INFLUENCE OF DOPPLER EFFECT AND DAMPING ON LINE-ABSORPTION COEFFICIENT AND ATMOSPHERIC RADIATION TRANSFER*

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ABSTRACT

The general term is given for the Taylor series and asymptotic expansions of the line-absorption coefficient when both the Doppler effect and damping contribute to the line shape. The influence of the Doppler effect on the fractional absorption of a line as measured in the laboratory is calculated and compared with the usual expressions which assume a Lorentz line shape. The radiation transfer in a planetary atmosphere is calculated when both the Doppler effect and damping contribute to the line shape and when the change of half-width with height is taken into account. It is shown that for the earth's atmosphere the radiative transfer calculated from the Lorentz shape alone is not changed appreciably by the Doppler effect for either weak or strong lines at heights up to at least 50 km, even at altitudes where the Doppler width is somethat greater than the Lorentz width.

INTRODUCTION

When both the Doppler effect and Lorentz collision damping contribute to the line width, a knowledge of the line-absorption coefficient is needed for problems in such diverse fields as stellar radiation, laboratory absorption measurement, and atmospheric radiation transfer. As is well known, the expression for the line-absorption coefficient is[1]

$$k(\nu, a) = \frac{k_0 a}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-x^2)}{a^2 + (\omega - x)^2}\, dx \tag{1}$$

$$= \frac{k_0}{\pi^{1/2}} \int_0^{\infty} \exp\left(-ax - \frac{x^2}{4}\right) \cos \omega x\, dx\,, \tag{2}$$

where $k_0 = (S/\Delta\nu_D)(\ln 2/\pi)^{1/2}$; $a = (\alpha/\Delta\nu_D)(\ln 2)^{1/2}$; $\omega = (\nu - \nu_0/\Delta\nu_D)(\ln 2)^{1/2}$; k is the absorption coefficient at the frequency ν; S is the total intensity of the line; $\Delta\nu_D$ is half the Doppler width at half-maximum; and α is half the Lorentz plus natural width at half-maximum.

The line-absorption coefficient given by equation (1) has been tabulated by a number of authors, including Mitchell and Zemansky,[1] Hjerting,[2] and Harris.[3] Here this work is extended to give the general term of the expansion of $k(\nu, a)$ in a Taylor series in powers of a and also in an asymptotic expansion in inverse powers of $\nu - \nu_0$.

In the second section we discuss the measurement of the fractional absorption from a beam of radiation in the laboratory when the Doppler effect makes an appreciable contribution to the line width. These results reduce in certain limiting cases to the well-known results of Ladenburg and Reiche[4] for the fractional absorption of a single line having the Lorentz shape.

The pressures and temperatures of a planetary atmosphere are such that both the Lorentz and the Doppler effects contribute to the line broadening. It was shown by Strong and Plass[5] that an isothermal atmosphere composed of radiating gases which have

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[1] Mitchell and Zemansky, *Resonance Radiation and Excited Atoms* (Cambridge: At the University Press, 1934), pp. 101 and 320.

[2] *Ap. J.*, **88**, 508, 1938.

[3] *Ap. J.*, **108**, 112, 1948.

[4] *Ann. d. Phys.*, **42**, 181, 1911.

[5] *Ap. J.*, **112**, 365, 1950.

Lorentz-broadened lines can emit very much more radiation from the lower layers than would be calculated from a model using a gray absorption coefficient. The radiation escaping to space from the lower layers of the atmosphere comes predominantly from the wings of the pressure-broadened lines, for, as the lines in the upper atmospheric layers are considerably narrower, they do not effectively absorb the radiation from the wings of the lines in the lower layers. The model of Strong and Plass assumed an isothermal atmosphere with a radiating gas having a constant fractional concentration. The absorption lines of this gas were assumed to have the Lorentz line shape and not to overlap appreciably at the atmospheric pressures considered. This model has been extended to include cases where the atmosphere is nonisothermal,[6] where the collision-broadened line shape is asymmetrical at frequencies far from the line center,[7, 8, 9] where the lines of the band overlap appreciably,[9, 10, 11, 12] and where the fractional concentration of the radiating gas varies with height.[6, 9, 10]

Although the above-mentioned calculations determine the radiation balance in the atmosphere from ground level to the middle stratosphere, they cannot be applied at higher altitudes without further consideration, since the collision-broadened line shape has been assumed. At higher altitudes the Doppler effect contributes appreciably to the half-width. The radiation balance under these conditions is examined in the last part of this paper.

