\mathfrak{R}\phi(x) = \gamma(x)$, so that $F(x) = \mathfrak{D}\mathfrak{R}\phi(x)$ and $G(x) = \mathfrak{D}\phi(x)$, to obtain the desired relation **3.2** of *Theorem 1*.

Lemma 3 permits us to replace ϕ by $\gamma = \mathfrak{R}\phi$ in this relation. Then $\mathfrak{R}\gamma = \mathfrak{R}^2\phi = \phi$, and we obtain relation **3.3**.

The convergence statements of *Theorem 1* follow directly from *Lemmas 2* and *3*.

Example: Because the function $\phi(x) = x/(1 + x^3)$ satisfies the conditions of *Theorem 1* and because $\mathfrak{R}\phi = \phi$, we find that the function

$$F(x) = \sum_{n=1}^\infty \frac{nx}{n^3 + x^3} - 3^{-3/2}2\pi$$

is its own Fourier cosine transform.

Section 4. Discussion

Poisson's formula, as expressed by Eq. **1.2**, has been studied by Wintner (3). He related it to the Euler–Maclaurin sum. In particular, he analyzed how rapidly the sum on the left of Eq. **1.2** approaches the integral as $x \to \infty$.

Analogs of *Theorem 1* for the sine transform have been obtained by Duffin (4–6). Analogs for the Hankel transform have been obtained by Weinberger (7). Boas (8) has shown that Duffin's relation and a relation similar to our dual Poisson relation can be obtained by a formal interchange of summation and integration in the Poisson relation **1.2**.

The fact that such relations arise as duals of the Poisson relation was not recognized in these previous studies.

1. Titchmarsh, E. C. (1937) *Introduction to the Theory of Fourier Integrals* (Oxford Univ. Press, Oxford), pp. 60–68.
2. Weinberger, H. F. (1965) *A First Course in Partial Differential Equations* (Wiley, New York).
3. Wintner, A. (1947) *Am. J. Math.* **69**, 685–708.
4. Duffin, R. J. (1945) *Bull. Am. Math. Soc.* **51**, 447–455.
5. Duffin, R. J. (1950) *Proc. Am. Math. Soc.* **1**, 250–255.
6. Duffin, R. J. (1957) *Proc. Am. Math. Soc.* **8**, 272–277.
7. Weinberger, H. F. (1950) Dissertation (Carnegie–Mellon University, Pittsburgh).
8. Boas, R. P., Jr. (1952) *Proc. Am. Math. Soc.* **3**, 444–447.