

MINOR ARCS FOR GOLDBACH'S PROBLEM

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ABSTRACT. The ternary Goldbach conjecture states that every odd number $n \geq 7$ is the sum of three primes. The estimation of sums of the form $\sum_{p \leq x} e(\alpha p)$, $\alpha = a/q + O(1/q^2)$, has been a central part of the main approach to the conjecture since (Vinogradov, 1937). Previous work required q or x to be too large to make a proof of the conjecture for all n feasible.

The present paper gives new bounds on minor arcs and the tails of major arcs. This is part of the author's proof of the ternary Goldbach conjecture.

The new bounds are due to several qualitative improvements. In particular, this paper presents a general method for reducing the cost of Vaughan's identity, as well as a way to exploit the tails of minor arcs in the context of the large sieve.

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1. INTRODUCTION

The ternary Goldbach conjecture (or *three-prime conjecture*) states that every odd number greater than 5 is the sum of three primes. I. M. Vinogradov [Vin37]

showed in 1937 that every odd integer larger than a very large constant C is indeed the sum of three primes. His work was based on the study of exponential sums

$$\sum_{n \leq N} \Lambda(n) e(\alpha n)$$

and their use within the circle method.

Unfortunately, further work has so far reduced C only to e^{3100} ([LW02]; see also [CW89]), which is still much too large for all odd integers up to C to be checked numerically. The main problem has been that existing bounds for (1.1) in the *minor arc* regime – namely, $\alpha = a/q + O(1/q^2)$, $\gcd(a, q) = 1$, q relatively large – have not been strong enough.

The present paper gives new bounds on smoothed exponential sums

$$(1.1) \quad S_\eta(\alpha, x) = \sum_n \Lambda(n) e(\alpha n) \eta(n/x).$$

These bounds are clearly stronger than those on smoothed or unsmoothed exponential sums in the previous literature, including the bounds of [Tao]. (See also work by Ramaré [Ram10].)

In particular, on all arcs around a/q , $q > 1.5 \cdot 10^5$ odd or $q > 3 \cdot 10^5$ even, the bounds are of the strength required for a full solution to the three-prime conjecture. The same holds on the tails of arcs around a/q for smaller q .

(The remaining arcs – namely, those around a/q , q small – are the *major arcs*; they are treated in the companion paper [Hela].)

The quality of the results here is due to several new ideas of general applicability. In particular, §4.1 introduces a way to obtain cancellation from Vaughan’s identity. Vaughan’s identity is a two-log gambit, in that it introduces two convolutions (each of them at a cost of log) and offers a great deal of flexibility in compensation. One of the ideas in the present paper is that at least one of two logs can be successfully recovered after having been given away in the first stage of the proof. This reduces the cost of the use of this basic identity in this and, presumably, many other problems.

We will also see how to exploit being on the tail of a major arc, whether in the large sieve (Lemma 4.3, Prop. 4.6) or in other contexts.

There are also several technical improvements that make a qualitative difference; see the discussions at the beginning of §3 and §4. Considering smoothed sums – now a common idea – also helps. (Smooth sums here go back to Hardy-Littlewood [HL23] – both in the general context of the circle method and in the context of Goldbach’s ternary problem. In recent work on the problem, they reappear in [Tao].)

1.1. Results. The main bound we are about to see is essentially proportional to $((\log q)/\sqrt{\phi(q)}) \cdot x$. The term δ_0 serves to improve the bound when we are on the tail of an arc.

Main Theorem. *Let $x \geq x_0$, $x_0 = 2.16 \cdot 10^{20}$. Let $S_\eta(\alpha, x)$ be as in (1.1), with η defined in (1.4). Let $2\alpha = a/q + \delta/x$, $q \leq Q$, $\gcd(a, q) = 1$, $|\delta/x| \leq 1/qQ$, where $Q = (3/4)x^{2/3}$. If $q \leq x^{1/3}/6$, then*

$$(1.2) \quad |S_\eta(\alpha, x)| \leq \frac{R_{x, \delta_0 q} \log \delta_0 q + 0.5}{\sqrt{\delta_0 \phi(q)}} \cdot x + \frac{2.5x}{\sqrt{\delta_0 q}} + \frac{2x}{\delta_0 q} \cdot L_{x, \delta_0, q} + 3.2x^{5/6},$$

where

$$(1.3) \quad \begin{aligned} \delta_0 &= \max(2, |\delta|/4), & R_{x,t} &= 0.27125 \log \left(1 + \frac{\log 4t}{2 \log \frac{9x^{1/3}}{2.004t}} \right) + 0.41415 \\ L_{x,\delta,q} &= \min \left(\frac{\log \delta^7 q^{\frac{13}{4}} + \frac{80}{9}}{\phi(q)/q}, \frac{5}{6} \log x + \frac{50}{9} \right) + \log q \frac{80}{9} \delta^{\frac{16}{9}} + \frac{111}{5}. \end{aligned}$$

If $q > x^{1/3}/6$, then

$$|S_\eta(\alpha, x)| \leq 0.2727x^{5/6}(\log x)^{3/2} + 1218x^{2/3} \log x.$$

The factor $R_{x,t}$ is small in practice; for instance, for $x = 10^{25}$ and $\delta_0 q = 5 \cdot 10^5$ (typical “difficult” values), $R_{x,\delta_0 q}$ equals $0.59648 \dots$.

The classical choice¹ for η in (1.1) is $\eta(t) = 1$ for $t \leq 1$, $\eta(t) = 0$ for $t > 1$, which, of course, is not smooth, or even continuous. We use

$$(1.4) \quad \eta(t) = \eta_2(t) = 4 \max(\log 2 - |\log 2t|, 0),$$

as in Tao [Tao], in part for purposes of comparison. (This is the multiplicative convolution of the characteristic function of an interval with itself.) Nearly all work should be applicable to any other sufficiently smooth function η of fast decay. It is important that $\hat{\eta}$ decay at least quadratically.

1.2. History. The following notes are here to provide some background; no claim to completeness is made.

Vinogradov’s proof [Vin37] was based on his novel estimates for exponential sums over primes. Most work on the problem since then, including essentially all work with explicit constants, has been based on estimates for exponential sums; there are some elegant proofs based on cancellation in other kinds of sums ([HB85], [IK04, §19]), but they have not been made to yield practical estimates.

The earliest explicit result is that of Vinogradov’s student Borodzín (1939). Vaughan [Vau77] greatly simplified the proof by introducing what is now called Vaughan’s identity.

The current record is that of Liu and Wang [LW02]: the best previous result was that of [CW89]. Other recent work falls into the following categories.

Conditional results. The ternary Goldbach conjecture has been proven under the assumption of the generalized Riemann hypothesis [DEtRZ97].

Ineffective results. An example is the bound given by Buttkewitz [But11]. The issue is usually a reliance on the Siegel-Walfisz theorem. In general, to obtain effective bounds with good constants, it is best to avoid analytic results on L -functions with large conductor. (The present paper implicitly uses known results on the Riemann ζ function, but uses nothing at all about other L -functions.)

Results based on Vaughan’s identity. Vaughan’s identity [Vau77] greatly simplified matters; most textbook treatments are by now based on it. The minor-arc treatment in [Tao] updates this approach to current technical standards (smoothing), while taking advantage of its flexibility (letting variable ranges depend on q).