LINE-ABSORPTION COEFFICIENT

The line-absorption coefficient can be evaluated as a power series in the parameter a if we expand the Reiche form of the integral as given by equation (2) in a Taylor series in a:

$$k(\nu, a) = \frac{k_0}{\pi^{1/2}} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{n+1} a^n}{n!} \int_0^{\infty} x^n \exp(-x^2) \cos 2\omega x \, dx . \tag{3}$$

First we evaluate the integrals that occur in equation (3) for even integral values of n. We note that the Hermite polynomial of the nth degree, $H_n(x)$, is defined as

$$H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2) . \tag{4}$$

Let

$$A(\omega) = \int_0^{\infty} \exp(-x^2) \cos 2\omega x \, dx = \tfrac{1}{2}\pi^{1/2} \exp(-\omega^2) . \tag{5}$$

Differentiating this equation n times, where n is an even integer, we obtain

$$\begin{aligned} \frac{d^n}{d\omega^n} A(\omega) &= (-1)^{n/2} 2^n \int_0^{\infty} x^n \exp(-x^2) \cos 2\omega x \, dx \\ &= \tfrac{1}{2}\pi^{1/2} \frac{d^n}{d\omega^n} \exp(-\omega^2) . \end{aligned} \tag{6}$$

If we combine this expression with the definition of the Hermite polynomial, the nth

[6] J. I. King, *J. Meteorol.*, **9**, 311, 1952.

[7] G. N. Plass and D. Warner, *Phys. Rev.*, **86**, 138, 1952.

[8] G. N. Plass and D. Warner, *J. Meteorol.*, **9**, 333, 1952.

[9] G. N. Plass, *J. Opt. Soc.*, **42**, 677, 1952.

[10] G. N. Plass, *J. Meteorol.* (in press).

[11] L. D. Kaplan, *J. Meteorol.*, **9**, 1, 139, 1952.

[12] J. I. King, *Transfer Theory for Purely Pressure-broadened Band Spectra* (A.F.C.R.C. Rept. No. 5 [Salt Lake City: University of Utah, 1952]).

term in the summation of equation (3) for even n is found to be

$$k_0 a^n (n!)^{-1} \exp(-\omega^2) H_n(\omega) \cos \tfrac{1}{2} n\pi \, .$$

The odd terms in the summation can be evaluated in a similar manner in terms of the function $F(\omega)$ defined by the equation[13]

$$F(\omega) = \int_0^\infty \exp(-x^2) \sin 2\omega x \, dx = \exp(-\omega^2) \int_0^\omega \exp(x^2) \, dx \, . \tag{7}$$

Now the nth derivative where n is an odd integer is

$$F^{(n)}(\omega) = (-1)^{(1/2)(n-1)} 2^n \int_0^\infty x^n \exp(-x^2) \cos 2\omega x \, dx \, . \tag{8}$$

Thus the nth term in the summation of equation (3) for odd n is found to be

$$-k_0 a^n (n!)^{-1} \frac{2}{\pi^{1/2}} F^{(n)}(\omega) \sin \tfrac{1}{2} n\pi \, .$$

The line-absorption coefficient can now be written as the sum of Hermite polynomials and derivatives of $F(\omega)$ in the form

$$k(\nu, a) = k_0 \sum_{n=0}^{\infty} \frac{a^n}{n!} \left[\exp(-\omega^2) H_n(\omega) \cos \tfrac{1}{2} n\pi - 2\pi^{-1/2} F^{(n)}(\omega) \sin \tfrac{1}{2} n\pi \right] . \tag{9}$$

This result agrees with the first five terms of the expansion as given by Harris.[3] However, the general term of the expansion is given here, and this result is used later.

