Scattering, determinants, hyperfunctions in relation to $\frac{\Gamma(1-s)}{\Gamma(s)}$

Jean-François Burnol

Abstract

The method of realizing certain self-reciprocal transforms as (absolute) scattering, previously presented in summarized form in the case of the Fourier cosine and sine transforms, is here applied to the self-reciprocal transform $f(y) \mapsto \mathcal{H}(f)(x) = \int_0^\infty J_0(2\sqrt{xy}) f(y)\,dy$, which is isometrically equivalent to the Hankel transform of order zero and is related to the functional equations of the Dedekind zeta functions of imaginary quadratic fields. This also allows to re-prove and to extend theorems of de Branges and V. Rovnyak regarding square integrable functions which are self-or-skew reciprocal under the Hankel transform of order zero. Related integral formulae involving various Bessel functions are all established internally to the method. Fredholm determinants of the kernel $J_0(2\sqrt{xy})$ restricted to finite intervals $(0, a)$ give the coefficients of first and second order differential equations whose associated scattering is (isometrically) the self-reciprocal transform $\mathcal{H}$, closely related to the function $\frac{\Gamma(1-s)}{\Gamma(s)}$. Remarkable distributions involved in this analysis are seen to have most natural expressions as (difference of) boundary values (*i.e.* hyperfunctions.) The present work is completely independent from the previous study by the author on the same transform $\mathcal{H}$, which centered around the Klein-Gordon equation and relativistic causality. In an appendix, we make a simple-minded observation regarding the resolvent of the Dirichlet kernel as a Hilbert space reproducing kernel.

keywords: Hankel transform; Scattering; Fredholm determinants.

Université Lille 1
UFR de Mathématiques
Cité scientifique M2
F-59655 Villeneuve d'Ascq
France
burnol@math.univ-lille1.fr

February 19, 2006.

January 2, 2008.: there was an i in (148c) and other equations leading to Theorem 15 which should not have been there. This only affected equation (156e). This is all folks. I could possibly have proposed other improvements if only the referee who kept my paper hostage for most of 2006 and 2007 had actually read it. Not reading it did not prevent from commenting upon it, unfortunately the substance of those inspired five lines is hard to transfer beneficially to my wide readership.

Contents

1 Introduction (the idea of co-Poisson)

We explain the underlying framework and the general contours of this work. Throughout the paper, we have tried to formulate the theorems in such a form that one can, for most of them, read their statements without having studied the preceeding material in its entirety, so a sufficiently clear idea of the results and methods is easily accessible. Setting up here all notations and necessary preliminaries for stating the results would have taken up too much space.

The Riemann zeta function $\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \dots$ is a meromorphic function in the entire complex plane with a simple pole at $s = 1$, residue 1. Its functional equation is usually written in one of the following two forms:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma(\frac{1-s}{2})\zeta(1-s) \tag{1a}$$

$$\zeta(s) = \chi_0(s)\zeta(1-s) \qquad \chi_0(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} \tag{1b}$$

The former is related to the expression of $\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$ as a left Mellin transform[1] and to the Jacobi identity:

$$\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s) = \frac{1}{2}\int_0^\infty (\theta(t)-1)t^{\frac{s}{2}-1}\,dt \qquad (\Re(s) > 1) \tag{2a}$$

$$= \frac{1}{2}\int_0^\infty (\theta(t)-1-\frac{1}{\sqrt{t}})t^{\frac{s}{2}-1}\,dt \qquad (0 < \Re(s) < 1) \tag{2b}$$

$$\theta(t) = 1 + 2\sum_{n\geq 1} q^{n^2} \qquad q = e^{-\pi t} \qquad \theta(t) = \frac{1}{\sqrt{t}}\theta(\frac{1}{t}) \tag{2c}$$

[1] in the left Mellin transform we use $s-1$, in the right Mellin transform we use $-s$.

The latter form of the functional equation is related to the expression of $\zeta(s)$ as the right Mellin transform of a tempered distribution with support in $[0, +\infty)$, which is self-reciprocal under the Fourier cosine transform:[2]

$$\zeta(s) = \int_0^\infty (\sum_{m\geq 1} \delta_m(x) - 1) x^{-s} dx \tag{3a}$$

$$\int_0^\infty 2\cos(2\pi xy)(\sum_{n\geq 1} \delta_n(y) - 1)\, dy = \sum_{m\geq 1} \delta_m(x) - 1 \qquad (x > 0) \tag{3b}$$

This last identity may be written in the more familiar form:

$$\int_{\mathbb{R}} e^{2\pi ixy} \sum_{n\in\mathbb{Z}} \delta_n(y) dy = \sum_{m\in\mathbb{Z}} \delta_m(x) \tag{4}$$

which expresses the invariance of the "Dirac comb" distribution $\sum_{m\in\mathbb{Z}} \delta_m(x)$ under the Fourier transform. As a linear functional on Schwartz functions ϕ , the invariance of $\sum_{m\in\mathbb{Z}} \delta_m(x)$ under Fourier is expressed as the Poisson summation formula:

$$\sum_{n\in\mathbb{Z}} \widetilde{\phi}(n) = \sum_{m\in\mathbb{Z}} \phi(m) \qquad\qquad \widetilde{\phi}(y) = \int_{\mathbb{R}} e^{2\pi ixy} \phi(x)\, dx \tag{5}$$

The Jacobi identity is the special instance with $\phi(x) = \exp(-\pi t x^2)$, and conversely the validity of (5) for Schwartz functions (and more) may be seen as a corollary to the Jacobi identity.

The *idea of co-Poisson* [4] leads to *another* formulation of the functional equation as an identity involving functions. The co-Poisson identity ((10) below) appeared in the work of Duffin and Weinberger [13]. In one of the approaches to this identity, we start with a function g on the positive half-line such that both $\int_0^\infty g(t)\, dt$ and $\int_0^\infty g(t)t^{-1}\, dt$ are absolutely convergent. Then we consider the averaged distribution $g * D(x) = \int_0^\infty g(t) D(\frac{x}{t}) \frac{dt}{t}$ where $D(x) = \sum_{n\geq 1} \delta_n(x) - \mathbf{1}_{x>0}(x)$. This gives (for $x > 0$):

$$g * D(x) = \sum_{n=1}^\infty \frac{g(x/n)}{n} - \int_0^\infty \frac{g(1/t)}{t}\, dt \tag{6}$$

If g is smooth with support in $[a, A]$, $0 < a < A < \infty$, then the co-Poisson sum $g * D$ has Schwartz decrease at $+\infty$ (easy from applying the Poisson formula to $\frac{g(1/t)}{t}$; *cf.* [8, 4.29] for a general statement). The right Mellin transform $\widehat{g * D}(s)$ is related to the right Mellin transform $\widehat{g}(s)$ of g via the identity:

$$\widehat{g * D}(s) = \int_0^\infty (g * D)(x) x^{-s}\, dx = \zeta(s) \int_0^\infty g(x) x^{-s}\, dx = \zeta(s)\widehat{g}(s) \tag{7}$$

This is because the right Mellin transform of a multiplicative convolution is the product of the right Mellin transforms. The necessary calculus of tempered distributions needed for this and other statements in this paragraph is detailed in [8]. The functional equation in the form of (1b) gives:[3]

$$\widehat{g * D}(s) = \chi_0(s)\zeta(1-s)\widehat{g}(s) = \chi_0(s)\widehat{I(g) * D}(1-s) \qquad I(g)(t) = \frac{g(1/t)}{t} \tag{8}$$

[2] of course, $\delta_m(x) = \delta(x - m)$.

[3] one observes that $\widehat{I(g)}(s) = \widehat{g}(1-s)$.

One may reinterpret this in a manner involving the cosine transform $\mathcal{C}$ acting on $L^2(0,+\infty;dx)$. The Mellin transform of a function $f(x)$ in $L^2(0,\infty;dx)$ is a function $\widehat{f}(s)$ on $\Re(s)=\frac{1}{2}$ which is nothing else than the Plancherel Fourier transform of $e^{\frac{1}{2}u}f(e^u)$: $\widehat{f}(\frac{1}{2}+i\gamma)=\int_0^\infty f(x)x^{-\frac{1}{2}-i\gamma}\,dx = \int_{-\infty}^\infty f(e^u)e^{\frac{u}{2}}e^{-i\gamma u}\,du$, $\int_0^\infty |f(x)|^2\,dx = \int_{-\infty}^\infty |f(e^u)e^{\frac{u}{2}}|^2\,du = \frac{1}{2\pi}\int_{\Re(s)=\frac{1}{2}}|\widehat{f}(s)|^2|ds|$. The unitary operator $\mathcal{C}I$ is scale invariant hence it is diagonalized by the Mellin transform: $\widehat{\mathcal{C}I(f)}(s)=\chi_0(s)\widehat{f}(s)$, $\widehat{\mathcal{C}(f)}(s)=\chi_0(s)\widehat{f}(1-s)$, where $\chi_0(s)$ is obtained for example using $f(x)=e^{-\pi x^2}$ and coincides with the chi-function defined in (1b). It has modulus 1 on the critical line as $\mathcal{C}$ is unitary. So (8) says that the *co-Poisson intertwining identity* holds:

$$\mathcal{C}(g*D) = I(g)*D \tag{9}$$

The co-Poisson intertwining (9) or explicitely:

$$\int_0^\infty 2\cos(2\pi xy)\left(\sum_{m=1}^\infty \frac{g(x/m)}{m} - \int_0^\infty \frac{g(1/t)}{t}\,dt\right)dx = \sum_{n=1}^\infty \frac{g(n/y)}{y} - \int_0^\infty g(t)\,dt \qquad (y>0) \tag{10}$$

is, when g is smooth with support in $[a,A]$, $0<a<A<\infty$, an identity of (even) Schwartz functions. If g is only supposed to be such that $\int_0^\infty |g(t)|(1+\frac{1}{t})\,dt<\infty$ then the co-Poisson intertwining $\mathcal{C}(g*D)=I(g)*D$ holds as an identity of distributions (either considered even or with support in $[0,\infty)$). Sufficient conditions for pointwise validity have been established [8]. The general statement of the intertwining is $\mathcal{C}(g*E)=I(g)*\mathcal{C}(E)$ where E is an arbitrary tempered distribution with support on $[0,\infty)$ (see footnote[4]) and it is proven directly. The co-Poisson identity (10) is another manner, not identical with the Poisson summation formula, to express the invariance of D under the cosine transform, or the invariance of the Dirac comb under the Fourier transform.

If the integrable function g has its support in $[a,A]$, $0<a<A<\infty$, then $g*D$ is constant in $(0,a)$ and its cosine transform (thanks to the co-Poisson intertwining) is constant in $(0,A^{-1})$. Up to a rescaling we may take $A=a^{-1}$, and then $a<1$ (if a non zero example is wanted.) Appropriate modifications allow to construct non zero even Schwartz functions constant in $(-a,a)$ and with Fourier transform again constant in $(-a,a)$ where $a>0$ is arbitrary [8].

Schwartz functions are square-integrable so here we have made contact with the investigation of de Branges [1], V Rovnyak [28] and J. and V. Rovnyak [29, 30] of square integrable functions on $(0,\infty)$ vanishing on $(0,a)$ and with Hankel transform of order ν vanishing on $(0,a)$. For $\nu=-\frac{1}{2}$ the Hankel transform of order ν is $f(y)\mapsto\sqrt{\frac{2}{\pi}}\int_0^\infty\cos(xy)f(y)\,dy$ and up to a scale change this is the cosine transform considered above. The co-Poisson idea allows to attach the zeta function to, among the spaces defined by de Branges [1], the spaces associated with the cosine transform: it has allowed the definition of some novel Hilbert spaces [3] of entire functions in relation with the Riemann zeta function and Dirichlet L-functions (the co-Poisson idea is in [4] on the adeles of an arbitrary algebraic number field K; then, the study of the related Hilbert spaces was begun for $K=\mathbb{Q}$. Further results were obtained in [7].)

[4] both sides in fact depend only on $E(x)+E(-x)$ as a distribution on the line, which may be identically 0, and this happens exactly when E is a linear combination of odd derivatives of the delta function.

The study of the function $\chi_0(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}$, of unit modulus on the critical line, is interesting. We proposed to realize the χ_0 function as a "scattering matrix". This is indeed possible and has been achieved in [6]. The distributions, functions, and differential equations involved are all related to, or expressed by, the Fredholm determinants of the finite cosine transform, which in turn are related to the Fredholm determinants of the finite Dirichlet kernels $\frac{\sin(t(x-y))}{\pi(x-y)}$ on $[-1,1]$. The study of the Dirichlet kernels is a topic with a vast literature. A minor remark will be made in an appendix.

We mentioned the Riemann zeta function and how it relates to $\chi_0(s) = \pi^{s-\frac{1}{2}}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}$ and to the cosine transform. Let us now briefly consider the Dedekind zeta function of the Gaussian number field $\mathbb{Q}(i)$ and how it relates to $\chi(s) = \frac{\Gamma(1-s)}{\Gamma(s)}$ and to the $\mathcal{H}$ transform. The $\mathcal{H}$ transform is

$$\mathcal{H}(g)(y) = \int_0^\infty J_0(2\sqrt{xy})g(x)\,dx \qquad\qquad J_0(2\sqrt{xy}) = \sum_{n=0}^\infty (-1)^n\frac{x^n y^n}{n!^2} \tag{11}$$

Up to the unitary transformation $g(x) = (2x)^{-\frac{1}{4}}f(\sqrt{2x})$, $\mathcal{H}(g)(y) = (2y)^{-\frac{1}{4}}k(\sqrt{2y})$, it becomes the Hankel transform of order zero $k(y) = \int_0^\infty \sqrt{xy}J_0(xy)f(x)\,dx$. It is a self-reciprocal, unitary, scale reversing operator ($\mathcal{H}(g(\lambda x))(y) = \frac{1}{\lambda}\mathcal{H}(g)(\frac{y}{\lambda})$). We shall also extend its action to tempered distributions on $\mathbb{R}$ with support in $[0,+\infty)$. At the level of right Mellin transforms of elements of $L^2(0,\infty;dx)$ it acts as:

$$\widehat{\mathcal{H}(g)}(s) = \chi(s)\widehat{g}(1-s) \qquad \chi(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \qquad \Re(s) = \frac{1}{2} \tag{12}$$

It has $e^{-x}\mathbf{1}_{x\geq 0}(x)$ as one among its self-reciprocal functions, as is verified directly by series expansion $\int_0^\infty J_0(2\sqrt{xy})e^{-y}\,dy = \sum_{n=0}^\infty \frac{(-1)^n}{n!^2}x^n\int_0^\infty y^n e^{-y}\,dy = e^{-x}$. The identity

$$\int_0^\infty J_0(2\sqrt{t})t^{-s}\,dt = \chi(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \tag{13}$$

is equivalent to a special case of well-known formulas of Weber, Sonine and Schafheitlin [33, 13.24.(1)]. Here we have an absolutely convergent integral for $\frac{3}{4} < \Re(s) < 1$ and in that range the identity may be proven as in: $e^{-x} = \int_0^\infty J_0(2\sqrt{xy})e^{-y}\,dy = \int_0^\infty J_0(2\sqrt{y})\frac{1}{x}e^{-\frac{y}{x}}\,dy$, $\Gamma(1-s) = \int_0^\infty J_0(2\sqrt{y})(\int_0^\infty x^{-s-1}e^{-\frac{y}{x}}\,dx)\,dy = \Gamma(s)\int_0^\infty J_0(2\sqrt{y})y^{-s}\,dy$. The integral is semi-convergent for $\Re(s) > \frac{1}{4}$, and of course (13) still holds. In particular on the critical line and writing $t = e^u$, $s = \frac{1}{2}+i\gamma$, we obtain the identities of tempered distributions $\int_{\mathbb{R}} e^{\frac{1}{2}u}J_0(2e^{\frac{1}{2}u})e^{-i\gamma u}\,du = \chi(\frac{1}{2}+i\gamma)$, $e^{\frac{1}{2}u}J_0(2e^{\frac{1}{2}u}) = \frac{1}{2\pi}\int_{\mathbb{R}}\chi(\frac{1}{2}+i\gamma)e^{+i\gamma u}\,du$.

We have $\zeta_{\mathbb{Q}(i)}(s) = \frac{1}{4}\sum_{(n,m)\neq(0,0)}\frac{1}{(n^2+m^2)^s} = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{4^s}+\frac{2}{5^s}+\frac{1}{8^s}+\cdots = \sum_{n\geq 1}\frac{c_n}{n^s}$ and it is a meromorphic function in the entire complex plane with a simple pole at $s=1$, residue $\frac{\pi}{4}$. Its functional equation assumes at least two convenient well-known forms:

$$(\sqrt{4})^s(2\pi)^{-s}\Gamma(s)\zeta_{\mathbb{Q}(i)}(s) = (\sqrt{4})^{1-s}(2\pi)^{-(1-s)}\Gamma(1-s)\zeta_{\mathbb{Q}(i)}(1-s) \tag{14a}$$

$$(\frac{1}{\pi})^s\zeta_{\mathbb{Q}(i)}(s) = \chi(s)(\frac{1}{\pi})^{1-s}\zeta_{\mathbb{Q}(i)}(1-s) \qquad \chi(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \tag{14b}$$

The former is related to the expression of $\pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}(i)}(s)$ as a left Mellin transform:

$$\pi^{-s}\Gamma(s)\zeta_{\mathbb{Q}(i)}(s) = \frac{1}{4}\int_0^\infty (\theta(t)^2 - 1)t^{s-1}\,dt \qquad (\Re(s) > 1) \tag{15a}$$

$$= \frac{1}{4}\int_0^\infty (\theta(t)^2 - 1 - \frac{1}{t})t^{s-1}\,dt \qquad (0 < \Re(s) < 1) \tag{15b}$$

$$\theta(t)^2 = \frac{1}{t}\theta(\frac{1}{t})^2 \tag{15c}$$

The latter form of the functional equation is related to the expression of $(\frac{1}{\pi})^s\zeta_{\mathbb{Q}(i)}(s)$ as the right Mellin transform of a tempered distribution which is supported in $[0,\infty)$ and which is self-reciprocal under the $\mathcal{H}$-transform:

$$(\frac{1}{\pi})^s\zeta_{\mathbb{Q}(i)}(s) = \int_0^\infty (\sum_{m\geq 1} c_m\delta_{\pi m}(x) - \frac{1}{4})x^{-s}dx \tag{16a}$$

$$\int_0^\infty J_0(2\sqrt{xy})(\sum_{n\geq 1} c_n\delta_{\pi n}(y) - \frac{1}{4})\,dy = \sum_{m\geq 1} c_m\delta_{\pi m}(x) - \frac{1}{4}\mathbf{1}_{x>0}(x) = E(x) \qquad (x > 0) \tag{16b}$$

The invariance of E under the $\mathcal{H}$-transform is equivalent to the validity of the functional equation of $(\frac{1}{\pi})^s\zeta_{\mathbb{Q}(i)}(s)$ and it having a pole with residue $\frac{1}{4}$ at $s = 1$. The co-Poisson intertwining becomes the assertion:

$$y > 0 \implies \int_0^\infty J_0(2\sqrt{xy})\left(\sum_{m=1}^\infty c_m\frac{g(x/\pi m)}{\pi m} - \frac{1}{4}\int_0^\infty g(\frac{1}{t})\frac{dt}{t}\right)dx = \sum_{n=1}^\infty c_n\frac{g(\pi n/y)}{y} - \frac{1}{4}\int_0^\infty g(t)\,dt \tag{17}$$

If g is smooth with support in $[b, B]$, $0 < b < B < \infty$, then we have on the right hand side a function of Schwartz decrease at $+\infty$ (compare to Theorem 3), and its $\mathcal{H}$-transform is also of Schwartz decrease at $+\infty$. The former is constant for $0 < y < \pi B^{-1}$ and the latter is constant for $0 < x < \pi b$. The supremum of the values obtainable for the product of the lengths of the intervals of constancy is π^2. But, as for the cosine and sine transforms, there does exist smooth functions which are constant on a given $(0, a)$ for arbitrary $a > 0$ with an $\mathcal{H}$ transform again constant on $(0, a)$ and have Schwartz decrease at $+\infty$ (the two constants being arbitrarily prescribed.)

De Branges and V. Rovnyak have obtained [1, 28] rather complete results in the study of the Hankel transform of order zero $f(x) \mapsto g(y) = \int_0^\infty \sqrt{xy}J_0(xy)f(x)\,dx$ from the point of view of understanding the support property of being zero and with transform again zero in a given interval $(0, b)$. They obtained an isometric expansion (Theorem 1 of section 2) and also the detailed description of the related spaces of entire functions ([1]). The more complicated case of the Hankel transforms of non-zero integer orders was treated by J. and V. Rovnyak [29, 30]. These, rather complete, results are an indication that the Hankel transform of order zero or of integer order is easier to understand than the cosine or sine transforms, and that doing so thoroughly could be useful to better understand how to try to understand the cosine and sine transforms.

The kernel $J_0(2\sqrt{uv})$ of the $\mathcal{H}$-transform satisfies the Klein-Gordon equation in the variables

$x = v - u$, $t = v + u$:

$$(\frac{\partial^2}{\partial u \partial v} + 1) J_0(2\sqrt{uv}) = (\Box + 1) J_0(2\sqrt{uv}) = (\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 1) J_0(\sqrt{t^2 - x^2}) = 0 \tag{18}$$

It is a noteworthy fact that the support condition, initially considered by de Branges and V. Rovnyak, and which, nowadays, is also seen to be in relation with the co-Poisson identities, has turned out to be related to the relativistic causality governing the propagation of solutions to the Klein-Gordon equation. This has been established in [9] where we obtained as an application of this idea the isometric expansion of [1, 28] in a novel manner. It was furthermore proven in [9] that the $\mathcal{H}$ transform is indeed an (absolute) scattering, in fact the scattering from the past boundary to the future boundary of the Rindler wedge $0 < |t| < x$ for solutions of a first order, two-component ("Dirac"), form of the KG equation.

In the present paper, which is completely independent from [9], we shall again study the $\mathcal{H}$-transform and show in particular how to recover in yet a different way the earlier results of [1, 28] and also we shall extend them. This will be based on the methods from [5, 6], and uses the techniques motivated by the study of the co-Poisson idea [8]. Our exposition will thus give a fully detailed account of the material available in summarized form in [5, 6]. Then we proceed with a development of these methods to provide the elucidation of the (two dimensions bigger) spaces of functions constant in $(0, a)$ and with $\mathcal{H}$-transforms constant in $(0, a)$.

The use of tempered distributions is an important point of our approach[5]; also one may envision the co-Poisson idea as asking not to completely identify a distribution with the linear functional it "is". In this regard it is of note that the distributions which arise following the method of [5] are seen in the present case of the study of the $\mathcal{H}$-transform to have a very natural formulation as differences of boundary values of analytic functions, that is, as hyperfunctions [23]. We do not use the theory of hyperfunctions as such, but could not see how not to mention that this is what these distributions seem to be in a natural manner.

The paper contains no number theory. And, the reader will need no prior knowledge of [2]; some familiarity with the m-function of Hermann Weyl [10, 21, 26] is necessary at one stage of the discussion (there is much common ground, in fact, between the properties of the m-function and the axioms of [2]). The reproducing kernel in any space with the axioms of [2] has a specific appearance (equation (109) below) which has been used as a guide to what we should be looking for. The validity of the formula is re-proven in the specific instance considered here[6]. Regarding the differential equations governing the deformation, with respect to the parameter $a > 0$ [7], of the

[5] at the bottom of page 456 of [1] the formulas given for $A(a, z)$ and $B(a, z)$ as completed Mellin transforms are lacking terms which would correspond to Dirac distributions; possibly related to this, the isometric expansion as presented in Theorem II of [1] is lacking corresponding terms. The exact isometric expansion appears in [28] and the exact formulas for $A(a, z)$ and $B(a, z)$ as completed Mellin transforms appear, in an equivalent form, in [30, eq.(37)].

[6] the critical line here plays the rôle of the real axis in [2], s is $\frac{1}{2} - iz$ and the use of the variable s is most useful in distinguishing the right Mellin transforms which need to be completed by a Gamma factor from the left Mellin transforms of "theta"-like functions.

[7] the a here corresponds to $\frac{1}{2}a^2$ in [1].

Hilbert spaces, we depart from the general formalism of [2] and obtain them in a canonical form, as defined in [21, §3]. Interestingly this is related to the fact that the A and B functions (connected to the reproducing kernel, equation (109)) which are obtained by the method of [5] turn out not to be normalized according to the rule in general use in [2]. Each rule of normalization has its own advantages; here the equations are obtained in the Schrödinger and Dirac forms familiar from the spectral theory of linear second order differential equations [10, 21, 26]. This allows to make reference to the well-known Weyl-Stone-Titchmarsh-Kodaira theory [10, 21, 26], and to understand $\mathcal{H}$ as a scattering. Regarding spaces with the axioms of [2], the articles of Dym [14] and Remling [27] will be useful to the reader interested in second order linear differential equations. And we refer the reader with number theoretical interests to the recent papers of Lagarias [18, 19].

The author has been confronted with a dilemma: a substantial portion of the paper (most of chapters 5, 6, 8) has a general validity for operators having a kernel of the multiplicative type $k(xy)$ possessing certain properties in common with the cosine, sine or $\mathcal{H}$ transforms. But on the other hand the (essentially) unique example where all quantities arising may be computed is the $\mathcal{H}$ transform (and transforms derived from it, or closely related to it, as the Hankel transforms of integer orders). We have tried to give proofs whose generality is obvious, but we also made full use of distributions, as this allows to give to the quantities arising very natural expressions. Also we never hesitate using arguments of analyticity although for some topics (for example, some aspects involving certain integral equations and Fredholm determinants) this is certainly not really needed.

2 Hardy spaces and the de Branges-Rovnyak isometric expansion

Let us state the isometric expansion of [1, 28] regarding the square integrable Hankel transforms of order zero. We reformulate the theorem to express it with the $\mathcal{H}$ transform (11) rather than the Hankel transform of order zero.

Theorem 1 ([1], [28])**.** *: Let $k \in L^2(0,\infty;dx)$. The functions f_1 and g_1, defined as the following integrals:*

$$f_1(y) = \int_y^\infty J_0(2\sqrt{y(x-y)})k(x)\,dx \ , \tag{19a}$$

$$g_1(y) = k(y) - \int_y^\infty \sqrt{\frac{y}{x-y}} J_1(2\sqrt{y(x-y)})k(x)\,dx \ , \tag{19b}$$

exist in L^2 in the sense of mean-square convergence, and they verify:

$$\int_0^\infty |f_1(y)|^2 + |g_1(y)|^2\,dy = \int_0^\infty |k(x)|^2\,dx \ . \tag{19c}$$

The function k is given in terms of the pair (f_1, g_1) as:

$$k(x) = g_1(x) + \int_0^x J_0(2\sqrt{y(x-y)})f_1(y)\,dy - \int_0^x \sqrt{\frac{y}{x-y}} J_1(2\sqrt{y(x-y)})g_1(y)\,dy \tag{19d}$$

The assignment $k \mapsto (f_1, g_1)$ is a unitary equivalence of $L^2(0,\infty;dx)$ with $L^2(0,\infty;dy) \oplus L^2(0,\infty;dy)$ such that the $\mathcal{H}$-transform acts as $(f_1,g_1) \mapsto (g_1,f_1)$. Furthermore k and $\mathcal{H}(k)$ both identically vanish in $(0,a)$ if and only if f_1 and g_1 both identically vanish in $(0,a)$.

Let us mention the following (which follows from the proof we have given of Thm. 1 in [9]): *if f_1, f_1', g_1, g_1' are in L^2 then k, k' and $\mathcal{H}(k)'$ are in L^2. Conversely if k, k' and $\mathcal{H}(k)'$ are in L^2 then the integrals defining $f_1(y)$ and $g_1(y)$ are convergent for each $y>0$ as improper Riemann integrals, and f_1' and g_1' are in L^2.*

It will prove convenient to work with $(f(x), g(x)) = \frac{1}{2}(g_1(\frac{x}{2}) + f_1(\frac{x}{2}), g_1(\frac{x}{2}) - f_1(\frac{x}{2}))$:

$$f(y) = \frac{1}{2}k(\frac{y}{2}) + \frac{1}{2}\int_{y/2}^{\infty}\left(J_0(\sqrt{y(2x-y)}) - \sqrt{\frac{y}{2x-y}}J_1(\sqrt{y(2x-y)})\right)k(x)\,dx \tag{20a}$$

$$g(y) = \frac{1}{2}k(\frac{y}{2}) - \frac{1}{2}\int_{y/2}^{\infty}\left(J_0(\sqrt{y(2x-y)}) + \sqrt{\frac{y}{2x-y}}J_1(\sqrt{y(2x-y)})\right)k(x)\,dx \tag{20b}$$

$$\begin{aligned} k(x) = f(2x) &+ \frac{1}{2}\int_0^{2x}\left(J_0(\sqrt{y(2x-y)}) - \sqrt{\frac{y}{2x-y}}J_1(\sqrt{y(2x-y)})\right)f(y)\,dy \\ &+ g(2x) - \frac{1}{2}\int_0^{2x}\left(J_0(\sqrt{y(2x-y)}) + \sqrt{\frac{y}{2x-y}}J_1(\sqrt{y(2x-y)})\right)g(y)\,dy \end{aligned} \tag{20c}$$

$$\int_0^\infty |k(x)|^2\,dx = \int_0^\infty |f(y)|^2 + |g(y)|^2\,dy \tag{20d}$$

The $\mathcal{H}$ transform on k acts as $(f,g) \mapsto (f,-g)$. The pair $(k,\mathcal{H}(k))$ identically vanishes on $(0,a)$ if and only if the pair (f,g) identically vanishes on $(0,2a)$. The structure of the formulas is more apparent after observing $(x,y>0)$:

$$\frac{\partial}{\partial x}\Big(\frac{1}{2}J_0(\sqrt{y(2x-y)})\mathbf{1}_{0<y<2x}(y)\Big) = \delta_{2x}(y) - \frac{1}{2}\sqrt{\frac{y}{2x-y}}J_1(\sqrt{y(2x-y)})\mathbf{1}_{0<y<2x}(y) \tag{21}$$

In this section I shall prove the existence of an isometric expansion $k \leftrightarrow (f,g)$ having the stated support properties and relation to the $\mathcal{H}$-transform; that this construction does give the equations (20a), (20b), (20c), will only be established in the last section (9) of the paper. The method followed in this section coincides partly with the one of V. Rovnyak [28]; we try to produce the most direct arguments, using the commonly known facts on Hardy spaces. The reader only interested in Theorem 1 is invited after having read the present section to then jump directly to section 9 for the conclusion of the proof.

To a function $k \in L^2(0,\infty;dx)$ we associate the analytic function

$$\widetilde{k}(\lambda) = \int_0^\infty e^{i\lambda x}k(x)\,dx \qquad (\Im(\lambda) > 0) \tag{22}$$

with boundary values for $\lambda \in \mathbb{R}$ again written $\widetilde{k}(\lambda)$, which defines an element of $L^2(\mathbb{R}, \frac{d\lambda}{2\pi})$, the assignment $k \mapsto \widetilde{k}$ being unitary from $L^2(0,\infty;dx)$ onto $\mathbb{H}^2(\Im(\lambda>0), \frac{d\lambda}{2\pi})$. Next we have the conformal equivalence and its associated unitary map from $\mathbb{H}^2(\Im(\lambda>0), \frac{d\lambda}{2\pi})$ to $\mathbb{H}^2(|w|<1, \frac{d\theta}{2\pi})$:

$$w = \frac{\lambda - i}{\lambda + i} \qquad K(w) = \frac{1}{\sqrt{2}}\frac{\lambda+i}{i}\,\widetilde{k}(\lambda) \tag{23}$$

It is well known that this indeed unitarily identifies the two Hardy spaces. With $k_0(x) = e^{-x}$, $\widetilde{k}_0(\lambda) = \frac{i}{\lambda+i}$, $K_0(w) = \frac{1}{\sqrt{2}}$, and $\|k_0\|^2 = \int_0^\infty e^{-2x}dx = \frac{1}{2} = \|K_0\|^2$. The functions $\widetilde{k}_n(\lambda) = (\frac{\lambda-i}{\lambda+i})^n \frac{i}{\lambda+i}$ correspond to $K_n(w) = \frac{1}{\sqrt{2}}w^n$. To obtain explicitely the orthogonal basis $(k_n)_{n\geq 0}$, we first observe that $w = 1 - 2\frac{i}{\lambda+i}$, so as a unitary operator it acts as:

$$w \cdot k(x) = k(x) - 2\int_0^x e^{-(x-y)}k(y)\,dy = k(x) - e^{-x}2\int_0^x e^y k(y)\,dy \tag{24}$$

Writing $k_n(x) = P_n(x)e^{-x}$ we thus obtain $P_{n+1}(x) = P_n(x) - 2\int_0^x P_n(y)\,dy$:

$$P_n(x) = \left(1 - 2\int_0^x\right)^n \cdot 1 = \sum_{j=0}^n \binom{n}{j}\frac{(-2x)^j}{j!} \tag{25}$$

So as is well-known $P_n(x) = L_n^{(0)}(2x)$ (in the notation of [31, §5]) where the Laguerre polynomials $L_n^{(0)}(x)$ are an orthonormal system for the weight $e^{-x}dx$ on $(0,\infty)$.