Results based on log-free identities. Using Vaughan’s identity implies losing a factor of $(\log x)^2$ (or $(\log q)^2$, as in [Tao]) in the first step. It thus makes sense to consider other identities that do not involve such a loss. Most approaches before

¹Or, more precisely, the choice made by Vinogradov and followed by most of the literature since him. Hardy and Littlewood [HL23] worked with $\eta(t) = e^{-t}$.

| q_0 | $\frac{ S_\eta(a/q, x) }{x}$, HH | $\frac{ S_\eta(a/q, x) }{x}$, Tao |
|------------------|-----------------------------------|------------------------------------|
| 10^9 | 0.04522 | 0.34475 |
| $1.5 \cdot 10^5$ | 0.03821 | 0.28836 |
| $2.5 \cdot 10^5$ | 0.03097 | 0.23194 |
| $5 \cdot 10^5$ | 0.02336 | 0.17416 |
| $7.5 \cdot 10^5$ | 0.01984 | 0.14775 |
| 10^6 | 0.01767 | 0.13159 |
| 10^7 | 0.00716 | 0.05251 |

TABLE 1. Worst-case upper bounds on $x^{-1}|S_\eta(a/2q, x)|$ for $q \geq q_0$, $|\delta| \leq 8$, $x = 10^{27}$. The trivial bound is 1.

Vaughan’s identity involved larger losses, but already [Vin37, §9] is relatively economical, at least for very large x . The work of Daboussi [Dab96] and Daboussi and Rivat [DR01] explores other identities. (A reading of [DR01] gave part of the initial inspiration for the present work.) Ramaré’s work [Ram10] – asymptotically the best to date – is based on the Diamond-Steinig inequality (for k large).

* * *

The author’s work on the subject leading to the present paper was at first based on the (log-free) Bombieri-Selberg identity ($k = 3$), but has now been redone with Vaughan’s identity in its foundations. This is feasible thanks to the factor of log regained in §4.1.

1.3. Comparison to earlier work. Table 1 compares the bounds for the ratio $|S_\eta(a/q, x)|/x$ given by this paper and by [Tao] for $x = 10^{27}$ and different values of q . We are comparing worst cases: $\phi(q)$ as small as possible (q divisible by $2 \cdot 3 \cdot 5 \cdots$) in the result here, and q divisible by 4 (implying $4\alpha \sim a/(q/4)$) in Tao’s result. The main term in the result in this paper improves slowly with increasing x ; the results in [Tao] worsen slowly with increasing x .

The qualitative gain with respect to [Tao] is about $\log(q)\sqrt{\phi(q)/q}$, which is $\sim \log(q)/\sqrt{e^\gamma(\log \log q)}$ in the worst case.

The results in [DR01] are unfortunately worse than the trivial bound in this range. Ramaré’s results ([Ram10, Thm. 3], [Ramd, Thm. 6]) are not applicable within the range, since neither of the conditions $\log q \leq (1/50)(\log x)^{1/3}$, $q \leq x^{1/48}$ is satisfied. Ramaré’s bound in [Ramd, Thm. 6] is

$$(1.5) \quad \left| \sum_{x < n \leq 2x} \Lambda(n)e(an/q) \right| \leq 13000 \frac{\sqrt{q}}{\phi(q)} x$$

for $20 \leq q \leq x^{1/48}$. We should underline that, while both the constant 13000 and the condition $q \leq x^{1/48}$ keep (1.5) from being immediately useful in the present context, (1.5) is asymptotically better than the results here as $q \rightarrow \infty$. (Indeed, qualitatively speaking, the form of (1.5) is the best one can expect from results derived by the family of methods stemming from [Vin37].) There is also unpublished work by Ramaré (ca. 1993) with better constants for $q \ll (\log x / \log \log x)^4$.

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2. PRELIMINARIES

2.1. Notation. Given positive integers m, n , we say $m|n^\infty$ if every prime dividing m also divides n . We say a positive integer n is *square-full* if, for every prime p dividing n , the square p^2 also divides n . (In particular, 1 is square-full.) We say n is *square-free* if $p^2 \nmid n$ for every prime p . For p prime, n a non-zero integer, we define $v_p(n)$ to be the largest non-negative integer α such that $p^\alpha | n$.

When we write \sum_n , we mean $\sum_{n=1}^\infty$, unless the contrary is stated. As usual, μ , Λ , τ and σ denote the Moebius function, the von Mangoldt function, the divisor function and the sum-of-divisors function, respectively.

As is customary, we write $e(x)$ for $e^{2\pi i x}$. We write $|f|_r$ for the L_r norm of a function f .

We write $O^*(R)$ to mean a quantity at most R in absolute value.

2.2. Fourier transforms and exponential sums. The Fourier transform on \mathbb{R} is normalized here as follows:

$$\widehat{f}(t) = \int_{-\infty}^{\infty} e(-xt) f(x) dx.$$

If f is compactly supported (or of fast decay) and piecewise continuous, $\widehat{f}(t) = \widehat{f}'(t)/(2\pi it)$ by integration by parts. Iterating, we obtain that, if f is compactly supported, continuous and piecewise C^1 , then

$$(2.1) \quad \widehat{f}(t) = O^* \left(\frac{|\widehat{f''}|_\infty}{(2\pi t)^2} \right) = O^* \left(\frac{|f''|_1}{(2\pi t)^2} \right),$$

and so \widehat{f} decays at least quadratically.

The following bound is standard (see, e.g., [Tao, Lemma 3.1]): for $\alpha \in \mathbb{R}/\mathbb{Z}$ and $f : \mathbb{R} \rightarrow \mathbb{C}$ compactly supported and piecewise continuous,

$$(2.2) \quad \left| \sum_{n \in \mathbb{Z}} f(n) e(\alpha n) \right| \leq \min \left(|f|_1 + \frac{1}{2} |f'|_1, \frac{\frac{1}{2} |f''|_1}{|\sin(\pi \alpha)|} \right).$$

(The first bound follows from $\sum_{n \in \mathbb{Z}} |f(n)| \leq |f|_1 + (1/2) |f'|_1$, which, in turn is a quick consequence of the fundamental theorem of calculus; the second bound is proven by summation by parts.) The alternative bound $(1/4) |f''|_1 / |\sin(\pi \alpha)|^2$ given in [Tao, Lemma 3.1] (for f continuous and piecewise C^1) can usually be improved by the following estimate.

Lemma 2.1. *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be compactly supported, continuous and piecewise C^1 . Then*

$$(2.3) \quad \left| \sum_{n \in \mathbb{Z}} f(n) e(\alpha n) \right| \leq \frac{\frac{1}{4} |\widehat{f''}|_\infty}{(\sin \alpha \pi)^2}$$

for every $\alpha \in \mathbb{R}$.

As usual, the assumption of compact support could be easily relaxed to an assumption of fast decay.

