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The polygamma function $\psi^{(k)}(x)$ for $x = \frac{1}{4}$ and $x = \frac{3}{4}$

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Abstract

Expressions for the polygamma function $\psi^{(k)}(x)$ for the arguments $x = \frac{1}{4}$ and $x = \frac{3}{4}$ are given in terms of Bernoulli numbers, Euler numbers, the Riemann zeta function for odd integer arguments, and the related series of reciprocal powers of integers $\beta(m)$.

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1. Introduction

The polygamma function

$$\psi^{(k)}(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k}\psi(x) = \frac{\mathrm{d}^{k+1}}{\mathrm{d}x^{k+1}} \ln \Gamma(x)$$

appears in a number of theoretical and practical applications, and a number of its properties are listed in the relevant handbooks. In particular, it is well-known that for $k \geqslant 1$ [1, 6.4.4]

$$\psi^{(k)}(1) = (-1)^k k!\, \zeta(k+1), \qquad \psi^{(k)}(\tfrac{1}{2}) = (-1)^{k+1} k!\, (2^{k+1} - 1)\, \zeta(k+1),$$

where $\zeta(n)$ is the Riemann zeta function for integer arguments. Together with the recursion formula [1, 6.4.6]

$$\psi^{(k)}(1+x) - \psi^{(k)}(x) = (-1)^k k!\, x^{-k-1} \tag{1}$$

and the reflection formula [1, 6.4.7]

$$\psi^{(x)}(1-x) + (-1)^{k+1}\psi^{(k)}(x) = (-1)^k\, \pi \frac{\mathrm{d}^k}{\mathrm{d}x^k} \cot \pi x, \tag{2}$$

it is easy to find expressions for $\psi^{(k)}(n)$ and $\psi^{(k)}(\frac{1}{2} \pm n)$, where $n \in \mathbb{N}$.

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It seems surprising that the values of $\psi^{(k)}(x)$ for $x=\frac{1}{4}$ and $x=\frac{3}{4}$ and hence for $x=\frac{1}{4}\pm n$ and $x=\frac{3}{4}\pm n$ have received less attention; at least these values are seldom found in the handbooks. Sometimes, e.g., in [2], one finds a relation like $\psi'(\frac{1}{4})-\psi'(\frac{3}{4})=16G$, where $G=0.91596\ldots$ is the Catalan constant, and Krupnikov [4] has evaluated $\psi'(x)$ and $\psi''(x)$ for $x=\frac{1}{4}$ and $x=\frac{3}{4}$.

It is the purpose of this note to present expressions for $\psi^{(k)}(\frac{1}{4})$ and $\psi^{(k)}(\frac{3}{4})$ in terms of the Bernoulli numbers, the Euler numbers, the Riemann zeta function for odd integer arguments, and the related series of reciprocal powers of integers $\beta(m)$.

2. Expressions for $\psi^{(k)}(\frac{1}{4})$ and $\psi^{(k)}(\frac{3}{4})$

For rational arguments $x=p/q$ the polygamma function $\psi^{(k)}(x)$ can be written as [3, 8.363 8]

$$\psi^{(k)}\left(\frac{p}{q}\right)=(-1)^{k+1}k!\,\zeta\left(k+1,\frac{p}{q}\right)$$
$$=(-1)^{k+1}k!\,q^{k+1}\sum_{n=0}^{\infty}\frac{1}{(p+qn)^{k+1}}\qquad(k\geqslant 1),\tag{3}$$

where $\zeta(k+1,x)$ is the generalized zeta function. In particular we obtain

$$\psi^{(k)}(\tfrac{1}{4})+(-1)^k\psi^{(k)}(\tfrac{3}{4})$$
$$=(-1)^{k+1}k!\,2^{2k+2}\left\{\sum_{n=0}^{\infty}\frac{1}{(4n+1)^{k+1}}+(-1)^k\sum_{n=0}^{\infty}\frac{1}{(4n+3)^{k+1}}\right\}.\tag{4}$$

From the reflection formula (2) we find for $x=\frac{1}{4}$

$$\psi^{(k)}(\tfrac{1}{4})-(-1)^k\psi^{(k)}(\tfrac{3}{4})=-\pi\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cot\pi x\bigg|_{x=\frac{1}{4}}.\tag{5}$$