One of the most common manner to be led to the $\mathcal{H}$-transform is to define it from the two-dimensional Fourier transform as:

$$\begin{aligned}\mathcal{H}(f)(\frac{1}{2}r^2) = \frac{1}{2\pi}\iint e^{i(x_1y_1+x_2y_2)}f(\frac{y_1^2+y_2^2}{2})dy_1dy_2 = \int_0^\infty \left(\int_0^{2\pi} e^{irs\cos\theta}\frac{d\theta}{2\pi}\right)f(\frac{1}{2}s^2)sds \\ \mathcal{H}(f)(\frac{1}{2}r^2) = \int_0^\infty J_0(rs)f(\frac{1}{2}s^2)\,sds \qquad r^2 = x_1^2+x_2^2, s^2 = y_1^2+y_2^2\end{aligned} \tag{26}$$

which proves its unitarity, self-adjointness, and self-reciprocal character and the fact that it has e^{-x} has self-reciprocal function. The direct verification of $\mathcal{H}(k_0) = k_0$ is immediate: $\mathcal{H}(k_0)(x) = \int_0^\infty J_0(2\sqrt{xy})e^{-y}\,dy = \sum_{n=0}^\infty \frac{(-1)^n}{n!^2}x^n\int_0^\infty y^n e^{-y}\,dy = e^{-x}$. Then, $\mathcal{H}(e^{-tx}) = t^{-1}e^{-\frac{x}{t}}$ for each $t > 0$. So $\int_0^\infty e^{-tx}\mathcal{H}(k)(x)\,dx = t^{-1}\int_0^\infty e^{-\frac{1}{t}x}k(x)\,dx$ hence:

$$\forall k \in L^2(0,\infty;dx) \qquad \widetilde{\mathcal{H}(k)}(\lambda) = \frac{i}{\lambda}\widetilde{k}(\frac{-1}{\lambda}) \tag{27}$$

With the notation $\mathcal{H}(K)$ for the function in $\mathbb{H}^2(|w|<1)$ corresponding to $\mathcal{H}(k)$, we obtain from (23), (27), an extremely simple result:[8]

$$\mathcal{H}(K)(w) = K(-w) \tag{28}$$

This obviously leads us to associate to $K(w) = \sum_{n=0}^\infty c_n w^n$ the functions:

$$F(w) := \sum_{n=0}^\infty c_{2n}w^n \tag{29a}$$

$$G(w) := \sum_{n=0}^\infty c_{2n+1}w^n \tag{29b}$$

$$K(w) = F(w^2) + wG(w^2) \tag{29c}$$

and to k the functions f and g in $L^2(0,+\infty;dx)$ corresponding to F and G. Certainly, $\|k\|^2 = \|f\|^2 + \|g\|^2$, and the assignment of (f,g) to k is an isometric identification. Furthermore, certainly

[8] we also take note of the operator identity $\mathcal{H}\cdot w = -w\cdot\mathcal{H}$.

the $\mathcal{H}$ transform acts in this picture as $(f,g)\mapsto(f,-g)$. Let us now check the support properties. Let $\alpha(m)$ be the leftmost point of the (essential) support of a given $m\in L^2(0,\infty;dx)$. As is well-known,

$$-\alpha(m)=\limsup_{t\to+\infty}\frac{1}{t}\log|\widetilde{m}(it)|\ , \tag{30}$$

If w corresponds to λ via (23) then w^2 corresponds to $\frac{1}{2}(\lambda-\frac{1}{\lambda})$, so if to a function f with corresponding $F(w)$ we associate the function $\psi(f)\in L^2(0,\infty;dx)$ which corresponds to $F(w^2)$,

$$(t+1)\widetilde{\psi(f)}(i\,t)=(\frac{t+\frac{1}{t}}{2}+1)\widetilde{f}(i\,\frac{t+\frac{1}{t}}{2})\ , \tag{31}$$

then we have the identity:

$$\alpha(\psi(f))=\frac{1}{2}\alpha(f) \tag{32}$$

Returning to F (resp. f) and G (resp. g) associated via (29a), (29b), to K (resp. k) we thus have $k=\psi(f)+w\cdot\psi(g)$, $\mathcal{H}(k)=\psi(f)-w\cdot\psi(g)$, hence if the pair (f,g) vanishes on $(0,2a)$ then the pair $(k,\mathcal{H}(k))$ vanishes on $(0,a)$ (clearly the unitary operator of multiplication by $w=\frac{\lambda-i}{\lambda+i}$ does not affect $\alpha(m)$.) Conversely, as $\alpha(f)=2\alpha(k+\mathcal{H}(k))$ and $\alpha(g)=2\alpha(k-\mathcal{H}(k))$, if the pair $(k,\mathcal{H}(k))$ vanishes on $(0,a)$ then the pair (f,g) vanishes on $(0,2a)$.

We have thus established the existence of an isometric expansion, its support properties, and its relation to the $\mathcal{H}$-transform. That there is indeed compatibility of (20a) and (20b) with (29a) and (29b), and with (20c), will be established in the last section (9) of the paper with a direct study of (31). In the meantime equations (20a), (20b), (20c) and (20d) will have been confirmed in another manner. Yet another proof of the isometric expansion has been given in [9].

3 Tempered distributions and their $\mathcal{H}$ and Mellin transforms

Any distribution D on $\mathbb{R}$ has a primitive. If the closed support of D is included in $[0,+\infty)$, then it has a unique primitive, which we will denote $\int_0^x D(x)\,dx$, or, more safely, $D^{(-1)}$, which also has its support in $[0,+\infty)$. The temperedness of such a D is equivalent to the fact that $D^{(-N)}$ for $N\gg 0$ is a continuous function with polynomial growth. With $D^{(-N)}(x)=(1+x^2)^M g_{(N,M)}(x)$, $M\gg 0$, we can express D as $P(x,\frac{d}{dx})(g)$ where P is a polynomial and $g\in L^2(0,\infty;dx)$. Conversely any such expression is a tempered distribution vanishing in $(-\infty,0)$. The Fourier transforms of such tempered distributions $\widetilde{D}(\lambda)$ appear thus as the boundary values of $Q(\frac{d}{d\lambda},\lambda)f(\lambda)$ where Q are polynomials and the f's belong to $\mathbb{H}^2(\Im(\lambda)>0)$. As taking primitives is allowed we know without further ado that this class of analytic functions is the same thing as the space of functions $g(\lambda)=R(\frac{d}{d\lambda},\lambda,\lambda^{-1})f(\lambda)$, R a polynomial and $f\in\mathbb{H}^2$. It is thus clearly left stable by the operation:

$$g\mapsto\mathcal{H}(g)(\lambda):=\frac{i}{\lambda}g(\frac{-1}{\lambda})\qquad(\Im(\lambda)>0) \tag{33}$$

which will serve to define the action of $\mathcal{H}$ on tempered distributions with support in $[0,+\infty)$.

Let us also use (33), where now $\lambda \in \mathbb{R}$, to define $\mathcal{H}$ as a unitary operator on $L^2(-\infty, +\infty; dx)$. It will anti-commute with $f(x) \to f(-x)$ so:

$$\mathcal{H}(f)(x) = \int_{-\infty}^{\infty} (J_0(2\sqrt{xy})\mathbf{1}_{x>0}(x)\mathbf{1}_{y>0}(y) - J_0(2\sqrt{xy})\mathbf{1}_{x<0}(x)\mathbf{1}_{y<0}(y))f(y)\,dy \tag{34}$$

Useful operator identities are easily established from (33):

$$x\frac{d}{dx} \cdot \mathcal{H} = -\mathcal{H} \cdot \frac{d}{dx}x \qquad \text{and} \qquad \frac{d}{dx}x \cdot \mathcal{H} = -\mathcal{H} \cdot x\frac{d}{dx} \tag{35a}$$

$$\frac{d}{dx} \cdot \mathcal{H} = \mathcal{H} \cdot \int_0^x \qquad \text{and} \qquad \int_0^x \cdot \mathcal{H} = \mathcal{H} \cdot \frac{d}{dx} \tag{35b}$$

$$x \cdot \mathcal{H} = -\mathcal{H} \cdot \frac{d}{dx}\,x\,\frac{d}{dx} \qquad \text{and} \qquad \mathcal{H} \cdot x = -\frac{d}{dx}\,x\,\frac{d}{dx} \cdot \mathcal{H} \tag{35c}$$

It is important that $\frac{d}{dx}$ is always taken in the distribution sense. It would actually be possible to define the action of $\mathcal{H}$ on distributions supported in $[0, +\infty)$ without mention of the Fourier transform, because these identities uniquely determine $\mathcal{H}(D)$ if D is written $(\frac{d}{dx})^N(1+x)^M g_{N,M}(x)$ with $g_{N,M} \in L^2(0, \infty; dx)$. But the proof needs some organizing then as it is necessary to check independence from the choice of N and M, and also to establish afterwards all identities above. So (33) provides the easiest road. Still, in this context, let us mention the following which relates to the restriction of $\mathcal{H}(D)$ to $(0, +\infty)$:

Lemma 2. *Let k be smooth on $\mathbb{R}$ with compact support in $[0, +\infty)$. Then $\mathcal{H}(k)$ is the restriction to $[0, +\infty)$ of an entire function γ which has Schwartz decrease as $x \to +\infty$. For any tempered distribution D with support in $[0, +\infty)$, there holds*

$$\int_0^\infty \mathcal{H}(D)(x)k(x)\,dx = \int_0^\infty D(x)\gamma(x)\,dx\ , \tag{36}$$

where in the right hand side in fact one has $\int_{-\epsilon}^\infty D(x)\gamma(x)\theta(x)\,dx$ where the smooth function θ is 1 for $x \geq -\frac{\epsilon}{3}$ and 0 for $x \leq -\frac{\epsilon}{2}$ and is otherwise arbitrary (as is ϵ).

Let us suppose $k = 0$ for $x > B$. Defining:

$$\gamma(x) = \int_0^B J_0(2\sqrt{xy})\,k(y)\,dy \tag{37}$$

we obtain an entire function and, according to our definitions, $\mathcal{H}(k)(x) = \gamma(x)\mathbf{1}_{x>0}(x)$ as a distribution or a square-integrable function. Using (35c) ($\mathcal{H} = -\frac{1}{x}\mathcal{H} \cdot \frac{d}{dx}\,x\,\frac{d}{dx}$ for $x > 0$) and bounding J_0 by 1 we see (induction) that γ is $O(x^{-N})$ for any N as $x \to +\infty$, and using (35a) ($\frac{d}{dx} \cdot \mathcal{H} = -\frac{1}{x}\mathcal{H} \cdot \frac{d}{dx}x$ for $x > 0$) the same applies to its derivative and also to its higher derivatives. So it is of the Schwartz class for $x \to +\infty$.

Replacing D by $\mathcal{H}(D)$ in (36) it will be more convenient to prove:

$$\int_0^\infty D(x)k(x)\,dx = \int_0^\infty \mathcal{H}(D)(x)\gamma(x)\,dx \tag{38}$$

If (38) holds for D (and all k's) then $< D', k >= - < D, k' >= - < \mathcal{H}(D), -\theta(x) \int_x^\infty \gamma(y)dy >$ (observe that $\int_0^x \gamma(y)\,dy = \mathcal{H}(k')(x)$ vanishes at $+\infty$) so $< D', k >= + < \int_0^x \mathcal{H}(D), \theta\gamma >=< \mathcal{H}(D'), \theta\gamma >$ hence (38) holds as well for D' (and all k's). So we may assume D to be a continuous function of polynomial growth. It is also checked using (35c) that if (38) holds for D it holds for xD. So we may reduce to D being square-integrable, and the statement then follows from the self-adjointness of $\mathcal{H}$ on L^2 (or we reduce to Fubini).

The behavior of $\mathcal{H}$ with respect to the translations $\tau_a : f(x) \mapsto f(x-a)$ is important. For $f \in L^2(\mathbb{R}; dx)$ the value of a is arbitrary and we can define

$$\tau_a^{\#} := \mathcal{H}\,\tau_a\,\mathcal{H} \tag{39a}$$

$$\widetilde{\tau_a(f)}(\lambda) = e^{ia\lambda}\widetilde{f}(\lambda) \tag{39b}$$

$$\widetilde{\tau_a^{\#}(f)}(\lambda) = e^{ia\frac{-1}{\lambda}}\widetilde{f}(\lambda) \tag{39c}$$

We observe the remarkable commutation relations (which would fail for the cosine or sine transforms):

$$\forall a, b \qquad \tau_a\tau_b^{\#} = \tau_b^{\#}\tau_a \tag{40}$$

For a distribution D the action of $\tau_a^{\#}$ is here defined only for $a \geq -\alpha(\mathcal{H}(D))$, where $\alpha(E)$ is the leftmost point of the closed support of the distribution E. On this topic from the validity of (30) when $f \in L^2(0, \infty; dx)$, and invariance of α under derivation[9], integration, and multiplication by x, one has:

$$-\alpha(E) = \limsup_{t\to+\infty} \frac{1}{t}\,\log|\widetilde{E}(it)| \tag{41}$$

We thus have the property, not shared by the cosine or sine transforms:

$$a \geq -\alpha(\mathcal{H}(D)) \implies \alpha(\tau_a^{\#}(D)) = \alpha(D) \tag{42}$$

We now consider D with $\alpha(D) > 0$ and $\alpha(\mathcal{H}(D)) > 0$ and prove that its Mellin transform is an entire function with trivial zeros at $0, -1, -2, \ldots$, following the method of regularization by multiplicative convolution and co-Poisson intertwining from [8]. The other, very classical in spirit, proof shall be presented later. The latter method is shorter but the former provides complementary information.

In [8, §4.A] the detailed explanations relative to the notion of multiplicative convolution are given:

$$(g * D)(x) \text{ “}=\text{” } \int_{\mathbb{R}} g(t)D(\frac{x}{t})\,\frac{dt}{|t|}\,, \tag{43}$$

where we will in fact always take g to have compact support in $(0, +\infty)$. It is observed that

$$g * xD = x(\frac{g}{x} * D) \qquad (g * D)' = \frac{g}{x} * D' \tag{44}$$

[9] It is important in order to avoid a possible confusion to insist on the fact that $\frac{d}{dx}$ is always taken in the distribution sense so for example $\frac{d}{dx}\mathbf{1}_{x>0} = \delta(x)$ indeed has the leftmost point of its support not affected by $\frac{d}{dx}$.

The notion of right Mellin transform $\int_0^\infty D(x)x^{-s}dx$ is developed in [8, §4.C], for D with support in $[a,+\infty)$, $a>0$:

$$\widehat{D}(s) = s(s+1)\cdots(s+N-1)\widehat{D^{(-N)}}(s+N)\ , \tag{45}$$

where $N \gg 0$. The meaning of $\widehat{D}$ is as the maximal possible analytic continuation to a half-plane $\Re(s) > \sigma$, where σ is as to the left as is possible. The notion is extended[10] in [8, §4.F] to the case where the restriction of D to $(-a,a)$ is "quasi-homogeneous". For example, if $D|_{(-a,a)} = \mathbf{1}_{0<x<a}$ (resp. δ), then $\widehat{D}$ is defined as $\widehat{D}_1$ with $D_1 = D - \mathbf{1}_{0<x<\infty}$ (resp. $D-\delta$.) Then, also in the extended case, the following holds:

$$\widehat{g*D}(s) = \widehat{g}(s)\widehat{D}(s) \tag{46}$$

where g in an integrable function with compact support in $(0,\infty)$ and $\widehat{g}(s)$ is the entire function $\int_0^\infty g(t)t^{-s}\,dt$. We then have the following theorem:

Theorem 3. *Let D a tempered distribution with support in $[a,+\infty)$, $a>0$ and such that $\mathcal{H}(D)$ also has a positive leftmost point of support. Let g be a smooth function with compact support in $(0,\infty)$. Then the multiplicative convolution $g*D$ belongs to the Schwartz class.*

This is the analog of [8, Thm 4.29]. The function $k(t) = (Ig)(t) = \frac{g(1/t)}{t}$ is defined and it is written as $k = \mathcal{H}(\gamma\mathbf{1}_{x>0})$ where γ is the entire function, of Schwartz decrease at $+\infty$ such that $\mathcal{H}(k) = \gamma\cdot\mathbf{1}_{x>0}$. Then it is observed that

$$t>0 \implies \quad (g*D)(t) = \int_0^\infty D(x)\frac{k(x/t)}{t}\,dx = \int_0^\infty \mathcal{H}(D)(x)\gamma(tx)\,dx \tag{47}$$

We have used Lemma 2. Then the Schwartz decrease of $\int_0^\infty \mathcal{H}(D)(x)\gamma(tx)\,dx$ as $t\to+\infty$ is established as is done at the end of the proof of [8, Thm 4.29], integrating by parts enough times to transform $\mathcal{H}(D)$ into a continuous function of polynomial growth, identically zero on $[0,c]$, $c>0$.

Theorem 4. *Let D a tempered distribution with a positive lefmost point of support and such that $\mathcal{H}(D)$ also has a positive leftmost point of support. Then $\widehat{D}(s)$ and $\Gamma(s)\widehat{D}(s)$ are entire functions and:*

$$\Gamma(s)\widehat{D}(s) = \Gamma(1-s)\widehat{\mathcal{H}(D)}(1-s) \tag{48}$$

We first establish:

Theorem 5 ("co-Poisson intertwining")**.** *Let D be a tempered distribution supported in $[0,+\infty)$ and let g be an integrable function with compact support in $(0,\infty)$. Then, with $(Ig)(t) = \frac{g(1/t)}{t}$:*

$$\mathcal{H}(g*D) = (Ig)*\mathcal{H}(D) \tag{49}$$

[10]if D is near the origin a function with an analytic character, then straightforward elementary arguments allow a complementary discussion. However if D is just an element of $L^2(0,\infty;dx)$ then $\widehat{D}$ is a square-integrable function on the critical line, and nothing more nor less.

Let us first suppose that D is an L^2 function. In that case, we will use the Mellin-Plancherel transform $f \mapsto \widehat{f}(s) = \int_0^\infty f(t)t^{-s}\,dt$, for f square integrable and $\Re(s) = \frac{1}{2}$. Then $\widehat{g * f}$ is, changing variables, the Fourier transform of an additive convolution where one of the two has compact support, well known to be the product $\widehat{g} \cdot \widehat{f}$. We need also to understand the Mellin transform of $\mathcal{H}(f)$. Let us suppose $f_t(x) = \exp(-tx)$. Then $\mathcal{H}(f_t) = \frac{1}{t} f_{\frac{1}{t}}$ has Mellin transform $\widehat{\mathcal{H}(f_t)}(s) = t^{-s}\Gamma(1-s)$ and $\widehat{f_t}(s) = t^{s-1}\Gamma(1-s)$, so we have the identity for such f's:

$$\widehat{\mathcal{H}(f)}(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \widehat{f}(1-s) \tag{50}$$

The linear combinations of the f_t's are dense in L^2, so (49) holds for all f's as an identity of square integrable functions on the critical line. We are now in a position to check the intertwining: $\widehat{\mathcal{H}(g * f)}(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \widehat{g}(1-s)\widehat{f}(1-s) = \widehat{Ig}(s)\widehat{\mathcal{H}(f)}(s) = \widehat{Ig * \mathcal{H}(f)}(s)$.

For the case of an arbitrary distribution it will then be sufficient to check that if (49) holds for D it holds for xD and for D'. This is easily done using (44). We have $g * (D') = (xg * D)'$, so $\mathcal{H}(g * D') = \int_0^x \mathcal{H}(xg * D) = \int_0^x (\frac{Ig}{x} * \mathcal{H}(D)) = Ig * (\int_0^x \mathcal{H}(D)) = Ig * \mathcal{H}(D')$. A similar proof is done for xD. This completes the proof of the intertwining.

The theorem 4 is then established as is [8, Thm 4.30]. We pick an arbitrary g smooth with compact support in $(0, \infty)$. We know by theorem 3 that $g * D$ is a Schwartz function as $x \to +\infty$, and certainly it vanishes identically in a neighborhood of the origin, so $\widehat{g * D}(s) = \widehat{g}(s)\widehat{D}(s)$ is an entire function. So $\widehat{D}(s)$ is a meromorphic function in the entire complex plane, in fact an entire function as g is arbitrary. We then use the intertwining and (50) for square integrable functions. This gives $\widehat{g}(1-s)\widehat{\mathcal{H}(D)}(s) = \widehat{Ig * \mathcal{H}(D)}(s) = \widehat{\mathcal{H}(g * D)}(s) = \frac{\Gamma(1-s)}{\Gamma(s)} \widehat{g}(1-s)\widehat{D}(1-s)$. Hence, indeed, after replacing s by $1-s$:

$$\Gamma(s)\widehat{D}(s) = \Gamma(1-s)\widehat{\mathcal{H}(D)}(1-s) \tag{51}$$

The left-hand side may have poles only at $0, -1, \ldots$, and the right-hand side only at $1, 2, \ldots$. So both sides are entire functions and $\widehat{D}(s)$ has trivial zeros at $0, -1, -2, \ldots$

We now give another proof of Theorem 4, which is more classical, as it is the descendant of the second of Riemann's proof, and is the familiar one from the theory of theory of L-series and modular functions. The existence of two complementary proofs is instructive, as it helps to better understand the rôle of the right Mellin transform $\int_0^\infty f(x)x^{-s}\,dx$ vs. the left Mellin transform $\int_0^\infty \theta(it)t^{s-1}\,dt$.

To the distribution D we associate its "theta" function[11] $\theta_D(\lambda) = \widetilde{D}(\lambda) = \int_0^\infty e^{i\lambda x} D(x)\,dx$, which is an analytic function for $\Im(\lambda) > 0$ [12]. Right from the beginning we have:

$$\theta_{\mathcal{H}(D)}(it) = \frac{1}{t}\theta_D(\frac{i}{t}) \tag{52}$$

If the leftmost point of the support of D is positive then $\theta_D(it)$ has exponential decrease as $t \to +\infty$ and $\int_1^\infty \theta_D(it)t^{s-1}\,dt$ is an entire function. If also the leftmost point of support of $\mathcal{H}(D)$ is positive

[11] the author hopes to be forgiven this temporary terminology in a situation where only the behavior under $\lambda \mapsto \frac{-1}{\lambda}$ is at work.

[12] we adopt the usual notation, and consider θ_D as a function of it rather than t.

then $\theta_{\mathcal{H}(D)}(it)$ has exponential decrease as $t \to +\infty$ and $\int_0^1 \theta_D(it)t^{s-1}\,dt = \int_1^\infty \theta_{\mathcal{H}(D)}(it)t^{-s}\,dt$ is an entire function. So, under the support property considered in Theorem 4 $\mathcal{D}(s) := \int_0^\infty \theta_D(it)t^{s-1}\,dt$ is indeed an entire function, and the functional equation is

$$\mathcal{D}(s) = \mathcal{D}^*(1-s) \tag{53}$$

with $\mathcal{D}^*(s) = \int_0^\infty \theta_{\mathcal{H}(D)}(t)t^{s-1}\,dt$.

To conclude we also need to establish:

$$\mathcal{D}(s) = \Gamma(s)\widehat{D}(s) \tag{54}$$

We shall prove this for $\Re(s) \gg 0$ under the hypothesis that D has support in $[a, +\infty)$, $a > 0$ (no hypothesis on $\mathcal{H}(D)$). In that case, as $\theta_D(it)$ is $O(t^{-N})$ for a certain N as $t \to 0$ ($t > 0$), and is of exponential decrease as $t \to +\infty$, we can define $\mathcal{D}(s) = \int_0^\infty \theta_D(it)t^{s-1}\,dt$ as an analytic function for $\Re(s) \gg 0$. Let us suppose that D is a continuous function which is $O(x^{-2})$ as $x \to +\infty$. Then, for, $\Re(s) > 0$, the identity (54) holds as an application of the Fubini theorem. We then apply our usual method to check that if (54) holds for D it also holds for xD and for D'. For this, obviously we need things such as $\widehat{D'}(s) = s\widehat{D}(s+1)$ [8, 4.15] and $\widehat{xD}(s) = \widehat{D}(s-1)$, the formulas $\theta_{D'} = -i\lambda\theta_D$, $\theta_{xD} = -i\frac{\partial}{\partial\lambda}\theta_D$, and $\Gamma(s+1) = s\Gamma(s)$. The verifications are then straightforward.

In summary we have seen how the support property for D and $\mathcal{H}(D)$ is related in two complementary manners to the functional equation, one using the right Mellin transform $\widehat{D}(s)$ of D and the idea of co-Poisson, the other using the left Mellin transform $\mathcal{D}(s)$ of the "theta" function θ_D associated to D as an analytic function on the upper half-plane and the behavior of $\theta_D(it)$ under $t \mapsto \frac{1}{t}$. It is possible to push further the analysis and to characterize the class of entire functions $\mathcal{D}(s) = \Gamma(s)\widehat{D}(s)$, as has been done in [8] in the case of the cosine and sine transforms. It is also explained in [8] how the discussion extends to allow finitely many poles. The proofs and statements given there are easily adapted to the case of the $\mathcal{H}$ transform. Only the case of poles at 1 and 0 will be needed here and this corresponds, either to the condition that D and $\mathcal{H}(D)$ both restrict in $(-a, a)$ for some $a > 0$ to multiples of the Dirac delta function, or, that they are both constant in $[0, a)$ for some $a > 0$. We recall that the Mellin transform $\widehat{D}(s)$ is defined in such a manner, that it is not affected from either substracting δ or $\mathbf{1}_{x>0}$ from D.

4 A group of distributions and related integral formulas

We now derive some integral identities which will prove central. The identities will be re-obtained later as the outcome of a less direct path. We are interested in the tempered distribution $g_a(x)$ whose Fourier transform is $\exp(ia\frac{-1}{\lambda})$. Indeed $\tau_a^{\#}(f)$ (equation (39a)) is the additive convolution of f with g_a: we note that g_a differs from $\delta(x)$ by a square integrable function as $1 - \exp(-ia\lambda^{-1}) = O_{|\lambda|\to\infty}(|\lambda|^{-1})$; so there is a convolution formula $\tau_a^{\#}(f) = f - f_a * f$ for a certain square integrable function f_a. For $f \in L^2$, the convolution $f_a * f$ as the Fourier transform of an L^1-function is continuous on $\mathbb{R}$. Starting from the identity $\exp(ia\frac{-1}{\lambda}) = -i\lambda\frac{i}{\lambda}\exp(ia\frac{-1}{\lambda})$ we identify g_a for $a \geq 0$ as $\frac{\partial}{\partial x}\mathcal{H}\delta_a$. It is important that $\frac{\partial}{\partial x}$ is taken in the distribution sense. So we have, simply:

$$g_a(x) = \delta(x) - \frac{aJ_1(2\sqrt{ax})}{\sqrt{ax}}\mathbf{1}_{x>0}(x) \qquad (a \geq 0) \tag{55}$$

If $a < 0$ then $g_a(x) = g_{-a}(-x)$, $f_a(x) = f_{-a}(-x)$. So:

$$g_{-a}(x) = \delta(x) - \frac{aJ_1(2\sqrt{-ax})}{\sqrt{-ax}}\mathbf{1}_{x<0}(x) \qquad (-a \leq 0) \tag{56}$$

The group property under the additive convolution $g_a * g_b = g_{a+b}$ leads to remarkable integral identities $f_{a+b} = f_a + f_b - f_a * f_b$ involving the Bessel functions. The pointwise validity is guaranteed by continuity; the Plancherel identity confirms the identity, where $f_a(x) = \frac{aJ_1(2\sqrt{ax})}{\sqrt{ax}}\mathbf{1}_{x>0}(x)$ for $a \geq 0$ and $f_{-a}(x) = f_a(-x)$:

$$f_{a+b} = f_a + f_b - f_a * f_b \tag{57}$$

At $x = 0$ the pointwise identity is obtained by continuity from either $x > 0$ or $x < 0$. We have essentially two cases: $g_a * g_b$ for $a, b \geq 0$ and $g_a * g_{-b}$ for $a \geq b \geq 0$. The following is obtained:

Proposition 6. *Let $a \geq b \geq 0$ and $x \geq 0$. There holds:*

$$\frac{(a+b)J_1(2\sqrt{(a+b)x})}{\sqrt{(a+b)x}} = \frac{aJ_1(2\sqrt{ax})}{\sqrt{ax}} + \frac{bJ_1(2\sqrt{bx})}{\sqrt{bx}} - \int_0^x \frac{aJ_1(2\sqrt{ay})}{\sqrt{ay}}\,\frac{bJ_1(2\sqrt{b(x-y)})}{\sqrt{b(x-y)}}\,dy \tag{58a}$$

$$\frac{(a-b)J_1(2\sqrt{(a-b)x})}{\sqrt{(a-b)x}} = \frac{aJ_1(2\sqrt{ax})}{\sqrt{ax}} - \int_x^\infty \frac{aJ_1(2\sqrt{ay})}{\sqrt{ay}}\,\frac{bJ_1(2\sqrt{b(y-x)})}{\sqrt{b(y-x)}}\,dy \tag{58b}$$

$$0 = \frac{bJ_1(2\sqrt{bx})}{\sqrt{bx}} - \int_0^\infty \frac{aJ_1(2\sqrt{ay})}{\sqrt{ay}}\,\frac{bJ_1(2\sqrt{b(y+x)})}{\sqrt{b(y+x)}}\,dy \tag{58c}$$

Exchanging a and b and changing variables we combine (58b) and (58c) into one single equation for $x \geq 0$ and $a, b \geq 0$:

$$\frac{(a-b)J_1(2\sqrt{(a-b)x})}{\sqrt{(a-b)x}}\mathbf{1}_{a-b\geq 0}(a-b) = \frac{aJ_1(2\sqrt{ax})}{\sqrt{ax}} - \int_0^\infty \frac{aJ_1(2\sqrt{a(y+x)})}{\sqrt{a(y+x)}}\,\frac{bJ_1(2\sqrt{by})}{\sqrt{by}}\,dy \tag{59}$$

The formula for $x = 0$ in (59) is obtained by continuity. It is equivalent to

$$\int_0^\infty J_1(u)J_1(cu)\,\frac{du}{u} = \frac{1}{2}\min(c, \frac{1}{c}) \qquad (c > 0) \tag{60}$$

which is a very special case of formulas of Weber, Sonine and Schafheitlin ([33, 13.42.(1)]). Another interesting special case of (59) is for $a = b$. The formula becomes

$$\frac{J_1(2\sqrt{x})}{\sqrt{x}} = \int_0^\infty \frac{J_1(2\sqrt{y})}{\sqrt{y}}\,\frac{J_1(2\sqrt{x+y})}{\sqrt{x+y}}\,dy \tag{61}$$

which is equivalent to a special case of a formula of Sonine ([33, 13.48.(12)]).

We already mentioned the equation $\frac{\partial^2}{\partial u \partial v} J_0(2\sqrt{uv}) = -J_0(2\sqrt{uv})$. New identities are obtained from (59) or (58a) after taking either the a or the b derivative. We investigate no further (59) as the corresponding semi-convergent integrals, in a form or another, are certainly among the formulas of [33, §13]. Let us rather focus more closely on the case $a, b \geq 0$ ((58a).) We have a function which is entire in a, b, and x and the identity holds for all complex values of a, b, and x. Let us take the derivative with respect to a:

$$J_0(2\sqrt{(a+b)x}) = J_0(2\sqrt{ax}) - \int_0^x J_0(2\sqrt{ay})\, \frac{bJ_1(2\sqrt{b(x-y)})}{\sqrt{b(x-y)}}\, dy \tag{62}$$

We replace b by $-b$ and then set $x = b$. This gives:

$$I_0(2\sqrt{b(b-a)}) = J_0(2\sqrt{ba}) + \int_0^b J_0(2\sqrt{ay})\, \frac{bI_1(2\sqrt{b(b-y)})}{\sqrt{b(b-y)}}\, dy \tag{63}$$

We take the derivative of (62) with respect to b:

$$-\frac{xJ_1(2\sqrt{(a+b)x})}{\sqrt{(a+b)x}} = -\int_0^x J_0(2\sqrt{ay}) J_0(2\sqrt{b(x-y)})\, dy \tag{64}$$

Then we replace b by $-b$ and set $x = b$:

$$\frac{bI_1(2\sqrt{b(b-a)})}{\sqrt{b(b-a)}} = \int_0^b J_0(2\sqrt{ay}) I_0(2\sqrt{b(b-y)})\, dy \tag{65}$$

Combining (63) and (65) by addition and substraction we discover that we have solved certain integral equations:

$$\phi_b^+(x) = I_0(2\sqrt{b(b-x)}) - \frac{bI_1(2\sqrt{b(b-x)})}{\sqrt{b(b-x)}} = (1 + \frac{\partial}{\partial x}) I_0(2\sqrt{b(b-x)}) \tag{66a}$$

$$\phi_b^-(x) = I_0(2\sqrt{b(b-x)}) + \frac{bI_1(2\sqrt{b(b-x)})}{\sqrt{b(b-x)}} = (1 - \frac{\partial}{\partial x}) I_0(2\sqrt{b(b-x)}) \tag{66b}$$

$$\phi_b^+(x) + \int_0^b J_0(2\sqrt{xy}) \phi_b^+(y)\, dy = J_0(2\sqrt{bx}) \tag{66c}$$

$$\phi_b^-(x) - \int_0^b J_0(2\sqrt{xy}) \phi_b^-(y)\, dy = J_0(2\sqrt{bx}) \tag{66d}$$

The significance will appear later in the paper and we leave the matter here. The method was devised after the importance of solving equations (66c) and (66d) had emerged and after the solutions (66a) and (66b) had been obtained as the outcome of a more indirect path. Of course, direct verification by replacement of the Bessel functions by their series expansions is possible and easy.

5 Orthogonal projections and Hilbert space evaluators

Let $a > 0$ and let P_a be the orthogonal projection on $L^2(0, a; dx)$ and $Q_a = \mathcal{H} P_a \mathcal{H}$ the orthogonal projection on $\mathcal{H}(L^2(0, a; dx))$ and let $K_a \subset L^2(0, \infty; dx)$ be the Hilbert space of square integrable functions f such that both f and $\mathcal{H}(f)$ have their supports in $[a,\infty)$. Also we shall write $H_a = P_a \mathcal{H} P_a$. Also we shall very often use $D_a = H_a^2 = P_a \mathcal{H} P_a \mathcal{H} P_a$. Using:

$$J_0(2\sqrt{xy}) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n y^n}{n!^2} \ , \tag{67}$$

we exhibit $H_a = P_a \mathcal{H} P_a$ as a limit in operator norm of finite rank operators so $P_a \mathcal{H} P_a$ is a compact (self-adjoint) operator. It is not possible for a non zero $f \in L^2(0,a;dx)$ to be such that $\|H_a(f)\| = \|f\|$, as this would imply that $H_a(f)$ vanishes identically for $x > a$, but $H_a(f)$ is an entire function. So the operator norm of H_a is strictly less than one, and $1 \pm H_a$ as well as $1 - D_a$ are invertible. We consider the equation

$$\phi = u + \mathcal{H}(v) \qquad u, v \in L^2(0,a;dx) \tag{68}$$

Hence:

$$u + H_a(v) = P_a(\phi) \tag{69a}$$

$$H_a(u) + v = P_a(\mathcal{H}(\phi)) \tag{69b}$$

$$u = (1 - D_a)^{-1}(P_a(\phi) - H_a P_a \mathcal{H}(\phi)) \tag{69c}$$

$$v = (1 - D_a)^{-1}(-H_a P_a(\phi) + P_a \mathcal{H}(\phi)) \tag{69d}$$

Then if $\phi_n = u_n + \mathcal{H}(v_n)$ is L^2-convergent, (u_n) and (v_n) will be convergent, and the vector space sum $L^2(0,a;dx) + \mathcal{H}(L^2(0,a;dx))$ is closed. Its elements are analytic functions for $x > a$ so certainly this is a proper subspace of L^2. Hence we obtain that each K_a is not reduced to $\{0\}$ and

$$K_a^\perp = L^2(0,a;dx) + \mathcal{H}(L^2(0,a;dx)) \tag{70}$$

We also mention that $\cup_{a>0} K_a$ is dense in but not equal to $L^2(0,\infty;dx)$, more generally that $\cup_{a>b} K_a$ is dense in but not equal to K_b, and also obviously $\cap_{a<\infty} K_a = \{0\}$, $\cap_{a<b} K_a = K_b$.