Proof. By the Poisson summation formula,

$$\sum_{n=-\infty}^{\infty} f(n) e(\alpha n) = \sum_{n=-\infty}^{\infty} \widehat{f}(n - \alpha).$$

Since $\widehat{f}(t) = \widehat{f'}(t)/(2\pi i t)$,

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n - \alpha) = \sum_{n=-\infty}^{\infty} \frac{\widehat{f'}(n - \alpha)}{2\pi i (n - \alpha)} = \sum_{n=-\infty}^{\infty} \frac{\widehat{f''}(n - \alpha)}{(2\pi i (n - \alpha))^2}.$$

By Euler's formula $\pi \cot s\pi = 1/s + \sum_{n=1}^{\infty} (1/(n+s) - 1/(n-s))$,

$$(2.4) \quad \sum_{n=-\infty}^{\infty} \frac{1}{(n+s)^2} = -(\pi \cot s\pi)' = \frac{\pi^2}{(\sin s\pi)^2}.$$

Hence

$$\left| \sum_{n=-\infty}^{\infty} \widehat{f}(n - \alpha) \right| \leq |\widehat{f''}|_\infty \sum_{n=-\infty}^{\infty} \frac{1}{(2\pi(n - \alpha))^2} = |\widehat{f''}|_\infty \cdot \frac{1}{(2\pi)^2} \cdot \frac{\pi^2}{(\sin \alpha \pi)^2}.$$

□

The trivial bound $|\widehat{f''}|_\infty \leq |f''|_1$, applied to (2.3), recovers the bound in [Tao, Lemma 3.1]. In order to do better, we will give a tighter bound for $|\widehat{f''}|_\infty$ when $f = \eta_2$ in Appendix A.

Integrals of multiples of f'' (in particular, $|f''|_1$ and $\widehat{f''}$) can still be made sense of when f'' is undefined at a finite number of points, provided f is understood as a distribution (and f' has finite total variation). This is the case, in particular, for $f = \eta_2$.

* * *

When we need to estimate $\sum_n f(n)$ precisely, we will use the Poisson summation formula:

$$\sum_n f(n) = \sum_n \widehat{f}(n).$$

We will not have to worry about convergence here, since we will apply the Poisson summation formula only to compactly supported functions f whose Fourier transforms decay at least quadratically.

2.3. Smoothing functions. For the smoothing function η_2 in (1.4),

$$(2.5) \quad |\eta_2|_1 = 1, \quad |\eta_2'|_1 = 8 \log 2, \quad |\eta_2''|_1 = 48,$$

as per [Tao, (5.9)–(5.13)]. Similarly, for $\eta_{2,\rho}(t) = \log(\rho t)\eta_2(t)$, where $\rho \geq 4$,

$$(2.6) \quad \begin{aligned} |\eta_{2,\rho}|_1 &< \log(\rho)|\eta_2|_1 = \log(\rho) \\ |\eta_{2,\rho}'|_1 &= 2\eta_{2,\rho}(1/2) = 2 \log(\rho/2)\eta_2(1/2) < (8 \log 2) \log \rho, \\ |\eta_{2,\rho}''|_1 &= 4 \log(\rho/4) + |2 \log \rho - 4 \log(\rho/4)| + |4 \log 2 - 4 \log \rho| \\ &\quad + |\log \rho - 4 \log 2| + |\log \rho| < 48 \log \rho. \end{aligned}$$

(In the first inequality, we are using the fact that $\log(\rho t)$ is always positive (and less than $\log(\rho)$) when t is in the support of η_2 .)

Write $\log^+ x$ for $\max(\log x, 0)$.

2.4. Bounds on sums of $\mu(m)$ and $\Lambda(n)$. We will need explicit bounds on $\sum_{n \leq N} \mu(n)/n$ and related sums involving μ . The situation here is less well-developed than for sums involving Λ . The main reason is that the complex-analytic approach to estimating $\sum_{n \leq N} \mu(n)$ would involve $1/\zeta(s)$ rather than $\zeta'(s)/\zeta(s)$, and thus strong explicit bounds on the residues of $1/\zeta(s)$ would be needed.

Fortunately all we need is a saving of $(\log n)$ or $(\log n)^2$ on the trivial bound. This is provided by the following.

(1) (Granville-Ramaré [GR96], Lemma 10.2)

$$(2.7) \quad \left| \sum_{n \leq x: \gcd(n,q)=1} \frac{\mu(n)}{n} \right| \leq 1$$

for all $x, q \geq 1$,

(2) (Ramaré [Ramc]; cf. El Marraki [EM95], [EM96])

$$(2.8) \quad \left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq \frac{0.03}{\log x}$$

for $x \geq 11815$.

(3) (Ramaré [Rama])

$$(2.9) \quad \sum_{n \leq x: \gcd(n,q)=1} \frac{\mu(n)}{n} = O^* \left(\frac{1}{\log x/q} \cdot \frac{4}{5} \frac{q}{\phi(q)} \right)$$

for all x and all $q \leq x$;

$$(2.10) \quad \sum_{n \leq x: \gcd(n,q)=1} \frac{\mu(n)}{n} \log \frac{x}{n} = O^* \left(1.00303 \frac{q}{\phi(q)} \right)$$

for all x and all q .

Improvements on these bounds would lead to improvements on type I estimates, but not in what are the worst terms overall at this point.

A computation carried out by the author has proven the following inequality for all real $x \leq 10^{12}$:

$$(2.11) \quad \left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq \sqrt{\frac{2}{x}}$$

The computation was rigorous, in that it used D. Platt's implementation [Pla11] of double-precision interval arithmetic based on Lambov's [Lam08] ideas. For the sake of verification, we record that

$$5.42625 \cdot 10^{-8} \leq \sum_{n \leq 10^{12}} \frac{\mu(n)}{n} \leq 5.42898 \cdot 10^{-8}.$$

Computations also show that the stronger bound

$$\left| \sum_{n \leq x} \frac{\mu(n)}{n} \right| \leq \frac{1}{2\sqrt{x}}$$

holds for all $3 \leq x \leq 7727068587$, but not for $x = 7727068588 - \epsilon$.

Earlier, numerical work carried out by Olivier Ramaré [Ramb] had shown that (2.11) holds for all $x \leq 10^{10}$.

We will make reference to various bounds on $\Lambda(n)$ in the literature. The following bound can be easily derived from [RS62, (3.23)], supplemented by a quick calculation of the contribution of powers of primes $p < 32$:

$$(2.12) \quad \sum_{n \leq x} \frac{\Lambda(n)}{n} \leq \log x.$$

We can derive a bound in the other direction from [RS62, (3.21)] (for $x > 1000$, adding the contribution of all prime powers ≤ 1000) and a numerical verification for $x \leq 1000$:

$$(2.13) \quad \sum_{n \leq x} \frac{\Lambda(n)}{n} \geq \log x - \log \frac{3}{\sqrt{2}}.$$

We also use the following older bounds:

- (1) By the second table in [RR96, p. 423], supplemented by a computation for $2 \cdot 10^6 \leq V \leq 4 \cdot 10^6$,

$$(2.14) \quad \sum_{n \leq y} \Lambda(n) \leq 1.0004y$$

for $y \geq 2 \cdot 10^6$.

(2)

$$(2.15) \quad \sum_{n \leq y} \Lambda(n) < 1.03883y$$

for every $y > 0$ [RS62, Thm. 12].

For all $y > 663$,

$$(2.16) \quad \sum_{n \leq y} \Lambda(n)n < 1.03884 \frac{y^2}{2},$$

where we use (2.15) and partial summation for $y > 200000$, and a computation for $663 < y \leq 200000$. Using instead the second table in [RR96, p. 423], together with computations for small $y < 10^7$ and partial summation, we get that

$$(2.17) \quad \sum_{n \leq y} \Lambda(n)n < 1.0008 \frac{y^2}{2}$$

for $y > 1.6 \cdot 10^6$.