It is interesting to note that the even terms in equation (9) can be summed exactly. The generating function of the Hermite polynomials is

$$\exp[-(t-\omega)^2] = \exp(-\omega^2) \sum_{n=0}^{\infty} (n!)^{-1} H_n(\omega) t^n \, . \tag{10}$$

In equation (10) replace t by it and add to this equation (10) with t replaced by $-it$. The resulting sum is the same as the sum over even n in equation (9), so that

$$k(\nu, a) = k_0 \left\{ \exp(a^2 - \omega^2) \cos 2\omega a - \frac{2}{\pi^{1/2}} \sum_{n=1}^{\infty} (n!)^{-1} F^{(n)}(\omega) a^n \sin \tfrac{1}{2} n\pi \right\} . \tag{11}$$

A similar procedure may be performed on the summation over odd n, but it does not lead to elementary functions. Equation (11) approaches the Doppler line shape as a approaches zero, as, of course, it should.

We shall also require the asymptotic expansion of $k(\nu, a)$ in inverse powers of ω. This can be obtained from an integral evaluated by Stokes:[14]

$$\int_0^v \exp(v^2 - t^2) \, dt = \frac{\pi^{1/2}}{2} \exp(v^2) - \frac{1}{2v} - \sum_{m=1}^{M} \frac{(-1)^m (2m-1)!}{2^{2m} (m-1)! \, v^{2m+1}} \, . \tag{12}$$

The substitution $v = -i\omega$ changes this integral into $-iF(\omega)$. Taking the nth derivative

[13] See, e.g., Mitchell and Zemansky, *op. cit.*, p. 321.

[14] E. T. Whittaker and G. N. Watson, *Modern Analysis* (4th ed.; Cambridge: At the University Press, 1940), p. 152.

and evaluating the derivatives of the exponential in the same manner as described between equations (10) and (11), we obtain

$$k(\nu, a) = k_0 \Big\{ (\cos 2\omega a + \sin 2\omega a) \exp(a^2 - \omega^2) + \frac{1}{\pi^{1/2}} \sum_{n=1}^{\infty} \sum_{m=0}^{M} \frac{(2m+n)! \sin(\frac{1}{2}n\pi) a^n}{2^{2m} n! m! \omega^{2m+n+1}} \Big\} . \tag{13}$$

If $|\nu - \nu_0| \gg a$ and $|\nu - \nu_0| \gg \Delta\nu_D$, then the exponential term can be neglected, so that

$$k(\nu, a) = \frac{S a}{\pi (\nu - \nu_0)^2} \Big\{ 1 + (\tfrac{3}{2} - a^2) \frac{1}{\omega^2} + (\tfrac{15}{4} - 5a^2 + a^4) \frac{1}{\omega^4} + \ldots \Big\} . \tag{14}$$

The first three terms of equation (14) agree with those given by Harris.[3]

LABORATORY ABSORPTION

As an example of the application of these expressions for the line-absorption coefficient, the absorption measured in the laboratory over a finite frequency interval is calculated when the Doppler effect contributes to the width of the line. The assumption will be made that the lines in the band do not overlap to an appreciable extent at the pressures used in the experiment. This is usually the case under the conditions when the Doppler effect makes a contribution to the absorption. If the lines do not overlap appreciably, the total absorption from the lines in a band in a certain frequency interval is merely the sum of the absorptions from each of the individual lines.

For a single line the fractional absorption, A, is defined[15] as

$$A \Delta\nu = \int_{-\infty}^{\infty} [1 - \exp(-kw)] \, d\nu , \tag{15}$$

where $\Delta\nu$ is the frequency interval in which the absorption is measured and w is the optical thickness of the absorbing gas. The limits of integration are extended to infinity for convenience, since the interval $\Delta\nu$ is chosen sufficiently large so that the absorption from this particular line is negligible outside this interval. We consider the evaluation of the fractional absorption in three different cases.

Case I—weak-line approximation.—A line is called a weak line when $kw \ll 1$ at all frequencies, including the line center for a particular optical thickness w. Then, by expansion of equation (15) in powers of kw, we obtain

$$A \Delta\nu = w \int_{-\infty}^{\infty} k(\nu, a) \, d\nu . \tag{16}$$

If $k(\nu, a)$ is replaced by its expression from equation (11), the definite integral from the first term gives

$$A \Delta\nu = S w . \tag{17}$$

All the terms from the summation in equation (11) vanish after integration, since, for large ω, $F(\omega) \cong (2\omega)^{-1}$ and therefore $F^{(n)}(\omega = \pm\infty) = 0$. If there are N nonoverlapping lines in the interval $\Delta\nu$, then

$$A \Delta\nu = w \sum_{j=1}^{N} S_j . \tag{18}$$

[15] See, e.g., W. M. Elsasser, *Heat Transfer by Infrared Radiation in the Atmosphere* (Cambridge: Harvard University Press, 1942).