In this section $a > 0$ will be fixed (all defined quantities and functions will depend on a, but this will not always be explicitely indicated.) We shall be occupied with understanding the vectors $X_s^a \in K_a$ such that

$$\forall f \in K_a \qquad \int_a^\infty f(x) X_s^a(x)\, dx = \widehat{f}(s) = \int_a^\infty f(x) x^{-s}\, dx \tag{71}$$

and in particular we are interested in computing

$$X_a(s,z) = \int_a^\infty X_s^a(x) X_z^a(x)\, dx \tag{72}$$

As a is fixed here, we shall drop the superscript a to lighten the notation. For the time being we shall restrict to $\Re(s) > \frac{1}{2}$ and we define X_s to be the orthogonal projection to K_a of $\mathbf{1}_{x>a}(x) x^{-s}$. As a preliminary to this study we need to say a few words regarding:

$$g_s(x) := \mathcal{H}(\mathbf{1}_{x>a}(x) x^{-s}) = \int_a^\infty J_0(2\sqrt{xy}) y^{-s}\, dy \tag{73}$$

The integral is absolutely convergent for $\Re(s) > \frac{3}{4}$, semi-convergent for $\Re(s) > \frac{1}{4}$, and g_s is defined by the equation as an L^2 function for $\Re(s) > \frac{1}{2}$ (it will prove to be entire in s for each $x > 0$). We need the following identity, which shows also that $g_s(x)$ is analytic in $x > 0$:

$$g_s(x) = \chi(s)x^{s-1} - \int_0^a J_0(2\sqrt{xy})y^{-s}\,dy = \chi(s)x^{s-1} - \sum_{n=0}^{\infty}(-1)^n \frac{x^n\, a^{n+1-s}}{n!^2(n+1-s)} \tag{74}$$

This is obtained first in the range $\frac{3}{4} < \Re(s) < 1$: $\int_a^\infty J_0(2\sqrt{xy})y^{-s}\,dy = x^{s-1}\int_{ax}^\infty J_0(2\sqrt{y})y^{-s}\,dy = x^{s-1}\left(\chi(s) - \int_0^{ax} J_0(2\sqrt{y})y^{-s}\,dy\right) = x^{s-1}\chi(s) - \int_0^a J_0(2\sqrt{xy})y^{-s}\,dy$. The poles at $s = 1$, $s = 2$, ...are only apparent. The identity is valid by analytic continuation in the entire plane $\Re(s) > \frac{1}{2}$. For each given $x > 0$ we have in fact an entire function of $s \in \mathbb{C}$. But we are here more interested in g_s as a function of x and we indeed see that it is analytic in $\mathbb{C}\setminus]-\infty, 0]$ (it is an entire function of x if $s \in -\mathbb{N}$). [13]

There are unique vectors u_s, v_s in $L^2(0, a; dx)$ such that

$$\mathbf{1}_{x>a}(x)x^{-s} = X_s(x) + u_s(x) + \mathcal{H}(v_s)(x) \tag{75}$$

and they are the solutions to the system of equations:

$$u_s + H_a(v_s) = 0 \tag{76a}$$

$$H_a(u_s) + v_s = P_a(g_s) \tag{76b}$$

From (76a) we see that u_s is in fact the restriction to $(0, a)$ of an entire funtion, and from (76b) that v_s is the restriction to $(0, a)$ of a function which is analytic in $\mathbb{C}\setminus]-\infty, 0]$. Redefining u_s and v_s to now refer to these analytic functions their defining equations become (on $(0, +\infty)$):

$$u_s + \mathcal{H}P_a(v_s) = 0 \tag{77a}$$

$$\mathcal{H}P_a(u_s) + v_s = g_s \tag{77b}$$

and (75) becomes (we set $X_s(a) = X_s(a+)$):

$$\mathbf{1}_{x\geq a}(x)x^{-s} = X_s(x) + \mathbf{1}_{0<x<a}(x)u_s(x) + \mathcal{H}P_a(v_s)(x) \tag{78a}$$

$$\mathbf{1}_{x\geq a}(x)x^{-s} = X_s(x) - \mathbf{1}_{x\geq a}(x)u_s(x) \tag{78b}$$

$$X_s(x) = \mathbf{1}_{x\geq a}(x)(x^{-s} + u_s(x)) \tag{78c}$$

The key to the next steps will be the idea to investigate the distribution $(x\frac{d}{dx}+s)X_s$ on the (positive) real line. Let D_s be $x\frac{d}{dx} + s$. There holds:

$$D_s\mathcal{H} = -\mathcal{H}D_{1-s} \tag{79}$$

To compute $\frac{d}{dx}P_a(v_s)$ we first suppose $\Re(s) > 1$, so (we know the behavior as $x \to 0$ from (74)) $\frac{d}{dx}P_a(v_s) = P_a(v_s') - v_s(a)\delta_a(x)$ and $x\frac{d}{dx}P_a(v_s) = P_a(xv_s') - av_s(a)\delta_a(x)$. This remains true for

[13] For some other transforms $k(xy)$, such as the cosine transform, the argument must be slightly modified in order to accomodate the fact $\int_0^\infty k(y)y^{-s}\,dy$ has no range of absolute convergence.

$\Re(s) > \frac{1}{2}$. Applying D_s to (77a) thus gives $D_s(u_s) - \mathcal{H}\left(P_a D_{1-s}(v_s) - a v_s(a)\delta_a(x)\right)$. We similarly apply D_{1-s} to (77b) and obtain the following system:

$$D_s(u_s)(x) - (\mathcal{H} P_a D_{1-s} v_s)(x) = -a v_s(a) J_0(2\sqrt{ax}) \tag{80a}$$

$$-(\mathcal{H} P_a D_s u_s)(x) + D_{1-s}(v_s)(x) = (D_{1-s} g_s)(x) - a u_s(a) J_0(2\sqrt{ax}) \tag{80b}$$

From (73), we have $D_{1-s} g_s = -\mathcal{H} D_s(\mathbf{1}_{x>a} x^{-s}) = -\mathcal{H}(a^{1-s}\delta_a(x)) = -a^{1-s} J_0(2\sqrt{ax})$. Let us define

$$J_0^a(x) = J_0(2\sqrt{ax}) \tag{81}$$

We have proven:

$$+D_s u_s - \mathcal{H} P_a D_{1-s} v_s = -a v_s(a) J_0^a \tag{82a}$$

$$-\mathcal{H} P_a D_s u_s + D_{1-s} v_s = -a(a^{-s} + u_s(a)) J_0^a \tag{82b}$$

Restricting to the interval $(0, a)$ and solving, we find:

$$P_a D_s u_s = -a(1 - D_a)^{-1}(v_s(a) J_0^a + (a^{-s} + u_s(a)) H_a J_0^a) \tag{83a}$$

$$P_a D_{1-s} v_s = -a(1 - D_a)^{-1}((a^{-s} + u_s(a)) J_0^a + v_s(a) H_a J_0^a) \tag{83b}$$

It is advantageous at this stage to define ϕ_a^+ and ϕ_a^- to be the solutions of the equations (in $L^2(0, a; dx)$):

$$\phi_a^+ + H_a \phi_a^+ = J_0^a \tag{84a}$$

$$\phi_a^- - H_a \phi_a^- = J_0^a \tag{84b}$$

We already know from (66a) and (66b) exactly what ϕ_a^+ and ϕ_a^- are (in this special case of the $\mathcal{H}$ transform), but we shall proceed as if we didn't. We see from (84a), (84b) that ϕ_a^+ and ϕ_a^- are entire functions, and we can rewrite the system as:[14]

$$\phi_a^+ + \mathcal{H} P_a \phi_a^+ = J_0^a \tag{85a}$$

$$\phi_a^- - \mathcal{H} P_a \phi_a^- = J_0^a \tag{85b}$$

We observe the identities:

$$(1 - D_a)^{-1} J_0^a = P_a \frac{\phi_a^+ + \phi_a^-}{2} \tag{86a}$$

$$(1 - D_a)^{-1} H_a J_0^a = P_a \frac{-\phi_a^+ + \phi_a^-}{2} \tag{86b}$$

So (83a) and (83b) become

$$D_s u_s = +a \frac{a^{-s} + u_s(a) - v_s(a)}{2} \phi_a^+ - a \frac{a^{-s} + u_s(a) + v_s(a)}{2} \phi_a^- \tag{87a}$$

$$D_{1-s} v_s = -a \frac{a^{-s} + u_s(a) - v_s(a)}{2} \phi_a^+ - a \frac{a^{-s} + u_s(a) + v_s(a)}{2} \phi_a^- \tag{87b}$$

[14] in conformity with our conventions, these are identities on $(0, \infty)$; to see them as identities on $\mathbb{C}$ one must read $\int_0^a J_0(2\sqrt{xy}) \phi_a^+(y)\, dy$ rather than $(\mathcal{H} P_a \phi_a^+)(x)$.

From (87a) we compute successively (again, these are identities on $(0, +\infty)$):

$$\mathcal{H}P_aD_su_s = a\frac{a^{-s} + u_s(a) - v_s(a)}{2}(J_0^a - \phi_a^+) - a\frac{a^{-s} + u_s(a) + v_s(a)}{2}(-J_0^a + \phi_a^-) \tag{88}$$

$$P_aD_su_s = a\frac{a^{-s} + u_s(a) - v_s(a)}{2}(\delta_a - \mathcal{H}\phi_a^+) - a\frac{a^{-s} + u_s(a) + v_s(a)}{2}(-\delta^a + \mathcal{H}\phi_a^-) \tag{89}$$

In (89), $\mathcal{H}\phi_a^+$ should perhaps be more precisely written as $\mathcal{H}(\phi_a^+\mathbf{1}_{x>0})$. From (85a) we know that $\phi_a^+\mathbf{1}_{x>0}$ is tempered as a distribution. From (78c) we compute $D_sX_s = \mathbf{1}_{x>a}D_s(u_s) + a(a^{-s} + u_s(a))\delta_a(x) = D_su_s - P_aD_su_s + a(a^{-s} + u_s(a))\delta_a(x)$. From (87a) and(89) then follows:

$$\begin{aligned} D_sX_s = +a\frac{a^{-s} + u_s(a) - v_s(a)}{2}(\phi_a^+ + \mathcal{H}\phi_a^+ - \delta_a) - a\frac{a^{-s} + u_s(a) + v_s(a)}{2}(\phi_a^- - \mathcal{H}\phi_a^- + \delta^a) \\ +a(a^{-s} + u_s(a))\delta_a(x) \end{aligned} \tag{90}$$

And the result of the computation is:

$$D_sX_s = +a\frac{a^{-s} + u_s(a) - v_s(a)}{2}(\phi_a^+ + \mathcal{H}\phi_a^+) + a\frac{a^{-s} + u_s(a) + v_s(a)}{2}(-\phi_a^- + \mathcal{H}\phi_a^-) \tag{91}$$

We then define the remarkable distributions:

$$A_a = \frac{\sqrt{a}}{2}(\phi_a^+ + \mathcal{H}\phi_a^+) \tag{92a}$$

$$-iB_a = \frac{\sqrt{a}}{2}(-\phi_a^- + \mathcal{H}\phi_a^-) \tag{92b}$$

$$E_a = A_a - iB_a \tag{92c}$$

From (84a) we observe that A_a has its support in $[a, \infty)$. Furthermore it is $\mathcal{H}$ invariant. Similarly, $-iB_a$, which is $\mathcal{H}$ anti invariant, also has its support in $[a, +\infty)$. We recover A_a and $-iB_a$ from E_a through taking the invariant and anti-invariant parts. We may also rewrite D_sX_s as:

$$D_sX_s = \sqrt{a}(a^{-s} + u_s(a))E_a - \sqrt{a}\,v_s(a)\mathcal{H}E_a \tag{93}$$

Some other manners of writing A_a and $-iB_a$ are useful: from (85a) $\mathcal{H}\phi_a^+ = \delta_a - P_a\phi_a^+$ and from (85b) $\mathcal{H}\phi_a^- = \delta_a + P_a\phi_a^-$, so:

$$A_a = \frac{\sqrt{a}}{2}(\delta_a + \phi_a^+\mathbf{1}_{x>a}) \tag{94a}$$

$$-iB_a = \frac{\sqrt{a}}{2}(\delta_a - \phi_a^-\mathbf{1}_{x>a}) \tag{94b}$$

And, we take also notice of the following definitions and identities:

$$j_a = \sqrt{a}(\delta_a - \phi_a^+\mathbf{1}_{0<x<a}) \qquad j_a = \sqrt{a}\mathcal{H}\phi_a^+ \qquad A_a = \frac{1}{2}(j_a + \mathcal{H}j_a) \tag{95a}$$

$$-ik_a = \sqrt{a}(\delta_a + \phi_a^-\mathbf{1}_{0<x<a}) \qquad -ik_a = \sqrt{a}\mathcal{H}\phi_a^- \qquad B_a = \frac{1}{2}(k_a - \mathcal{H}k_a) \tag{95b}$$

From (85a) and (85b) we know that ϕ_a^+ and ϕ_a^- are bounded, so the right Mellin transforms are defined directly for $\Re(s) > 1$ by:[15]

$$\widehat{A_a}(s) = \frac{\sqrt{a}}{2}\left(a^{-s} + \int_a^\infty \phi_a^+(x)x^{-s}\,dx\right) \tag{96a}$$

$$-i\widehat{B_a}(s) = \frac{\sqrt{a}}{2}\left(a^{-s} - \int_a^\infty \phi_a^-(x)x^{-s}\,dx\right) \tag{96b}$$

$$\widehat{E_a}(s) = \sqrt{a}\left(a^{-s} + \frac{1}{2}\int_a^\infty (\phi_a^+(x) - \phi_a^-(x))x^{-s}\,dx\right) \tag{96c}$$

$$\widehat{\mathcal{H}(E_a)}(s) = \sqrt{a}\,\frac{1}{2}\int_a^\infty (\phi_a^+(x) + \phi_a^-(x))x^{-s}\,dx \tag{96d}$$

From (85a) and (85b) we know that $\phi_a^+ - \phi_a^-$ is square-integrable at $+\infty$, so, using $\mathcal{H}\phi_a^+ = \delta_a - P_a\phi_a^+$ and $\mathcal{H}\phi_a^- = \delta_a + P_a\phi_a^-$ we compute:

$$\int_a^\infty (\phi_a^+(x) - \phi_a^-(x))x^{-s}\,dx = \int_0^\infty (\mathcal{H}\phi_a^+ - \mathcal{H}\phi_a^-)g_s(x)\,dx = -\int_0^a (\phi_a^+(x) + \phi_a^-(x))g_s(x)\,dx \tag{97}$$

Then using (86a):

$$\int_0^a \frac{\phi_a^+(x) + \phi_a^-(x)}{2}g_s(x)\,dx = \int_0^a (1 - D_a)^{-1}(J_0^a)(x)g_s(x)\,dx = \int_0^a J_0^a(x)((1 - D_a)^{-1}(g_s))(x)\,dx \tag{98}$$

Comparing with (76a) and (76b) the right-most term of (98) may be written as $\int_0^a J_0(2\sqrt{ax})v_s(x)\,dx$ which in turn we recognize from (77a) to be $-u_s(a)$. We have thus proven the identity:

$$\widehat{E_a}(s) = \sqrt{a}(a^{-s} + u_s(a)) \tag{99}$$

In a similar manner we have:

$$\int_a^\infty \frac{\phi_a^+(x) + \phi_a^-(x)}{2}x^{-s}\,dx = \int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx + \int_0^a \frac{-\phi_a^+(x) + \phi_a^-(x)}{2}g_s(x)\,dx \tag{100}$$

$$\begin{aligned}\int_0^a \frac{-\phi_a^+(x) + \phi_a^-(x)}{2}g_s(x)\,dx &= \int_0^a ((1 - D_a)^{-1}H_aJ_0^a)(x)g_s(x)\,dx = -\int_0^a J_0(2\sqrt{ax})u_s(x)\,dx \\ &= v_s(a) - g_s(a) = v_s(a) - \int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx\end{aligned} \tag{101}$$

$$\widehat{A_a}(s) + i\widehat{B_a}(s) = \widehat{\mathcal{H}(E_a)}(s) = \sqrt{a}\int_a^\infty \frac{\phi_a^+(x) + \phi_a^-(x)}{2}x^{-s}\,dx = \sqrt{a}v_s(a) \tag{102}$$

Then, we obtain the reformulation of (93) as:

$$D_sX_s = \widehat{E_a}(s)E_a - \widehat{\mathcal{H}(E_a)}(s)\mathcal{H}E_a \tag{103}$$

[15] the integral for $\widehat{E_a}(s)$ is certainly absolutely convergent for $\Re(s) > \frac{1}{2}$ as $\phi_a^+ - \phi_a^-$ is square integrable on $(0,\infty)$, and in fact it is absolutely convergent for $\Re(s) > \frac{1}{4}$. As we know already completely explicitely ϕ_a^+ and ϕ_a^-, we do not pause on this here. A general argument suitable to establish in more general cases absolute convergence for $\Re(s) > \sigma$ for some $\sigma < \frac{1}{2}$ will be given later.

And, noting $\widehat{D_s X_s}(z) = (s+z-1)\widehat{X_s}(z) = (s+z-1)\int_a^\infty X_s(x)X_z(x)\,dx$ we are finally led to the remarkable result:

$$X_a(s,z) = \int_a^\infty X_s^a(x)X_z^a(x)\,dx = \frac{\widehat{E_a}(s)\widehat{E_a}(z) - \widehat{\mathcal{H}(E_a)}(s)\widehat{\mathcal{H}E_a}(z)}{s+z-1} \tag{104}$$

This equation has been proven under the assumption $\Re(s) > 1$, and $\Re(z) > \frac{1}{2}$. To complete the discussion we need to know that the evaluators $f \mapsto \widehat{f}(s)$, $s \in \mathbb{C}$ are indeed continuous linear forms on K_a. For $\Re(s) > \frac{1}{2}$, we have $\widehat{f}(s) = \int_a^\infty f(x)x^{-s}\,dx$. For $\Re(s) < \frac{1}{2}$ we have $\widehat{f}(s) = \frac{\Gamma(1-s)}{\Gamma(s)}\widehat{\mathcal{H}(f)}(1-s)$. For $\Re(s) = \frac{1}{2}$ continuity follows by the Banach-Steinhaus theorem, and of course more elementary proofs exist (as in [3] for the cosine or sine transform). So we do have unique Hilbert space vectors $X_s^a \in K_a$ such that $\forall f \in K_a \forall s \in \mathbb{C}\ \widehat{f}(s) = \int_a^\infty X_s^a(x)f(x)\,dx$. Then (104) holds throughout $\mathbb{C}\times\mathbb{C}$ by analytic continuation.

The vectors X_s^a are zero for $s \in -\mathbb{N}$, and it is more precise to use vectors $\mathcal{X}_s^a = \Gamma(s)X_s^a$ which are non-zero for all $s \in \mathbb{C}$. These vectors are the evaluators[16] for $f \mapsto \mathcal{F}(s)$, $\mathcal{F}(s) = \Gamma(s)\widehat{f}(s)$. We recapitulate some of the results in the following theorem, whose analog for the cosine (or sine) transform was given in [5] (up to changes of variables and notations, the first paragraph as well as equation (108) are theorems from [1]; the equations (105), (106), (107) are our contributions. In this specific case of $\mathcal{H}$ we shall later identify exactly ϕ_a^+ and ϕ_a^- and E_a and $\mathcal{E}_a$. As we shall explain the analog of the $\mathcal{E}_a$-function in [1] has value 1 at $s = \frac{1}{2}$, and is not identical with the $\mathcal{E}_a$ here):

Theorem 7. *For a given $a > 0$ let K_a be the Hilbert space of square integrable functions $f(x)$ on $[a,+\infty)$ whose $\mathcal{H}$-transforms $\int_0^\infty J_0(2\sqrt{xy})f(y)\,dy$ (in the L^2-sense) again vanish for $0 < x < a$. The completed right Mellin transforms $\Gamma(s)\widehat{f}(s) = \Gamma(s)\int_a^\infty f(x)x^{-s}\,dx$ are entire functions and evaluations at $s \in \mathbb{C}$ are continuous linear forms.*
Let $\mathcal{X}_s^a$ for each $s \in \mathbb{C}$ be the unique vector in K_a such that $\forall f \in K_a\ \Gamma(s)\widehat{f}(s) = \int_a^\infty f(x)\mathcal{X}_s^a(x)\,dx$. Let ϕ_a^+ and ϕ_a^- be the entire functions which are the solutions to:

$$\phi_a^+(x) + \int_0^a J_0(2\sqrt{xy})\phi_a^+(y)\,dy = J_0(2\sqrt{ax}) \tag{105}$$

$$\phi_a^-(x) - \int_0^a J_0(2\sqrt{xy})\phi_a^-(y)\,dy = J_0(2\sqrt{ax}) \tag{106}$$

Then

$$\widehat{E_a}(s) = \sqrt{a}\Big(a^{-s} + \frac{1}{2}\int_a^\infty (\phi_a^+(x) - \phi_a^-(x))x^{-s}\,dx\Big) \tag{107}$$

is an entire function with trivial zeros at $-\mathbb{N}$ and, defining $\mathcal{E}_a(s) = \Gamma(s)\widehat{E_a}(s)$, we have:

$$\forall s,z \in \mathbb{C} \quad \int_a^\infty \mathcal{X}_s^a(x)\mathcal{X}_z^a(x)\,dx = \frac{\mathcal{E}_a(s)\mathcal{E}_a(z) - \mathcal{E}_a(1-s)\mathcal{E}_a(1-z)}{s+z-1} \tag{108}$$

We knew in advance that we had to end up with a formula such as (108) (with a $\mathcal{E}$ function to be discovered[17]), and this is why we started investigating $(x\frac{d}{dx} + s)X_s^a(x)$ in the first place! The

[16] evaluators for the "euclidean" product $\int fg\,dx$, not the "hilbertian" $\int f\overline{g}\,dx$.

[17] the method was initially developed by the author for the cosine and sine transforms [5, 6] and leads for them to the only known "explicit" formulas for $\mathcal{E}$; for the zero order Hankel transform the problem of computing the reproducing kernel had been already solved by de Branges [1].

reason is this: the Hilbert space of the entire functions $\mathcal{F}(s)$, $f \in K_a$ (the Hilbert structure is the one from K_a, or $(\mathcal{F}_1, \mathcal{F}_2) = \frac{1}{2\pi}\int_{\Re(s)=\frac{1}{2}} \mathcal{F}_1(s)\overline{\mathcal{F}_2(s)}\frac{|ds|}{|\Gamma(s)|^2}$) verifies the de Branges axioms [2], up to the change of variable $s = \frac{1}{2} - iz$. Let us recall the axioms of [2] for a (non-zero) Hilbert space of entire functions $F(z)$:

(H1) for each z, evalution at z is a continuous linear form,

(H2) for each F, $z \mapsto \overline{F(\overline{z})}$ belongs to the Hilbert space and has the same norm as F,

(H3) if $F(w) = 0$ then $G(z) = \frac{z-\overline{w}}{z-w}F(z)$ belongs to the space and has the same norm as F

Let $K(z, w)$ be defined as the evaluator at z: $\forall F\ F(z) = (F, K(z, \cdot))$. It is anti-analytic in z and analytic in w (the scalar product is complex linear in its first entry, and conjugate linear in its second entry). It is a reproducing kernel: $K(z, w) = (K(z, \cdot), K(w, \cdot))$. It is proven in [2] that (H1), (H2), (H3) entail the existence of an entire function $E(z)$ with $|E(z)| > |E(\overline{z})|$ for $\Im(z) > 0$, such that the space is exactly the set of entire functions $F(z)$ such that both $\frac{F(z)}{E(z)}$ and $\frac{\overline{F(\overline{z})}}{E(z)}$ belong to $\mathbb{H}^2(\Im(z) > 0)$, and the Hilbert space norm of F is $\frac{1}{2\pi}\int_{\mathbb{R}} |F(t)|^2 \frac{dt}{|E(t)|^2}$.[18] We have incorporated a 2π for easier comparison with our conventions. Then the reproducing kernel is expressed as:

$$K(z, w) = \frac{\overline{E(z)}E(w) - E(\overline{z})\overline{E(\overline{w})}}{i(\overline{z} - w)} \tag{109}$$

The function E is not unique; if the space has the isometric symmetry $F(z) \mapsto F(-z)$, a function E exists which is real on the imaginary axis and writing $E = A - iB$ where A and B are real on the real axis, the pair (A, B) is unique up to $A \mapsto kA$, $B \mapsto k^{-1}B$, A is even and B is odd. If $A(0) \neq 0$ (this happens exactly when the space contains at least one element not vanishing at 0) then it may be uniquely normalized so that $A(0) = 1$. Then E is uniquely determined.

Model examples are the Paley-Wiener spaces of entire functions $F(z)$ of exponential type at most τ with $\|F\|^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |F(t)|^2\, dt < \infty$. Then $E(z) = e^{-i\tau z}$ is a possible E function. The Paley-Wiener spaces are related to the study of the differential operator $-\frac{d^2}{dx^2}$ on the positive half-line, and an important class of spaces verifying the axioms of [2] is associated with the theory of the eigenfunction expansions for Schrödinger operators $-\frac{d^2}{dx^2} + V(x)$ ([27]). In these examples the spaces are indexed by a parameter τ (the Schrödinger operator is first studied on a finite interval $(0, \tau)$) and they are ordered by isometric inclusions (the E-function of a bigger space may be used in the computation of the norm of an element of a smaller space). Typically indeed, de Branges spaces are studied included in one fixed space $L^2(\mathbb{R}, \frac{1}{2\pi}d\nu)$, are ordered by isometric inclusion and indexed by a parameter[19]. Obviously this theory is intimately related with the Weyl-Stone-Titchmarsh-Kodaira theory of the spectral measure. The articles of Dym [14] and Remling [27], the book of Dym and McKean [15], will be useful to the interested reader. In the case of the study of $\mathcal{H}$ we will have

[18] the conditions on $F(z)$ are not formulated in [2] as Hardy space conditions, but they are exactly equivalent.

[19] the axioms allow for "jumps" in the isometric chain of inclusions, as occur in the theory of the Krein strings [15], discrete Schrödinger equations being special cases.

$d\nu(\gamma) = |\Gamma(\frac{1}{2} + i\gamma)|^{-2} d\gamma$. It is an important flexibility of the axioms not to be limited to functions of finite exponential type, and also the spectral measures are not necessarily such that $(1+\gamma^2)^{-1}$ is integrable. It has turned out in our study of the spaces associated with the $\mathcal{H}$-transform that the naturally occurring E function is not the one normalized to take value 1 at $z = 0$. Rather the normalization will prove to be $\lim_{\sigma\to+\infty} \frac{-iB(i\sigma)}{A(i\sigma)} = 1$. This has an important impact on the aspect of the differential equations which will govern the deformation of the K_a's with respect to a: they will take the form of a first order linear differential system in canonical form (as generally studied in [21, §3].)

The space of the functions $\mathcal{F}(s) = \Gamma(s)\widehat{f}(s)$, $f \in K_a$ verify (this is easy) the de Branges axioms, with $s = \frac{1}{2} - iz$ and they were defined[20] in [1]. The spaces K_a have the real structure, which is manifest in the s variable through the isometry $\mathcal{F}(s) \mapsto \overline{\mathcal{F}(\overline{s})}$. Rather than with the reproducing kernel $K(z_1, z_2)$ we work mainly with $\mathcal{X}(s_1, s_2) = K(-\overline{z_1}, z_2)$ which is analytic in both variables. Of course then it is $\mathcal{X}(\overline{s}, s)$ which gives the squared norm of the evaluator at s. Writing $\mathcal{E}(s) = E(z)$ we obtain from (109):

$$\mathcal{X}(s_1, s_2) = \frac{\mathcal{E}(s_1)\mathcal{E}(s_2) - \mathcal{E}(1-s_1)\mathcal{E}(1-s_2)}{s_1 + s_2 - 1} \tag{110}$$

which is indeed what has appeared on the right hand side of (108). With $\mathcal{E}(s) = \mathcal{A}(s) - i\mathcal{B}(s)$, $\mathcal{A}$ (resp. $\mathcal{B}$) even (resp. odd) under $s \mapsto 1 - s$, this is also:

$$\mathcal{X}(s_1, s_2) = 2\frac{-i\mathcal{B}(s_1)\mathcal{A}(s_2) + \mathcal{A}(s_1)(-i\mathcal{B}(s_2))}{s_1 + s_2 - 1} \tag{111}$$

and for $\Re(s) \neq \frac{1}{2}$, $0 < \mathcal{X}(\overline{s}, s) = 2\frac{\Im(\mathcal{B}(s)\overline{\mathcal{A}(s)})}{\Re(s) - \frac{1}{2}}$ so both $\mathcal{A}$ and $\mathcal{B}$ have all their zeros on the critical line.[21]

The method in this chapter has been developed in [5, 8, 6] for the case of the cosine and sine transforms, and it leads to the currently only known "explicit" formulae[22] for the structural elements $\mathcal{E}$, $\mathcal{A}$, $\mathcal{B}$ and reproducing kernels for the spaces for the cosine and sine transforms. So far, almost nothing very specific to $\mathcal{H}$ has been used apart from it being self-adjoint self-reciprocal with an entire multiplicative kernel $k(xy)$. The next section is still of a very general validity.

As was mentioned in the Introduction the realization of the structural elements of the spaces as right Mellin transforms of distributions is a characteristic aspect of the method; the Dirac delta's in the expressions for $A_a(x)$ and $-iB_a(x)$ could have been overlooked if we had only been prepared to use functions, and the whole development was based on the computation of $(x\frac{d}{dx} + s)X_s(x)$ as a distribution. This aspect will be further reinforced in the concluding chapter of the paper (section 9) where it will be seen that the distributions $A_a(x)$ and $-iB_a(x)$ are very naturally differences of boundary values of analytic functions, so they are hyperfunctions [23] in a natural manner.

[20] in the variable z, and associated with the Hankel transform of order zero, rather than with the $\mathcal{H}$ transform.

[21] this is also seen from $2\mathcal{A}(s) = \mathcal{E}(s) + \mathcal{E}(1-s)$ as $|\mathcal{E}(s)| > |\mathcal{E}(1-s)|$ for $\Re(s) > \frac{1}{2}$. As $\mathcal{X}(\overline{s}, s) = \frac{|\mathcal{E}(s)|^2 - |\mathcal{E}(1-s)|^2}{2\Re(s) - 1}$ this is in fact the same argument.

[22] as "explicit" as the Fredholm determinants of the finite Dirichlet kernels are "explicit".

Let us consider the behavior of $\widehat{A_a}(s)$, $\widehat{B_a}(s)$, $\widehat{E_a}(s)$ and $\widehat{\mathcal{H}(E_a)}(s)$ for $\Re(s) \geq \frac{1}{2}$. Let us first look at $\widehat{E_a}(s) = \sqrt{a}\Big(a^{-s} + \frac{1}{2}\int_a^\infty (\phi_a^+(x) - \phi_a^-(x))x^{-s}\,dx\Big)$. We remark that $\phi_a^+(x) - \phi_a^-(x)$ is the $\mathcal{H}$-transform of $-(\phi_a^+(x) + \phi_a^-(x))\mathbf{1}_{0<x<a}(x)$.

Lemma 8. *Let $k(x)$ a continuous function on $[0, +\infty)$ and $A \in [0,1]$ be such that $k_1(x) = \int_0^x k(t)\,dt = O(x^A)$ as $x \to \infty$. Let $a > 0$ and let $f(x)$ be an absolutely continuous function on $[0, a]$. Then $\int_0^a k(xy)f(y)\,dy = O(x^{A-1})$ as $x \to +\infty$.*

There exists $C < \infty$ such that $\forall x > 0$ $|k_1(x)| \leq C\,x^A$. Then $\int_0^a k(xy)f(y)\,dy = \frac{1}{x}k_1(xa)f(a) - \frac{1}{x}\int_0^a k_1(xy)f'(y)\,dy$, and $|\int_0^a k_1(xy)f'(y)\,dy| \leq Cx^A\int_0^a y^A|f'(y)|\,dy$. This was easy...

With $k(x) = J_0(2\sqrt{x})$, one has $k_1(x) = \sqrt{x}J_1(2\sqrt{x}) = O(x^{\frac{1}{4}})$. We have $\phi_a^+(x) - \phi_a^-(x) = -\int_0^a J_0(2\sqrt{xy})(\phi_a^+(x) + \phi_a^-(x))\,dy$ and from Lemma 8 this is $O(x^{-\frac{3}{4}})$. So the integral in the expression for $\widehat{E_a}(s)$ is absolutely convergent for $\Re(s) > \frac{1}{4}$. In particular $\widehat{E_a}$ is bounded on the critical line. But then $\widehat{\mathcal{H}(E_a)}(s) = \chi(s)\widehat{E_a}(1-s)$ is also bounded. Hence:

Proposition 9. *The functions $\widehat{A_a}$ and $\widehat{B_a}$ are bounded on the critical line.*

Let us turn to the situation regarding $\Re(s) = \sigma \to +\infty$.