Similarly,

$$(2.18) \quad \sum_{n \leq y} \Lambda(n) < 2 \cdot 1.0004 \sqrt{y}$$

for all $y \geq 1$.

It is also true that

$$(2.19) \quad \sum_{y/2 < p \leq y} (\log p)^2 \leq \frac{1}{2} y (\log y)$$

for $y \geq 117$: this holds for $y \geq 2 \cdot 758699$ by [RS75, Cor. 2] (applied to $x = y$, $x = y/2$ and $x = 2y/3$) and for $117 \leq y < 2 \cdot 758699$ by direct computation.

2.5. Basic setup. We begin by applying Vaughan's identity [Vau77]: for any function $\eta : \mathbb{R} \rightarrow \mathbb{R}$, any completely multiplicative function $f : \mathbb{Z}^+ \rightarrow \mathbb{C}$ and any $x > 0$, $U, V \geq 0$,

$$(2.20) \quad \sum_n \Lambda(n) f(n) e(\alpha n) \eta(n/x) = S_{I,1} - S_{I,2} + S_{II} + S_{0,\infty},$$

where

$$(2.21) \quad \begin{aligned} S_{I,1} &= \sum_{m \leq U} \mu(m) f(m) \sum_n (\log n) e(\alpha m n) f(n) \eta(mn/x), \\ S_{I,2} &= \sum_{d \leq V} \Lambda(d) f(d) \sum_{m \leq U} \mu(m) f(m) \sum_n e(\alpha d m n) f(n) \eta(dmn/x), \\ S_{II} &= \sum_{m > U} f(m) \left(\sum_{\substack{d > U \\ d|m}} \mu(d) \right) \sum_{n > V} \Lambda(n) e(\alpha m n) f(n) \eta(mn/x), \\ S_{0,\infty} &= \sum_{n \leq V} \Lambda(n) e(\alpha n) f(n) \eta(n/x). \end{aligned}$$

The proof is essentially an application of the Möbius inversion formula; see, e.g., [IK04, §13.4]. In practice, we will use the function

$$(2.22) \quad f(n) = \begin{cases} 1 & \text{if } \gcd(n, v) = 1, \\ 0 & \text{otherwise,} \end{cases}$$

where v is a small, positive, square-free integer. (Our final choice will be $v = 2$.) Then

$$(2.23) \quad S_\eta(x, \alpha) = S_{I,1} - S_{I,2} + S_{II} + S_{0,\infty} + S_{0,w},$$

where $S_\eta(x, \alpha)$ is as in (1.1) and

$$S_{0,w} = \sum_{n|v} \Lambda(n) e(\alpha n) \eta(n/x).$$

The sums $S_{I,1}$, $S_{I,2}$ are called “of type I” (or linear), the sum S_{II} is called “of type II” (or bilinear). The sum S_0 is in general negligible; for our later choice of V and η , it will be in fact 0. The sum $S_{0,w}$ will be negligible as well.

3. TYPE I

There are here three main improvements in comparison to standard treatments:

- (1) The terms with m divisible by q get taken out and treated separately by analytic means. This all but eliminates what would otherwise be the main term.
- (2) For large m , the other terms get handled by improved estimates on trigonometric sums.
- (3) The “error” term $\delta/x = \alpha - a/q$ is used to our advantage. This happens both through the Poisson summation formula and through the use of two successive approximations.

3.1. Trigonometric sums. The following lemmas on trigonometric sums improve on the best Vinogradov-type lemmas in the literature. (By this, we mean results of the type of Lemma 8a and Lemma 8b in [Vin04, Ch. I]. See, in particular, the work of Daboussi and Rivat [DR01, Lemma 1].) The main idea is to switch between different types of approximation within the sum, rather than just choosing between bounding all terms either trivially (by A) or non-trivially (by $C/|\sin(\pi\alpha n)|^2$). There will also² be improvements in our applications stemming from the fact that Lemmas 3.1 and Lemma 3.2 take quadratic ($|\sin(\pi\alpha n)|^2$) rather than linear ($|\sin(\pi\alpha n)|$) inputs.

Lemma 3.1. *Let $\alpha = a/q + \beta/qQ$, $\gcd(a, q) = 1$, $|\beta| \leq 1$, $q \leq Q$. Then, for any $A, C \geq 0$,*

$$(3.1) \quad \sum_{y < n \leq y+q} \min \left(A, \frac{C}{|\sin(\pi\alpha n)|^2} \right) \leq \min \left(2A + \frac{6q^2}{\pi^2} C, 3A + \frac{4q}{\pi} \sqrt{AC} \right).$$

Proof. We start by letting $m_0 = \lfloor y \rfloor + \lfloor (q+1)/2 \rfloor$, $j = n - m_0$, so that j ranges in the interval $(-q/2, q/2]$. We write

$$\alpha n = \frac{aj + c}{q} + \delta_1(j) + \delta_2 \pmod{1},$$

where $|\delta_1(j)|$ and $|\delta_2|$ are both $\leq 1/2q$; we can assume $\delta_2 \geq 0$. The variable $r = aj + c \pmod{q}$ occupies each residue class \pmod{q} exactly once.

One option is to bound the terms corresponding to $r = 0, -1$ by A each and all the other terms by $C/|\sin(\pi\alpha n)|^2$. The terms corresponding to $r = -k$ and $r = k - 1$ ($2 \leq k \leq q/2$) contribute at most

$$\frac{1}{\sin^2 \frac{\pi}{q} (k - \frac{1}{2} - q\delta_2)} + \frac{1}{\sin^2 \frac{\pi}{q} (k - \frac{3}{2} + q\delta_2)} \leq \frac{1}{\sin^2 \frac{\pi}{q} (k - \frac{1}{2})} + \frac{1}{\sin^2 \frac{\pi}{q} (k - \frac{3}{2})},$$

since $x \mapsto \frac{1}{(\sin x)^2}$ is convex-up on $(0, \infty)$. Hence the terms with $r \neq 0, 1$ contribute at most

$$\frac{1}{\left(\sin \frac{\pi}{2q}\right)^2} + 2 \sum_{2 \leq r \leq \frac{q}{2}} \frac{1}{\left(\sin \frac{\pi}{q} (r - 1/2)\right)^2} \leq \frac{1}{\left(\sin \frac{\pi}{2q}\right)^2} + 2 \int_1^{q/2} \frac{1}{\left(\sin \frac{\pi}{q} x\right)^2},$$

²This is a change with respect to the first version of this paper’s preprint [Helb]. The version of Lemma 3.1 there has, however, the advantage of being immediately comparable to results in the literature.

where we use again the convexity of $x \mapsto 1/(\sin x)^2$. (We can assume $q > 2$, as otherwise we have no terms other than $r = 0, 1$.) Now

$$\int_1^{q/2} \frac{1}{\left(\sin \frac{\pi}{q} x\right)^2} dx = \frac{q}{\pi} \int_{\frac{\pi}{q}}^{\frac{\pi}{2}} \frac{1}{(\sin u)^2} du = \frac{q}{\pi} \cot \frac{\pi}{q}.$$

Hence

$$\sum_{y < n \leq y+q} \min \left(A, \frac{C}{\left(\sin \pi \alpha n\right)^2} \right) \leq 2A + \frac{C}{\left(\sin \frac{\pi}{2q}\right)^2} + C \cdot \frac{2q}{\pi} \cot \frac{\pi}{q}.$$