Thus the fractional absorption varies linearly with w for a weak line, regardless of the value of a, i.e., of the relative contributions of the Doppler effect or natural and collision damping to the half-width. This result states that the total line intensity always determines the absorption from a weak line.

Case II—Doppler width alone.—At sufficiently low pressures, when the half-width due to damping is very small compared to the Doppler half-width, the fractional absorption can be calculated by substituting the first term of equation (9) in equation (15). The result has been given by Struve and Elvey.[16]

Case III—strong-line approximation.—A line is called a strong line when almost all the incident radiation is absorbed at frequencies within the total half-width of the line center for a particular optical thickness w. Obviously, when the optical thickness is increased sufficiently, a weak line becomes a strong line. The fractional absorption for a strong line is obtained from equation (14), since it accurately represents the line-absorption coefficient in the wings of the line. It is inaccurate only at frequencies inside the half-width, where both the approximate and exact expressions give the same virtually complete absorption. The result of the substitution of equation (14) in equation (15) is

$$A\Delta\nu = 2\left(\frac{Saw}{\pi}\right)^{1/2}\int_0^\infty \left\{1 - \exp\left[-\frac{1}{x^2}\left(1+\frac{b}{x^2}+\frac{c}{x^4}+\ldots\right)\right]\right\}dx\,, \tag{19}$$

where

$$b = \left(\tfrac{3}{2} - a^2\right)\beta\,,\qquad c = \left(\tfrac{15}{4} - 5a^2 + a^4\right)\beta^2\,,\qquad \beta = \frac{\pi\Delta\nu_D^2}{Saw\ \ln 2}\,.$$

This integral can be evaluated by expanding the $\exp(-b/x^4)$ and $\exp(-c/x^6)$ into a power series. The resulting integrals are all well known. The fractional absorption is thus found to be

$$A\Delta\nu = 2\,(Saw)^{1/2}\left\{1+\tfrac{1}{4}b+\tfrac{3}{8}c-\tfrac{15}{32}b^2-\tfrac{105}{8}bc-\tfrac{945}{128}c^2+\ldots\right\}. \tag{20}$$

The fractional absorption for a line that is sufficiently strong so that $b \ll 1$ and $c \ll 1$ is given by the familiar square-root law, $A\Delta\nu = 2(Saw)^{1/2}$, usually derived from the Lorentz line shape.[4, 15] If $\alpha \gg \Delta\nu_D$, the square-root law results when $Sw \gg \alpha$, from the definitions of b, c, and β in equation (19). On the other hand, if $\Delta\nu_D \gg \alpha$, the square-root law still results if $Saw \gg \Delta\nu_D^2$. In other cases the fractional absorption due to both Doppler and Lorentz widths can be calculated from equation (20). It is interesting to note that even when $\Delta\nu_D \gg \alpha$ the main contribution to the absorption for a strong line comes from the wings of the Lorentz line shape and not from the Doppler shape, since the latter falls off exponentially from the line center.

ATMOSPHERIC RADIATION TRANSFER

In the calculation of radiation transfer in the atmosphere, the information usually desired is the amount of radiation transferred over a frequency interval large compared to the half-width of a particular line, rather than the amount transferred at a single particular frequency. It is well known[5, 6, 8, 9, 10, 11] that the transfer of radiation between any two atmospheric layers, between the ground and an atmospheric layer, and between an atmospheric layer and space can be calculated if the transmission function, $\tau(u_0, u_1)$, is known, where

$$\tau(u_0, u_1) = \exp\left\{-\sec\theta\int_{u_0}^{u_1} k(\nu, u, a)\,du\right\}, \tag{21}$$

θ being the angle included between the beam and the vertical. The optical thickness in

[16] O. Struve and C. T. Elvey, *Ap. J.*, **79**, 409, 1934. The result for strong lines is their equation (6) and for weak lines is equation (5). In the latter equation the first factor 2 is incorrect and should be omitted.

the atmosphere, u, is defined as

$$u = \int_z^{\infty} c \rho d z = \int_z^{\infty} \rho_r d z , \tag{22}$$

where z is the vertical distance and the fractional concentration, c, is the ratio of the density of the radiating gas, ρ_r, to the total density, ρ. It should be noted that the optical thickness defined here decreases with height and is zero at the top of the atmosphere.