Let $f(x)$ be a function of class C^2 on $[0, a]$ and $e(x) = \int_0^a J_0(2\sqrt{xy})f(y)\,dy$. It is $O(x^{-\frac{3}{4}})$. There holds $\frac{d}{dx}xe(x) = \int_0^a(\frac{d}{dy}yJ_0(2\sqrt{xy}))f(y)\,dy = af(a)J_0(2\sqrt{ax}) - \int_0^a J_0(2\sqrt{xy})\,yf'(y)\,dy$. Let $k(x) = \int_0^a J_0(2\sqrt{xy})\,yf'(y)\,dy$. By the Lemma 8 it is $O(x^{-\frac{3}{4}})$. For $\Re(s) > \frac{3}{4}$, with absolutely convergent integrals:

$$af(a)\int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx - \int_a^\infty k(x)x^{-s}\,dx = -ae(a)a^{-s} + s\int_a^\infty e(x)x^{-s}\,dx \tag{112}$$

We show that the left hand side of (112) is $O(a^{-s}\frac{1}{s})$ for $\Re(s) > \frac{5}{4}$. We apply to k what we did for e, $\frac{d}{dx}xk(x) = a^2f'(a)J_0(2\sqrt{ax}) - \int_0^a J_0(2\sqrt{xy})\,y(yf')'(y)\,dy$. This is $O(1)$ (using $|J_0| \leq 1$). So for $\Re(s) > 1$, we can compute $\int_0^\infty(\frac{d}{dx}xk(x))x^{-s}\,dx$ by integration by parts, this gives $-ak(a)a^{-s} + s\int_a^\infty k(x)x^{-s}\,dx$. So for $\Re(s) \geq 1 + \epsilon$ we have $\int_a^\infty k(x)x^{-s}\,dx = O(a^{-s}\frac{1}{s})$. Then regarding $\int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx$ we note that $\frac{d}{dx}xJ_0(2\sqrt{ax}) = J_0(2\sqrt{ax}) - \sqrt{ax}J_1(2\sqrt{ax})$, so for $\Re(s) \geq \frac{5}{4} + \epsilon$ we can apply the same method of integration by parts, and prove that $\int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx = O(a^{-s}\frac{1}{s})$. So the left hand side of (112) is indeed $O(a^{-s}\frac{1}{s})$ for $\Re(s) \geq \frac{3}{2}$ and we have:

Lemma 10. *Let $f(x)$ be a function of class C^2 on $[0, a]$ and let $e(x) = \int_0^a J_0(2\sqrt{xy})f(y)\,dy$. One has*

$$\int_a^\infty e(x)x^{-s}\,dx = \frac{a^{-s}}{s}\big(ae(a) + O(\frac{1}{s})\big) \qquad (\Re(s) \geq \frac{3}{2}) \tag{113}$$

Let us return to $\int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx = \frac{1}{s}\big(J_0(2a)a^{1-s} + \int_a^\infty(J_0(2\sqrt{ax}) - \sqrt{ax}J_1(2\sqrt{ax}))x^{-s}\,dx\big)$. We want to iterate so we also need $x\frac{d}{dx}\sqrt{ax}J_1(2\sqrt{ax}) = \frac{1}{4}2\sqrt{ax}\frac{d}{d2\sqrt{ax}}2\sqrt{ax}J_1(2\sqrt{ax}) = axJ_0(2\sqrt{ax})$. So we can integrate by parts and obtain that the last Mellin integral is $O(a^{-s}\frac{1}{s})$ for $\Re(s) \geq \frac{7}{4} + \epsilon$.

So, certainly:

$$\int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx = \frac{a^{-s}}{s}\big(aJ_0(2a)+O(\frac{1}{s})\big) \qquad (\Re(s)\geq \frac{5}{2}) \tag{114}$$

Using $\phi_a^+ = J_0^a - \mathcal{H}P_a\phi_a^+$ and $\phi_a^- = J_0^a + \mathcal{H}P_a\phi_a^-$ and combining (113) and (114) we obtain:

Proposition 11. *One has for $\Re(s)\geq\sigma_0$ (here $\sigma_0=\frac{5}{2}$ for example):*

$$\widehat{E_a}(s) = a^{\frac{1}{2}-s}(1+\frac{a\phi^+(a)-a\phi^-(a)}{2s}+O(\frac{1}{s^2})) \qquad \widehat{A_a}(s) = \frac{\sqrt{a}}{2}a^{-s}(1+\frac{a\phi_a^+(a)}{s}+O(\frac{1}{s^2})) \tag{115a}$$

$$\widehat{\mathcal{H}(E_a)}(s) = a^{\frac{1}{2}-s}(\frac{a\phi^+(a)+a\phi^-(a)}{2s}+O(\frac{1}{s^2})) \qquad -i\widehat{B_a}(s) = \frac{\sqrt{a}}{2}a^{-s}(1-\frac{a\phi_a^-(a)}{s}+O(\frac{1}{s^2})) \tag{115b}$$

Theorem 12. *One has*

$$\lim_{\sigma\to+\infty}\frac{-i\mathcal{B}_a(\sigma)}{\mathcal{A}_a(\sigma)} = 1 \qquad \textit{and} \qquad \frac{\mathcal{E}_a(1-\sigma)}{\mathcal{E}_a(\sigma)} \sim_{\sigma\to+\infty} \frac{a\phi_a^+(a)+a\phi_a^-(a)}{2\sigma} \tag{116}$$

So the functions $\mathcal{A}_a$ and $\mathcal{B}_a$ are not normalized as is usually done in [2] which is to impose (when possible) to the E function to have value 1 at the origin (which for us is $s=\frac{1}{2}$; the exact value of $\mathcal{A}_a(\frac{1}{2})$ will be obtained later.) This difference in normalization is related to the realization of the differential equations governing the deformation of the spaces K_a as a first order differential system in "canonical" form, as in the classical spectral theory of linear differential equations ([21, 10].) This allows to realize the self-reciprocal scale reversing operator as a scattering [6].

6 Fredholm determinants, the first order differential system, and scattering

Let us return to the defining equations for the entire functions ϕ_a^+ and ϕ_a^-:

$$\phi_a^+ + \mathcal{H}P_a\phi_a^+ = J_0^a \tag{117a}$$

$$\phi_a^- - \mathcal{H}P_a\phi_a^- = J_0^a \tag{117b}$$

Either we read these equations as identities on $(0,\infty)$, or we decide that $\mathcal{H}P_a\phi_a^\pm$ in fact stands for $\int_0^a J_0(2\sqrt{xy})\phi_a^\pm(y)\,dy$, and the equation holds for $x\in\mathbb{C}$; the latter option slightly conflicts with our earlier definition of $\mathcal{H}$ as an operator on functions or distributions. But whatever choice is made this has no impact on what comes next. We shall apply to the equations the operators $a\frac{\partial}{\partial a}$ and $x\frac{\partial}{\partial x}$. As $J_0^a(x)=J_0(2\sqrt{ax})$ we have $a\frac{\partial}{\partial a}J_0^a = x\frac{\partial}{\partial x}J_0^a$. We write $\delta_x = x\frac{\partial}{\partial x}+\frac{1}{2} = \frac{\partial}{\partial x}x - \frac{1}{2}$. First we have:

$$a\frac{\partial}{\partial a}\phi_a^+ + \mathcal{H}P_a a\frac{\partial}{\partial a}\phi_a^+ = -a\phi_a^+(a)J_0^a + a\frac{\partial}{\partial a}J_0^a \tag{118a}$$

$$a\frac{\partial}{\partial a}\phi_a^- - \mathcal{H}P_a a\frac{\partial}{\partial a}\phi_a^- = +a\phi_a^-(a)J_0^a + a\frac{\partial}{\partial a}J_0^a \tag{118b}$$

Then, as $x\frac{\partial}{\partial x}\mathcal{H} = -\mathcal{H}\frac{\partial}{\partial x}x$, $\delta_x\mathcal{H} = -\mathcal{H}\delta_x$, $\delta_x P_a f = (P_a\delta_x f) - af(a)\delta_a(x)$, $\delta_x J_0^a = a\frac{\partial}{\partial a}J_0^a + \frac{1}{2}J_0^a$:

$$\delta_x\phi_a^+ - \mathcal{H}P_a\delta_x\phi_a^+ = (\frac{1}{2} - a\phi_a^+(a))J_0^a + a\frac{\partial}{\partial a}J_0^a \tag{119a}$$

$$\delta_x\phi_a^- + \mathcal{H}P_a\delta_x\phi_a^- = (\frac{1}{2} + a\phi_a^-(a))J_0^a + a\frac{\partial}{\partial a}J_0^a \tag{119b}$$

Combining we obtain:

$$a\frac{\partial}{\partial a}\phi_a^+ - \delta_x\phi_a^- + \mathcal{H}P_a(a\frac{\partial}{\partial a}\phi_a^+ - \delta_x\phi_a^-) = -(a\phi_a^+(a) + a\phi_a^-(a) + \frac{1}{2})J_0^a \tag{120a}$$

$$a\frac{\partial}{\partial a}\phi_a^- - \delta_x\phi_a^+ - \mathcal{H}P_a(a\frac{\partial}{\partial a}\phi_a^- - \delta_x\phi_a^+) = +(a\phi_a^+(a) + a\phi_a^-(a) - \frac{1}{2})J_0^a \tag{120b}$$

Comparing with (117a) and (117b), and as there is uniqueness:

$$a\frac{\partial}{\partial a}\phi_a^+ - \delta_x\phi_a^- = -(a\phi_a^+(a) + a\phi_a^-(a) + \frac{1}{2})\phi_a^+ \tag{121a}$$

$$a\frac{\partial}{\partial a}\phi_a^- - \delta_x\phi_a^+ = +(a\phi_a^+(a) + a\phi_a^-(a) - \frac{1}{2})\phi_a^- \tag{121b}$$

The quantity $a\phi_a^+(a) + a\phi_a^-(a)$ will play a fundamental rôle and we shall denote it by $\mu(a)$.[23] So:

$$(a\frac{\partial}{\partial a} + \frac{1}{2} + \mu(a))\phi_a^+ = \delta_x\phi_a^- \tag{122a}$$

$$(a\frac{\partial}{\partial a} + \frac{1}{2} - \mu(a))\phi_a^- = \delta_x\phi_a^+ \tag{122b}$$

It follows easily from this that $a\frac{\partial}{\partial a}(\phi_a^+\phi_a^-) = -\phi_a^+\phi_a^- + \frac{1}{2}\frac{\partial}{\partial x}x((\phi_a^+)^2 + (\phi_a^-)^2)$. So

$$a\frac{d}{da}\int_0^a \phi_a^+(x)\phi_a^-(x)\,dx = a\phi_a^+(a)\phi_a^-(a) - \int_0^a \phi_a^+(x)\phi_a^-(x)\,dx + \frac{1}{2}a(\phi_a^+(a)^2 + \phi_a^-(a)^2) \tag{123}$$

$$a\frac{d}{da}\,a\int_0^a \phi_a^+(x)\phi_a^-(x)\,dx = \frac{1}{2}\mu(a)^2 \tag{124}$$

We then compute:

$$\int_0^a \phi_a^+(x)\phi_a^-(x)\,dx = \int_0^a ((1 - D_a)^{-1}J_0^a)(x)J_0^a(x)\,dx\ , \tag{125}$$

where we recall $\phi_a^+ = (1 + H_a)^{-1}J_0^a$, $\phi_a^- = (1 - H_a)^{-1}J_0^a$, $D_a = H_a^2$. The operator D_a acts on $L^2(0, a; dx)$ with kernel $D_a(x, z) = \int_0^a J_0(2\sqrt{xy})J_0(2\sqrt{yz})\,dy$. After the change of variables $x = at$, $y = au$, $z = av$ this becomes the operator d_a on $L^1(0, 1; dt)$ with kernel $d_a(t, v) = \int_0^1 aJ_0(2a\sqrt{tu})\,aJ_0(2a\sqrt{uv})\,du$. We compute the derivative with respect to a:

$$\frac{\partial}{\partial a}\int_0^1 aJ_0(2a\sqrt{tu})\,aJ_0(2a\sqrt{uv})\,du \tag{126a}$$

$$= \int_0^1 ((2u\frac{\partial}{\partial u} + 1)J_0(2a\sqrt{tu}))\,aJ_0(2a\sqrt{uv})\,du + \int_0^1 aJ_0(2a\sqrt{tu}))((2\frac{\partial}{\partial u}u - 1)J_0(2a\sqrt{uv}))\,du \tag{126b}$$

$$= 2aJ_0(2a\sqrt{t})J_0(2a\sqrt{v}) \tag{126c}$$

[23] maybe it would be unfair to hide the fact that $\mu(a) = 2a$, in this study of $\mathcal{H}$! In a later section a further mu function, associated with a variant of $\mathcal{H}$, will also be found explicitely and it will be quite more complicated.

So $\frac{d}{da}d_a$ is a rank one operator, with range $\mathbb{C}J_0(2a\sqrt{t})\mathbf{1}_{0<t<1}(t)$. We now use the well-known formula

$$\frac{d}{da}\log\det(1-d_a) = -\mathrm{Tr}((1-d_a)^{-1}\frac{d}{da}d_a) \tag{127}$$

The rank one operator $(1-d_a)^{-1}\frac{d}{da}d_a$ has the function $(1-d_a)^{-1}2aJ_0(2a\sqrt{t})$ as eigenvector and the eigenvalue is $\int_0^1 J_0(2a\sqrt{t})((1-d_a)^{-1}2aJ_0(2a\sqrt{v}))(t)\,dt$. Going back to $(0,a)$ we obtain $2\int_0^a J_0(2\sqrt{ax})((1-D_a)^{-1}J_0(2\sqrt{az}))(x)\,dx$ and in view of (125) we have proven:

$$\frac{d}{da}\log\det(1-D_a) = -2\int_0^a \phi_a^+(x)\phi_a^-(x)\,dx \tag{128}$$

Then, using (124), we have the important formula:

$$\mu(a)^2 = -a\frac{d}{da}\,a\frac{d}{da}\,\log\det(1-D_a) \tag{129}$$

We shall now also relate $\phi_a^+(a)$ and $\phi_a^-(a)$ to Fredholm determinants. In fact the following holds:

$$a\phi_a^+(a) = +a\frac{d}{da}\log\det(1+H_a) \tag{130a}$$

$$a\phi_a^-(a) = -a\frac{d}{da}\log\det(1-H_a) \tag{130b}$$

This is the application of a well-known general theorem, for any continuous kernel $k(x,y)$: if $w(x)+\int_0^a k(x,y)w(y)\,dy = k(x,a)$ for $0\le x\le a$ then $w(a) = +\frac{d}{da}\log\det_{(0,a)}(\delta(x-y)+k(x,y))$. A proof may be given which is of a somewhat similar kind as the one given above for (128), or one may more directly use the Fredholm's formulas for the determinant and the resolvent.[24] The theorem is proven in the book of P. Lax [20], Theorem 12 of Chapter 24 (Lax treats the case of a kernel on $(a,+\infty)$, here we have the simpler case of a finite interval $(0,a)$.) This means that $\mu(a)$ has another expression in terms of Fredholm determinants:

$$\mu(a) = a\frac{d}{da}\log\frac{\det(1+H_a)}{\det(1-H_a)} \tag{131}$$

Combining (129) and (131) we obtain:

$$-2a\frac{d}{da}\,a\frac{d}{da}\log\det(1+H_a) = \left(a\frac{d}{da}\log\frac{\det(1+H_a)}{\det(1-H_a)}\right)^2 - a\frac{d}{da}a\frac{d}{da}\log\frac{\det(1+H_a)}{\det(1-H_a)} \tag{132a}$$

$$-2a\frac{d}{da}\,a\frac{d}{da}\log\det(1-H_a) = \left(a\frac{d}{da}\log\frac{\det(1+H_a)}{\det(1-H_a)}\right)^2 + a\frac{d}{da}a\frac{d}{da}\log\frac{\det(1+H_a)}{\det(1-H_a)} \tag{132b}$$

$$2a\frac{d}{da}a\phi_a^+(a) = -\mu(a)^2 + a\mu'(a) \tag{132c}$$

$$2a\frac{d}{da}a\phi_a^-(a) = +\mu(a)^2 + a\mu'(a) \tag{132d}$$

$$\frac{d}{da}a(\phi_a^-(a)-\phi_a^+(a)) = a(\phi_a^+(a)+\phi_a^-(a))^2 \tag{132e}$$

These Fredholm determinants identities are reminiscent of certain well-known Gaudin identities [22, App. A16], which apply to the even and odd parts of an additive (Toeplitz) convolution kernel on

[24] let us recall that for a continuous kernel on a finite interval, the formula of Fredholm for a determinant as a convergent series always applies, even if the operator given by the kernel is not trace class, which may happen.

an interval $(-a,a)$; here the situation is with kernels $k(xy)$ which have a multiplicative look, and reduction to the additive case would give $g(t+u)$ type kernels on semi-infinite intervals.

We have defined $A_a = \frac{\sqrt{a}}{2}(\phi_a^+ + \mathcal{H}\phi_a^+)$ and $-iB_a = \frac{\sqrt{a}}{2}(-\phi_a^- + \mathcal{H}\phi_a^-)$. Let us recall that here $\phi_a^\pm$ is restricted to $[0,+\infty)$ and is then tempered as a distribution. Using the differential equations (122a) and (122b) and the commutation property $\delta_x \mathcal{H} = -\mathcal{H}\delta_x$, $\delta_x = x\frac{\partial}{\partial x} + \frac{1}{2}$, we have $\delta_x A_a = \frac{\sqrt{a}}{2}(\delta_x\phi_a^+ - \mathcal{H}\delta_x\phi_a^+) = \frac{\sqrt{a}}{2}(a\frac{\partial}{\partial a} + \frac{1}{2} - \mu(a))(\phi_a^- - \mathcal{H}\phi_a^-) = -(a\frac{\partial}{\partial a} - \mu(a))(-iB_a)$ and $\delta_x(-iB_a) = \frac{\sqrt{a}}{2}(-(a\frac{\partial}{\partial a} + \frac{1}{2} + \mu(a))\phi_a^+ - (a\frac{\partial}{\partial a} + \frac{1}{2} + \mu(a))\mathcal{H}\phi_a^+) = -(a\frac{\partial}{\partial a} + \mu(a))A_a$. The following first order system of differential equations therefore holds:

$$a\frac{\partial}{\partial a}A_a = -\mu(a)A_a - \delta_x(-iB_a) \tag{133a}$$

$$a\frac{\partial}{\partial a}(-iB_a) = +\mu(a)(-iB_a) - \delta_x A_a \tag{133b}$$

Then we also have the second order differential equations $(a\frac{\partial}{\partial a} - \mu(a))(a\frac{\partial}{\partial a} + \mu(a))A_a = +\delta_x^2 A_a$ and $(a\frac{\partial}{\partial a} + \mu(a))(a\frac{\partial}{\partial a} - \mu(a))(-iB_a) = +\delta_x^2(-iB_a)$, or, taking the right Mellin transforms, and writing $s = \frac{1}{2} + i\gamma$, $\delta_x = i\gamma$:

$$-a\frac{\partial}{\partial a}a\frac{\partial}{\partial a}\widehat{A_a} + (\mu(a)^2 - a\mu'(a))\widehat{A_a} = \gamma^2\widehat{A_a} \tag{134a}$$

$$-a\frac{\partial}{\partial a}a\frac{\partial}{\partial a}(-i\widehat{B_a}) + (\mu(a)^2 + a\mu'(a))(-i\widehat{B_a}) = \gamma^2(-i\widehat{B_a}) \tag{134b}$$

With the new variable $u = \log(a)$ we obtain Dirac and Schrödinger equations which are associated with this study of $\mathcal{H}$, modeled on the study of the cosine and sine transforms summarized in [5, 6]. All quantities in the statement of the theorem will be completely explicited later in terms of Bessel functions, but we keep the notation sufficiently general to allow, if an interesting other case arises, to write down the identical results:

Theorem 13. *For each $a > 0$ let ϕ_a^+ and ϕ_a^- be the entire functions which are the solutions to:*

$$\phi_a^+(x) + \int_0^a J_0(2\sqrt{xy})\phi_a^+(y)\,dy = J_0(2\sqrt{ax}) \tag{135a}$$

$$\phi_a^-(x) - \int_0^a J_0(2\sqrt{xy})\phi_a^-(y)\,dy = J_0(2\sqrt{ax}) \tag{135b}$$

Let H_a be the integral operator on $L^2(0,a;dx)$ with kernel $J_0(2\sqrt{xy})$. There holds:

$$\phi_a^+(a) = +\frac{d}{da}\log\det(1+H_a) \tag{135c}$$

$$\phi_a^-(a) = -\frac{d}{da}\log\det(1-H_a)\,. \tag{135d}$$

The tempered distributions $A_a = \frac{\sqrt{a}}{2}(1+\mathcal{H})(\phi_a^+\mathbf{1}_{0<x<\infty})$ and $B_a = i\frac{\sqrt{a}}{2}(-1+\mathcal{H})(\phi_a^-\mathbf{1}_{0<x<\infty})$ vanish on $(-\infty,a)$ and are respectively self-reciprocal and skew-reciprocal under $\mathcal{H}$. Their completed right Mellin transforms $\mathcal{A}_a(s) = \Gamma(s)\widehat{A_a}(s)$ and $\mathcal{B}_a(s) = \Gamma(s)\widehat{B_a}(s)$ are entire functions with all their zeros on the critical line, they are respectively even and odd for $s \leftrightarrow 1-s$, and they verify the following

Dirac and Schrödinger types of differential equations in the variable $u = \log(a)$, $-\infty < u < +\infty$,

$$\frac{d}{du}\mathcal{A}_a = -\mu(a)\mathcal{A}_a - \gamma\mathcal{B}_a \tag{135e}$$

$$\frac{d}{du}\mathcal{B}_a = +\mu(a)\mathcal{B}_a + \gamma\mathcal{A}_a \tag{135f}$$

$$\gamma^2\mathcal{A}_a = \left(-\frac{d^2}{du^2} + V_+(u)\right)\mathcal{A}_a \tag{135g}$$

$$\gamma^2\mathcal{B}_a = \left(-\frac{d^2}{du^2} + V_-(u)\right)\mathcal{B}_a \tag{135h}$$

$$V_+(\log a) = \mu(a)^2 - \frac{d\,\mu(a)}{du} = -2\frac{d^2 \log\det(1+H_a)}{du^2} \tag{135i}$$

$$V_-(\log a) = \mu(a)^2 + \frac{d\,\mu(a)}{du} = -2\frac{d^2 \log\det(1-H_a)}{du^2} \tag{135j}$$

$$\mu(a) = \frac{d}{du}\log\frac{\det(1+H_a)}{\det(1-H_a)} = a\phi_a^+(a) + a\phi_a^-(a) \tag{135k}$$

where $s = \frac{1}{2} + i\gamma$.

Let us consider the Hilbert space of pairs $\begin{bmatrix}\alpha(u)\\ \beta(u)\end{bmatrix}$ on $\mathbb{R}$ with squared norms $\int_{-\infty}^{\infty} |\alpha(u)|^2 + |\beta(u)|^2 \frac{du}{2}$, and the two equivalent differential systems in canonical forms:

$$\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{d}{du} - \begin{bmatrix}0 & \mu(e^u)\\ \mu(e^u) & 0\end{bmatrix}\right)\begin{bmatrix}\alpha(u)\\ \beta(u)\end{bmatrix} = \gamma\begin{bmatrix}\alpha(u)\\ \beta(u)\end{bmatrix} \tag{136}$$

$$\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\frac{d}{du} + \begin{bmatrix}-\mu(e^u) & 0\\ 0 & \mu(e^u)\end{bmatrix}\right)\begin{bmatrix}\alpha(u)+\beta(u)\\ -\alpha(u)+\beta(u)\end{bmatrix} = \gamma\begin{bmatrix}\alpha(u)+\beta(u)\\ -\alpha(u)+\beta(u)\end{bmatrix} \tag{137}$$

The components obey the corresponding Schrödinger equations:

$$-\alpha''(u) + V_+(u)\alpha(u) = \gamma^2\alpha \qquad V_+(u) = \mu(e^u)^2 - \frac{d\,\mu(e^u)}{du} \tag{138a}$$

$$-\beta''(u) + V_-(u)\beta(u) = \gamma^2\beta \qquad V_-(u) = \mu(e^u)^2 + \frac{d\,\mu(e^u)}{du} \tag{138b}$$

Regarding the behavior at $-\infty$, we are in the limit-point case for each of the Schrödinger equations (138a) and (138b) because clearly (say, from the defining integral equations for ϕ_a^+ and ϕ_a^-) one has $\phi_a^+(a) \to_{a\to 0} J_0(0) = 1$, $\phi_a^-(a) \to_{a\to 0} 1$, $\mu(a) \sim_{a\to 0} 2a$, so the potentials are exponentially vanishing as $u \to -\infty$. Perhaps we should reveal that one has exactly $\mu(a) = 2a = 2e^u$ so we are dealing here with quite concrete Schrödinger equations and Dirac systems whose exact solutions will later be written explicitely in terms of modified Bessel functions, but we delay using any information which would be too specific of the $\mathcal{H}$-transform.

For each $\gamma \in \mathbb{C}$

$$u \mapsto \begin{bmatrix}\mathcal{A}_{\exp(u)}(\frac{1}{2}+i\gamma)\\ \mathcal{B}_{\exp(u)}(\frac{1}{2}+i\gamma)\end{bmatrix} \tag{139}$$

is a (non-zero) solution of the system (136), and we now show that it is square-integrable (with respect to $du = d\log(a)$) at $+\infty$. Let us recall the equation (111) $(s+z-1)\mathcal{X}_a(s,z) = -2i\mathcal{B}_a(s)\mathcal{A}_a(z)-$

$2i\mathcal{A}_a(s)\mathcal{B}_a(z)$, from which we deduce

$$a\frac{\partial}{\partial a}\mathcal{X}_a(s,z) = -2\mathcal{A}_a(s)\mathcal{A}_a(z) - 2(i\mathcal{B}_a(s))(i\mathcal{B}_a(z)) \tag{140}$$

We have[25] $\|\mathcal{X}_s^a\|^2 = \mathcal{X}_a(s,\overline{s})$, $\mathcal{A}_a(\overline{s}) = \overline{\mathcal{A}_a(s)}$, $i\mathcal{B}_a(\overline{s}) = \overline{i\mathcal{B}_a(s)}$, so

$$a\frac{\partial}{\partial a}\|\mathcal{X}_s^a\|^2 = -2|\mathcal{A}_a(s)|^2 - 2|\mathcal{B}_a(s)|^2 \tag{141}$$

and as of course $\lim_{a\to+\infty}\|\mathcal{X}_s^a\|^2 = 0$ (we have $\|\mathcal{X}_s^a\|^2 \leq \int_a^\infty |\mathcal{X}_s^1|^2(x)\,dx$ for $a \geq 1$) we obtain:

$$\forall s \in \mathbb{C} \quad \|\mathcal{X}_s^a\|^2 = 2\int_a^\infty (|\mathcal{A}_a(s)|^2 + |\mathcal{B}_a(s)|^2)\frac{da}{a} \tag{142}$$

This establishes the square-integrability at $+\infty$ of $\left[\begin{smallmatrix}\mathcal{A}_{\exp(u)}(s)\\ \mathcal{B}_{\exp(u)}(s)\end{smallmatrix}\right]$, for any $s \in \mathbb{C}$.

The solutions of (136) with eigenvalue $\gamma = 0$ are $\left[\begin{smallmatrix}\mathcal{A}_a(\frac{1}{2})\\ 0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\ \mathcal{A}_a(\frac{1}{2})^{-1}\end{smallmatrix}\right]$. The former is square-integrable, so from $2 \leq t + t^{-1}$ the latter then necessarily is not. This confirms that the Dirac system (136) is in the limit point case at $+\infty$ (according to a general theorem of Levitan [21, §13, Thm 7.1] any first order differential operator $\left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right]\frac{d}{du} + \left[\begin{smallmatrix}a(u) & b(u)\\ c(u) & d(u)\end{smallmatrix}\right]$ with continuous coefficients is in the limit point case at infinity). So the pair (139) is in fact, for any $\gamma \in \mathbb{C}$, the unique solution of (136) which is square-integrable at $+\infty$. Also the Schrödinger equation (138b) is in the limit point case as not all of its solutions are square integrable at $+\infty$. Whether the limit-point case at $+\infty$ holds for equation (138a) is less evident. Let us recall from [10, §9, Thm 2.4] and [26, §X, Thm X.8] that a sufficient condition for this is the existence of a lowerbound $\liminf_{u\to+\infty} V_+(u)/u^2 > -\infty$. We will prove in the next chapter that $\mu(a) = 2a = 2e^u$ so this is certainly the case here. In the present chapter only the fact that the Dirac system is known to be in the limit-point case will be used.

We now take $u_0 = \log a_0$ and apply on (u_0, ∞) the Weyl-Stone-Titchmarsh-Kodaira theory ([10, §9], [21, §3]). Let $\psi(u,s)$ be the unique solution of the system (136) for the eigenvalue γ, $s = \frac{1}{2} + i\gamma$, and with the initial condition $\psi(u_0,s) = \left[\begin{smallmatrix}1\\ 0\end{smallmatrix}\right]$ and let $\phi(u,s)$ be the unique solution with the initial condition $\phi(u_0,s) = \left[\begin{smallmatrix}0\\ 1\end{smallmatrix}\right]$. Let $m(\gamma)\psi(u,s) + \phi(u,s)$ for $\Im\gamma > 0$ be the unique solution which is square-integrable on $(u_0,+\infty)$. So $m(\gamma) = \frac{\mathcal{A}_{a_0}(s)}{\mathcal{B}_{a_0}(s)}$, $s = \frac{1}{2} + i\gamma$, $\Re(s) < \frac{1}{2}$. It is a fundamental general property of the m function from Hermann Weyl's theory that $\Im(m(\gamma)) > 0$ (for $\Im(\gamma) > 0$.) Here, we have a case where the m-function is found to be meromorphic on all of $\mathbb{C}$; so we see that its poles and zeros on $\mathbb{R}$ are simple. Furthermore, the spectral measure ν is obtained via the formula $\nu(a,b) = \lim_{\epsilon\to 0^+}\frac{1}{\pi}\int_a^b \Im m(\gamma + i\epsilon)\,d\gamma$ (under the condition $\nu\{a,b\} = 0$). We obtain:

$$d\nu(\gamma) = \sum_{\mathcal{B}_{a_0}(\rho)=0} \frac{\mathcal{A}_{a_0}(\rho)}{-i\mathcal{B}'_{a_0}(\rho)}\delta(\gamma - \Im\rho) \tag{143}$$

The spectrum is thus purely discrete and the general theory tells us further that the finite linear combinations $\sum_\rho c_\rho \frac{\mathcal{A}_{a_0}(\rho)}{-i\mathcal{B}'_{a_0}(\rho)}\psi(u,\rho)$ have squared norms $\sum_\rho \frac{\mathcal{A}_{a_0}(\rho)}{-i\mathcal{B}'_{a_0}(\rho)}|c_\rho|^2$ and also that they are

[25] let us recall the notation $\mathcal{X}_s^a = \Gamma(s)X_s^a \in L^2(a,+\infty;dx)$.

dense in $L^2((u_0,\infty)\to\mathbb{C}^2; du)$. For $\mathcal{B}_{a_0}(\rho)=0$, $\psi(u,\rho)=\mathcal{A}_{a_0}(\rho)^{-1}\left[\begin{smallmatrix}\mathcal{A}_{\exp(u)}(\rho)\\ \mathcal{B}_{\exp(u)}(\rho)\end{smallmatrix}\right]\mathbf{1}_{u\geq u_0}(u)$, so the vectors $Z_\rho^{a_0}=\left[\begin{smallmatrix}2\mathcal{A}_{\exp(u)}(\rho)\\ 2\,\mathcal{B}_{\exp(u)}(\rho)\end{smallmatrix}\right]\mathbf{1}_{u\geq u_0}(u)$ are an orthogonal basis of $L^2((u_0,\infty)\to\mathbb{C}^2; \frac{1}{2}du)$ and they satisfy $\|Z_\rho^{a_0}\|^2=-2\,\mathcal{A}_{a_0}(\rho)\,i\,\mathcal{B}'_{a_0}(\rho)$. Similarly a spectral interpretation is given to the zeros of $\mathcal{A}_{a_0}$ if one looks at the initial condition $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. The factors of 2 and $\frac{1}{2}$, have been incorporated so that the statement may be translated (taking into account results established later) into the fact that the evaluators $K_{a_0}(\rho,z)$, for $\mathcal{B}_{a_0}(\rho)=0$, are an orthogonal basis of the Hilbert space of the functions $\Gamma(z)\widehat{f}(z)$, $f\in K_{a_0}$. This last statement is a general theorem (under a certain condition) for spaces with the de Branges axioms [2, §22].