Now, by [AS64, (4.3.68)] and [AS64, (4.3.70)], for $t \in (-\pi, \pi)$,

$$(3.2) \quad \begin{aligned} \frac{t}{\sin t} &= 1 + \sum_{k \geq 0} a_{2k+1} t^{2k+2} = 1 + \frac{t^2}{6} + \dots \\ t \cot t &= 1 - \sum_{k \geq 0} b_{2k+1} t^{2k+2} = 1 - \frac{t^2}{3} - \frac{t^4}{45} - \dots, \end{aligned}$$

where $a_{2k+1} \geq 0$, $b_{2k+1} \geq 0$. Thus, for $t \in [0, t_0]$, $t_0 < \pi$,

$$(3.3) \quad \left(\frac{t}{\sin t} \right)^2 = 1 + \frac{t^2}{3} + c_0(t) t^4 \leq 1 + \frac{t^2}{3} + c_0(t_0) t^4,$$

where

$$c_0(t) = \frac{1}{t^4} \left(\left(\frac{t}{\sin t} \right)^2 - \left(1 + \frac{t^2}{3} \right) \right),$$

which is an increasing function because $a_{2k+1} \geq 0$. For $t_0 = \pi/4$, $c_0(t_0) \leq 0.074807$. Hence,

$$\begin{aligned} \frac{t^2}{\sin^2 t} + t \cot 2t &\leq \left(1 + \frac{t^2}{3} + c_0 \left(\frac{\pi}{4} \right) t^4 \right) + \left(\frac{1}{2} - \frac{2t^2}{3} - \frac{8t^4}{45} \right) \\ &= \frac{3}{2} - \frac{t^2}{3} + \left(c_0 \left(\frac{\pi}{4} \right) - \frac{8}{45} \right) t^4 \leq \frac{3}{2} - \frac{t^2}{3} \leq \frac{3}{2} \end{aligned}$$

for $t \in [0, \pi/4]$.

Therefore, the left side of (3.1) is at most

$$2A + C \cdot \left(\frac{2q}{\pi} \right)^2 \cdot \frac{3}{2} = 2A + \frac{6}{\pi^2} C q^2.$$

The following is an alternative approach yielding the other estimate in (3.1). We bound the terms corresponding to $r = 0$, $r = -1$, $r = 1$ by A each. We let $r = \pm r'$ for r' ranging from 2 to $q/2$. We obtain that the sum is at most

$$(3.4) \quad \begin{aligned} &3A + \sum_{2 \leq r' \leq q/2} \min \left(A, \frac{C}{\left(\sin \frac{\pi}{q} \left(r' - \frac{1}{2} - q\delta_2 \right)\right)^2} \right) \\ &+ \sum_{2 \leq r' \leq q/2} \min \left(A, \frac{C}{\left(\sin \frac{\pi}{q} \left(r' - \frac{1}{2} + q\delta_2 \right)\right)^2} \right). \end{aligned}$$

We bound a term $\min(A, C/\sin((\pi/q)(r' - 1/2 \pm q\delta_2))^2)$ by A if and only if $C/\sin((\pi/q)(r' - 1 \pm q\delta_2))^2 \geq A$. The number of such terms is

$$\leq \max(0, \lfloor (q/\pi) \arcsin(\sqrt{C/A}) \mp q\delta_2 \rfloor),$$

and thus at most $(2q/\pi) \arcsin(\sqrt{C/A})$ in total. (Recall that $q\delta_2 \leq 1/2$.) Each other term gets bounded by the integral of $C/\sin^2(\pi\alpha/q)$ from $r' - 1 \pm q\delta_2$ ($\geq (q/\pi) \arcsin(\sqrt{C/A})$) to $r' \pm q\delta_2$, by convexity. Thus (3.4) is at most

$$\begin{aligned} & 3A + \frac{2q}{\pi} A \arcsin \sqrt{\frac{C}{A}} + 2 \int_{\frac{q}{\pi} \arcsin \sqrt{\frac{C}{A}}}^{q/2} \frac{C}{\sin^2 \frac{\pi t}{q}} dt \\ & \leq 3A + \frac{2q}{\pi} A \arcsin \sqrt{\frac{C}{A}} + \frac{2q}{\pi} C \sqrt{\frac{A}{C} - 1} \end{aligned}$$

We can easily show (taking derivatives) that $\arcsin x + x(1 - x^2) \leq 2x$ for $0 \leq x \leq 1$. Setting $x = C/A$, we see that this implies that

$$3A + \frac{2q}{\pi} A \arcsin \sqrt{\frac{C}{A}} + \frac{2q}{\pi} C \sqrt{\frac{A}{C} - 1} \leq 3A + \frac{4q}{\pi} \sqrt{AC}.$$

(If $C/A > 1$, then $3A + (4q/\pi)\sqrt{AC}$ is greater than Aq , which is an obvious upper bound for the left side of (3.1).) \square

Lemma 3.2. *Let $\alpha = a/q + \beta/qQ$, $\gcd(a, q) = 1$, $|\beta| \leq 1$, $q \leq Q$. Let $y_2 > y_1 \geq 0$. If $y_2 - y_1 \leq q$ and $y_2 \leq Q/2$, then, for any $A, C \geq 0$,*

$$(3.5) \quad \sum_{\substack{y_1 < n \leq y_2 \\ q \nmid n}} \min \left(A, \frac{C}{|\sin(\pi\alpha n)|^2} \right) \leq \min \left(\frac{20}{3\pi^2} Cq^2, 2A + \frac{4q}{\pi} \sqrt{AC} \right).$$

Proof. Clearly, αn equals $an/q + (n/Q)\beta/q$; since $y_2 \leq Q/2$, this means that $|\alpha n - an/q| \leq 1/2q$ for $n \leq y_2$; moreover, again for $n \leq y_2$, the sign of $\alpha n - an/q$ remains constant. Hence the left side of (3.5) is at most

$$\sum_{r=1}^{q/2} \min \left(A, \frac{C}{(\sin \frac{\pi}{q}(r - 1/2))^2} \right) + \sum_{r=1}^{q/2} \min \left(A, \frac{C}{(\sin \frac{\pi}{q}r)^2} \right).$$

Proceeding as in the proof of Lemma 3.1, we obtain a bound of at most

$$C \left(\frac{1}{(\sin \frac{\pi}{2q})^2} + \frac{1}{(\sin \frac{\pi}{q})^2} + \frac{q}{\pi} \cot \frac{\pi}{q} + \frac{q}{\pi} \cot \frac{3\pi}{2q} \right)$$

for $q \geq 2$. (If $q = 1$, then the left-side of (3.5) is trivially zero.) Now, by (3.2),

$$\begin{aligned} \frac{t^2}{(\sin t)^2} + \frac{t}{2} \cot 2t & \leq \left(1 + \frac{t^2}{3} + c_0 \left(\frac{\pi}{4} \right) t^4 \right) + \frac{1}{4} \left(1 - \frac{4t^2}{3} - \frac{16t^4}{45} \right) \\ & \leq \frac{5}{4} + \left(c_0 \left(\frac{\pi}{4} \right) - \frac{4}{45} \right) t^4 \leq \frac{5}{4} \end{aligned}$$

for $t \in [0, \pi/4]$, and

$$\begin{aligned} \frac{t^2}{(\sin t)^2} + t \cot \frac{3t}{2} & \leq \left(1 + \frac{t^2}{3} + c_0 \left(\frac{\pi}{2} \right) t^4 \right) + \frac{2}{3} \left(1 - \frac{3t^2}{4} - \frac{81t^4}{2^4 \cdot 45} \right) \\ & \leq \frac{5}{3} + \left(-\frac{1}{6} + \left(c_0 \left(\frac{\pi}{2} \right) - \frac{27}{360} \right) \left(\frac{\pi}{2} \right)^2 \right) t^2 \leq \frac{5}{3} \end{aligned}$$

for $t \in [0, \pi/2]$. Hence,

$$\left(\frac{1}{(\sin \frac{\pi}{2q})^2} + \frac{1}{(\sin \frac{\pi}{q})^2} + \frac{q}{\pi} \cot \frac{\pi}{q} + \frac{q}{\pi} \cot \frac{3\pi}{2q} \right) \leq \left(\frac{2q}{\pi} \right)^2 \cdot \frac{5}{4} + \left(\frac{q}{\pi} \right)^2 \cdot \frac{5}{3} \leq \frac{20}{3\pi^2} q^2.$$