Consider the solution of Schwarzschild's equation for the intensity, I, of a beam directed upward at an angle θ to the vertical, at the height where the optical thickness has the value u_0

$$I(u_0) = I_0(u_1)\, \tau(u_0, u_1) + \sec\theta \int_{u_0}^{u_1} k(\nu, u, a)\, I_b(u)\, \tau(u_0, u)\, du , \tag{23}$$

where $I_0(u_1)$ is the incident intensity at the level u_1 and I_b is the black-body intensity, which is a function of u if the temperature varies with height. From equation (23) and a similar equation for radiation directed downward, expressions can be derived for the solution of any problem involving radiation exchange. Since all these expressions can be written in terms of the transmission function, $\tau(u_0, u_1)$, a knowledge of this function enables one to calculate the answer to any atmospheric radiation-transfer problem.

Here we calculate the contribution of the Doppler effect to the atmospheric radiation exchange. The Doppler width makes an appreciable contribution to the total line width for the absorbing bands in the atmosphere only at heights greater than 20 km. The overlapping of the lines in the band can be neglected for almost all lines at heights greater than 20 km because of the very low pressures. Therefore, we calculate the transmission function only for a single line. The contribution from the entire band is the sum of the contributions from each line when the lines do not overlap appreciably. In order to simplify the following calculations, we also assume that the atmosphere is isothermal. Since the Doppler width varies as the square root of the temperature, the width is only about 30 per cent greater at the temperature maximum near 60 km than it is at the temperature minimum in the stratosphere. The results below show that the atmospheric transmission in many cases would be very insensitive to a change of the Doppler width of this order of magnitude.

The integrated absorption for a line, Λ, defined by

$$\Lambda(u_0, u_1) = \int_{-\infty}^{\infty} [1 - \tau(u_0, u_1)]\, d\nu , \tag{24}$$

will be calculated for each of three cases. As discussed above, the radiation exchange over a frequency range large compared to the half-width can be calculated, once the integrated absorption, Λ, is known. As an example of the usefulness of Λ, the radiation loss to space, dR, into a solid angle $d\Omega$ at an angle θ to the vertical from an atmospheric layer between u_0 and u_1 is calculated under the assumptions of constant temperature with height and negligible overlapping of the absorption lines. Under these conditions, it has been shown that[5, 9]

$$dR = I_b \cos\theta\, [\Lambda(0, u_1) - \Lambda(0, u_0)]\, d\Omega . \tag{25}$$

According to kinetic theory, the Lorentz half-width is proportional to the total pressure, provided that the partial pressure of the absorbing gas is small compared to the total pressure. This condition is satisfied for all the absorbing gases in the earth's atmosphere. Therefore, the half-width,

$$a = \left(\frac{a_s}{p_s}\right) p , \tag{26}$$

where α_s is the half-width at the standard pressure p_s. A slightly more complicated equation applies if the absorbing gas is a major constituent of the atmosphere. The natural breadth is negligible for planetary atmospheres.

The dimensionless constant, γ, introduced by Strong and Plass[17] is defined by

$$\gamma = \frac{S c p_s}{2\pi \alpha_s g} . \tag{27}$$

When the Doppler width is negligible, the strong or weak-line approximation applies when γ is much greater or less than unity, respectively.[5]

Case I—weak-line approximation.—It will be assumed that the argument of the exponential in equation (21) is less than 1, even at the line center. In this case the exponential may be expanded, the order of integration reversed, and k replaced by the expression from equation (11). Then by an integration similar to that described in the section on laboratory absorption (case I), the result is obtained that

$$\Lambda (u_0, u_1) = S (u_1 - u_0) \sec \theta . \tag{28}$$

This result is valid even if the temperature varies with height (provided that S does not change appreciably over this temperature range) and even if the fractional concentration of the absorbing gas, c, varies with height.