To discuss in a self-contained manner the generalized Parseval identity which is associated with the differential system on the full line, it is convenient to make a preliminary majoration of $\|X_s^a\|^2$, $\Re(s)=\frac{1}{2}$. From (108) we have, for $\Re(s)=\frac{1}{2}$: $\|\mathcal{X}_s^a\|^2=2\Re(\mathcal{E}_a(s)\overline{\mathcal{E}_a'(s)})$. And $\mathcal{E}_a(s)=\Gamma(s)\widehat{E_a}(s)$. And $\widehat{E_a}(s)=\sqrt{a}\Big(a^{-s}+\frac{1}{2}\int_a^\infty(\phi_a^+(x)-\phi_a^-(x))x^{-s}\,dx\Big)$. We know from the discussion of Lemma 9 that the integral in the expression for $\widehat{E_a}(s)$ is absolutely convergent for $\Re(s)>\frac{1}{4}$. Hence by the Riemann-Lebesgue lemma $\widehat{E_a}(\frac{1}{2}+i\gamma)\sim a^{-i\gamma}$ as $|\gamma|\to\infty$, $\gamma\in\mathbb{R}$. Similarly, $\widehat{E_a}'(\frac{1}{2}+i\gamma)\sim-\log(a)a^{-i\gamma}$. So, with $\|\mathcal{X}_s^a\|^2=|\Gamma(s)|^2\|X_s^a\|^2$ and using Stirling's formula we obtain:

Lemma 14. *For each given $a>0$ one has $\|X_s^a\|^2\sim 2\log|s|$ as $|s|\to\infty$, $\Re(s)=\frac{1}{2}$.*

From (104) expressed using A_a and B_a we see that $\frac{\widehat{B_a}(s)}{s-\frac{1}{2}}$ is square integrable, so $s^{-1}\widehat{B_a}(s)$ is square integrable on the critical line (with respect to $|ds|$). Then using again (104) we see that $s^{-1}\widehat{A_a}(s)$ is also square integrable on the critical line.[26] Let us pick a function $F(s)$ on the critical line which is such that $sF(s)$ is square integrable. Then $F(s)\overline{\widehat{A_a}(s)}$ and $F(s)\overline{\widehat{B_a}(s)}$ are absolutely integrable on the critical line and $(\int_{\Re(s)=\frac{1}{2}}|F(s)\overline{\widehat{A_a}(s)}|\frac{|ds|}{2\pi})^2\leq C\int_{\Re(s)=\frac{1}{2}}\frac{|\widehat{A_a}(s)|^2}{|s|^2}\frac{|ds|}{2\pi}$ and similarly with B_a. If we define $\alpha_F(u)=2\int_{\Re(s)=\frac{1}{2}}F(s)\overline{\widehat{A_a}(s)}\frac{|ds|}{2\pi}$ and $\beta_F(u)=2\int_{\Re(s)=\frac{1}{2}}F(s)\overline{\widehat{B_a}(s)}\frac{|ds|}{2\pi}$ we then compute:

$$\int_{u_0}^\infty|\alpha_F(u)|^2+|\beta_F(u)|^2\,du\leq C\int_{\Re(s)=\frac{1}{2}}\frac{\int_{u_0}^\infty(|\widehat{A_a}(s)|^2+|\widehat{B_a}(s)|^2)\,du}{|s|^2}\,\frac{|ds|}{2\pi}$$
$$=\frac{C}{2}\int_{\Re(s)=\frac{1}{2}}\frac{\|X_s^a\|^2}{|s|^2}\frac{|ds|}{2\pi}<\infty \quad (144)$$

So $\alpha_F(u)$ and $\beta_F(u)$ are square integrable at $+\infty$. More precisely the above upper bound holds as well for $\int_{\Re(s)=\frac{1}{2}}|F(s)\widehat{A_a}(s)|\frac{|ds|}{2\pi}$ and $\int_{\Re(s)=\frac{1}{2}}|F(s)\widehat{B_a}(s)|\frac{|ds|}{2\pi}$. So the double integrals

$$\iint_{u_0<u<\infty,\Re(s)=\frac{1}{2}}\mathcal{A}_{\exp(u)}(z)\mathcal{A}_{\exp(u)}(s)\mathcal{F}(s)\;\frac{|ds|}{2\pi|\Gamma(s)|^2}\;du \quad (145a)$$
$$\iint_{u_0<u<\infty,\Re(s)=\frac{1}{2}}\mathcal{B}_{\exp(u)}(z)\mathcal{B}_{\exp(u)}(s)\mathcal{F}(s)\;\frac{|ds|}{2\pi|\Gamma(s)|^2}\;du \quad (145b)$$

[26] we know in fact according to proposition 11 that $\widehat{A_a}$ and $\widehat{B_a}$ are bounded on the critical line.

where $z \in \mathbb{C}$ is arbitrary, and $\mathcal{F}(s) = \Gamma(s)F(s)$, are absolutely convergent and Fubini may be employed. Using (140):

$$\mathcal{X}_{\exp(u_0)}(z, \overline{s}) = 2\int_{u_0}^{\infty} (\mathcal{A}_{\exp(u)}(z)\overline{\mathcal{A}_{\exp(u)}(s)} + \mathcal{B}_{\exp(u)}(z)\overline{\mathcal{B}_{\exp(u)}(s)})\, du \tag{146}$$

And we obtain the following identity of absolutely convergent integrals, for any $\mathcal{F}(s) = \Gamma(s)F(s)$ with $s\,F(s) \in L^2(\Re(s) = \frac{1}{2}; \frac{|ds|}{2\pi})$:

$$\int_{\Re(s)=\frac{1}{2}} \mathcal{X}_{\exp(u_0)}(z, \overline{s})\mathcal{F}(s)\, \frac{|ds|}{2\pi|\Gamma(s)|^2} = \int_{u_0}^{\infty} (\mathcal{A}_{\exp(u)}(z)\alpha_F(u) + \mathcal{B}_{\exp(u)}(z)\beta_F(u))\, du \tag{147}$$

We shall prove that this identity holds under the weaker hypothesis $F(s) \in L^2(\Re(s) = \frac{1}{2}; \frac{|ds|}{2\pi})$. First, still with $s\,F(s)$ square integrable we suppose additionally that $F = \widehat{f}$ with $f \in K_{\exp(u_0)}$[27]. The hilbertian kernel $K_{\exp(u_0)}(z, s)$ is $\mathcal{X}_{\exp(u_0)}(\overline{z}, s)$ so $\overline{K_{\exp(u_0)}(z, s)} = \mathcal{X}_{\exp(u_0)}(z, \overline{s})$. The equations give then:

$$\mathcal{F}(z) = \int_{u_0}^{\infty} (\mathcal{A}_{\exp(u)}(z)\alpha_F(u) + \mathcal{B}_{\exp(u)}(z)\beta_F(u))\, du \tag{148a}$$

$$\alpha_F(u) = 2\int_{\Re(s)=\frac{1}{2}} \mathcal{F}(s)\mathcal{A}_a(s)\frac{|ds|}{2\pi\,|\Gamma(s)|^2} \tag{148b}$$

$$\beta_F(u) = 2\int_{\Re(s)=\frac{1}{2}} \mathcal{F}(s)\mathcal{B}_a(s)\frac{|ds|}{2\pi\,|\Gamma(s)|^2} \tag{148c}$$

We have worked under the hypothesis that $sF(s)$ is square integrable. To show that the formulae extend in the L^2 sense, we first examine:

$$|\alpha_F(u)|^2 = 4\int_{\Re(s_1)=\frac{1}{2}} \mathcal{F}(s_1)\mathcal{A}_a(s_1)\frac{|ds_1|}{2\pi\,|\Gamma(s_1)|^2} \int_{\Re(s_2)=\frac{1}{2}} \overline{\mathcal{F}(s_2)}\mathcal{A}_a(s_2)\frac{|ds_2|}{2\pi\,|\Gamma(s_2)|^2} \tag{149}$$

$$\int_{u_0}^{\infty} |\alpha_F(u)|^2 + |\beta_F(u)|^2\, \frac{du}{2} = \iint_{\Re(s_i)=\frac{1}{2}} \mathcal{F}(s_1)\overline{\mathcal{F}(s_2)}\mathcal{X}_{\exp(u_0)}(s_1, \overline{s}_2)\frac{|ds_1|}{2\pi\,|\Gamma(s_1)|^2}\frac{|ds_2|}{2\pi\,|\Gamma(s_2)|^2} \tag{150}$$

There was absolute convergence in the triple integral used as an intermediate. Also $\mathcal{X}_{\exp(u_0)}(s_1, \overline{s}_2) = \mathcal{X}_{\exp(u_0)}(s_2, \overline{s}_1)$ and $\int_{\Re(s_1)=\frac{1}{2}} \mathcal{F}(s_1)\mathcal{X}_{\exp(u_0)}(s_2, \overline{s}_1)\frac{|ds_1|}{2\pi\,|\Gamma(s_1)|^2} = \mathcal{F}(s_2)$. Hence:

$$\int_{u_0}^{\infty} (|\alpha_F(u)|^2 + |\beta_F(u)|^2)\, \frac{du}{2} = \int_{\Re(s)=\frac{1}{2}} |F(s)|^2\frac{|ds|}{2\pi} = \int_{\exp(u_0)}^{\infty} |f(x)|^2\, dx \tag{151}$$

So with an arbitrary $f \in K_a$, $F = \widehat{f}$, $\mathcal{F}(s) = \Gamma(s)\widehat{f}(s)$, the assignment $f \mapsto (\alpha_F, \beta_F)$ exists in the sense of L^2 convergence when one approximates f by a sequence f_n in K_a such that $s\widehat{f_n}(s)$ is in $L^2(\Re(s) = \frac{1}{2}; \frac{|ds|}{2\pi})$, and $f \mapsto (\alpha_F, \beta_F)$ is linear and isometric. We check that its range is all of $L^2(u_0, \infty; \frac{du}{2}) \oplus L^2(u_0, \infty; \frac{du}{2})$. For this let us identify the functions $\alpha_w(u)$ and $\beta_w(u)$ which will correspond to $\mathcal{F}(s) = \mathcal{X}_{a_0}(w, s)$ ($a_0 = \exp(u_0)$.) On one hand from (147) it must be the case that:

$$\forall z \in \mathbb{C} \quad \int_{\Re(s)=\frac{1}{2}} \mathcal{X}_{a_0}(z, \overline{s})\mathcal{X}_{a_0}(w, s)\, \frac{|ds|}{2\pi|\Gamma(s)|^2} = \int_{u_0}^{\infty} (\mathcal{A}_{\exp(u)}(z)\alpha_w(u) + \mathcal{B}_{\exp(u)}(z)\beta_w(u))\, du \tag{152}$$

[27] this is certainly possible as we know that the $f(x)$ which are smooth, vanishing on $(0, a)$ and of Schwartz decrease as $x \to +\infty$ are dense in K_a.

The left hand side is $\mathcal{X}_{a_0}(w,z)$ which on the other hand is given by the formula $2\int_{u_0}^\infty(\mathcal{A}_a(z)\mathcal{A}_a(w)-\mathcal{B}_a(z)\mathcal{B}_a(w))\,du$. The functions $u\mapsto\left[\begin{smallmatrix}\mathcal{A}_a(z)\\ \mathcal{B}_a(z)\end{smallmatrix}\right]$, $z\in\mathbb{C}$ are certainly dense in $L^2((u_0,\infty)\to\mathbb{C}^2;\frac{du}{2})$ as we know in particular that the pairs for the ρ's such that $\mathcal{B}_{a_0}(\rho)=0$ give an orthogonal basis. So we have the identification on $(u_0,+\infty)$:

$$\alpha_w(u)=2\,\mathcal{A}_{\exp(u)}(w)\qquad \beta_w(u)=-2\mathcal{B}_{\exp(u)}(w)\tag{153}$$

This proves that the range is all of $L^2((u_0,\infty)\to\mathbb{C}^2;\frac{du}{2})$. Let us note that in this identification the hilbertian evaluator $K_{a_0}(w,\cdot)$ is sent to the pair $u\mapsto 2\,\mathbf{1}_{u>u_0}(u)(\overline{\mathcal{A}_a(w)}\,,\overline{\mathcal{B}_a(w)})$. To check if all is coherent we compute the hilbertian scalar product $(K_{a_0}(w,\cdot),K_{a_0}(z,\cdot))$. We obtain $4\int_{u_0}^\infty\overline{\mathcal{A}_a(w)}\mathcal{A}_a(z)+\overline{\mathcal{B}_a(w)}\mathcal{B}_a(z)\frac{du}{2}=2\int_{u_0}^\infty\mathcal{A}_a(\overline{w})\mathcal{A}_a(z)-\mathcal{B}_a(\overline{w})\mathcal{B}_a(z)\,du=\mathcal{X}_{exp(u_0)}(\overline{w},z)$, which is indeed $K_{a_0}(w,z)$.

Let us return to the consideration of a general $F(s)\in L^2(\Re(s)=\frac12;\frac{|ds|}{2\pi})$. Under the hypothesis that $s\,F(s)$ is square integrable we have assigned to F the functions

$$\alpha_F(u)=2\int_{\Re(s)=\frac12}\mathcal{F}(s)\mathcal{A}_a(s)\frac{|ds|}{2\pi\,|\Gamma(s)|^2}=\int_{\Re(s)=\frac12}F(s)\overline{2\widehat{A_a}(s)}\frac{|ds|}{2\pi}\tag{154a}$$

$$\beta_F(u)=2\int_{\Re(s)=\frac12}\mathcal{F}(s)\mathcal{B}_a(s)\frac{|ds|}{2\pi\,|\Gamma(s)|^2}=\int_{\Re(s)=\frac12}F(s)\overline{2\widehat{\mathcal{B}_a}(s)}\frac{|ds|}{2\pi}\tag{154b}$$

which are square-integrable at $+\infty$. From (147) there holds, for any $a_0=\exp(u_0)$:

$$\int_{\Re(s)=\frac12}X_{\exp(u_0)}(z,\overline{s})F(s)\,\frac{|ds|}{2\pi}=\int_{u_0}^\infty(2\widehat{A_{\exp(u)}}(z)\alpha_F(u)+2\widehat{B_{\exp(u)}}(z)\beta_F(u))\,\frac{du}{2}\tag{155}$$

The function of z on the left side is the orthogonal projection F_{a_0} of F to the space $\widehat{K_{a_0}}$. So, we deduce by unicity $\alpha_F(u)\mathbf{1}_{u\geq u_0}(u)=\alpha_{F_{u_0}}(u)$ and $\beta_F(u)\mathbf{1}_{u\geq u_0}(u)=\beta_{F_{u_0}}(u)$. We then obtain $\|F_{a_0}\|^2=\int_{u_0}^\infty(|\alpha_F(u)|^2+|\beta_F(u)|^2)\frac{du}{2}$ so α_F and β_F are square-integrable on $(-\infty,+\infty)$, and as $\cup_a K_a$ is dense in $L^2(0,\infty;dx)$ the assignment $F\mapsto(\alpha_F,\beta_F)$ is isometric, and also it has a dense range in $L^2(\mathbb{R}\to\mathbb{C}^2;\frac{du}{2})$. We can then remove the hypothesis that $s\,F(s)$ is square integrable and define the functions α_F and β_F to be the limit in the L^2 sense of functions α_n and β_n associated with F_n's such that $\|F-F_n\|\to 0$ and the $s\,F_n$ are square-integrable. Summing up:

Theorem 15. *There are unitary identifications $L^2(0,\infty;dx)\xrightarrow{\sim}L^2(\Re(s)=\frac12;\frac{|ds|}{2\pi})\xrightarrow{\sim}L^2(\mathbb{R}\to\mathbb{C}^2;\frac{du}{2})$ given in the L^2 sense by the formulas, where $\Re(s)=\frac12$:*

$$F(s)=\widehat{f}(s)=\int_0^\infty f(x)x^{-s}\,dx\qquad\qquad f(x)=\int_{\Re(s)=\frac12}F(s)x^{s-1}\,\frac{|ds|}{2\pi}\tag{156a}$$

$$\alpha(u)=\lim_{n\to\infty}\int_{\Re(s)=\frac12}F_n(s)\,\overline{2\widehat{A_{\exp(u)}}(s)}\,\frac{|ds|}{2\pi}\qquad(F_n\to_{L^2}F;\qquad s\,F_n(s)\in L^2)\tag{156b}$$

$$\beta(u)=\lim_{n\to\infty}\int_{\Re(s)=\frac12}F_n(s)\,\overline{2\widehat{B_{\exp(u)}}(s)}\,\frac{|ds|}{2\pi}\tag{156c}$$

$$F(s)=\lim_{a_0\to0}\int_{\log(a_0)}^\infty(\alpha(u)\,2\widehat{A_{\exp(u)}}(s)+\beta(u)\,2\widehat{B_{\exp(u)}}(s))\,\frac{du}{2}\tag{156d}$$

The orthogonal projection of f to K_{a_0} corresponds to the replacement of $\alpha(u)$ by $\alpha(u)\mathbf{1}_{u>u_0}(u)$ and of $\beta(u)$ by $\beta(u)\mathbf{1}_{u>u_0}(u)$ ($u_0 = \log(a_0)$.). The unitary operators $f \mapsto \mathcal{H}(f)$, $F(s) \mapsto \chi(s)F(1-s)$, correspond to $(\alpha, \beta) \mapsto (\alpha, -\beta)$. For $f = X_z^{a_0}$ one has $\alpha(u) = 2\widehat{A_{\exp(u)}}(z)\mathbf{1}_{u>\log(a_0)}(u)$ and $\beta(u) = -2\widehat{B_{\exp(u)}}(z)\mathbf{1}_{u>\log(a_0)}(u)$. The self-adjoint operator $F(s) \mapsto \gamma F(s)$ ($s = \frac{1}{2} + i\gamma$) corresponds to the canonical operator:

$$H = \begin{bmatrix} 0 & \frac{d}{du} \\ -\frac{d}{du} & 0 \end{bmatrix} - \begin{bmatrix} 0 & \mu(e^u) \\ \mu(e^u) & 0 \end{bmatrix} \tag{156e}$$

which, in $L^2(\mathbb{R} \to \mathbb{C}^2; \frac{du}{2})$, is essentially self-adjoint when defined on the domain of the functions of class C^1 (or even C^∞) with compact support. The unitary operator $e^{i\tau H}$ acts on $L^2(0, \infty; dx)$ as $f(x) \mapsto e^{\frac{1}{2}\tau} f(e^\tau x)$.

For the statement of self-adjointness we start with α and β of class C^1 with compact support, define F by (156d) and integrate by parts to confirm that $\gamma F(s)$ corresponds to $H\left(\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]\right)$. We know by Hermann Weyl's theorem that in the limit point case the pairs $\left[\begin{smallmatrix}\alpha\\ \beta\end{smallmatrix}\right]$ of class C^1 with compact support are a core of self-adjointness (*cf.* [21, §13].) On the other hand we know that multiplication by γ on $L^2(\Re(s) = \frac{1}{2}; \frac{|ds|}{2\pi})$ with maximal domain is a self-adjoint operator. So the two self-adjoint operators are the same.

Having discussed the matter from the point of view of the isometric expansion we now turn to another topic, the topic of the scattering, or rather total reflection against the potential barrier at $+\infty$. Another pair of solutions of the first order system (136) (hence also of the second order differential equations) is known. Let us recall from equations (95a), (95b) that we defined $j_a = \sqrt{a}(\delta_a - \phi_a^+ \mathbf{1}_{0<x<a}) = \sqrt{a}\mathcal{H}\phi_a^+$ and $-ik_a = \sqrt{a}(\delta_a + \phi_a^- \mathbf{1}_{0<x<a}) = \sqrt{a}\mathcal{H}\phi_a^-$. Using again (122a) and (122b) it is checked that j_a and k_a verify the exact same differential system as A_a and B_a:

$$a\frac{\partial}{\partial a} j_a = -\mu(a) j_a + i\delta_x k_a \tag{157a}$$

$$a\frac{\partial}{\partial a} k_a = +\mu(a) k_a - i\delta_x j_a \tag{157b}$$

The right Mellin transforms $\widehat{j_a}(s)$ and $-i\widehat{k_a}(s)$ are defined as

$$\widehat{j_a}(s) = a^{\frac{1}{2}-s} - \sqrt{a}\int_0^a \phi_a^+(x) x^{-s}\, dx \tag{158a}$$

$$-i\widehat{k_a}(s) = a^{\frac{1}{2}-s} + \sqrt{a}\int_0^a \phi_a^-(x) x^{-s}\, dx \tag{158b}$$

As ϕ_a^+ and ϕ_a^- are analytic these are meromorphic functions in $\mathbb{C}$ with possible[28] pole locations at $s = 1, 2, \dots$ From the point of view of the Schrödinger equation (138a) and as $u = \log(a) \to -\infty$, $a \to 0$, we thus see that, for $s = \frac{1}{2} + i\gamma$, $\gamma \in \mathbb{R}$, $\widehat{j_a}(\frac{1}{2}+i\gamma)$ and $\widehat{j_a}(\frac{1}{2}-i\gamma)$ are two (linearly independent for $\gamma \neq 0$) solutions, differing from $e^{-i\gamma u}$ and $e^{i\gamma u}$ by an exponentially small (in $u = \log(a)$) quantity (and similarly with $-i\widehat{k_a}$ with respect to the Schrödinger equation (135h)). So we have identified the unique solutions which verify the Jost conditions at $-\infty$.

[28] the poles do exist.

As $P_a\phi_a^+$ is square integrable, and also using (85a), we have on the critical line:

$$\Gamma(s)\widehat{j_a}(s) + \Gamma(1-s)\widehat{j_a}(1-s) \tag{159a}$$

$$= \Gamma(s)\widehat{j_a}(s) + \Gamma(1-s)a^{s-\frac{1}{2}} - \Gamma(1-s)\sqrt{a}\int_0^a \phi_a^+(x)x^{s-1}\,dx \tag{159b}$$

$$= \Gamma(s)\widehat{j_a}(s) + \Gamma(1-s)a^{s-\frac{1}{2}} - \Gamma(s)\sqrt{a}\int_0^\infty (\mathcal{H}P_a\phi_a^+)(x)x^{-s}\,dx \tag{159c}$$

$$= \Gamma(s)\widehat{j_a}(s) + \Gamma(1-s)a^{s-\frac{1}{2}} + \Gamma(s)\sqrt{a}\int_0^\infty (\phi_a^+(x) - J_0(2\sqrt{ax}))x^{-s}\,dx \tag{159d}$$

$$= \Gamma(s)a^{\frac{1}{2}-s} + \Gamma(1-s)a^{s-\frac{1}{2}} - \Gamma(s)\sqrt{a}\int_0^a J_0(2\sqrt{ax})x^{-s}\,dx$$

$$+ \Gamma(s)\sqrt{a}\int_a^\infty (\phi_a^+(x) - J_0(2\sqrt{ax}))x^{-s}\,dx \tag{159e}$$

As $J_0(2\sqrt{ax}) - \phi_a^+(x)$ is square integrable both integrals are simultaneously absolutely convergent at least for $\frac{1}{2} < \Re(s) < 1$ (the $\frac{1}{2}$ can be improved, but this does not matter). As the boundary values on the critical line coincide we have an identity of analytic functions. We recognize in $\int_a^\infty J_0(2\sqrt{ax})x^{-s}\,dx$, which is absolutely convergent for $\Re(s) > \frac{3}{4}$, the quantity $g_s(a)$ (equation (73)). And from equation (74) we know $g_s(a) = \chi(s)a^{s-1} - \int_0^a J_0(2\sqrt{ax})x^{-s}\,dx$. So

$$\Gamma(s)\widehat{j_a}(s) + \Gamma(1-s)\widehat{j_a}(1-s) = \Gamma(s)a^{\frac{1}{2}-s} + \Gamma(s)\sqrt{a}\int_a^\infty \phi_a^+(x)x^{-s}\,dx\ , \tag{160}$$

which is indeed $2\mathcal{A}_a(s)$. From the equation (158a) $\widehat{j_a}(s) = a^{\frac{1}{2}}(a^{-s} - \int_0^a \phi_a^+(x)x^{-s}\,dx)$ (valid as is for $\Re(s) < 1$) the function $u \mapsto \widehat{j_a}(s)$ differs from $u \mapsto e^{-i\gamma u}$ by an error which is relatively exponentially smaller (we write $s = \frac{1}{2} + i\gamma$, $\Im(\gamma) > -\frac{1}{2}$). So $\widehat{j_a}$ is the Jost solution at $-\infty$ of the Schrödinger equation (135g). The identity relating $\mathcal{B}_a(s)$ and $\widehat{k_a}(s) = i\,a^{\frac{1}{2}}(a^{-s} + \int_0^a \phi_a^-(x)x^{-s}\,dx)$ is proven similarly.

Theorem 16. *The unique*[29] *solution, square integrable at $u = +\infty$, of the Schrödinger equation* (135g) *(resp.* (135h)*; $\gamma \neq 0$) is expressed in terms of the functions $\widehat{j_a}(\frac{1}{2}+i\gamma)$ (resp. $-i\,\widehat{k_a}(\frac{1}{2}+i\gamma)$) satisfying at $-\infty$ the Jost condition $\widehat{j_a} \sim_{u\to-\infty} e^{-i\gamma u}$ (resp. $-i\widehat{k_a} \sim_{u\to-\infty} e^{-i\gamma u}$) as:*

$$\mathcal{A}_a(s) = \frac{1}{2}\Big(\Gamma(s)\widehat{j_a}(s) + \Gamma(1-s)\widehat{j_a}(1-s)\Big) \tag{161a}$$

$$\mathcal{B}_a(s) = \frac{1}{2}\Big(\Gamma(s)\widehat{k_a}(s) - \Gamma(1-s)\widehat{k_a}(1-s)\Big) \tag{161b}$$

Let us add a time parameter t and consider the wave equation:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial u^2} + \mu^2 - \frac{d\mu}{du}\right)\Phi(t,u) = 0 \tag{162}$$

Then $\Phi(t,u) = e^{i\gamma t}\widehat{j_{\exp(u)}}(\frac{1}{2}+i\gamma)$ is a solution which behaves as $e^{i\gamma(t-u)}$ as $u \to -\infty$. This wave is thus right-moving, it is an incoming wave from $u = -\infty$ at $t = -\infty$. For a given frequency γ there

[29]here we make use of the fact that (135g) is in the limit point case at $+\infty$, because it is proven in the next chapter, or known from (66a), (66b), that $\mu(a) = 2a = 2\,e^u$, in this study of the $\mathcal{H}$ transform.

is a unique, up to multiplicative factor, wave which respects the condition of being at each time square integrable at $u = +\infty$. This wave is $e^{i\gamma t}\mathcal{A}_{\exp(u)}(\frac{1}{2}+i\gamma)$. So the equation (161a) represents the decomposition in incoming and reflected components. There is in the reflected component a phase shift $\theta_\gamma = \arg\chi(s)$, the solution behaving approximatively at $u \to -\infty$ as $C(\gamma)\cos(\gamma u + \frac{1}{2}\theta_\gamma)$. This is an absolute scattering, as there is nothing a priori to compare it too. We will thus declare that equation (162) has realized $\chi(s)$ as an (absolute) scattering. Similarly the Schrödinger equation (135h) realizes $-\chi(s)$ as an absolute scattering.

We have $2\mathcal{A}_a(\frac{1}{2}) = 2\Gamma(\frac{1}{2})\widehat{j_a}(\frac{1}{2})$ and $\widehat{j_a}(\frac{1}{2}) = 1 - a^{\frac{1}{2}}\int_0^a \phi_a^+(x)x^{-\frac{1}{2}}\,dx$. So $\lim_{a\to 0}\mathcal{A}_a(\frac{1}{2}) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$. On the other hand $a\frac{d}{da}\mathcal{A}_a(\frac{1}{2}) = -\mu(a)\mathcal{A}_a(\frac{1}{2})$ and $\mu(a) = a\frac{d}{da}\log\frac{\det(1+H_a)}{\det(1-H_a)}$. so:

$$\mathcal{A}_a(\frac{1}{2}) = \sqrt{\pi}\,\widehat{A_a}(\frac{1}{2}) = \sqrt{\pi}\,\frac{\det(1-H_a)}{\det(1+H_a)} \tag{163}$$

We have $a\frac{d}{da}\|\mathcal{X}^a_{\frac{1}{2}}\|^2 = -2\mathcal{A}_a(\frac{1}{2})^2$. And $\mathcal{X}^a_{\frac{1}{2}} = \Gamma(\frac{1}{2})X^a_{\frac{1}{2}}$. So:

Theorem 17. *The squared-norm of the evaluator $f \mapsto X^a_{\frac{1}{2}}(f) = \int_a^\infty \frac{f(x)}{\sqrt{x}}\,dx$ on the Hilbert space K_a of square integrable functions vanishing on $(0,a)$ and with $\mathcal{H}$ transforms again vanishing on $(0,a)$ is:*

$$\|X^a_{\frac{1}{2}}\|^2 = 2\int_a^\infty \left(\det\frac{1-H_b}{1+H_b}\right)^2\frac{db}{b} \tag{164}$$

where H_a is the restriction of $\mathcal{H}$ to $L^2(0,a;dx)$.

It will be seen that $\det(1+H_a) = e^{a-\frac{1}{2}a^2}$ and $\det(1-H_a) = e^{-a-\frac{1}{2}a^2}$. Having spent a long time in the general set-up we now turn to determine explicitely what the functions ϕ_a^+, ϕ_a^-, etc... are.

7 The K-Bessel function in the theory of the $\mathcal{H}$ transform

Let us recall that we may define the $\mathcal{H}$ transform on all of $L^2(\mathbb{R};dx)$ through the formula $\widetilde{\mathcal{H}(f)}(\lambda) = \frac{i}{\lambda}\widetilde{f}(\frac{-1}{\lambda})$. This anticommutes with $f(x) \mapsto f(-x)$, and $\mathcal{H}$ leaves separately invariant $L^2(0,+\infty;dx)$ and $L^2(-\infty,0;dx)$. We defined the groups $\tau_a : f(x) \mapsto f(x-a)$ and $\tau_a^\# = \mathcal{H}\tau_a\mathcal{H}$. We observed that the two groups are mutually commuting, and that if the leftmost point of the support of f is at $\alpha(f) \geq 0$ then the leftmost point of the support of $\tau_b^\#(f)$, for any $b \geq 0$, more precisely for any $b \geq -\alpha(\mathcal{H}(f))$, is still exactly at $\alpha(f)$. From this we obtain the exact description of K_a:

Lemma 18. *One has $K_a = \tau_a\tau_a^\# L^2(0,+\infty;dx)$.*

Let now Q be the orthogonal projection $L^2(\mathbb{R};dx) \mapsto L^2(0,+\infty;dx)$. The orthogonal projection Q_a from $L^2(0,\infty;dx)$ to K_a is thus exactly $\tau_a\tau_a^\# Q\tau_{-a}\tau_{-a}^\#$. It will be easier to work with $R_a = Q\tau_{-a}\tau_{-a}^\#$, especially as we are interested in scalar products so we can skip the $\tau_a\tau_a^\#$ isometry. First, we obtain $g_t^a(x) = R_a(f_t)(x)$, for $f_t(x) = e^{-tx}$. The part of $\tau_{-a}(f_t)$ supported in $x < 0$ will be sent by $\tau_{-a}^\#$ to a function supported again in $x < 0$. We can forget about it and we have thus first

$e^{-ta}e^{-tx}\mathbf{1}_{x>0}(x)$, whose $\mathcal{H}$ transform is $e^{-ta}\frac{1}{t}\exp(-\frac{x}{t})$, which we translate to the left, again we cut the part in $x<0$, and we reapply $\mathcal{H}$, this gives $g_t^a(x) = e^{-a(t+\frac{1}{t})}e^{-tx}\mathbf{1}_{x>0}(x)$. In other words we have used in this computation:

$$Q\tau_{-a}\tau_{-a}^{\#} = \mathcal{H}Q\tau_{-a}\mathcal{H}Q\tau_{-a} \qquad (a\geq 0) \tag{165}$$

The orthogonal projection $f_t^a := Q_a(f_t)$ of $f_t(x) = e^{-tx}\mathbf{1}_{x>0}(x)$ to K_a is thus $\tau_a\tau_a^{\#}(g_t^a)$. We can then compute exactly the Fourier transform of f_t^a as $f_t^a(i\tau) = (e^{-\tau x}, \tau_a\tau_a^{\#}(g_t^a))_{L^2(\mathbb{R})}$ which is also $(\tau_{-a}^{\#}\tau_{-a}e^{-\tau x}, g_t^a)_{L^2(\mathbb{R})} = (g_\tau, g_t^a) = e^{-a(t+\frac{1}{t})}e^{-a(\tau+\frac{1}{\tau})}\frac{1}{t+\tau}$. Hence:

Lemma 19. *The orthogonal projection f_t^a to K_a of $e^{-tx}\mathbf{1}_{x>0}(x)$ has its Fourier transform $\widetilde{f_t^a}(\lambda)$ which is given as:*

$$\widetilde{f_t^a}(i\tau) = e^{-a(t+\frac{1}{t}+\tau+\frac{1}{\tau})}\frac{1}{t+\tau} \tag{166}$$

The Gamma completed right Mellin transform $\mathcal{F}_t^a(s)$ of f_t^a is the left Mellin transform of $\widetilde{f_t^a}(i\tau)$.

$$\int_a^\infty f_t^a(x)\mathcal{X}_s^a(x)\,dx = \mathcal{F}_t^a(s) = e^{-a(t+\frac{1}{t})}\int_0^\infty e^{-a(\tau+\frac{1}{\tau})}\frac{\tau^{s-1}}{t+\tau}\,d\tau \tag{167}$$

Let us write W_s^a for the element of $L^2(0,+\infty;dx)$ such that $\tau_a\tau_a^{\#}W_s^a = \mathcal{X}_s^a$. We have $\mathcal{F}_t^a(s) = (\mathcal{X}_s^a, f_t^a) = (W_s^a, g_t^a) = e^{-a(t+\frac{1}{t})}\int_0^\infty W_s^a(x)e^{-tx}\,dx$. So the Laplace transform of $W_s^a(x)$ is exactly:

$$\int_0^\infty W_s^a(x)e^{-tx}\,dx = \int_0^\infty e^{-a(\tau+\frac{1}{\tau})}\frac{\tau^{s-1}}{t+\tau}\,d\tau \tag{168}$$

Writing $\frac{1}{t+\tau} = \int_0^\infty e^{-(t+\tau)x}\,dx$, we recover $W_s^a(x)$ as:

$$W_s^a(x) = \int_0^\infty e^{-a(\tau+\frac{1}{\tau})}\tau^{s-1}e^{-\tau x}\,d\tau \tag{169}$$

Then we obtain $\int_0^\infty W_s^a(x)W_z^a(x)\,dx$ which is nothing else than $\mathcal{X}_a(s,z)$:

Theorem 20. *The (analytic) reproducing kernel associated with the space of the completed right Mellin transforms of the elements of K_a is*

$$\mathcal{X}_a(s,z) = \iint_{[0,+\infty)^2} e^{-a(t+\frac{1}{t}+u+\frac{1}{u})}\frac{t^{s-1}u^{z-1}}{t+u}\,dtdu \tag{170}$$

Here is a shortened argument: the analytic reproducing kernel $\mathcal{X}_a(s,z)$ is the completed right Mellin transform of $\mathcal{X}_s^a(x)$, so this is $\int_0^\infty(\mathcal{X}_s^a, e^{-tx})t^{s-1}\,dt$. But for $\Re(s) > \frac{1}{2}$, $(\mathcal{X}_s^a, e^{-tx}) = \Gamma(s)(Q_a(x^{-s}\mathbf{1}_{x>a}), e^{-tx}) = \Gamma(s)(x^{-s}\mathbf{1}_{x>a}, f_t^a) = \mathcal{F}_t^a(s)$ (Q_a is the orthogonal projection to K_a). This gives again (170).