Alternatively, we can follow the second approach in the proof of Lemma 3.1, and obtain an upper bound of $2A + (4q/\pi)\sqrt{AC}$. \square

The following bound will be useful when the constant A in an application of Lemma 3.2 would be too large. (This tends to happen for n small.)

Lemma 3.3. *Let $\alpha = a/q + \beta/qQ$, $\gcd(a, q) = 1$, $|\beta| \leq 1$, $q \leq Q$. Let $y_2 > y_1 \geq 0$. If $y_2 - y_1 \leq q$ and $y_2 \leq Q/2$, then, for any $B, C \geq 0$,*

$$(3.6) \quad \sum_{\substack{y_1 < n \leq y_2 \\ q|n}} \min \left(\frac{B}{|\sin(\pi\alpha n)|}, \frac{C}{|\sin(\pi\alpha n)|^2} \right) \leq 2B \frac{q}{\pi} \max \left(2, \log \frac{Ce^3 q}{B\pi} \right).$$

The upper bound $\leq (2Bq/\pi) \log(2e^2 q/\pi)$ is also valid.

Proof. As in the proof of Lemma 3.2, we can bound the left side of (3.6) by

$$2 \sum_{r=1}^{q/2} \min \left(\frac{B}{\sin \frac{\pi}{q} \left(r - \frac{1}{2} \right)}, \frac{C}{\sin^2 \frac{\pi}{q} \left(r - \frac{1}{2} \right)} \right).$$

Assume $B \sin(\pi/q) \leq C \leq B$. By the convexity of $1/\sin(t)$ and $1/\sin(t)^2$ for $t \in (0, \pi/2]$,

$$\begin{aligned} & \sum_{r=1}^{q/2} \min \left(\frac{B}{\sin \frac{\pi}{q} \left(r - \frac{1}{2} \right)}, \frac{C}{\sin^2 \frac{\pi}{q} \left(r - \frac{1}{2} \right)} \right) \\ & \leq \frac{B}{\sin \frac{\pi}{2q}} + \int_1^{\frac{q}{\pi} \arcsin \frac{C}{B}} \frac{B}{\sin \frac{\pi}{q} t} dt + \int_{\frac{q}{\pi} \arcsin \frac{C}{B}}^{q/2} \frac{1}{\sin^2 \frac{\pi}{q} t} dt \\ & \leq \frac{B}{\sin \frac{\pi}{2q}} + \frac{q}{\pi} \left(B \left(\log \tan \left(\frac{1}{2} \arcsin \frac{C}{B} \right) - \log \tan \frac{\pi}{2q} \right) + C \cot \arcsin \frac{C}{B} \right) \\ & \leq \frac{B}{\sin \frac{\pi}{2q}} + \frac{q}{\pi} \left(B \left(\log \cot \frac{\pi}{2q} - \log \frac{C}{B - \sqrt{B^2 - C^2}} \right) + \sqrt{B^2 - C^2} \right). \end{aligned}$$

Now, for all $t \in (0, \pi/2)$,

$$\frac{2}{\sin t} + \frac{1}{t} \log \cot t < \frac{1}{t} \log \left(\frac{e^2}{t} \right);$$

we can verify this by comparing series. Thus

$$\frac{B}{\sin \frac{\pi}{2q}} + \frac{q}{\pi} B \log \cot \frac{\pi}{2q} \leq B \frac{q}{\pi} \log \frac{2e^2 q}{\pi}$$

for $q \geq 2$. (If $q = 1$, the sum on the left of (3.6) is empty, and so the bound we are trying to prove is trivial.) We also have

$$(3.7) \quad t \log(t - \sqrt{t^2 - 1}) + \sqrt{t^2 - 1} < -t \log 2t + t$$

for $t \geq 1$ (as this is equivalent to $\log(2t^2(1 - \sqrt{1 - t^{-2}})) < 1 - \sqrt{1 - t^{-2}}$, which we check easily after changing variables to $\delta = 1 - \sqrt{1 - t^{-2}}$). Hence

$$\begin{aligned} & \frac{B}{\sin \frac{\pi}{2q}} + \frac{q}{\pi} \left(B \left(\log \cot \frac{\pi}{2q} - \log \frac{C}{B - \sqrt{B^2 - C^2}} \right) + \sqrt{B^2 - C^2} \right) \\ & \leq B \frac{q}{\pi} \log \frac{2e^2 q}{\pi} + \frac{q}{\pi} \left(B - B \log \frac{2B}{C} \right) \leq B \frac{q}{\pi} \log \frac{Ce^3 q}{B\pi} \end{aligned}$$

for $q \geq 2$.

Given any C , we can apply the above with $C = B$ instead, as, for any $t > 0$, $\min(B/t, C/t^2) \leq B/t \leq \min(B/t, B/t^2)$. (We refrain from applying (3.7) so as to avoid worsening a constant.) If $C < B \sin \pi/q$ (or even if $C < (\pi/q)B$), we relax the input to $C = B \sin \pi/q$ and go through the above. \square

3.2. Type I estimates. Our main type I estimate is the following.³ One of the main innovations is the manner in which the “main term” (m divisible by q) is separated; we are able to keep error terms small thanks to the particular way in which we switch between two different approximations.

(These are *not* necessarily successive approximations in the sense of continued fractions; we do not want to assume that the approximation a/q we are given arises from a continued fraction, and at any rate we need more control on the denominator q' of the new approximation a'/q' than continued fractions would furnish.)