In order to calculate the higher derivatives of $\cot\pi x$ at $x=\frac{1}{4}$, we make use of the trigonometric relation

$$\tan\pi(x+\tfrac{1}{4})=\sec 2\pi x+\tan 2\pi x.$$

It is then not difficult to obtain the derivatives of the cotangent function at $x=\frac{1}{4}\pi$ from the power series expansions for the tangent and secant functions [1, 4.3.67, 69], namely

$$\tan z=\sum_{n=0}^{\infty}\frac{2^{2n}(2^{2n}-1)\,|B_{2n}|}{(2n)!}z^{2n-1}\qquad(|z|<\tfrac{1}{2}\pi)$$

and

$$\sec z=\sum_{n=0}^{\infty}\frac{|E_{2n}|}{(2n)!}z^{2n}\qquad(|z|<\tfrac{1}{2}\pi),$$

by noting that

$$\frac{d^k}{dx^k}\cot \pi x\bigg|_{x=\frac{1}{4}} = (-1)^k \frac{d^k}{dx^k}\tan \pi x\bigg|_{x=\frac{1}{4}},$$

where B_{2n} and E_{2n} are the Bernoulli and Euler numbers, respectively. This leads to

$$\frac{d^{2k-1}}{dx^{2k-1}}\cot \pi x\bigg|_{x=\frac{1}{4}} = -(2\pi)^{2k-1}\, 2^{2k}\,(2^{2k}-1)\,\frac{|B_{2k}|}{2k}.$$

and

$$\frac{d^{2k}}{dx^{2k}}\cot \pi x\bigg|_{x=\frac{1}{4}} = (2\pi)^{2k}\,|E_{2k}|.$$

By using the two series [1, 23.2.20, 21]

$$\sum_{j=0}^{\infty}\frac{1}{(2j+1)^m} = (1-2^{-m})\,\zeta(m) \quad \text{and} \quad \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^m} = \beta(m)$$

we obtain from (4) and (5), for odd $k = 2n-1$,

$$\psi^{(2n-1)}(\tfrac{1}{4}) + \psi^{(2n-1)}(\tfrac{3}{4}) = \pi^{2n}\, 2^{4n-1}\,(2^{2n}-1)\,\frac{|B_{2n}|}{2n},$$
$$\psi^{(2n-1)}(\tfrac{1}{4}) - \psi^{(2n-1)}(\tfrac{3}{4}) = (2n-1)!\, 2^{4n}\,\beta(2n),$$

and for even $k = 2n$

$$\psi^{(2n)}(\tfrac{1}{4}) - \psi^{(2n)}(\tfrac{3}{4}) = -\pi(2\pi)^{2n}\,|E_{2n}|,$$
$$\psi^{(2n)}(\tfrac{1}{4}) + \psi^{(2n)}(\tfrac{3}{4}) = -(2n)!\, 2^{2n+1}\,(2^{2n+1}-1)\,\zeta(2n+1).$$

Hence for $n \in \mathbb{N}$,

$$\left.\begin{matrix}\psi^{(2n-1)}(\frac{1}{4})\\ \psi^{(2n-1)}(\frac{3}{4})\end{matrix}\right\} = \frac{4^{2n-1}}{2n}\left\{\pi^{2n}\,(2^{2n}-1)\,|B_{2n}| \pm 2(2n)!\,\beta(2n)\right\}$$

and

$$\left.\begin{matrix}\psi^{(2n)}(\frac{1}{4})\\ \psi^{(2n)}(\frac{3}{4})\end{matrix}\right\} = \mp 2^{2n-1}\left\{\pi^{2n+1}\,|E_{2n}| \pm 2(2n)!\,(2^{2n+1}-1)\,\zeta(2n+1)\right\}.$$

With the exception of $\beta(2) = G$, which is merely a definition, no expressions for $\zeta(2n+1)$ or $\beta(2n)$ in terms of other well-known constants are known.

The following list gives a few examples for $\psi^{(k)}(\frac{1}{4})$ and $\psi^{(k)}(\frac{3}{4})$:

$$\psi'(\tfrac{1}{4}) = \pi^2 + 8G, \qquad \psi'(\tfrac{3}{4}) = \pi^2 - 8G,$$
$$\psi''(\tfrac{1}{4}) = -2\,[\pi^3 + 28\zeta(3)], \qquad \psi''(\tfrac{3}{4}) = 2\,[\pi^3 - 28\zeta(3)],$$

$$\psi'''(\tfrac{1}{4}) = 8[\pi^4 + 96\beta(4)], \qquad \psi'''(\tfrac{3}{4}) = 8[\pi^4 - 96\beta(4)],$$
$$\psi^{(4)}(\tfrac{1}{4}) = -8[5\pi^5 + 1488\zeta(5)], \qquad \psi^{(4)}(\tfrac{3}{4}) = 8[5\pi^5 - 1488\zeta(5)],$$
$$\psi^{(5)}(\tfrac{1}{4}) = 256[\pi^6 + 960\beta(6)], \qquad \psi^{(5)}(\tfrac{3}{4}) = 256[\pi^6 - 960\beta(6)],$$
$$\psi^{(6)}(\tfrac{1}{4}) = -32[61\pi^7 + 182880\zeta(7)], \qquad \psi^{(6)}(\tfrac{3}{4}) = 32[61\pi^7 - 182880\zeta(7)].$$

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