To proceed further, we compute $(s+z-1)\mathcal{X}_a(s,z)$. Using integration by parts, multiplication by s (resp. z) is converted into $-t\frac{d}{dt}$ (resp. $-u\frac{d}{du}$; there are no boundary terms.)

$$s\mathcal{X}_a(s,z) = \iint_{[0,+\infty)^2}(a(t-\frac{1}{t}) + \frac{t}{t+u})e^{-a(t+\frac{1}{t}+u+\frac{1}{u})}\frac{t^{s-1}u^{z-1}}{t+u}\,dtdu \tag{171a}$$

$$z\mathcal{X}_a(s,z) = \iint_{[0,+\infty)^2}(a(u-\frac{1}{u}) + \frac{u}{t+u})e^{-a(t+\frac{1}{t}+u+\frac{1}{u})}\frac{t^{s-1}u^{z-1}}{t+u}\,dtdu \tag{171b}$$

$$
\begin{aligned}
(s+z-1)\mathcal{X}_a(s,z) &= a \iint_{[0,+\infty)^2} (t-\frac{1}{t}+u-\frac{1}{u}) e^{-a(t+\frac{1}{t}+u+\frac{1}{u})} \frac{t^{s-1}u^{z-1}}{t+u}\, dtdu \\
&= a\int_0^\infty e^{-a(t+\frac{1}{t})} t^{s-1}\, dt \int_0^\infty e^{-a(u+\frac{1}{u})} u^{z-1}\, du \\
&\quad - a\int_0^\infty e^{-a(t+\frac{1}{t})} t^{s-2}\, dt \int_0^\infty e^{-a(u+\frac{1}{u})} u^{z-2}\, du
\end{aligned} \tag{171c}
$$

The K-Bessel function is $K_s(x) = \frac{1}{2}\int_0^\infty e^{-x\frac{1}{2}(t+\frac{1}{t})} t^{s-1}\, dt = \int_0^\infty e^{-x\cosh u}\cosh(su)\, du$. It is an even function of s. It has, for each $x > 0$, all its zeros on the imaginary axis, and was used by Pólya in a famous work on functions inspired by the Riemann ξ-function and for which he proved the validity of the Riemann hypothesis [24, 25]. We have obtained the formula

$$
\mathcal{X}_a(s,z) = \frac{E(s)E(z) - E(1-s)E(1-z)}{s+z-1} \qquad E(s) = 2\sqrt{a}K_s(2a) \tag{172}
$$

To confirm $\mathcal{E}_a(s) = 2\sqrt{a}K_s(2a)$, let us define temporarily $A(s) = \frac{1}{2}\sqrt{a}\int_0^\infty e^{-a(t+\frac{1}{t})}(1+\frac{1}{t})t^{s-1}\, dt$ and $-iB(s) = \frac{1}{2}\sqrt{a}\int_0^\infty e^{-a(t+\frac{1}{t})}(1-\frac{1}{t})t^{s-1}\, dt$ which are respectively even and odd under $s \mapsto 1-s$ and are such that $E(z) = A(z) - iB(z)$. We have $\forall s,z \in \mathbb{C}\quad -iB(s)A(z) + A(s)(-iB(z)) = -i\mathcal{B}_a(s)\mathcal{A}_a(z) + \mathcal{A}_a(s)(-i\mathcal{B}_a(z))$ and considering separately the even and odd parts in z, we find that there exists a constant $k(a)$ such that $A(s) = k(a)\mathcal{A}_a(s)$ and $B(s) = k(a)^{-1}\mathcal{B}_a(s)$. Let us check that $\lim_{\sigma\to\infty} \frac{-iB(\sigma)}{A(\sigma)} = 1$. It is a corollary to $\lim_{\sigma\to\infty} K_\sigma(x)/K_{\sigma+1}(x) = 0$ which is elementary: $\int_{-\infty}^0 \exp(-x\cosh u)e^{\sigma u}\, du = O(1)$ $(\sigma \to +\infty)$, and for each $T > 0$, $\int_T^\infty \exp(-x\cosh u)e^{\sigma u}\, du \geq T\exp(-x\cosh 3T)e^{2\sigma T}$, $\int_0^T \exp(-x\cosh u)e^{\sigma u}\, du \leq Te^{\sigma T}$, and combining we get $K_\sigma(x) = (1+o(1))\frac{1}{2}\int_T^\infty \exp(-x\cosh u)e^{\sigma u}\, du$. So $\limsup_{\sigma\to+\infty} \frac{K_\sigma(x)}{K_{\sigma+1}(x)} \leq e^{-T}$ for each $T > 0$. Using (116), we then conclude $k(a) = 1$.

Let us examine the equality $\mathcal{E}_a(s) = 2\sqrt{a}K_s(2a) = \sqrt{a}\int_0^\infty \exp(-a(t+\frac{1}{t}))t^{s-1}\, dt$. It exhibits $\mathcal{E}_a$ as the left Mellin transform of $\sqrt{a}\exp(-a(t+\frac{1}{t}))$, so the distribution E_a is determined as the distribution whose Fourier transform is $\sqrt{a}\exp(i\,a(\lambda-\lambda^{-1}))$. Using τ_a and $\tau_a^\#$, this means exactly:

$$
E_a = \sqrt{a}\,\tau_a^\#\,\tau_a\delta = \sqrt{a}\,\mathcal{H}\tau_a\mathcal{H}\delta(x-a) \tag{173}
$$

We exploit the symmetry $K_s = K_{-s}$, which corresponds to $\widetilde{E_a}(\lambda) = \widetilde{E_a}(-\lambda^{-1}) = -i\lambda\widetilde{\mathcal{H}E_a}(\lambda)$, so the unexpected identity appears:

$$
E_a = \frac{d}{dx}\mathcal{H}E_a \tag{174}
$$

From (173) we read $\mathcal{H}E_a = \sqrt{a}\tau_a J_0(2\sqrt{ax}) = \sqrt{a}J_0(2\sqrt{a(x-a)})\mathbf{1}_{x>a}(x)$. Using (174), and recalling equations (96c) and (96d) we deduce:

$$
\frac{\phi_a^+(x) + \phi_a^-(x)}{2} = J_0(2\sqrt{a(x-a)}) = I_0(2\sqrt{a(a-x)} \tag{175a}
$$

$$
\frac{\phi_a^+(x) - \phi_a^-(x)}{2} = \frac{\partial}{\partial x}J_0(2\sqrt{a(x-a)}) = \frac{\partial}{\partial x}I_0(2\sqrt{a(a-x)}) \tag{175b}
$$

We knew already from equations (66a), (66b)! Summing up we have proven:

Theorem 21. *Let $\mathcal{H}$ be the self-reciprocal operator with kernel $J_0(2\sqrt{xy})$ on $L^2(0,\infty;dx)$. Let H_a be the restriction of $\mathcal{H}$ to $L^2(0,a;dx)$. The solutions to the integral equations $\phi_a^+ + H_a\phi_a^+ = J_0(2\sqrt{ax})$ and $\phi_a^- - H_a\phi_a^- = J_0(2\sqrt{ax})$ are the entire functions:*

$$\phi_a^+(x) = I_0(2\sqrt{a(a-x)}) + \frac{\partial}{\partial x} I_0(2\sqrt{a(a-x)}) \tag{176a}$$

$$\phi_a^-(x) = I_0(2\sqrt{a(a-x)}) - \frac{\partial}{\partial x} I_0(2\sqrt{a(a-x)}) \tag{176b}$$

One has $1-a = \phi_a^+(a) = \frac{d}{da}\log\det(1+H_a)$ and $1+a = \phi_a^-(a) = -\frac{d}{da}\log\det(1-H_a)$, and

$$\det(1+H_a) = e^{+a-\frac{1}{2}a^2} \tag{176c}$$

$$\det(1-H_a) = e^{-a-\frac{1}{2}a^2} \tag{176d}$$

The tempered distributions $A_a = \frac{\sqrt{a}}{2}(1+\mathcal{H})\phi_a^+$ and $-i\,B_a = \frac{\sqrt{a}}{2}(-1+\mathcal{H})\phi_a^-$, respectively invariant and anti-invariant under $\mathcal{H}$, are also given as:

$$A_a(x) = \frac{\sqrt{a}}{2}\left(\delta_a(x) + \mathbf{1}_{x>a}(x)\big(J_0(2\sqrt{a(x-a)}) + \frac{\partial}{\partial x}J_0(2\sqrt{a(x-a)})\big)\right) \tag{176e}$$

$$-iB_a(x) = \frac{\sqrt{a}}{2}\left(\delta_a(x) - \mathbf{1}_{x>a}(x)\big(J_0(2\sqrt{a(x-a)}) - \frac{\partial}{\partial x}J_0(2\sqrt{a(x-a)})\big)\right) \tag{176f}$$

Their Fourier transforms are $\int_{\mathbb{R}} e^{i\lambda x}A_a(x)\,dx = \frac{\sqrt{a}}{2}(1+\frac{i}{\lambda})\exp(ia(\lambda-\frac{1}{\lambda}))$ and $-i\int_{\mathbb{R}} e^{i\lambda x}B_a(x)\,dx = \frac{\sqrt{a}}{2}(1-\frac{i}{\lambda})\exp(ia(\lambda-\frac{1}{\lambda}))$. The Gamma completed right Mellin transforms are:

$$\Gamma(s)\widehat{A_a}(s) = \mathcal{A}_a(s) = \sqrt{a}(K_s(2a) + K_{s-1}(2a)) \tag{176g}$$

$$-i\Gamma(s)\widehat{B_a}(s) = -i\mathcal{B}_a(s) = \sqrt{a}(K_s(2a) - K_{s-1}(2a)) \tag{176h}$$

$$\mathcal{A}_a(s) - i\mathcal{B}_a(s) = \mathcal{E}_a(s) = 2\sqrt{a}\,K_s(2a) = \sqrt{a}\int_0^\infty e^{-a(t+\frac{1}{t})}t^{s-1}\,dt \tag{176i}$$

They verify the first order system, where $\mu(a) = a\phi^+(a) + a\phi^-(a) = 2a$:

$$\left(\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} a\frac{d}{da} - \begin{bmatrix}0 & \mu(a)\\ \mu(a) & 0\end{bmatrix}\right)\begin{bmatrix}\mathcal{A}_a(s)\\ \mathcal{B}_a(s)\end{bmatrix} = -i(s-\frac{1}{2})\begin{bmatrix}\mathcal{A}_a(s)\\ \mathcal{B}_a(s)\end{bmatrix} \tag{176j}$$

The pair $\left[\begin{smallmatrix}\mathcal{A}_a(s)\\ \mathcal{B}_a(s)\end{smallmatrix}\right]$ is the unique solution of the first order system which is square-integrable with respect to $d\log(a)$ at $+\infty$. The total reflection against the exponential barriers at $\log(a)\to+\infty$ of the associated Schrödinger equations realizes $+\frac{\Gamma(1-s)}{\Gamma(s)}$ and $-\frac{\Gamma(1-s)}{\Gamma(s)}$ ($\Re(s) = \frac{1}{2}$) as scattering matrices.

From (163) we have $\mathcal{A}_a(\frac{1}{2}) = \sqrt{\pi}\,e^{-2a}$. To normalize $\mathcal{A}_a$ according to $\mathcal{A}_a(\frac{1}{2}) = 1$, we would have to make the replacement $\mathcal{A}_a \to \pi^{-\frac{1}{2}}e^{2a}\mathcal{A}_a$ and $\mathcal{B}_a \to \sqrt{\pi}\,e^{-2a}\mathcal{B}_a$ and the expression of $\mathcal{E}_a$ in terms of the K-Bessel function would be less simple. Let us also note that according to (116) we must have $\frac{K_{\sigma-1}(2a)}{K_\sigma(2a)} \sim_{\sigma\to+\infty} \frac{a}{\sigma}$.

Regarding the isometric expansion, as given in theorem 15, we apply it to a function $F(s) = \int_0^\infty k(x)x^{-s}\,dx$ such that $\frac{d}{dx}x\,k(x)$ as a distribution on $\mathbb{R}$ is in L^2. Using the L^2-function $\frac{1}{s}\widehat{A_a}(s)$, which is the Mellin transform of the function $C_a(x) = \frac{1}{x}\int_0^x A_a(x)\,dx$, and the Parseval identity we obtain $\alpha(u) = 2\int_0^\infty (-x\frac{d}{dx}k(x))C_a(x)\,dx$ as an absolutely convergent integral. The square integrable function $C_a(x)$ is explicitely:

$$C_a(x) = \frac{\sqrt{a}}{2x}\mathbf{1}_{x>a}(x)\Big(J_0(2\sqrt{a(x-a)}) + \sqrt{\frac{x-a}{a}}J_1(2\sqrt{a(x-a)})\Big) \tag{177}$$

And under the hypothesis made on $x\frac{d}{dx}k(x)$ we obtain the existence of:

$$\alpha(u) = \lim_{X\to\infty} \sqrt{a}k(a) - 2Xk(X)C_a(X) + \sqrt{a}\int_a^X k(x)(1+\frac{\partial}{\partial x})J_0(2\sqrt{a(x-a)})\,dx \tag{178}$$

Let us observe that $k(x) = \int_x^\infty \frac{l(y)}{y}\,dy$ with $l(y)\in L^2$, so $|k(x)|^2 \le \frac{C}{x}$ and $Xk(X)C_a(X) = O(X^{-\frac{1}{4}})$. Hence:

$$\alpha(u) = \sqrt{a}k(a) + \sqrt{a}\int_a^{\to\infty} k(x)(1+\frac{\partial}{\partial x})J_0(2\sqrt{a(x-a)})\,dx \tag{179}$$

Comparing with equation (20a) we see that the $f(y)$ defined there is related to $\alpha(u)$, $u = \log(a)$ by the formula $f(y) = \frac{1}{2\sqrt{a}}\alpha(\log(a))$, $a = \frac{y}{2}$, so $|f(y)|^2\,dy = \frac{1}{4a}|\alpha(\log(a))|^2 2\,da = |\alpha(\log(a))|^2\frac{1}{2}d\log(a)$. Similarly we obtain $\beta(u)$:

$$\beta(u) = \sqrt{a}k(a) + \sqrt{a}\int_a^{\to\infty} k(x)(-1+\frac{\partial}{\partial x})J_0(2\sqrt{a(x-a)})\,dx \tag{180}$$

and comparing with (20b), $g(y) = \frac{1}{2\sqrt{a}}\beta(\log(a))$, $|g(y)|^2\,dy = |\beta(\log(a))|^2\frac{1}{2}d\log(a)$. So according to theorem 15 we do have equation (20d):

$$\int_0^\infty (|f(y)|^2 + |g(y)|^2)\,dy = \int_0^\infty |k(x)|^2\,dx \tag{181}$$

From 15 the assignment $k\to(\alpha,\beta)$ extends to a unitary identification $L^2(\Re(s)=\frac{1}{2};\frac{|ds|}{2\pi})\widetilde{\to}L^2(\mathbb{R}\to\mathbb{C}^2;\frac{du}{2})$, which has the property $\mathcal{H}(k)\to(\alpha,-\beta)$. In order to complete the proof of the isometric expansion, it remains to check the equation (20c) which expresses k in terms of f and g. According to 15 we recover $k(x)$ has the inverse Mellin transform of $\int_{\mathbb{R}}(\alpha(u)\,2\widehat{A_a}(s) + \beta(u)\,2(-i\widehat{B_a}(s)))\,\frac{du}{2}$. Expressing this in terms of $f(y)$ and $g(y)$, $y = 2a$, $u = \log(a)$, this means the identity of distributions, where we suppose for simplicity that $f(y)$ and $g(y)$ have compact support in $(0,+\infty)$ (as usual, this means having support away from 0 as well as ∞.):

$$\begin{aligned} k(x) &= \int_0^\infty \left(2\sqrt{\frac{y}{2}}f(y)\,2A_{\frac{y}{2}}(x) + 2\sqrt{\frac{y}{2}}g(y)\,2(-iB_{\frac{y}{2}}(x))\right)\frac{dy}{2y} \\ &= 2\int_0^\infty (\sqrt{y}f(2y)\,A_y(x) + \sqrt{y}g(2y)\,2(-iB_y(x)))\,\frac{dy}{y} \end{aligned} \tag{182}$$

Then imagining that we are integrating against a test function $\psi(x)$ and using Fubini we obtain:

$$\begin{aligned} &2\int_0^\infty \sqrt{y}f(2y)\,A_y(x)\frac{dy}{y} \\ &= \int_0^\infty f(2y)\left(\delta(x-y) + \mathbf{1}_{x>y}\Big(J_0(2\sqrt{y(x-y)}) - \sqrt{\frac{y}{x-y}}J_1(2\sqrt{y(x-y)})\Big)\right)dy \end{aligned} \tag{183}$$

$$
\begin{aligned}
&= f(2x) + \int_0^x \Big(J_0(2\sqrt{y(x-y)}) - \sqrt{\frac{y}{x-y}}J_1(2\sqrt{y(x-y)})\Big) f(y)\,dy \\
&= f(2x) + \frac{1}{2}\int_0^{2x} \Big(J_0(\sqrt{y(2x-y)}) - \sqrt{\frac{y}{2x-y}}J_1(\sqrt{y(2x-y)})\Big) f(y)\,dy
\end{aligned} \tag{184}
$$

Proceeding similarly with $g(y)$ one obtains for $2\int_0^\infty \sqrt{y}g(2y)(-i\,B_y(x))\frac{dy}{y}$:

$$
g(2x) - \frac{1}{2}\int_0^{2x} \Big(J_0(\sqrt{y(2x-y)}) + \sqrt{\frac{y}{2x-y}}J_1(\sqrt{y(2x-y)})\Big) f(y)\,dy \tag{185}
$$

Combining (183) and (185) in the formula (182) for $k(x)$ we obtain equation (20c).

8 The reproducing kernel and differential equations for the extended spaces

Let $L_a \subset L^2(0,\infty;dx)$ be the Hilbert space of square integrable functions f which are constant in $(0,a)$ and with their $\mathcal{H}$-transforms again constant in $(0,a)$. The distribution $x\frac{d}{dx}\frac{d}{dx}x\,f = \frac{d}{dx}x\,x\frac{d}{dx}f$ vanishes in $(0,a)$ and its $\mathcal{H}$ transform does too. So $s(s-1)\widehat{f}(s)$ is an entire function with trivial zeros at $-\mathbb{N}$. The Hilbert space of the functions $s(s-1)\Gamma(s)\widehat{f}(s)$ satisfies the axioms of [2]; we prove everything according to the methods developed in the earlier chapters. Our goal is to determine the evaluators and reproducing kernel for the spaces L_a.

For $f \in L_a$, $\widehat{f}(s)$ is a meromorphic function with at most a pole at $s=1$ and also $\widehat{f}(0)$ does not necessarily vanish. The Mellin-Plancherel transform $\int_0^\infty f(x)x^{-s}\,dx = \int_0^a c(f)x^{-s}\,dx + \int_a^\infty f(x)x^{-s}\,dx$ has polar part $-\frac{c(f)}{s-1}$. Let us write $(f, Y_1) = -c(f) = \mathrm{Res}(\widehat{f}(s), s=1) = s(s-1)\Gamma(s)\widehat{f}(s)|_{s=1}$. This defines an element $Y_1 \in L_a$. We define also $\mathcal{Y}_1^a = \Gamma(1)Y_1 = Y_1$. Then $(f,\mathcal{Y}_1^a) = s(s-1)\Gamma(s)\widehat{f}(s)|_{s=1}$. We also define $\mathcal{Y}_0^a$ as the vector such that $(f,\mathcal{Y}_0^a) = s(s-1)\Gamma(s)\widehat{f}(s)|_{s=0} = -\widehat{f}(0)$. One observes $(f,\mathcal{H}(Y_1)) = (\mathcal{H}(f), Y_1) = s(s-1)\Gamma(s)\widehat{\mathcal{H}(f)}(s)|_{s=1} = s(s-1)\Gamma(1-s)\widehat{f}(1-s)|_{s=1} = -\widehat{f}(0) = (f,\mathcal{Y}_0^a)$ so $\mathcal{Y}_0^a = \mathcal{H}(\mathcal{Y}_1^a)$. To lighten the notation we sometimes write $\mathcal{Y}_1$ and $\mathcal{Y}_0$ instead $\mathcal{Y}_1^a$ and $\mathcal{Y}_0^a$ when no confusion can arise.

We will also consider the vectors $X_s^\times \in L_a$ such that $\forall f \in L_a$ $\widehat{f}(s) = (X_s^\times, f)$.[30] The orthogonal projection of $X_s^\times$ to $K_a \subset L_a$ is X_s. Let us look more closely at this orthogonal projection. First let N_a be the (closed) vector space sum $L^2(0,a;dx) + \mathcal{H}L^2(0,a;dx)$. Inside N_a we have the codimension two space M_a defined as the sum of $(\mathbf{1}_{0<x<a})^\perp \cap L^2(0,a;dx)$ and of its image under $\mathcal{H}$. Finally, let R_a be the orthogonal complement in N_a of M_a, which has dimension two. For a function f to belong to L_a it is necessary and sufficient that its orthogonal projection to N_a be perpendicular to the functions in $L^2(0,a;dx)$ which are perpendicular to $\mathbf{1}_{0<x<a}$, and the same for the $\mathcal{H}$-transform, so this means exactly that its orthogonal projection to N_a belongs to R_a. So we have the orthogonal decomposition of $L^2(0,\infty;dx)$ into the sum of the three spaces K_a, R_a and M_a and $L_a = K_a \oplus R_a$. For $f \in L_a$ to be in K_a it is necessary and sufficient that $c(f) = -(f,\mathcal{Y}_1^a) = 0$ and the same for $c(\mathcal{H}(f))$, so this means that $\{\mathcal{Y}_1^a, \mathcal{Y}_0^a\}$ is a basis of R_a. The function $\mathcal{Y}_1^a$ belongs

[30] sometimes written $X_s^{a\times}$.

to $N_a = L^2(0, a; dx) + \mathcal{H}L^2(0, a; dx)$ and as such is uniquely written as $u_1 + \mathcal{H}v_1$. As $\mathcal{Y}_1^a \in L_a$ we have constants $\alpha, \beta \in \mathbb{C}$ such that:

$$u_1 + H_a v_1 = -\alpha \tag{186a}$$

$$H_a u_1 + v_1 = -\beta \tag{186b}$$

where we recall that P_a is the restriction to $(0, a)$ and $H_a = P_a \mathcal{H} P_a$, $D_a = H_a^2$. From what was said previously $\alpha = (\mathcal{Y}_1^a, \mathcal{Y}_1^a)$ and $\beta = (\mathcal{H}(\mathcal{Y}_1^a), \mathcal{Y}_1^a) = (\mathcal{Y}_0^a, \mathcal{Y}_1^a)$. We have thus:

$$u_1 = (1 - D_a)^{-1}(-\alpha \mathbf{1}_{0<x<a} + \beta H_a(\mathbf{1}_{0<x<a})) \tag{186c}$$

$$v_1 = (1 - D_a)^{-1}(+\alpha H_a(\mathbf{1}_{0<x<a}) - \beta \mathbf{1}_{0<x<a}) \tag{186d}$$

Also the function $\mathcal{Y}_1^a$ may be obtained as the orthogonal projection to L_a of $-\frac{1}{a}\mathbf{1}_{0<x<a}$. Indeed it follows from what has been seen above that for any element $f \in L_a$, $(f, \mathcal{Y}_1^a) = -\frac{1}{a}\int_0^a f(x)\,dx$. As the function $-\frac{1}{a}\mathbf{1}_{0<x<a}$ already belongs to N_a, we have

$$-\frac{1}{a}\mathbf{1}_{0<x<a} = u_1 + \mathcal{H}v_1 + u_2 + \mathcal{H}v_2 \tag{187}$$

where $u_2 + \mathcal{H}v_2$ belongs to M_a, which means that $u_2 \in L^2(0, a; dx)$ verifies $\int_0^a u_2(x)\,dx = 0$ and $v_2 \in L^2(0, a; dx)$ verifies $\int_0^a v_2(x)\,dx = 0$. But there is unicity so we have exactly

$$u_1 + u_2 = -\frac{1}{a}\mathbf{1}_{0<x<a} \qquad v_1 + v_2 = 0 \tag{188}$$

And we deduce:

$$\int_0^a u_1(x)\,dx = -1 \qquad \int_0^a v_1(x)\,dx = 0 \tag{189}$$

So α and β are determined as the solutions of the system:

$$\alpha(\mathbf{1}_{0<x<a}, (1 - D_a)^{-1}\mathbf{1}_{0<x<a}) - \beta(\mathbf{1}_{0<x<a}, (1 - D_a)^{-1}H_a\mathbf{1}_{0<x<a}) = 1 \tag{190a}$$

$$\alpha(\mathbf{1}_{0<x<a}, (1 - D_a)^{-1}H_a\mathbf{1}_{0<x<a}) - \beta(\mathbf{1}_{0<x<a}, (1 - D_a)^{-1}\mathbf{1}_{0<x<a}) = 0 \tag{190b}$$

We thus have:

Proposition 22. *Let $p(a)$ and $q(a)$ be defined as*

$$p(a) = \int_0^a (1 - D_a)^{-1}(\mathbf{1}_{0<x<a})(x)\,dx \tag{191a}$$

$$q(a) = \int_0^a (1 - D_a)^{-1}H_a(\mathbf{1}_{0<x<a})(x)\,dx \tag{191b}$$

then:

$$\begin{bmatrix} p(a) & -q(a) \\ -q(a) & p(a) \end{bmatrix} \begin{bmatrix} (\mathcal{Y}_1, \mathcal{Y}_1) & (\mathcal{Y}_1, \mathcal{Y}_0) \\ (\mathcal{Y}_0, \mathcal{Y}_1) & (\mathcal{Y}_0, \mathcal{Y}_0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{191c}$$

The evaluators $\mathcal{Y}_1^a$ and $\mathcal{Y}_0^a = \mathcal{H}(\mathcal{Y}_1^a)$ are given as $u_1 + \mathcal{H}v_1$ and $\mathcal{H}u_1 + v_1$ with:

$$u_1 = -(\mathcal{Y}_1, \mathcal{Y}_1)(1 - D_a)^{-1}(\mathbf{1}_{0<x<a}) + (\mathcal{Y}_0, \mathcal{Y}_1)(1 - D_a)^{-1}H_a(\mathbf{1}_{0<x<a}) \tag{191d}$$

$$v_1 = -(\mathcal{Y}_0, \mathcal{Y}_1)(1 - D_a)^{-1}(\mathbf{1}_{0<x<a}) + (\mathcal{Y}_0, \mathcal{Y}_0)(1 - D_a)^{-1}H_a(\mathbf{1}_{0<x<a}) \tag{191e}$$

We have introduced, for $s \neq 0, 1$, $X_s^\times$ as the evaluator $\widehat{f}(s)$ for functions in L_a. We shall write $\mathcal{X}_s^\times = \Gamma(s) X_s^\times$ and then $\mathcal{Y}_s = s(s-1)\mathcal{X}_s^\times$. This is compatible with our previous definitions of $\mathcal{Y}_1^a$ and $\mathcal{Y}_0^a$. We note that the orthogonal projection of $\mathcal{X}_s^\times$ to K_a is $\mathcal{X}_s$. So we may write

$$\mathcal{X}_s^\times = \mathcal{X}_s + \lambda(s)\mathcal{Y}_1^a + \mu(s)\mathcal{Y}_0^a \tag{192}$$

We shall also write $\mathcal{Y}_a(s,z) = \int_0^\infty \mathcal{Y}_s^a(x)\mathcal{Y}_z^a(x)\,dx = z(z-1)\Gamma(z)\widehat{\mathcal{Y}_s^a}(z)$. One has $\mathcal{H}(\mathcal{Y}_s^a) = \mathcal{Y}_{1-s}^a$ and $\mathcal{Y}_a(s,z) = \mathcal{Y}_a(1-s, 1-z) = \mathcal{Y}_a(z,s)$. Taking the scalar products with $\mathcal{Y}_1^a$ and $\mathcal{Y}_0^a$ in (192) we obtain

$$\frac{1}{s(s-1)}\mathcal{Y}_a(1,s) = \lambda(s)\alpha + \mu(s)\beta \tag{193a}$$

$$\frac{1}{s(s-1)}\mathcal{Y}_a(1,1-s) = \lambda(s)\beta + \mu(s)\alpha \tag{193b}$$

$$\lambda(s) = \frac{1}{s(s-1)}(p\mathcal{Y}_a(1,s) - q\mathcal{Y}_a(1,1-s)) \tag{193c}$$

$$\mu(s) = \frac{1}{s(s-1)}(-q\mathcal{Y}_a(1,s) + p\mathcal{Y}_a(1,1-s)) = \lambda(1-s) \tag{193d}$$

Combining with (192), this gives:

$$\mathcal{Y}_s = s(s-1)\mathcal{X}_s + \mathcal{Y}_a(1,s)(p\mathcal{Y}_1^a - q\mathcal{Y}_0^a) + \mathcal{Y}_a(1,1-s)(-q\mathcal{Y}_1^a + p\mathcal{Y}_0^a) \tag{194}$$

$$\text{Let} \quad T_a(s) = p(a)\mathcal{Y}_a(1,s) - q(a)\mathcal{Y}_a(1,1-s) \tag{195}$$

Proposition 23. *The (analytic) reproducing kernel $\mathcal{Y}_a(s,z)$ of the extended space L_a is given by each of the following expressions:*

$$s(s-1)z(z-1)\mathcal{X}_a(s,z) + \begin{bmatrix}\mathcal{Y}_a(1,s) & \mathcal{Y}_a(1,1-s)\end{bmatrix}\begin{bmatrix}p(a) & -q(a)\\ -q(a) & p(a)\end{bmatrix}\begin{bmatrix}\mathcal{Y}_a(1,z)\\ \mathcal{Y}_a(1,1-z)\end{bmatrix} \tag{196a}$$

$$= s(s-1)z(z-1)\mathcal{X}_a(s,z) + \begin{bmatrix}T_a(s) & T_a(1-s)\end{bmatrix}\begin{bmatrix}\alpha(a) & \beta(a)\\ \beta(a) & \alpha(a)\end{bmatrix}\begin{bmatrix}T_a(z)\\ T_a(1-z)\end{bmatrix} \tag{196b}$$

$$= s(s-1)z(z-1)\mathcal{X}_a(s,z) + T_a(s)\mathcal{Y}_a(1,z) + T_a(1-s)\mathcal{Y}_a(1-z) \tag{196c}$$

A very important observation, before turning to the determination of the quantities $p(a)$ and $q(a)$ shall now be made. Let L be the unitary operator:

$$L(f)(x) = f(x) - \frac{1}{x}\int_0^x f(y)\,dy \tag{197}$$

It is the operator of multiplication by $\frac{s-1}{s}$ at the level of right Mellin transforms. Obviously it converts functions constant on $(0,a)$ into functions vanishing on $(0,a)$. Let us now consider the operator

$$\mathcal{H}^\diamond = L\,\mathcal{H}\,L^{-1} = L^2\,\mathcal{H} = \mathcal{H}L^{-2} \tag{198}$$

It is a unitary, self-adjoint, self-reciprocal, scale reversing operator whose kernel is easily computed to be

$$k^\diamond(xy) = J_0(2\sqrt{xy}) - 2\frac{J_1(2\sqrt{xy})}{\sqrt{xy}} + \frac{1-J_0(2\sqrt{xy})}{xy} = \sum_{n=0}^\infty (-1)^n \frac{n^2 x^n y^n}{(n+1)!^2} \tag{199}$$

It has $L(e^{-x}) = (1+\frac{1}{x})e^{-x} - \frac{1}{x}$ as self-reciprocal function; the Mellin transform is $\frac{s-1}{s}\Gamma(1-s)$ which, multiplied by $s(s-1)$ gives $(1-s)^2\Gamma(1-s)$ which is the Mellin transform of a more convenient invariant function for $\mathcal{H}^\diamond$, the function $x(x-1)e^{-x}$. This function is the analog for $\mathcal{H}^\diamond$ of e^{-x} for $\mathcal{H}$. Let us now consider the space $L(L_a)$. It consists of the square integrable functions vanishing identically on $(0,a)$ and having $\mathcal{H}^\diamond$ transforms also identically zero on $(0,a)$. But then the entire theory applies to $\mathcal{H}^\diamond$ exactly as it did for $\mathcal{H}$, up to some minor details in the proofs where the function J_0 was really used like in Lemma 9 or proposition 11. We have for $\mathcal{H}^\diamond$ versions of all quantities previously considered for $\mathcal{H}$. To check that the proof of 11 may be adapted, we need to look at $k_1^\diamond(x) = \int_0^x k^\diamond(t)\,dt = \sqrt{x}J_1(2\sqrt{x}) + 2(J_0(2\sqrt{x})-1) + 2\int_0^{2\sqrt{x}} \frac{1-J_0(u)}{u}\,du = O_{x\to+\infty}(x^{\frac{1}{4}})$. So we may employ lemma 8 as was done for $\mathcal{H}$. Proposition 11 and Theorem 12 thus hold. We must be careful that the operator L^{-1} is always involved when comparing functions or distributions related to $\mathcal{H}^\diamond$ with those related to $\mathcal{H}$. For example, one has $X_s^\times = \frac{s}{s-1}L^{-1}X_s^\diamond$ and $\mathcal{Y}_s = s(s-1)\mathcal{X}_s^\times = L^{-1}\mathcal{X}_s^\diamond$. The two types of Gamma completed Mellin transforms differ: for $\mathcal{H}$ we consider $\Gamma(s)\widehat{f}(s)$ while for $\mathcal{H}^\diamond$ we consider $s^2\Gamma(s)\widehat{g}(s)$. Indeed this is quite the coherent thing to do in order that:

$$s^2\Gamma(s)\widehat{g}(s) = s(s-1)\Gamma(s)\widehat{f}(s) \tag{200}$$

for $g = L(f)$. The bare Mellin transforms of elements of spaces $K_a^\diamond$ are not always entire in the complex plane: they may have a pole at $s=0$. After multiplying by $s^2\Gamma(s)$ which is the *left* Mellin transform of the self-invariant function $x(x-1)e^{-x}$, as $\Gamma(s)$ is the *left* Mellin transform of e^{-x}, we do obtain entire functions, whose trivial zeros are at -1, -2, $\dots$ (0 is not a trivial zero anymore.) From equation (200) we see that the (analytic) reproducing kernel $\mathcal{X}_a^\diamond(s,z)$ exactly coincides with the function $\mathcal{Y}_a(s,z)$ whose initial computation has been given in Proposition 23. Also the Schrödinger equations will realize $\pm\left(\frac{1-s}{s}\right)^2\frac{\Gamma(1-s)}{\Gamma(s)}$ as scattering matrices, and there will be an isometric expansion generalizing the de Branges-Rovnyak expansion to the spaces L_a. We will determine exactly the functions $\mathcal{A}_a^\diamond(s)$, $\mathcal{B}_a^\diamond(s)$, $\mathcal{E}_a^\diamond(s)$ and especially the function $\mu^\diamond(a)$. It will be seen that this is a more complicated function than the simple-minded $\mu(a) = 2a\dots$

The key now is to obtain the functions $p(a)$ and $q(a)$ defined in Proposition 22, and the function $\mathcal{Y}_a(1,s)$. It turns out that their computation also involves the quantities (we recall that $J_0^a(x) = J_0(2\sqrt{ax})$):

$$r(a) = 1 + \int_0^a ((1-D_a)^{-1}H_a \cdot J_0^a)(x)\,dx \tag{201a}$$

$$s(a) = \int_0^a ((1-D_a)^{-1} \cdot J_0^a)(x)\,dx \tag{201b}$$

In order to compute r, s, p, q we shall need the already defined functions ϕ_a^+ $(= (1+H_a)^{-1}J_0^a$ on

$(0,a))$, ϕ_a^-, $(= (1-H_a)^{-1}J_0^a)$ as well as the entire functions ψ_a^+ and ψ_a^- verifying:

$$\psi_a^+ + \mathcal{H}P_a\psi_a^+ = 1 \tag{202a}$$

$$\psi_a^- - \mathcal{H}P_a\psi_a^- = 1 \tag{202b}$$

We have $r(a) = 1 + \frac{1}{2}\int_0^a (-\phi_a^+(x) + \phi_a^-(x))\,dx$, and we know explicitely $\phi_a^\pm$. But, we shall proceed in a more general manner. First we recall the differential equations (121a), (121b) which are verified by $\phi_a^\pm$ (where $\delta_x = x\frac{\partial}{\partial x} + \frac{1}{2}$):

$$a\frac{\partial}{\partial a}\phi_a^+ = +\delta_x\phi_a^- - (\mu(a) + \frac{1}{2})\phi_a^+ \tag{203a}$$

$$a\frac{\partial}{\partial a}\phi_a^- = +\delta_x\phi_a^+ + (\mu(a) - \frac{1}{2})\phi_a^- \tag{203b}$$

We compute $ar'(a) = a\frac{-\phi_a^+(a)+\phi_a^-(a)}{2} + \frac{1}{2}\int_0^a (x\frac{\partial}{\partial x}+1)(\phi_a^+(x) - \phi_a^-(x)) + \mu(a)(\phi_a^+(x) + \phi_a^-(x))\,dx$ and simplifying this gives exactly $ar'(a) = \mu(a)\frac{1}{2}\int_0^a (\phi_a^+(x) + \phi_a^-(x))\,dx = s(a)$. Similarly starting with $s(a) = \frac{1}{2}\int_0^a (\phi_a^+(x)+\phi_a^-(x))\,dx$ we obtain $as'(a) = \frac{1}{2}\mu(a) + \frac{1}{2}\int_0^a (x\frac{\partial}{\partial x}(\phi_a^+(x)+\phi_a^-(x)) + \mu(a)(-\phi_a^+(x)+\phi_a^-(x)))\,dx$ which gives $\mu(a)r(a) - s(a)$. So the quantities r and s verify the system:

$$ar'(a) = \mu(a)s(a) \tag{204a}$$

$$(as)'(a) = \mu(a)r(a) \tag{204b}$$

Either solving the system taking into account the behavior as $a \to 0$ or using the explicit formulas for $\phi_a^\pm$ we obtain in this specific instance of the study of $\mathcal{H}$, for which $\mu(a) = 2a$, that $r(a) = I_0(2a)$ and $s(a) = I_1(2a)$.