Lemma 3.4. *Let $\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $|\delta/x| \leq 1/qQ_0$, $q \leq Q_0$, $Q_0 \geq 16$. Let η be continuous, piecewise C^2 and compactly supported, with $|\eta|_1 = 1$ and $\eta'' \in L_1$. Let $c_0 \geq |\hat{\eta}''|_\infty$.*

Let $1 \leq D \leq x$. Then, if $|\delta| \leq 1/2c_2$, where $c_2 = (3\pi/5\sqrt{c_0})(1 + \sqrt{13/3})$, the absolute value of

$$(3.8) \quad \sum_{m \leq D} \mu(m) \sum_n e(\alpha mn) \eta \left(\frac{mn}{x} \right)$$

is at most

$$(3.9) \quad \frac{x}{q} \min \left(1, \frac{c_0}{(2\pi\delta)^2} \right) \left| \sum_{\substack{m \leq \frac{M}{q} \\ \gcd(m, q) = 1}} \frac{\mu(m)}{m} \right| + O^* \left(c_0 \left(\frac{1}{4} - \frac{1}{\pi^2} \right) \left(\frac{D^2}{2xq} + \frac{D}{2x} \right) \right)$$

plus

$$(3.10) \quad \begin{aligned} & \frac{2\sqrt{c_0 c_1}}{\pi} D + 3c_1 \frac{x}{q} \log^+ \frac{D}{c_2 x/q} + \frac{\sqrt{c_0 c_1}}{\pi} q \log^+ \frac{D}{q/2} \\ & + \frac{|\eta'|_1}{\pi} q \cdot \max \left(2, \log \frac{c_0 e^3 q^2}{4\pi |\eta'|_1 x} \right) + \left(\frac{2\sqrt{3c_0 c_1}}{\pi} + \frac{3c_1}{c_2} + \frac{55c_0 c_2}{12\pi^2} \right) q, \end{aligned}$$

where $c_1 = 1 + |\eta'|_1/(2x/D)$ and $M \in [\min(Q_0/2, D), D]$. The same bound holds if $|\delta| \geq 1/2c_2$ but $D \leq Q_0/2$.

³The current version of Lemma 3.4 is an improvement over that included in the first preprint of this paper.

In general, if $|\delta| \geq 1/2c_2$, the absolute value of (3.8) is at most (3.9) plus

$$(3.11) \quad \begin{aligned} & \frac{2\sqrt{c_0c_1}}{\pi} \left(D + (1 + \epsilon) \min \left(\left\lfloor \frac{x}{|\delta|q} \right\rfloor + 1, 2D \right) \left(\varpi_\epsilon + \frac{1}{2} \log^+ \frac{2D}{\frac{x}{|\delta|q}} \right) \right) \\ & + 3c_1 \left(2 + \frac{(1 + \epsilon)}{\epsilon} \log^+ \frac{2D}{\frac{x}{|\delta|q}} \right) \frac{x}{Q_0} + \frac{35c_0c_2}{6\pi^2} q, \end{aligned}$$

for $\epsilon \in (0, 1]$ arbitrary, where $\varpi_\epsilon = \sqrt{3 + 2\epsilon} + ((1 + \sqrt{13/3})/4 - 1)/(2(1 + \epsilon))$.

In (3.9), $\min(1, c_0/(2\pi\delta)^2)$ always equals 1 when $|\delta| \leq 1/2c_2$ (since $(3/5)(1 + \sqrt{13/3}) > 1$).

Proof. Let $Q = \lfloor x/|\delta q| \rfloor$. Then $\alpha = a/q + O^*(1/qQ)$ and $q \leq Q$. (If $\delta = 0$, we let $Q = \infty$ and ignore the rest of the paragraph, since then we will never need Q' or the alternative approximation a'/q' .) Let $Q' = \lceil (1 + \epsilon)Q \rceil \geq Q + 1$. Then α is *not* $a/q + O^*(1/qQ')$, and so there must be a different approximation a'/q' , $\gcd(a', q') = 1$, $q' \leq Q'$ such that $\alpha = a'/q' + O^*(1/q'Q')$ (since such an approximation always exists). Obviously, $|a/q - a'/q'| \geq 1/qQ'$, yet, at the same time, $|a/q - a'/q'| \leq 1/qQ + 1/q'Q' \leq 1/qQ + 1/((1 + \epsilon)q'Q)$. Hence $q'/Q + q/((1 + \epsilon)Q) \geq 1$, and so $q' \geq Q - q/(1 + \epsilon) \geq (\epsilon/(1 + \epsilon))Q$. (Note also that $(\epsilon/(1 + \epsilon))Q \geq (2|\delta q|/x) \cdot \lfloor x/\delta q \rfloor > 1$, and so $q' \geq 2$.)

Lemma 3.2 will enable us to treat separately the contribution from terms with m divisible by q and m not divisible by q , provided that $m \leq Q/2$. Let $M = \min(Q/2, D)$. We start by considering all terms with $m \leq M$ divisible by q . Then $e(\alpha mn)$ equals $e((\delta m/x)n)$. By Poisson summation,

$$\sum_n e(\alpha mn) \eta(mn/x) = \sum_n \widehat{f}(n),$$

where $f(u) = e((\delta m/x)u) \eta((m/x)u)$. Now

$$\widehat{f}(n) = \int e(-un) f(u) du = \frac{x}{m} \int e\left(\left(\delta - \frac{xn}{m}\right)u\right) \eta(u) du = \frac{x}{m} \widehat{\eta}\left(\frac{x}{m}n - \delta\right).$$

By assumption, $m \leq M \leq Q/2 \leq x/2|\delta q|$, and so $|x/m| \geq 2|\delta q| \geq 2\delta$. Thus, by (2.1),

$$(3.12) \quad \begin{aligned} \sum_n \widehat{f}(n) &= \frac{x}{m} \left(\widehat{\eta}(-\delta) + \sum_{n \neq 0} \widehat{\eta}\left(\frac{nx}{m} - \delta\right) \right) \\ &= \frac{x}{m} \left(\widehat{\eta}(-\delta) + O^* \left(\sum_{n \neq 0} \frac{1}{(2\pi(\frac{nx}{m} - \delta))^2} \right) \cdot \left| \widehat{\eta}' \right|_\infty \right) \\ &= \frac{x}{m} \widehat{\eta}(-\delta) + \frac{m}{x} \frac{c_0}{(2\pi)^2} O^* \left(\max_{|r| \leq \frac{1}{2}} \sum_{n \neq 0} \frac{1}{(n-r)^2} \right). \end{aligned}$$

Since $x \mapsto 1/x^2$ is convex on \mathbb{R}^+ ,

$$\max_{|r| \leq \frac{1}{2}} \sum_{n \neq 0} \frac{1}{(n-r)^2} = \sum_{n \neq 0} \frac{1}{(n - \frac{1}{2})^2} = \pi^2 - 4.$$

Therefore, the sum of all terms with $m \leq M$ and $q|m$ is

$$\begin{aligned} & \sum_{\substack{m \leq M \\ q|m}} \frac{x}{m} \widehat{\eta}(-\delta) + \sum_{\substack{m \leq M \\ q|m}} \frac{m}{x} \frac{c_0}{(2\pi)^2} (\pi^2 - 4) \\ &= \frac{x\mu(q)}{q} \cdot \widehat{\eta}(-\delta) \cdot \sum_{\substack{m \leq \frac{M}{q} \\ \gcd(m,q)=1}} \frac{\mu(m)}{m} \\ &+ O^* \left(\mu(q)^2 c_0 \left(\frac{1}{4} - \frac{1}{\pi^2} \right) \left(\frac{D^2}{2xq} + \frac{D}{2x} \right) \right). \end{aligned}$$

We bound $|\widehat{\eta}(-\delta)|$ by (2.1).