If equation (28) is substituted in equation (25) and the result integrated over the hemisphere, the radiation loss to space from the layer between u_0 and u_1 for an isothermal atmosphere with constant c is found to be

$$R = 2 f_b S (u_1 - u_0) = 4\pi f_b \gamma (\alpha_1 - \alpha_0) , \tag{29}$$

where f_b is the black-body flux, $f_b = \pi I_b$. Thus the radiation loss is the same as though there were only Lorentz and no Doppler broadening.

Case II—Doppler width alone.—The result of the substitution of the first term of equation (9) in equation (24) is

$$\Lambda (u_0, u_1) = 2\Delta\nu_D (\ln 2)^{-1/2} \int_0^\infty \{ 1 - \exp [- k_0 (u_0 - u_1) \sec \theta \exp (- \omega^2)] \} d\omega . \tag{30}$$

This integral is similar in form to the one considered by Struve and Elvey.[16] By the use of this result, we find, for weak lines,

$$\Lambda (u_0, u_1) = S (u_0 - u_1) \sec \theta \sum_{n=0}^{\infty} \frac{(- 1)^n [k_0 (u_0 - u_1) \sec \theta]^n}{(n+1)!(n+1)^{1/2}} \tag{31}$$

and, for strong lines,

$$\Lambda (u_0, u_1) = \frac{2\Delta\nu_D}{(\ln 2)^{1/2}} \{ \ln [k_0 (u_0 - u_1) \sec \theta] \}^{1/2} \times \left\{ 1 + \frac{C}{2 \ln [k_0 (u_0 - u_1) \sec \theta} - \ldots \right\} , \tag{32}$$

where C is Euler's constant. The radiation loss to space into the solid angle $d\Omega$ can be obtained immediately from equation (25). The total radiation loss to space can be obtained by integrating equation (32) over the hemisphere. The radiation exchange between the highest layers of a planetary atmosphere can be calculated from these expressions for the integrated absorption.

Case III—strong-line approximation.—The majority of radiation is transferred by the

[17] The constant, γ, defined here differs by a factor $\cos \theta$ from the constant used by Strong and Plass, *op. cit.*

strong lines of the radiating gases in the atmosphere.[5, 6] In order to calculate the influence of the Doppler effect on this radiation transfer when the change of the Lorentz half-width is taken into account, we substitute equation (14) in equation (21). With the help of equations (26) and (27) and after an elementary integration, the result is obtained that

$$\tau(u_0, u_1) = \exp\left\{-\frac{1}{x^2}\left[1+\frac{f\cos\theta}{x^2}+\frac{g\cos^2\theta}{x^4}+\ldots\right]\right\}, \tag{33}$$

where

$$x^2 = \frac{(\nu-\nu_0)^2}{\gamma \sec\theta\,(\alpha_1^2-\alpha_0^2)^2},$$

$$f = \frac{1}{\gamma(\alpha_1^2-\alpha_0^2)}\left[\frac{3\Delta\nu_D^2}{2\ln 2}-\frac{\alpha_1^2+\alpha_0^2}{2}\right],$$

$$g = \frac{1}{\gamma^2(\alpha_1^2-\alpha_0^2)^2}\left[\frac{15\Delta\nu_D^4}{4(\ln 2)^2}-\frac{5(\alpha_1^2+\alpha_0^2)\Delta\nu_D^2}{2\ln 2}+\frac{\alpha_1^4+\alpha_1^2\alpha_0^2+\alpha_0^4}{3}\right],$$

α_0 and α_1 being the Lorentz half-widths at the heights where the optical thicknesses are u_0 and u_1, respectively.

In this case the integrated absorption, Λ, from equation (24) is

$$\Lambda(u_0, u_1) = 2\gamma^{1/2}(\alpha_1^2-\alpha_0^2)^{1/2}\sec^{1/2}\theta\int_0^\infty [1-\tau(u_0, u_1)]\,dx. \tag{34}$$

Substituting for τ from equation (33), the resulting integral has the identical form of equation (19). From the previous evaluation of this integral as given by equation (20), the result in the present case can be immediately written as

$$\Lambda(u_0, u_1) = 2\pi^{1/2}\gamma^{1/2}(\alpha_1^2-\alpha_0^2)^{1/2}\sec^{1/2}\theta\,[1+\tfrac{1}{4}f\cos\theta+\tfrac{3}{8}g\cos^2\theta - \tfrac{15}{32}f^2\cos^2\theta-\tfrac{105}{8}fg\cos^3\theta-\tfrac{945}{128}g^2\cos^4\theta+\ldots]. \tag{35}$$