From (202a) and (202b) we obtain two types of differential equations, either involving $x\frac{\partial}{\partial x}$ or $a\frac{\partial}{\partial a}$. From $\psi_a^+(x) + \int_0^a J_0(2\sqrt{xy})\psi_a^+(x)\,dx = 1$, we obtain $(1+\mathcal{H}P_a)a\frac{\partial}{\partial a}\psi_a^+(x) = -a\psi_a^+(a)J_0^a$. We do similarly with ψ_a^- and deduce:

$$a\frac{\partial}{\partial a}\psi_a^+(x) = -a\psi_a^+(a)\phi_a^+(x) \tag{205a}$$

$$a\frac{\partial}{\partial a}\psi_a^-(x) = +a\psi_a^-(a)\phi_a^-(x) \tag{205b}$$

Regarding the differential equations with $x\frac{\partial}{\partial x}$, which we shall actually not use, the computation is done using only the fact that the kernel is a function of xy so $x\frac{\partial}{\partial x}J_0(2\sqrt{xy}) = y\frac{\partial}{\partial y}J_0(2\sqrt{xy})$. We only state the result:

$$(x\frac{\partial}{\partial x} + \frac{1}{2})\psi_a^+(x) = \frac{1}{2}\psi_a^-(x) - a\psi_a^+(a)\phi_a^-(x) \tag{206a}$$

$$(x\frac{\partial}{\partial x} + \frac{1}{2})\psi_a^-(x) = \frac{1}{2}\psi_a^+(x) + a\psi_a^-(a)\phi_a^+(x) \tag{206b}$$

Let us now turn to the quantities $p(a)$ and $q(a)$. We have $p(a) = \int_0^a (1-D_a)^{-1}(\mathbf{1}_{0<x<a})(x)\,dx = \frac{1}{2}\int_0^a (\psi_a^+(x)+\psi_a^-(x))\,dx$. So $p'(a) = \frac{1}{2}(\psi_a^+(a)+\psi_a^-(a)) - \frac{1}{2}\psi_a^+(a)\int_0^a \phi_a^+(x)\,dx + \frac{1}{2}\psi_a^-(a)\int_0^a \phi_a^-(x)\,dx$. Reorganizing this gives:

$$p'(a) = \frac{\psi_a^+(a)+\psi_a^-(a)}{2}(1 + \int_0^a \frac{-\phi_a^+(x)+\phi_a^-(x)}{2}\,dx) + \frac{-\psi_a^+(a)+\psi_a^-(a)}{2}\int_0^a \frac{+\phi_a^+(x)+\phi_a^-(x)}{2}\,dx \tag{207}$$

We remark that from the integral equations defining $\psi_a^\pm$ we have $\psi_a^+(a) = 1-\int_0^a J_0(2\sqrt{ax})\psi_a^+(x)\,dx = 1-\int_0^a \phi_a^+(x)\,dx$ and $\psi_a^-(a) = 1+\int_0^a J_0(2\sqrt{ax})\psi_a^-(x)\,dx = 1+\int_0^a \phi_a^-(x)\,dx$. So $\frac{\psi_a^+(a)+\psi_a^-(a)}{2} = r(a)$ and $\frac{-\psi_a^+(a)+\psi_a^-(a)}{2} = s(a)$. Hence the quantity $p(a)$ verifies:

$$p'(a) = r(a)^2 + s(a)^2 \tag{208}$$

With exactly the same method one obtains:

$$q'(a) = 2r(a)s(a) \tag{209}$$

Let us observe that $q(a) = \frac{1}{2}\int_0^a((1-H_a)^{-1} - (1+H_a)^{-1})(1)\,dx = O(a^2)$ and $p(a) = \frac{1}{2}\int_0^a((1+H_a)^{-1} + (1-H_a)^{-1})(1)\,dx \sim_{a\to 0} a$. So $(p\pm q)\sim_{a\to 0} a$. Also $r(a)\sim_{a\to 0} 1$ and $s(a)\sim_{a\to 0} a$. The equation for $p(a)$ can be integrated:

$$p(a) = a(r(a)^2 - s(a)^2) \tag{210}$$

Indeed this has the correct derivative. Regarding $q(a)$ the situation is different, one has $q' = 2rs = \frac{2a}{\mu}rr'$ so in the special case considered here, and only in that case we have $q(a) = \frac{1}{2}(r(a)^2 - 1)$. Summing up:

Proposition 24. *The quantities $r(a)$, $s(a)$, $p(a)$ and $q(a)$ verify the differential equations $ar'(a) = \mu(a)s(a)$, $as'(a)+s(a) = \mu(a)r(a)$, $p'(a) = r(a)^2+s(a)^2$, $q'(a) = 2r(a)s(a)$, $p(a) = a(r(a)^2-s(a)^2)$. In the special case of the $\mathcal{H}$ transform one has:*

$$r(a) = I_0(2a) \tag{211a}$$
$$s(a) = I_1(2a) \tag{211b}$$
$$p(a) = a(I_0^2(2a) - I_1^2(2a)) \tag{211c}$$
$$q(a) = \frac{1}{2}(I_0^2(2a) - 1) \tag{211d}$$

We now need to determine $\mathcal{Y}_a(1,s) = s(s-1)\Gamma(s)\widehat{Y_1}(s)$. There holds $Y_1 = u_1 + \mathcal{H}v_1 = -\alpha\mathbf{1}_{0<x<a} + \mathbf{1}_{x>a}\mathcal{H}v_1$. So $\widehat{Y_1}(s) = \alpha\frac{a^{1-s}}{s-1} + \int_a^\infty(\mathcal{H}v_1)(x)x^{-s}\,dx$. Then $\int_a^\infty(\mathcal{H}v_1)(x)x^{-s}\,dx = \int_0^\infty v_1(x)g_s(x)\,dx = \int_0^a v_1(x)g_s(x)\,dx$, where the function g_s from (73) has been used. Recalling from (76a), (76b) the analytic functions u_s, equal to $-(1-D_a)^{-1}H_a(g_s)$ on $(0,a)$, and v_s, equal to $(1-D_a)^{-1}P_a(g_s)$ on $(0,a)$, and using (186d) and self-adjointness we obtain

$$\int_0^a v_1(x)g_s(x)\,dx = -\alpha\int_0^a u_s(x)\,dx - \beta\int_0^a v_s(x)\,dx \tag{212}$$

Let us now recall that we computed ((83a)) $(x\frac{\partial}{\partial x}+s)u_s$ and found it to be on the interval $(0,a)$ given as $-av_s(a)(1-D_a)^{-1}(J_0^a) - a(a^{-s}+u_s(a))(1-D_a)^{-1}H_a(J_0^a)$. Integrating and also using equations (99) and (102) we obtain

$$\sqrt{a}\widehat{E_a}(s) - a^{1-s} + (s-1)\int_0^a u_s(x)\,dx = -\sqrt{a}\widehat{\mathcal{H}(E_a)}(s)s(a) - \sqrt{a}\widehat{E_a}(s)(r(a)-1) \tag{213}$$

$$\int_0^a u_s(x)\,dx = \sqrt{a}\,\frac{a^{\frac{1}{2}-s} - \widehat{E_a}(s)r(a) - \widehat{\mathcal{H}(E_a)}(s)s(a)}{s-1} \tag{214}$$

We have similarly ((83b)) $(x\frac{\partial}{\partial x}+1-s)v_s = -\sqrt{a}\widehat{E_a}(s)(1-D_a)^{-1}(J_0^a) - \sqrt{a}\widehat{\mathcal{H}(E_a)}(s)(1-D_a)^{-1}H_a(J_0^a)$ so integration gives $av_s(a) - s\int_0^a v_s(x)\,dx = -\sqrt{a}\widehat{E_a}(s)s(a) - \sqrt{a}\widehat{\mathcal{H}(E_a)}(s)(r(a)-1)$ hence

$$\int_0^a v_s(x)\,dx = \sqrt{a}\,\frac{\widehat{E_a}(s)s(a) + \widehat{\mathcal{H}(E_a)}(s)r(a)}{s} \tag{215}$$

Combining (214), (215) with (212), and using $\mathcal{Y}_a(1,s) = s(s-1)\Gamma(s)\widehat{Y_1^a}(s)$:

$$\widehat{Y_1}(s) = \sqrt{a}\,\widehat{E_a}(s)\Big(\frac{\alpha(a)r(a)}{s-1} - \frac{\beta(a)s(a)}{s}\Big) + \sqrt{a}\,\widehat{\mathcal{H}(E_a)}(s)\Big(\frac{\alpha(a)s(a)}{s-1} - \frac{\beta(a)r(a)}{s}\Big) \tag{216a}$$

$$\mathcal{Y}_a(1,s) = \sqrt{a}\,\Big(s\alpha(a)(\mathcal{E}_a(s)r(a) + \mathcal{E}_a(1-s)s(a)) + (1-s)\beta(a)(\mathcal{E}_a(s)s(a) + \mathcal{E}_a(1-s)r(a))\Big) \tag{216b}$$

Proposition 25. *The functions $\mathcal{Y}_a(1,s)$ and $\mathcal{Y}_a(1,1-s)$ verify*

$$\begin{bmatrix}\mathcal{Y}_a(1,s)\\ \mathcal{Y}_a(1,1-s)\end{bmatrix} = \sqrt{a}\begin{bmatrix}\alpha(a) & \beta(a)\\ \beta(a) & \alpha(a)\end{bmatrix}\begin{bmatrix} s(\mathcal{E}_a(s)r(a) + \mathcal{E}_a(1-s)s(a))\\ (1-s)(\mathcal{E}_a(s)s(a) + \mathcal{E}_a(1-s)r(a))\end{bmatrix} \tag{217}$$

Comparing with equation (195) we get: $T_a(s) = \sqrt{a}s(\mathcal{E}_a(s)r(a) + \mathcal{E}_a(1-s)s(a))$. So:

Theorem 26. *The analytic reproducing kernel $\mathcal{Y}_a(s,z)$ associated with the extended spaces L_a is:*

$$\mathcal{Y}_a(s,z) = s(s-1)z(z-1)\mathcal{X}_a(s,z) + \begin{bmatrix}T_a(s) & T_a(1-s)\end{bmatrix}\begin{bmatrix}\alpha(a) & \beta(a)\\ \beta(a) & \alpha(a)\end{bmatrix}\begin{bmatrix}T_a(z)\\ T_a(1-z)\end{bmatrix} \tag{218a}$$

$$\mathcal{X}_a(s,z) = \frac{\mathcal{E}_a(s)\mathcal{E}_a(z) - \mathcal{E}_a(1-s)\mathcal{E}_a(1-z)}{s+z-1} \qquad \mathcal{E}_a(s) = 2\sqrt{a}K_s(2a) \tag{218b}$$

$$T_a(s) = \sqrt{a}\,s\,(\mathcal{E}_a(s)r(a) + \mathcal{E}_a(1-s)s(a)) \qquad r(a) = I_0(2a) \quad s(a) = I_1(2a) \tag{218c}$$

$$\alpha(a) = \frac{p(a)}{p(a)^2 - q(a)^2} \qquad p(a) = a(I_0^2(2a) - I_1^2(2a)) \tag{218d}$$

$$\beta(a) = \frac{q(a)}{p(a)^2 - q(a)^2} \qquad q(a) = \frac{1}{2}(I_0^2(2a) - 1) \tag{218e}$$

We proceed now to the determination of $\mathcal{A}_a^\diamond$, $\mathcal{B}_a^\diamond$ and $\mathcal{E}_a^\diamond = \mathcal{A}_a^\diamond(s) - i\mathcal{B}_a^\diamond(s)$. The function $\mathcal{A}_a^\diamond(s)$ is even under $s \to 1-s$ and $\mathcal{B}_a^\diamond(s)$ is odd. We must have:

$$z\mathcal{Y}_a(1,z) = 2(-i\mathcal{B}_a^\diamond(1))\mathcal{A}_a^\diamond(z) + 2\mathcal{A}_a^\diamond(1)(-i\mathcal{B}_a^\diamond(z)) \tag{219}$$

On the other hand from (195) we have $\mathcal{Y}_a(1,z) = \alpha T_a(z) + \beta T_a(1-z)$. Let us write

$$\begin{aligned} z\,T_a(z) = \sqrt{a}(&z(z-1)r(a)\mathcal{E}_a(z) + zr(a)\mathcal{E}_a(z)\\ &+ z(z-1)s(a)\mathcal{E}_a(1-z) + zs(a)\mathcal{E}_a(1-z)) \end{aligned} \tag{220}$$

$$z\,T_a(1-z) = \sqrt{a}(-z(z-1)s(a)\mathcal{E}_a(z) - z(z-1)r(a)\mathcal{E}_a(1-z)) \tag{221}$$

$$z\mathcal{Y}_a(1,z) = \sqrt{a}\Big(z(z-1)((\alpha r-\beta s)\mathcal{E}_a(z)+(\alpha s-\beta r)\mathcal{E}_a(1-z))$$
$$+ z\alpha(r\,\mathcal{E}_a(z)+s\mathcal{E}_a(1-z))\Big) \tag{222}$$

Extracting the even part $(z\mathcal{Y}_a(1,z))^+$ and the odd part $(z\mathcal{Y}_a(1,z))^-$:

$$(z\mathcal{Y}_a(1,z))^+ = \sqrt{a}\Big(z(z-1)(\alpha-\beta)(r+s)\mathcal{A}_a + (z-\frac{1}{2})\alpha(r-s)(-i\mathcal{B}_a) + \frac{1}{2}\alpha(r+s)\mathcal{A}_a\Big) \tag{223}$$

$$(z\mathcal{Y}_a(1,z))^- = \sqrt{a}\Big(z(z-1)(\alpha+\beta)(r-s)(-i\mathcal{B}_a) + (z-\frac{1}{2})\alpha(r+s)\mathcal{A}_a + \frac{1}{2}\alpha(r-s)(-i\mathcal{B}_a)\Big) \tag{224}$$

We have $(z\mathcal{Y}_a(1,z))^+ = 2(-i\mathcal{B}_a^\diamond(1))\mathcal{A}_a^\diamond(z)$ and $(z\mathcal{Y}_a(1,z))^- = 2\mathcal{A}_a^\diamond(1)(-i\mathcal{B}_a^\diamond(z))$. Let us define $K(a) = (2(-i\mathcal{B}_a^\diamond(1)))^{-1}$ and $L(a) = (2\mathcal{A}_a^\diamond(1))^{-1}$. We know that:

$$\lim_{\sigma\to+\infty} \frac{-i\mathcal{B}_a^\diamond(\sigma)}{\mathcal{A}_a^\diamond(\sigma)} = 1 \tag{225}$$

So it must be that

$$K(a)(\alpha-\beta)(r+s) = L(a)(\alpha+\beta)(r-s) \tag{226}$$

Also, taking $z=1$ in (223) we have $\frac{1}{KL} = 2\sqrt{a}\frac{1}{2}\alpha\,(r\mathcal{E}_a(1)+s\mathcal{E}_a(0)) = \alpha T_a(1)$. But referring to (195) one has $T_a(1) = p\alpha - q\beta = 1$. So:

$$K(a)L(a) = \frac{1}{\alpha(a)} \tag{227}$$

Then:

$$K(a)^2 = \frac{1}{\alpha(a)}\frac{(\alpha+\beta)(r-s)}{(\alpha-\beta)(r+s)} = \frac{\alpha^2-\beta^2}{\alpha(a)}\frac{r^2-s^2}{(\alpha-\beta)^2(r+s)^2} = \frac{1}{p}\frac{p}{a}\frac{1}{(\alpha-\beta)^2(r+s)^2} \tag{228}$$

$$(\alpha-\beta)(r+s)K(a) = a^{-\frac{1}{2}} \tag{229}$$

We conclude:

$$\mathcal{A}_a^\diamond = z(z-1)\mathcal{A}_a + (z-\frac{1}{2})\frac{\alpha(r-s)}{(\alpha-\beta)(r+s)}(-i\mathcal{B}_a) + \frac{\alpha}{2(\alpha-\beta)}\mathcal{A}_a \tag{230a}$$

$$-i\mathcal{B}_a^\diamond = z(z-1)(-i\mathcal{B}_a) + (z-\frac{1}{2})\frac{\alpha(r+s)}{(\alpha+\beta)(r-s)}\mathcal{A}_a + \frac{\alpha}{2(\alpha+\beta)}(-i\mathcal{B}_a) \tag{230b}$$

Let us now observe that $\frac{\alpha}{\alpha\pm\beta} = \frac{p}{p\pm q}$ and further:

$$\frac{\alpha}{\alpha-\beta}\frac{r-s}{r+s} = \frac{p}{p-q}\frac{a(r-s)^2}{p} = a\frac{p'-q'}{p-q} = a\frac{d}{da}\log(p-q) \tag{231a}$$

$$\frac{\alpha}{\alpha+\beta}\frac{r+s}{r-s} = \frac{p}{p+q}\frac{a(r+s)^2}{p} = a\frac{p'+q'}{p+q} = a\frac{d}{da}\log(p+q) \tag{231b}$$

$$\mathcal{A}_a^\diamond(z) = (z(z-1)+\frac{1}{2}\frac{p}{p-q})\mathcal{A}_a(z) + a\frac{d}{da}\log(p-q)(z-\frac{1}{2})(-i\mathcal{B}_a(z)) \tag{232a}$$

$$-i\mathcal{B}_a^\diamond(z) = (z(z-1)+\frac{1}{2}\frac{p}{p+q})(-i\mathcal{B}_a(z)) + a\frac{d}{da}\log(p+q)(z-\frac{1}{2})\mathcal{A}_a(z) \tag{232b}$$

Combining we get finally:

Theorem 27. *The E function associated with the entire functions* $s(s-1)\Gamma(s)\widehat{f}(s)$*,* $f \in L_a$ *is:*

$$\begin{aligned}\mathcal{E}^\diamond_a(z) = \Big(z(z-1) + \frac{1}{2}a\frac{d}{da}\log(p(a)^2 - q(a)^2)(z-\frac{1}{2}) + \frac{1}{2}p(a)\alpha(a)\Big)\mathcal{E}_a(z)\\ + \Big(\frac{1}{2}a\frac{d}{da}\log\frac{p(a)+q(a)}{p(a)-q(a)}(z-\frac{1}{2}) + \frac{1}{2}p(a)\beta(a)\Big)\mathcal{E}_a(1-z)\end{aligned} \tag{233}$$

where $p(a) = a(I_0^2(2a) - I_1^2(2a))$*,* $q(a) = \frac{1}{2}(I_0^2(2a) - 1)$*,* $\alpha(a) = \frac{p(a)}{p(a)^2 - q(a)^2}$*,* $\beta(a) = \frac{q(a)}{p(a)^2 - q(a)^2}$*, and* $\mathcal{E}_a(z) = 2\sqrt{a}K_z(2a)$*.*

We shall now obtain by two methods the function $\mu^\diamond(a)$. First, we compute $\mathcal{E}^\diamond_a(\frac{1}{2}) = (-\frac{1}{4} + \frac{1}{2}p(\alpha+\beta))\mathcal{E}_a(\frac{1}{2}) = \frac{1}{4}\frac{p+q}{p-q}\mathcal{E}_a(\frac{1}{2})$ and invoke $a\frac{d}{da}\mathcal{E}^\diamond_a(\frac{1}{2}) = -\mu^\diamond(a)\mathcal{E}^\diamond_a(\frac{1}{2})$. We thus have:

Theorem 28. *The mu function for the chain of spaces* L_a*,* $0 < a < \infty$ *is*

$$\mu^\diamond(a) = \mu(a) + a\frac{d}{da}\log\frac{p-q}{p+q} \tag{234}$$

$$= 2a + a\frac{d}{da}\log\frac{(2a-1)I_0^2(2a) - 2aI_1^2(2a) + 1}{(2a+1)I_0^2(2a) - 2aI_1^2(2a) - 1} \tag{235}$$

$$= 2a - 2 + o(1) \qquad (a \to +\infty) \tag{236}$$

The asymptotic behavior is a corollary to $\lim_{a\to\infty} 2a\frac{-pq'+qp'}{p^2-q^2} = -2$ which itself follows from $p^2 - q^2 \sim \frac{1}{4}I_0(2a)^4\frac{1}{16a^2}$ and $(\frac{q}{p})' \sim +\frac{1}{16a^3}$ which are easily deduced from the asymptotic expansion $I_0(x) = \frac{e^x}{\sqrt{2\pi x}}(1 + \frac{1}{8x} + \frac{9}{128x^2} + \dots)$ ([33]). Of course the $o(1)$ is in fact an $O(a^{-1})$.

The second method to obtain $\mu^\diamond(a)$ relies on $\frac{\mathcal{E}^\diamond_a(1-\sigma)}{\mathcal{E}^\diamond_a(\sigma)} \sim_{\sigma\to+\infty} \frac{\mu^\diamond(a)}{2\sigma}$ ((116)). We have:

$$\frac{\mathcal{E}^\diamond_a(\sigma)}{\sigma^2\mathcal{E}_a(\sigma)} = 1 + \frac{\frac{1}{2}a\frac{d}{da}\log(p^2-q^2) - 1}{\sigma} + O(\frac{1}{\sigma^2}) \tag{237a}$$

$$\frac{\mathcal{E}^\diamond_a(1-\sigma)}{\sigma^2\mathcal{E}_a(1-\sigma)} \to_{\sigma\to\infty} 1 - \frac{1}{2}(a\frac{d}{da}\log\frac{p+q}{p-q})\frac{2}{\mu(a)} \tag{237b}$$

so $\frac{\mu^\diamond(a)}{\mu(a)} = 1 - \frac{1}{\mu(a)}a\frac{d}{da}\log\frac{p+q}{p-q}$ and (234) is confirmed. We can use this method to gather more information. From (115a) we have, as $\Re(s) \to +\infty$:

$$\widehat{E_a}(s) = a^{\frac{1}{2}-s}(1 + \frac{a\phi^+(a) - a\phi^-(a)}{2s} + O(\frac{1}{s^2})) \tag{238a}$$

$$\widehat{E^\diamond_a}(s) = a^{\frac{1}{2}-s}(1 + \frac{a\phi^{\diamond+}(a) - a\phi^{\diamond-}(a)}{2s} + O(\frac{1}{s^2})) \tag{238b}$$

Let us be careful that $\mathcal{E}_a(s) = \Gamma(s)\widehat{E_a}(s)$ while $\mathcal{E}^\diamond_a(s) = s^2\Gamma(s)\widehat{E^\diamond_a}(s)$. We obtain:

$$a\phi^{\diamond+}(a) - a\phi^{\diamond-}(a) = a\phi^+(a) - a\phi^-(a) + a\frac{d}{da}\log(p^2-q^2) - 2 \tag{239a}$$

$$a\phi^{\diamond+}(a) = a\phi^+(a) + a\frac{d}{da}\log\frac{p-q}{a} \tag{239b}$$

$$a\phi^{\diamond-}(a) = a\phi^-(a) - a\frac{d}{da}\log\frac{p+q}{a} \tag{239c}$$

We recall that $(p\pm q)\sim_{a\to 0} a$. We integrate (239b) and (239c) using (130a), (130b) and this gives $\det(1+H_a^\diamond)=\frac{p-q}{a}\det(1+H_a)$ and $\det(1-H_a^\diamond)=\frac{p+q}{a}\det(1-H_a)$.

$$\det(1+H_a^\diamond)=\frac{p-q}{a}\det(1+H_a)=\det(1+H_a)\ \frac{1}{a}\int_0^a (r-s)^2\,da \tag{240a}$$

$$\det(1-H_a^\diamond)=\frac{p+q}{a}\det(1-H_a)=\det(1-H_a)\ \frac{1}{a}\int_0^a (r+s)^2\,da \tag{240b}$$

Theorem 29. *Let $\mathcal{H}^\diamond = L\,\mathcal{H}\,L^{-1}$ be the self-reciprocal operator on $L^2(0,\infty;dx)$ with kernel:*

$$J_0(2\sqrt{xy})-2\frac{J_1(2\sqrt{xy})}{\sqrt{xy}}+\frac{1-J_0(2\sqrt{xy})}{xy}=\sum_{n=0}^\infty(-1)^n\frac{n^2x^ny^n}{(n+1)!^2} \tag{241a}$$

and let $H_a^\diamond$ be the restriction to $L^2(0,a;dx)$. Then:

$$\det(1+H_a^\diamond)=e^{+a-\frac12 a^2}\ \frac{1}{a}\int_0^a (I_0(2a)-I_1(2a))^2\,da = e^{+a-\frac12 a^2}\big(I_0^2(2a)-I_1^2(2a)-\frac{I_0^2(2a)-1}{2a}\big) \tag{241b}$$

$$\det(1-H_a^\diamond)=e^{-a-\frac12 a^2}\ \frac{1}{a}\int_0^a (I_0(2a)+I_1(2a))^2\,da = e^{-a-\frac12 a^2}\big(I_0^2(2a)-I_1^2(2a)+\frac{I_0^2(2a)-1}{2a}\big) \tag{241c}$$

From theorem 26 $\|\mathcal{Y}_{\frac12}\|^2=\frac{1}{16}\|\mathcal{X}_{\frac12}\|^2+2(\alpha+\beta)T_a(\frac12)^2$ and $T_a(\frac12)=\frac12\sqrt{a}\,(r+s)\mathcal{E}_a(\frac12)$. Also, $(\alpha+\beta)(r+s)^2=\frac{1}{p-q}(r+s)^2$. Furthermore $\|\mathcal{Y}_{\frac12}\|^2=\frac{1}{16}\Gamma(\frac12)^2\|X_{\frac12}^\diamond\|^2=\frac{\pi}{16}\|X_{\frac12}^\diamond\|^2$ and $\|\mathcal{X}_{\frac12}\|^2=\pi\|X_{\frac12}\|^2$. And also from (163) $\mathcal{E}_a(\frac12)=\sqrt{\pi}\ \frac{\det(1-H_a)}{\det(1+H_a)}$ and from theorem 17 one has $\|X_{\frac12}^a\|^2=2\int_a^\infty\left(\det\frac{1-H_b}{1+H_b}\right)^2\frac{db}{b}$ and the analog holds for $X_{\frac12}^{\diamond a}$. Let us observe that $X_{\frac12}^{\diamond a}=-LX_{\frac12}^\times$ so $\|X_{\frac12}^{\diamond a}\|=\|X_{\frac12}^{a\times}\|$. So

$$\|X_{\frac12}^{a\times}\|^2=2\int_a^\infty\left(\det\frac{1-H_b^\diamond}{1+H_b^\diamond}\right)^2\frac{db}{b}=2\int_a^\infty\left(\det\frac{1-H_b}{1+H_b}\right)^2\frac{db}{b}+8a\,\frac{p'+q'}{p-q}\left(\det\frac{1-H_a}{1+H_a}\right)^2 \tag{242}$$

Theorem 30. *Let L_a be the Hilbert space of square integrable functions on $f\in L^2(0,\infty;dx)$ such that both f and $\mathcal{H}(f)=\int_0^\infty J_0(2\sqrt{xy})f(y)\,dy$ are constant on $(0,a)$. Then the squared norm of the linear form $f\mapsto\int_0^\infty\frac{f(x)}{\sqrt{x}}\,dx$ is given by either one of the following two expressions:*

$$2\int_a^\infty\left(\frac{(2b+1)I_0^2(2b)-2bI_1^2(2b)-1}{(2b-1)I_0^2(2b)-2bI_1^2(2b)+1}\right)^2\ \frac{e^{-4b}}{b}\,db \tag{243a}$$

$$=2\int_a^\infty\frac{e^{-4b}}{b}\,db+2\frac{8a(I_0(2a)+I_1(2a))^2}{(2a-1)I_0^2(2a)-2aI_1^2(2a)+1}e^{-4a} \tag{243b}$$

The squared norm of the restriction of the linear form to the subspace K_a of functions vanishing on $(0,a)$ and with $\mathcal{H}(f)$ also vanishing on $(0,a)$ is $2\int_a^\infty\frac{e^{-4b}}{b}\,db$.

One may express the wish to verify explicitely from equations (230a) and (230b), or in the equivalent form

$$\mathcal{A}_a^\diamond(z) = \Big((z-\frac{1}{2})^2 + \frac{1}{4}\frac{p+q}{p-q}\Big)\mathcal{A}_a(z) + (a\frac{d}{da}\log(p-q))(z-\frac{1}{2})(-i\mathcal{B}_a(z)) \tag{244a}$$

$$-i\mathcal{B}_a^\diamond(z) = \Big((z-\frac{1}{2})^2 + \frac{1}{4}\frac{p-q}{p+q}\Big)(-i\mathcal{B}_a(z)) + (a\frac{d}{da}\log(p+q))(z-\frac{1}{2})\mathcal{A}_a(z) \tag{244b}$$

the differential system:

$$a\frac{\partial}{\partial a}\mathcal{A}_a^\diamond(z) = -\mu^\diamond(a)\mathcal{A}_a^\diamond(z) - (z-\frac{1}{2})(-i\mathcal{B}_a^\diamond(z)) \tag{245a}$$

$$a\frac{\partial}{\partial a}(-i\mathcal{B}_a^\diamond(z)) = +\mu^\diamond(a)(-i\mathcal{B}_a^\diamond(z)) - (z-\frac{1}{2})\mathcal{A}_a^\diamond(z) \tag{245b}$$

and also to verify explicitely the reproducing kernel formula

$$\mathcal{Y}_a(s,z) = \frac{\mathcal{E}_a^\diamond(s)\mathcal{E}_a^\diamond(z) - \mathcal{E}_a^\diamond(1-s)\mathcal{E}_a^\diamond(1-z)}{s+z-1} \tag{246}$$

The interested reader will see that the algebra has a tendency to become slightly involved if one does not benefit from the following preliminary observations: using $p' = r^2+s^2$, $q' = 2rs$, $ar' = \mu r$, $as' = \mu r - s$, $p = a(r^2-s^2)$ one first establishes $aq''+q' = 2\mu p'$, $ap''+p'-\frac{p}{a} = 2\mu q'$. Using this one checks easily:

$$\left(\frac{p'+q'}{p+q}\right)' + \left(\frac{p'+q'}{p+q}\right)^2 = \frac{1}{a^2}\frac{p}{p+q} + \frac{2\mu-1}{a}\frac{p'+q'}{p+q} \tag{247a}$$

$$\left(\frac{p'-q'}{p-q}\right)' + \left(\frac{p'-q'}{p-q}\right)^2 = \frac{1}{a^2}\frac{p}{p-q} - \frac{2\mu+1}{a}\frac{p'-q'}{p-q} \tag{247b}$$

Also the identity

$$\frac{p^2}{p^2-q^2} = a\frac{p'+q'}{p+q}a\frac{p'-q'}{p-q} = p\alpha \tag{247c}$$

is useful. The verifications may then be done.

9 Hyperfunctions in the study of the $\mathcal{H}$ transform

In this final section we return to the equation (31):

$$\widetilde{\psi(f)}(i\,t) = \frac{t+1}{2t}\widetilde{f}(i\,\frac{t+\frac{1}{t}}{2})\ , \tag{248}$$

Let us recall that $f \in L^2(0,\infty;dx)$ and $\psi : L^2(0,\infty;dx) \to L^2(0,\infty;dx)$ is the isometry which corresponds to $F(w) \mapsto F(w^2)$, where $F(w) = \sum_{n=0}^\infty c_n w^n$, $f(x) = \sum_{n=0}^\infty c_n P_n(x)e^{-x}$, $P_n(x) = L_n^{(0)}(2x)$. Let $g = \psi(f)$. Using $\lambda = it$, in the L^2 sense:

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\lambda+i}{2\lambda}\widetilde{f}(\frac{\lambda-\frac{1}{\lambda}}{2})e^{-i\lambda x}\,d\lambda \tag{249}$$

It is natural to consider separately $\lambda > 0$ and $\lambda < 0$. So let us define:

$$G_+(x) = \frac{1}{2\pi}\int_{-\infty}^{0} \frac{\lambda+i}{2\lambda}\widetilde{f}(\frac{\lambda-\frac{1}{\lambda}}{2})e^{-i\lambda x}\,d\lambda \tag{250a}$$

$$G_-(x) = -\frac{1}{2\pi}\int_{0}^{\infty} \frac{\lambda+i}{2\lambda}\widetilde{f}(\frac{\lambda-\frac{1}{\lambda}}{2})e^{-i\lambda x}\,d\lambda \tag{250b}$$

We observe that G_+ is in the Hardy space of $\Im(x) > 0$ and G_- is in the Hardy space of $\Im(x) < 0$. Their boundary values must coincide on $(-\infty, 0)$ as $g \in L^2(0, +\infty; dx)$. So we have a single analytic function $G(z)$ on $\mathbb{C} \setminus [0, +\infty)$ with $G = G_+$ for $\Im(x) > 0$ and $G = G_-$ for $\Im(x) < 0$. Then $g = \psi(f) = G_+ - G_-$ is computed as

$$g(x) = G(x + i0) - G(x - i0) \tag{251}$$

In other words g is most naturally seen as a hyperfunction [23], as a difference of boundary values of analytic functions. We shall now compute it explicitely, and also we will show later that this observation extends to the distributions $A_a(x)$, $-iB_a(x)$, $E_a(x)$ which are associated with the study of the $\mathcal{H}$ transform. The point of course is that the corresponding functions G will for them have a simple natural expression.

