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NOTE

THE DERIVATIVES OF DAWSON'S FUNCTION

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Abstract—Higher-order derivatives of Dawson's function are given in terms of Hermite and other polynomials.

HIGH order derivatives of Dawson's function,

$$F(t) \equiv e^{-t^2}\int_0^t e^{x^2}\,dx = t\,{}_1F_1(1,\tfrac{3}{2},-t^2), \tag{1}$$

occur in radiative transfer,[1,2] as well as in many other areas. The purpose of this note is to present an explicit formula for the nth derivative of $F(t)$ in terms of Hermite polynomials and an additional set of polynomials.

If we repeatedly differentiate equation (1), then we may show by induction that

$$\frac{d^n}{dt^n}F(t) = (-1)^n[H_n(t)F(t) - G_{n-1}(t)] \tag{2}$$

where $H_n(t)$ is the Hermite polynomial

$$H_n(t) = (-1)^n e^{t^2}\frac{d^n}{dt^n}e^{-t^2}, \tag{3}$$

$G_n(t)$ is a polynomial of degree n in t of which the first eight are $G_0(t) = 1$, $G_1(t) = 2t$, $G_2(t) = 4t^2-4$, $G_3(t) = 8t^3-20t$, $G_4(t) = 16t^4-72t^2+32$, $G_5(t) = 32t^5-224t^3+264t$, $G_6(t) = 64t^6-640t^4+1392t^2-384$, $G_7(t) = 128t^7-1728t^5+5520t^3-4464t$. These G-polynomials satisfy the three-term recurrence relation

$$G_n(t) = 2tG_{n-1}(t) - 2nG_{n-2}(t) \tag{4}$$

so that higher-order polynomials are easily generated.

Equation (2) is much easier to handle than the usual formula for the higher derivatives, viz.,

$$\frac{d^{k+1}}{dt^{k+1}}F(t) = -2t\frac{d^k}{dt^k}F(t) - 2k\frac{d^{k-1}}{dt^{k-1}}F(t), \qquad (k \geq 1). \tag{5}$$

The above results allow us to evaluate two integrals related to derivatives of Dawson's function. We recall that $F(t)$ can also be written in the form

$$F(t) = \int_0^\infty e^{-x^2} \sin 2tx \, dx. \quad (6)$$

Differentiating equation (6) an even or odd number of times and utilizing equation (2), one has

$$\int_0^\infty x^{2n} e^{-x^2} \sin 2tx \, dx = \frac{(-1)^n}{2^{2n}}[H_{2n}(t)F(t) - G_{2n-1}(t)], \quad (7)$$

$$\int_0^\infty x^{2n-1} e^{-x^2} \cos 2tx \, dx = \frac{(-1)^n}{2^{2n-1}}[H_{2n-1}(t)F(t) - G_{2n-2}(t)], \quad (8)$$

with $n = 1, 2, \ldots$. MIDDLETON[3] has evaluated these integrals in terms of confluent hypergeometric functions. If we compare the right-hand side of equations (7) and (8) with his results, we obtain the following interesting relations:

$${}_1F_1(n+1, \tfrac{3}{2}, -t^2) = \frac{(-1)^n}{2^{2n} n! t}[H_{2n}(t)F(t) - G_{2n-1}(t)], \quad (9)$$

$${}_1F_1(n, \tfrac{1}{2}, -t^2) = \frac{(-1)^n}{2^{2n-2}(n-1)!}[H_{2n-1}(t)F(t) - G_{2n-2}(t)]. \quad (10)$$

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