The radiation loss from an atmospheric layer to space in the solid angle $d\Omega$ is obtained by the substitution of this expression in equation (25). The total radiation loss from the top of the atmosphere down to the level $u = u_1$, obtained by integrating this expression over the hemisphere, is

$$R = \tfrac{8}{3}f_b\pi^{1/2}\gamma^{1/2}\alpha_1\,[1+\tfrac{3}{20}f_0+\tfrac{9}{56}g_0-\tfrac{45}{224}f_0^2-\tfrac{35}{8}f_0g_0-\tfrac{2835}{1408}g_0^2+\ldots], \tag{36}$$

where f_0 and g_0 are the values of f and g when $\alpha_0 = 0$. Similarly, the radiation exchange between any two layers can be calculated from equation (35).

The terms outside the brackets in equations (35) and (36) are identical with the expressions for the radiation loss from a strong Lorentz-broadened line as derived by Strong and Plass.[5] This term is obtained for the Lorentz line shape if the line absorption at large distances from the line center is assumed to vary as $(\nu - \nu_0)^{-2}$. The terms inside the brackets of equations (35) and (36) represent the change in the radiation loss due to the fact that the actual line shape does not vary as $(\nu - \nu_0)^{-2}$ as the line center is approached. Again these terms agree with the result of Strong and Plass[5] when the Doppler width is zero.

The additional terms in equations (35) and (36) show clearly when the radiation loss is different from the loss calculated by the use of the Lorentz line shape alone. The radiation loss is clearly unchanged from the Lorentz result when $f_0 \ll 1$ and $g_0 \ll 1$. For these inequalities to be satisfied, it follows from the definitions of these quantities that it is sufficient that $(\Delta\nu_D^2/\alpha_1^2) \ll \gamma$ and $\gamma \gg 1$. The major part of the atmospheric radiation is transferred by the very strong lines in the atmospheric bands. There are many lines in our atmosphere with $\gamma > 100$. For these most important lines, the Doppler width can become somewhat larger than the Lorentz width and still satisfy the above inequality. Therefore, we conclude that, *for the very strong atmospheric lines* ($\gamma > 100$), *the radiative transfer is the same as would be calculated from the Lorentz shape alone for heights up to 50 km*. The physical reason for this fact is that only the radiation transfer in the far wings is important for a very strong line. However, in the far wings the line shape is accurately represented by the Lorentz equation, even when the Doppler width is larger than the Lorentz width, since the Doppler line shape falls off exponentially. The results of case I show that, for a weak line, *the integrated absorption and radiation loss are the same, regardless of the relative contributions of the Lorentz and Doppler widths to the total width*. The physical reason for this result is that the total emission of a weak line determines the radiation loss, since the absorption by higher layers is very small.

The above results make possible an enormous simplification in the radiation calculations near the temperature maximum in the stratosphere (60 km). For a very strong line, the radiation is not influenced to any appreciable extent by the Doppler width. Thus the change in the Doppler width with temperature need not be taken into account. Similarly for the weak lines, the relative contributions of the Doppler and Lorentz widths to the total width need not concern us. It is only for the lines of intermediate strength at altitudes less than 50 km that the Doppler effect alters the radiative transfer calculated from the Lorentz shape alone.

In calculating the radiation transfer from an atmospheric band, the band is usually divided into several frequency intervals, each sufficiently large that it contains a number of individual lines. For the intervals at or near the center of the band, either the lines are all very strong lines, or the very strong lines completely dominate the weak lines in the radiation transferred. For such intervals the conclusions drawn above for very strong lines are valid. For the intervals near the edge of the band, all the lines usually satisfy the weak-line criterion. It is only intervals between the edge and center of the band that usually contain lines of intermediate strength and for which more elaborate calculations are necessary. However, since the frequency intervals may be chosen as large as desired, provided that the black-body intensity is approximately constant in the interval, it is usually possible to choose the intervals so that the majority of the lines in a given interval satisfy either the strong- or the weak-line criterion. When it is possible to divide the band into frequency intervals in this manner, the contribution of the Doppler effect to radiative transfer can be neglected for heights up to at least 50 km.