We have, for $\Im(z) > 0$:

$$G(z) = \frac{1}{2\pi}\int_0^\infty \frac{\lambda - i}{2\lambda}\widetilde{f}(\frac{-\lambda + \frac{1}{\lambda}}{2})e^{+i\lambda z}\, d\lambda \tag{252a}$$

$$G(z) = \frac{1}{2\pi}\int_0^\infty \frac{\lambda - i}{2\lambda}\left(\int_0^\infty e^{i\frac{1}{2}x(-\lambda+\frac{1}{\lambda})} f(x)\, dx\right) e^{+i\lambda z}\, d\lambda \tag{252b}$$

Let $\mu = \frac{1}{2}(\lambda - \frac{1}{\lambda})$, $\lambda = \mu + \sqrt{1+\mu^2}$, $\frac{\lambda - i}{2\lambda}\, d\lambda = \frac{\lambda}{\lambda+i}\, d\mu$, with, for $0 < \lambda < \infty$, $-\infty < \mu < \infty$.

$$G(z) = \frac{1}{2\pi}\int_{-\infty}^\infty \frac{\lambda}{\lambda + i}\left(\int_0^\infty e^{-i\mu x} f(x)\, dx\right) e^{+i\lambda z}\, d\mu \tag{252c}$$

For $\mu \to +\infty$, $\lambda = 2\mu + \frac{1}{2\mu} + \dots$, and for $\mu \to -\infty$, $\lambda = -\frac{1}{2\mu} + \dots$, and $\frac{\lambda}{\lambda+i} \sim \frac{i}{2\mu}$ as $\mu \to -\infty$. So far the inner integral is in the L^2 sense. We shall now suppose that f and f' are in L^1 (so $\lim_{x\to\infty} f(x) = 0$) and write $\int_0^\infty e^{-i\mu x} f(x)\, dx = \int_0^\infty e^{-i\mu x - x} e^x f(x)\, dx = \frac{f(0)}{i\mu+1} + \frac{1}{i\mu+1}\int_0^\infty e^{-i\mu x}(f(x) + f'(x))\, dx$.

$$G(z) = \frac{1}{2\pi}\left(f(0)\int_{-\infty}^\infty \frac{\lambda}{\lambda + i}\frac{e^{+i\lambda z}}{i\mu + 1}\, d\mu + \int_{-\infty}^\infty \frac{\lambda}{\lambda + i}\frac{e^{+i\lambda z}}{i\mu + 1}\left(\int_0^\infty e^{-i\mu x}(f(x) + f'(x))\, dx\right) d\mu\right) \tag{252d}$$

In this manner, with $f \in L^1$, $f' \in L^1$, $\Im(z) > 0$, we have an absolutely convergent double integral.

$$G(z) = \frac{1}{2\pi}\left(f(0)\int_{-\infty}^\infty \frac{\lambda}{\lambda + i}\frac{e^{+i\lambda z}}{i\mu + 1}\, d\mu + \int_0^\infty \left(\int_{-\infty}^\infty \frac{\lambda}{\lambda + i}\frac{e^{-i\mu x + i\lambda z}}{i\mu + 1}\, d\mu\right)(f(x) + f'(x))\, dx\right) \tag{252e}$$

Observing $\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\lambda}{\lambda+i}\frac{e^{+i\lambda z}}{i\mu+1}\, d\mu = \frac{1}{2\pi}\int_0^\infty \frac{\lambda - i}{2\lambda}\frac{1}{i\mu+1}e^{+i\lambda z}\, d\lambda = \frac{1}{2\pi i}\int_0^\infty \frac{1}{\lambda - i}e^{+i\lambda z}\, d\lambda$, we then suppose $\Re(z) < 0$, $\Im(z) > 0$ (or $\Im(z) \geq 0$) so that we may rotate the contour to $\lambda = -it$, $0 \leq t < \infty$. This procedure gives thus:

$$\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\lambda}{\lambda + i}\frac{e^{+i\lambda z}}{i\mu + 1}\, d\mu = \frac{1}{2\pi i}\int_0^\infty \frac{e^{tz}}{1 + t}\, dt \tag{252f}$$

Also:

$$\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\lambda}{\lambda + i}\frac{e^{-i\mu x + i\lambda z}}{i\mu + 1}\, d\mu = \frac{1}{2\pi i}\int_0^\infty \frac{1}{\lambda - i}e^{-i\frac{x}{2}(\lambda - \frac{1}{\lambda}) + i\lambda z}\, d\lambda \tag{252g}$$

We rotate the contour to $\lambda \in i[0,-\infty)$, which is licit as $x \geq 0$ and, for $\Re(z) < 0$, $x \geq 0$, we obtain:

$$\frac{1}{2\pi i}\int_0^\infty \frac{e^{zt-\frac{x}{2}(t+\frac{1}{t})}}{1+t}\,dt \tag{252h}$$

Going back this allows to write (252e), for $\Re(z) < 0$, $\Im(z) > 0$ as:

$$G(z) = \frac{1}{2\pi i}\left(f(0)\int_0^\infty \frac{e^{zt}}{1+t}\,dt + \int_0^\infty \left(\int_0^\infty \frac{e^{zt-\frac{x}{2}(t+\frac{1}{t})}}{1+t}\,dt\right)(f(x)+f'(x))\,dx\right) \tag{252i}$$

and finally, after integrating by parts:

$$G(z) = \frac{1}{2\pi i}\int_0^\infty \left(\int_0^\infty \frac{1}{2}(1+\frac{1}{t})e^{zt-\frac{1}{2}y(t+\frac{1}{t})}\,dt\right) f(y)\,dy \tag{252j}$$

This last expression (still temporarily under the hypothesis $f, f' \in L^1$) is certainly a priori absolutely convergent for $\Re(z) < 0$ and gives $G(z)$ in this half-plane.

We are led to the study of:

$$a(z,y) = \frac{1}{2\pi i}\int_0^\infty \frac{1}{2}(1+\frac{1}{t})e^{zt-\frac{1}{2}y(t+\frac{1}{t})}\,dt \tag{253a}$$

We still temporarily assume $\Re(z) < 0$. We even suppose $z < 0$ and make a change of variable:

$$a(z,y) = \frac{1}{2\pi i}\left(\sqrt{\frac{y}{y-2z}}\,\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\sqrt{y(y-2z)}(u+\frac{1}{u})}\,du + \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\sqrt{y(y-2z)}(u+\frac{1}{u})}\frac{1}{u}\,du\right) \tag{253b}$$

$$a(z,y) = \frac{1}{2\pi i}\left(\sqrt{\frac{1}{y-2z}}\,\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\sqrt{y-2z}(v+y\frac{1}{v})}\,dv + \frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\sqrt{y-2z}(v+y\frac{1}{v})}\frac{1}{v}\,dv\right) \tag{253c}$$

$$a(z,y) = \frac{1}{2\pi i}\left(\sqrt{\frac{y}{y-2z}}\,K_1(\sqrt{y(y-2z)}) + K_0(\sqrt{y(y-2z)})\right) \tag{253d}$$

For any $z \in \mathbb{C}\setminus[0,+\infty)$ and any $y \geq 0$ the integrals in (253c) converge absolutely and define an analytic function of z. Furthermore the K Bessel functions decrease exponentially as $y \to +\infty$ in (253d). For fixed z, $a(z,y)$ is certainly a square-integrable function of y (also at the origin), locally uniformly in z so the equation (252j) defines G as an analytic function on the entire domain $\mathbb{C}\setminus[0,+\infty)$. Then by an approximation argument (252j) applies to any $f \in L^2(0,\infty;dx)$ and any $z \in \mathbb{C}\setminus[0,+\infty)$.

We now study the boundary values $a(x+i0,y)$, $a(x-i0,y)$, $x,y \geq 0$. We could use the expression of the K Bessel functions in terms of the Hankel functions $H^{(1)}$ and $H^{(2)}$, go to the boundary, and then recover the Bessel functions J_0 and J_1. But we shall proceed in a more direct manner. Let us first examine

$$\begin{aligned} d(z,y) &= \frac{1}{2\pi i}\frac{1}{2}\int_0^\infty e^{zt-\frac{1}{2}y(t+\frac{1}{t})}\,\frac{1}{t}\,dt \\ &= \frac{1}{2\pi i}\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\sqrt{y(y-2z)}(u+\frac{1}{u})}\frac{1}{u}\,du = \frac{1}{2\pi i}\int_1^\infty e^{-\sqrt{y(y-2z)}\,t}\frac{dt}{\sqrt{t^2-1}} \end{aligned} \tag{254a}$$

$$= \frac{1}{2\pi i}\int_1^\infty e^{-\sqrt{y(y-2z)}\,t}\,t^{-1}\,dt + \frac{1}{2\pi i}\int_1^\infty e^{-\sqrt{y(y-2z)}\,t}\left(\frac{1}{\sqrt{t^2-1}} - \frac{1}{t}\right)dt \tag{254b}$$

$$= \frac{1}{2\pi i} \frac{e^{-\sqrt{y(y-2z)}} - \int_1^\infty e^{-\sqrt{y(y-2z)}\,t}\, t^{-2}\, dt}{\sqrt{y(y-2z)}} + \frac{1}{2\pi i} \int_1^\infty e^{-\sqrt{y(y-2z)}\,t} (\frac{1}{\sqrt{t^2-1}} - \frac{1}{t})\, dt \tag{254c}$$

We now look at the (distributional) boundary values $z \to x$ with $z = x + i\epsilon$, $\epsilon \to 0^+$ or $z = x - i\epsilon$ and $\epsilon \to 0^+$. We shall take $x > 0$. Here the singularities at $y = 2x$ and at $y = 0$ are integrable and we need only take the limit in the naive sense. We distinguish $y > 2x$ from $0 < y < 2x$. In the former case, nothing happens:

$$d(x + i0, y) = d(x - i0, y) = \frac{1}{2\pi i} \int_1^\infty e^{-\sqrt{y(y-2x)}\,t} \frac{dt}{\sqrt{t^2-1}} \tag{255a}$$

In the latter case:

$$d(x + i0, y) = \frac{1}{2\pi} \frac{e^{+i\sqrt{y(2x-y)}} - \int_1^\infty e^{+i\sqrt{y(2x-y)}\,t}\, t^{-2}\, dt}{\sqrt{y(2x-y)}} + \frac{1}{2\pi i} \int_1^\infty e^{+i\sqrt{y(2x-y)}\,t} (\frac{1}{\sqrt{t^2-1}} - \frac{1}{t})\, dt \tag{255b}$$

$$d(x - i0, y) = -\frac{1}{2\pi} \frac{e^{-i\sqrt{y(2x-y)}} - \int_1^\infty e^{-i\sqrt{y(2x-y)}\,t}\, t^{-2}\, dt}{\sqrt{y(2x-y)}} + \frac{1}{2\pi i} \int_1^\infty e^{-i\sqrt{y(2x-y)}\,t} (\frac{1}{\sqrt{t^2-1}} - \frac{1}{t})\, dt \tag{255c}$$

So $d(x + i0, y) - d(x - i0, y)$ is supported in $(0, 2x)$ and has values there

$$\frac{1}{\pi} \frac{\cos\sqrt{y(2x-y)} - \int_1^\infty \cos(\sqrt{y(2x-y)}\,t)\, t^{-2}\, dt}{\sqrt{y(2x-y)}} + \frac{1}{\pi} \int_1^\infty \sin(\sqrt{y(2x-y)}\,t) (\frac{1}{\sqrt{t^2-1}} - \frac{1}{t})\, dt \tag{255d}$$

We used this method to have a clear control not only of the pointwise behavior but also of the limit as a distribution. There is no necessity now to keep working with absolutely convergent integrals and we have the simple result, using the very classical Mehler formula:[31]

$$d(x + i0, y) - d(x - i0, y) = \mathbf{1}_{0<y<2x}(y) \frac{1}{\pi} \int_1^\infty \frac{\sin(\sqrt{y(2x-y)}\,t)}{\sqrt{t^2-1}}\, dt = \frac{1}{2} \mathbf{1}_{0<y<2x}(y) J_0(\sqrt{y(2x-y)}) \tag{256}$$

Let us now consider the behavior of

$$e(z, y) = \frac{1}{2\pi i} \frac{1}{2} \int_0^\infty e^{zt - \frac{1}{2} y (t + \frac{1}{t})}\, dt = \frac{1}{2\pi i} \sqrt{\frac{y}{y-2z}}\, \frac{1}{2} \int_0^\infty e^{-\frac{1}{2}\sqrt{y(y-2z)}(u+\frac{1}{u})}\, du \tag{257}$$

We make the simple observation that $e(z, y) = \frac{\partial}{\partial z} d(z, y)$. So we shall have (as is confirmed by a more detailed examination):

$$\begin{aligned} e(x + i0, y) - e(x - i0, y) &= \frac{\partial}{\partial x} \frac{1}{2} \mathbf{1}_{0<y<2x}(y) J_0(\sqrt{y(2x-y)}) \\ &= \delta_{2x}(y) - \frac{1}{2} \mathbf{1}_{0<y<2x}(y) \sqrt{\frac{y}{2x-y}} J_1(\sqrt{y(2x-y)}) \end{aligned} \tag{258}$$

Combining all those elements we obtain that the function $k(x) = \psi(f)(x)$ is given as:

$$k(x) = f(2x) + \frac{1}{2} \int_0^{2x} J_0(\sqrt{y(2x-y)}) f(y)\, dy - \frac{1}{2} \int_0^{2x} \sqrt{\frac{y}{2x-y}} J_1(\sqrt{y(2x-y)}) f(y)\, dy \tag{259}$$

[31] we are mainly interested in the boundary value as a distribution and we skip the discussion of the pointwise behavior at the borders $y = 0$ and $y = 2x$.

Some pointwise regularity of f at x is necessary to fully justify the formula; in order to check if continuity of f at $2x$ is enough we can not avoid examining $e(z,y)$ more closely as $z\to x$.

$$\begin{aligned} e(z,y) &= \sqrt{\frac{y}{y-2z}}\frac{1}{2\pi i}\int_1^\infty e^{-\sqrt{y(y-2z)}\,t}\frac{t\,dt}{\sqrt{t^2-1}} \\ &= \frac{1}{2\pi i}\frac{e^{-\sqrt{y(y-2z)}}}{y-2z} + \sqrt{\frac{y}{y-2z}}\frac{1}{2\pi i}\int_1^\infty e^{-\sqrt{y(y-2z)}\,t}\,\frac{t-\sqrt{t^2-1}}{\sqrt{t^2-1}}\,dt \end{aligned} \tag{260}$$

The integral term on the right causes no problem at all. And writing $\frac{e^{-\sqrt{y(y-2z)}}}{y-2z} = \frac{1}{y-2z} + \frac{e^{-\sqrt{y(y-2z)}}-1}{y-2z}$, again the term on the right has no problem, so there only remains $\frac{1}{y-2z}$, and of course, this is very well-known, the difference between $+i0$ and $-i0$ gives the Poisson kernel, so for non-tangential convergence, continuity of f at $2x$ is enough. Of course this discussion was quite superfluous if we wanted to understand k as an L^2 function, here we have the information that non tangential boundary value of $G(x+i0)-G(x-i0)$ does give pointwise the formula (259) if f is continuous at $y=2x$. We can also rewrite (259) as:

$$k(x) = (1+\frac{d}{dx})\frac{1}{2}\int_0^{2x} J_0(\sqrt{y(2x-y)})f(y)\,dy \tag{261}$$

This is exactly one half of equation (20c), where k was obtained from (f,g) as $\psi(f)+w\cdot\psi(g)$. Let us observe that $w=\frac{\lambda-i}{\lambda+i}$ verifies, as an operator, $(\frac{d}{dx}+1)\cdot w = w\cdot(\frac{d}{dx}+1) = \frac{d}{dx}-1$. So the isometry corresponding to $g(w)\mapsto w\,G(w^2)$, which is the composite $w\cdot\psi$, sends g to $(-1+\frac{d}{dx})\frac{1}{2}\int_0^{2x} J_0(\sqrt{y(2x-y)})f(y)\,dy$. This is indeed the second half of equation (20c).

The formulas (20a) and (20b) may be established in an exactly analogous manner (taking k with compact support to simplify the discussion). But this would be a repetition of the arguments we just went through, so rather I will conclude the paper with a method allowing to go directly from $\mathcal{A}_a(s)$, $-i\mathcal{B}_a(s)$, $\mathcal{E}_a(s)$ to the distributions $A_a(x)$, $-iB_a(x)$, $E_a(x)$, and this will show that they are in a natural manner (differences of) boundary values of an analytic function.

From the expression $\mathcal{E}_a(s) = \Gamma(s)\widehat{E_a}(s) = 2\sqrt{a}K_s(2a) = \sqrt{a}\int_0^\infty e^{-a(t+\frac{1}{t})}t^{s-1}\,dt$, we shall recover $\widehat{E_a}(s)$ as a right Mellin transform with the help of the Hankel formula $\Gamma(s)^{-1} = \int_{\mathcal{C}} e^v v^{-s}\,dv$, where $\mathcal{C}$ is a contour coming from $-\infty$ along the lower edge of the cut along $(-\infty,0]$ turning counterclockwise around the origin and going back to $-\infty$ along or slightly above the upper edge of the cut. Let us write the Hankel formula as

$$\frac{t^{s-1}}{\Gamma(s)} = \frac{1}{2\pi i}\int_{\mathcal{C}} e^{tv}v^{-s}\,dv \qquad (t>0) \tag{262}$$

So we have:

$$\widehat{E_a}(s) = \sqrt{a}\frac{1}{2\pi i}\int_0^\infty\left(\int_{\mathcal{C}} e^{tv}v^{-s}\,dv\right)e^{-a(t+\frac{1}{t})}\,dt \tag{263}$$

Let us suppose $\Re(s)>1$. Then the contour $\mathcal{C}$ can be deformed into the contour $\mathcal{C}_\epsilon$ coming from $-i\infty$ to $-i\epsilon$, then turning counterclockwise from $e^{-i\frac{\pi}{2}}\epsilon$ to $e^{i\frac{\pi}{2}}\epsilon$, then going to $+i\infty$. Also we impose

$0 < \epsilon < a$. The integrals may then be permuted:

$$\widehat{E_a}(s) = \sqrt{a}\frac{1}{2\pi i}\int_{\mathcal{C}_\epsilon}\left(\int_0^\infty e^{tv}e^{-a(t+\frac{1}{t})}\,dt\right)v^{-s}\,dv \tag{264}$$

and using $e(z,y)$ from (257) this gives:

$$\Re(s) > 1 \implies \quad \widehat{E_a}(s) = \sqrt{a}\int_{\mathcal{C}_\epsilon} 2e(v,2a)v^{-s}\,dv \tag{265}$$

We have previously studied $e(z,y)$, which is also expressed as in (260). We see on this basis and simple estimates that we may deform $\mathcal{C}_\epsilon$ into a contour $\mathcal{C}_{a,\eta}$ going from $+\infty$ to $a+\eta$ along the lower border, turning clockwise around a from $a+\eta-i0$ to $a+\eta+i0$, then going from $a+\eta$ to $+\infty$ on the upper border ($\eta \ll 1$). We will have in particular from (260) a term $\frac{2}{2\pi i}\int_{a+\eta-i0}^{a+\eta+i0}\frac{v^{-s}}{2a-2v}\,dv$ which is a^{-s}. The final result is obtained:

$$\Re(s) > 1 \implies \quad \widehat{E_a}(s) = \sqrt{a}\left(a^{-s} - \int_a^\infty \sqrt{\frac{a}{x-a}}J_1(2\sqrt{a(x-a)})x^{-s}\,dx\right) \tag{266}$$

This identifies $\widehat{E_a}(s)$ as the right Mellin transform of the distribution

$$E_a(x) = \sqrt{a}\left(\delta_a(x) + \mathbf{1}_{x>a}(x)\frac{\partial}{\partial x}J_0(2\sqrt{a(x-a)})\right) = \sqrt{a}\frac{\partial}{\partial x}\left(\mathbf{1}_{x>a}(x)J_0(2\sqrt{a(x-a)})\right) \tag{267}$$

This proof reveals that the distribution $E_a(x)$ is expressed in a natural manner as the difference of boundary values $\sqrt{a}(2e(x+i0,2a) - 2e(x-i0,2a))$, with

$$\sqrt{a}\,2e(z,2a) = \sqrt{a}\,\frac{1}{2\pi i}\int_0^\infty e^{zt-a(t+\frac{1}{t})}\,dt = \sqrt{a}\,\frac{1}{2\pi i}2\sqrt{\frac{a}{a-z}}\,K_1(2\sqrt{a(a-z)}) \tag{268}$$

The formulas (176e) and (176f) are recovered in the same manner.

Theorem 31. *The distribution* $A_a(x) = \frac{\sqrt{a}}{2}(1+\mathcal{H})(\phi_a^+\mathbf{1}_{0<x<\infty})$, $\phi_a^+(x) + \int_0^a J_0(2\sqrt{xy})\phi_a^+(y)\,dy = J_0(2\sqrt{ax})$, *is the difference of boundary values* $\sqrt{a}(a(x+i0,2a) - a(x-i0,2a))$, *with:*

$$\begin{aligned}\sqrt{a}\,a(z,2a) &= \sqrt{a}\,\frac{1}{2\pi i}\int_0^\infty \frac{1}{2}(1+\frac{1}{t})e^{zt-a(t+\frac{1}{t})}\,dt \\ &= \sqrt{a}\,\frac{1}{2\pi i}\left(\sqrt{\frac{a}{a-z}}\,K_1(2\sqrt{a(a-z)}) + K_0(2\sqrt{a(a-z)})\right)\end{aligned} \tag{269}$$

The distribution $-iB_a(x) = \frac{\sqrt{a}}{2}(-1+\mathcal{H})(\phi_a^-\mathbf{1}_{0<x<\infty})$, $\phi_a^-(x) - \int_0^a J_0(2\sqrt{xy})\phi_a^-(y)\,dy = J_0(2\sqrt{ax})$, *is the difference of boundary values* $\sqrt{a}(-ib(x+i0,2a) - (-ib(x-i0,2a)))$, *with:*

$$\begin{aligned}\sqrt{a}(-ib(z,2a)) &= \sqrt{a}\,\frac{1}{2\pi i}\int_0^\infty \frac{1}{2}(1-\frac{1}{t})e^{zt-a(t+\frac{1}{t})}\,dt \\ &= \sqrt{a}\,\frac{1}{2\pi i}\left(\sqrt{\frac{a}{a-z}}\,K_1(2\sqrt{a(a-z)}) - K_0(2\sqrt{a(a-z)})\right)\end{aligned} \tag{270}$$

10 Appendix: a remark on the resolvent of the Dirichlet kernel

In this paper we have studied a special transform on the positive half-line with a kernel of a multiplicative type $k(xy)$, following the method summarized in [5, 6]. We have associated to the kernel the investigation of its Fredholm determinants on finite intervals $(0, a)$, and have related them with first and second order differential equations leading to problems of spectral and scattering theory. There is a vast literature on kernels of the additive type $k(x-y)$, and on the related Fredholm determinants on finite intervals. The Dirichlet kernel on $L^2(-s, s; dx)$:

$$K_s(x,y) = \frac{\sin(x-y)}{\pi(x-y)} \tag{271}$$

has been the subject of many works (only a few references will be mentioned here.) The Fredholm determinant $\det(1-K_s)$, as a function of s (or more generally as a function of the endpoints of finitely many intervals), has many properties, and is related to the study of random matrices [22]. The Fredholm determinants of the even and odd parts

$$K_s^{\pm}(x,y) = \frac{\sin(x-y)}{\pi(x-y)} \pm \frac{\sin(x+y)}{\pi(x+y)} \tag{272}$$

on $L^2(0, s; dx)$ have been studied by Dyson [16]. He used the second derivatives of their logarithms to construct potentials for Schrödinger equations on the half-line, and studied their asymptotics with the tools of scattering theory. Jimbo, Miwa, Môri, and Sato [17] related $\det(1-K_s)$ to a Painlevé equation. Widom [34] obtained the leading asymptotics using the Krein continuous analog of orthogonal polynomials. Deift, Its, and Zhou [11] justified the Dyson asymptotic expansions using tools developed for Riemann-Hilbert problems. Tracy and Widom [32] established partial differential equations for the Fredholm determinants of integral operators arising in the study of the scaling limit of the distribution functions of eigenvalues of random matrices. We refer the reader to the cited references and to [12] for recent results and we apologize for not providing any more detailed information here.

We have, in the present paper, been talking a lot of scattering and determinants and one might wonder if this is not a re-wording of known things. In fact, our work is with the multiplicative kernels $k(xy)$, and (direct) reduction to additive kernels would lead to (somewhat strange) $g(t+u)$ kernels on semi-infinite intervals $(-\infty, \log(a)]$. So we are indeed doing something different; one may also point out that the entire functions arising in the present study are not of finite exponential type; and the scattering matrices do not at all tend to 1 as the frequency goes to infinity. In the case of the cosine and sine kernels the flow of information will presumably go from the additive to the multiplicative, as the additive situation is more flexible, and has stimulated the development of powerful tools, with relation to Painlevé equations, Riemann-Hilbert problems, Integrable systems [11].

Nevertheless, one may ask if the framework of reproducing kernels in Hilbert spaces of entire functions also may be used in the additive situation. This is the case indeed and it is very much connected to the method of Krein in inverse scattering theory, and his continuous analog of orthogonal polynomials (used by Widom in the context of the Dirichlet kernel in [34].) The Gaudin identities for convolution kernels ([22, App. A16]) play a rôle very analogous to the identities in the present paper (132a), (132b) involved in the study of multiplicative kernels. Widom in his proof [34] of the main term of the asymptotics as $s \to +\infty$ studied the Krein functions associated with the complement of the interval $(-1, +1)$ and he mentioned the interest of extremal properties. In this appendix, I shall point out that the resolvent of the Dirichlet kernel indeed does have an extremal property: it coincides exactly (up to complex conjugation in one variable) with the reproducing kernel of a certain (interesting) Hilbert space of entire functions. This could be a new observation, obviously closely related to the method of Widom [34].

The space mPW_s we shall use is, as a set, the Paley-Wiener space PW_s, but the norm is different:

$$\begin{aligned} mPW_s &= \{f(z) \text{ entire of exponential type at most } s \text{ with } \|f\| < \infty\} \\ \|f\|^2 &= \int_{\mathbb{R}\setminus(-1,1)} |f(t)|^2 \, dt \end{aligned} \tag{273}$$

Let $X_s(z, w)$ be the element of mPW_s which is the evaluator at z: $\forall f \in mPW_s$ $(f, X_s(z, \cdot)) = f(z)$. We shall compare $X_s(z, w)$ with the resolvent of the kernel

$$D_s(x, y) = \frac{\sin(s(x-y))}{\pi(x-y)} \tag{274}$$

on $L^2(-1, 1; dx)$.

Let $f \in mPW_s$. It belongs to PW_s so

$$f(z) = \int_{\mathbb{R}} f(t) \frac{e^{is(t-z)} - e^{-is(t-z)}}{2\pi i(t-z)} \, dt = \int_{\mathbb{R}} f(t) \frac{\sin(s(t-z))}{\pi(t-z)} \, dt \tag{275}$$

On the other hand:

$$f(z) = \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) f(t) \overline{X_s(z, t)} \, dt \tag{276}$$

As $\overline{f(\overline{z})} = \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) \overline{f(t)} \; \overline{X_s(z, t)} \, dt = \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) \overline{f(t)} X_s(\overline{z}, t) \, dt$ one has $\overline{X_s(z, t)} = X_s(\overline{z}, t)$ for $t \in \mathbb{R}$. We have for y_1 and y_2 real

$$X_s(y_1, y_2) = \int_{\mathbb{R}\setminus(-1,1)} X_s(y_1, t) \overline{X_s(y_2, t)} \, dt = \int_{\mathbb{R}\setminus(-1,1)} X_s(y_1, t) X_s(y_2, t) \, dt = X_s(y_2, y_1) \tag{277}$$

so more generally $X_s(\overline{z_1}, z_2) = X_s(\overline{z_2}, z_1)$.

We apply (276) to $f(z) = \frac{\sin(s(z-y))}{\pi(z-y)}$ for some $y \in \mathbb{C}$:

$$\frac{\sin(s(z-y))}{\pi(z-y)} = \int_{\mathbb{R}\setminus(-1,1)} \frac{\sin(s(t-y))}{\pi(t-y)} X_s(\overline{z}, t) \, dt \tag{278}$$

We apply (275) to $f(y) = X_s(\overline{z}, y)$ for some $z \in \mathbb{C}$:

$$X_s(\overline{z}, y) = \int_{\mathbb{R}} X_s(\overline{z}, t) \frac{\sin(s(t-y))}{\pi(t-y)}\, dt \tag{279}$$

Combining we obtain:

$$X_s(\overline{z}, y) - \frac{\sin(s(z-y))}{\pi(z-y)} = \int_{-1}^{1} X_s(\overline{z}, t)\, \frac{\sin(s(t-y))}{\pi(t-y)}\, dt \tag{280}$$

Restricting to $y \in (-1, 1)$, $z = x \in (-1, 1)$, this says exactly:

$$X_s(x, y) = R_s(x, y) \tag{281}$$

where $R_s(x, y)$ is the kernel of the resolvent: $1 + R_s = (1 - D_s)^{-1}$, $R_s - D_s = R_s D_s$. The resolvent $R_s(x, y)$ is entire in (x, y) and the general formula is thus:

$$\forall z, w \in \mathbb{C} \qquad R_s(z, w) = X_s(\overline{z}, w)\ . \tag{282}$$

References

[1] L. de Branges, Self-reciprocal functions, J. Math. Anal. Appl. **9** (1964) 433–457.

[2] L. de Branges, *Hilbert spaces of entire functions*, Prentice Hall Inc., Englewood Cliffs, 1968.

[3] J.-F. Burnol, Sur certains espaces de Hilbert de fonctions entières, liés à la transformation de Fourier et aux fonctions L de Dirichlet et de Riemann, C. R. Acad. Sci. Paris, Ser. I **333** (2001), 201–206.

[4] J.-F. Burnol, On Fourier and Zeta ('s), talk at University of Nice (18 Dec. 2001), Forum Mathematicum **16** (2004), 789–840.

[5] J.-F. Burnol, Sur les "espaces de Sonine" associés par de Branges à la transformation de Fourier, C. R. Acad. Sci. Paris, Ser. I **335** (2002), 689–692.

[6] J.-F. Burnol, Des équations de Dirac et de Schrödinger pour la transformation de Fourier, C. R. Acad. Sci. Paris, Ser. I **336** (2003), 919–924.

[7] J.-F. Burnol, Two complete and minimal systems associated with the zeros of the Riemann zeta function, Jour. Th. Nb. Bord. **16** (2004).

[8] J.-F. Burnol, *Entrelacement de co-Poisson*, Ann. Inst. Fourier, **57** no. 2 (2007), 525–602.

[9] J.-F. Burnol, Spacetime causality in the study of the Hankel transform, Ann. Henri Poincaré **7** (2006), 1013–1034.

[10] E. Coddington, N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955.

[11] P. Deift, A. Its, X. Zhou, A Riemann-Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, Ann. Math. **146** (1997), 149–235.

[12] P. Deift, A. Its, I. Krasovsky, X. Zhou, The Widom-Dyson constant for the gap probability in random matrix theory, arXiv:math.FA/0601535

[13] R. J. Duffin, H. F. Weinberger, Dualizing the Poisson summation formula, Proc. Natl. Acad. Sci. USA **88** (1991), 7348–7350.

[14] H. Dym, An introduction to de Branges spaces of entire functions with applications to differential equations of the Sturm-Liouville type, Advances in Math. **5** (1971), 395–471.

[15] H. Dym, H.P. McKean, *Gaussian processes, function theory, and the inverse spectral problem*, Probability and Mathematical Statistics, Vol. 31. Academic Press, New York-London, 1976.

[16] F. Dyson, Fredholm determinants and inverse scattering problems, Comm. Math. Phys. **47** (1976), 171–183

[17] M. Jimbo, T. Miwa, Y. Môri, M. Sato, Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent, Physica **1D** (1980), 80-158.

[18] J. C. Lagarias, Hilbert spaces of entire functions and Dirichlet L-functions, in: *Frontiers in Number Theory, Physics and Geometry: On Random Matrices, Zeta Functions and Dynamical Systems* (P. E. Cartier, B. Julia, P. Moussa and P. van Hove, Eds.), Springer-Verlag: Berlin 2006, to appear.

[19] J. C. Lagarias, Zero spacing distributions for differenced L-functions, Acta Arithmetica 120 (2005), No. 2, 159–184.

[20] P. Lax, *Functional Analysis*, Wiley, 2002.

[21] B. M. Levitan, I. S. Sargsjan, *Introduction to Spectral Theory*, Transl. of Math. Monographs **39**, AMS 1975.

[22] M. L. Mehta, *Random Matrices*, Academic Press, 2nd ed., 1991.

[23] M. Morimoto, *An introduction to Sato's Hyperfunctions*, Transl. of Math. Monographs **129**, AMS 1993.

[24] G. Pólya, Bemerkung über die Integraldarstellung der Riemannschen ξ-Funktion, Acta Math. **48** (1926), 305-317.

[25] G. Pólya, Über trigonometrische Integralen mit nur reellen Nullstellen, J. Reine u. Angew. Math. **158** (1927), 6-18.

[26] M. Reed, B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press, New York-London, 1975.

[27] C. Remling, Schrödinger operators and de Branges spaces, J. Funct. Anal. **196** (2002), 323–394.

[28] V. Rovnyak, Self-reciprocal functions, Duke Math. J. **33** (1966) 363–378.

[29] J. Rovnyak, V. Rovnyak, Self-reciprocal functions for the Hankel transformation of integer order, Duke Math. J. **34** (1967) 771–785.

[30] J. Rovnyak, V. Rovnyak, Sonine spaces of entire functions, J. Math. Anal. Appl. **27** (1969) 68–100.

[31] G. Szegö, *Orthogonal Polynomials*, AMS Colloquium Publications **23**, 1939.

[32] C. Tracy, H. Widom, Fredholm determinants, differential equations, and matrix models, Commun. Math. Phys. **163** (1994), 33–72.

[33] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, 2nd ed., Cambridge University Press, 1944.

[34] H. Widom, The asymptotics of a continuous analogue of orthogonal polynomials, J. Approx. Theory **77** (1994), 51–64