Let

$$T_m(\alpha) = \sum_n e(\alpha mn) \eta\left(\frac{mn}{x}\right).$$

Then, by (2.2) and Lemma 2.1,

$$(3.13) \quad |T_m(\alpha)| \leq \min \left(\frac{x}{m} + \frac{1}{2} |\eta'|_1, \frac{\frac{1}{2} |\eta'|_1}{|\sin(\pi m \alpha)|}, \frac{m c_0}{x} \frac{1}{4 (\sin \pi m \alpha)^2} \right).$$

For any $y_2 > y_1 > 0$ with $y_2 - y_1 \leq q$ and $y_2 \leq Q/2$, (3.13) gives us that

$$(3.14) \quad \sum_{\substack{y_1 < m \leq y_2 \\ q|m}} |T_m(\alpha)| \leq \sum_{\substack{y_1 < m \leq y_2 \\ q|m}} \min \left(A, \frac{C}{(\sin \pi m \alpha)^2} \right)$$

for $A = (x/y_1)(1 + |\eta'|_1/(2(x/y_1)))$ and $C = (c_0/4)(y_2/x)$. We must now estimate the sum

$$(3.15) \quad \sum_{\substack{m \leq M \\ q|m}} |T_m(\alpha)| + \sum_{\frac{Q}{2} < m \leq D} |T_m(\alpha)|.$$

To bound the terms with $m \leq M$, we can use Lemma 3.2. The question is then which one is smaller: the first or the second bound given by Lemma 3.2? A brief calculation gives that the second bound is smaller (and hence preferable) exactly when $\sqrt{C/A} > (3\pi/10q)(1 + \sqrt{13/3})$. Since $\sqrt{C/A} \sim (\sqrt{c_0}/2)m/x$, this means that it is sensible to prefer the second bound in Lemma 3.2 when $m > c_2 x/q$, where $c_2 = (3\pi/5\sqrt{c_0})(1 + \sqrt{13/3})$.

It thus makes sense to ask: does $Q/2 \leq c_2 x/q$ (so that $m \leq M$ implies $m \leq c_2 x/q$)? This question divides our work into two basic cases.

Case (a). δ large: $|\delta| \geq 1/2c_2$, where $c_2 = (3\pi/5\sqrt{c_0})(1 + \sqrt{13/3})$. Then $Q/2 \leq c_2 x/q$; this will induce us to bound the first sum in (3.15) by the very first bound in Lemma 3.2.

Recall that $M = \min(Q/2, D)$, and so $M \leq c_2x/q$. By (3.14) and Lemma 3.2, (3.16)

$$\begin{aligned} \sum_{\substack{1 \leq m \leq M \\ q \nmid m}} |T_m(\alpha)| &\leq \sum_{j=0}^{\infty} \sum_{jq < m \leq \min((j+1)q, M)} \min \left(\frac{x}{jq+1} + \frac{|\eta'|_1}{2}, \frac{\frac{c_0}{4} \frac{(j+1)q}{x}}{(\sin \pi m \alpha)^2} \right) \\ &\leq \frac{20}{3\pi^2} \frac{c_0 q^3}{4x} \sum_{0 \leq j \leq \frac{M}{q}} (j+1) \leq \frac{20}{3\pi^2} \frac{c_0 q^3}{4x} \cdot \left(\frac{1}{2} \frac{M^2}{q^2} + \frac{3}{2} \frac{c_2 x}{q^2} + 1 \right) \\ &\leq \frac{5c_0 c_2}{6\pi^2} M + \frac{5c_0 q}{3\pi^2} \left(\frac{3}{2} c_2 + \frac{q^2}{x} \right) \leq \frac{5c_0 c_2}{6\pi^2} M + \frac{35c_0 c_2}{6\pi^2} q, \end{aligned}$$

where, to bound the smaller terms, we are using the inequality $Q/2 \leq c_2x/q$, and where we are also using the observation that, since $|\delta/x| \leq 1/qQ_0$, the assumption $|\delta| \geq 1/2c_2$ implies that $q \leq 2c_2x/Q_0$; moreover, since $q \leq Q_0$, this gives us that $q^2 \leq 2c_2x$. In the main term, we are bounding qM^2/x from above by $M \cdot qQ/2x \leq M/2\delta \leq c_2M$.

If $D \leq (Q+1)/2$, then $M \geq \lfloor D \rfloor$ and so (3.16) is all we need. Assume from now on that $D > (Q+1)/2$. The first sum in (3.15) is then bounded by (3.16) (with $M = Q/2$). To bound the second sum in (3.15), we use the approximation a'/q' instead of a/q . By (3.14) (without the restriction $q \nmid m$) and Lemma 3.1,

$$\begin{aligned} \sum_{Q/2 < m \leq D} |T_m(\alpha)| &\leq \sum_{j=0}^{\infty} \sum_{jq' + \frac{Q}{2} < m \leq \min((j+1)q' + Q/2, D)} |T_m(\alpha)| \\ &\leq \sum_{j=0}^{\lfloor \frac{D-(Q+1)/2}{q'} \rfloor} \left(3c_1 \frac{x}{jq' + \frac{Q+1}{2}} + \frac{4q'}{\pi} \sqrt{\frac{c_1 c_0}{4} \frac{x}{jq' + (Q+1)/2} \frac{(j+1)q' + Q/2}{x}} \right) \\ &\leq \sum_{j=0}^{\lfloor \frac{D-(Q+1)/2}{q'} \rfloor} \left(3c_1 \frac{x}{jq' + \frac{Q+1}{2}} + \frac{4q'}{\pi} \sqrt{\frac{c_1 c_0}{4} \left(1 + \frac{q'}{jq' + (Q+1)/2} \right)} \right), \end{aligned}$$

where we recall that $c_1 = 1 + |\eta'|_1/(2x/D)$. Since $q' \geq (\epsilon/(1+\epsilon))Q$,

$$(3.17) \quad \sum_{j=0}^{\lfloor \frac{D-(Q+1)/2}{q'} \rfloor} \frac{x}{jq' + \frac{Q+1}{2}} \leq \frac{x}{Q/2} + \frac{x}{q'} \int_{\frac{Q+1}{2}}^D \frac{1}{t} dt \leq \frac{2x}{Q} + \frac{(1+\epsilon)x}{\epsilon Q} \log^+ \frac{D}{\frac{Q+1}{2}}.$$

Recall now that $q' \leq (1+\epsilon)Q + 1 \leq (1+\epsilon)(Q+1)$. Therefore,

$$(3.18) \quad \begin{aligned} q' \sum_{j=0}^{\lfloor \frac{D-(Q+1)/2}{q'} \rfloor} \sqrt{1 + \frac{q'}{jq' + (Q+1)/2}} &\leq q' \sqrt{1 + \frac{(1+\epsilon)Q+1}{(Q+1)/2}} + \int_{\frac{Q+1}{2}}^D \sqrt{1 + \frac{q'}{t}} dt \\ &\leq q' \sqrt{3+2\epsilon} + \left(D - \frac{Q+1}{2} \right) + \frac{q'}{2} \log^+ \frac{D}{\frac{Q+1}{2}}. \end{aligned}$$

We conclude that $\sum_{Q/2 < m \leq D} |T_m(\alpha)|$ is at most

$$(3.19) \quad \frac{2\sqrt{c_0 c_1}}{\pi} \left(D + \left((1+\epsilon)\sqrt{3+2\epsilon} - \frac{1}{2} \right) (Q+1) + \frac{(1+\epsilon)Q+1}{2} \log^+ \frac{D}{\frac{Q+1}{2}} \right) \\ + 3c_1 \left(2 + \frac{(1+\epsilon)}{\epsilon} \log^+ \frac{D}{\frac{Q+1}{2}} \right) \frac{x}{Q}$$

We sum this to (3.16) (with $M = Q/2$), and obtain that (3.15) is at most

$$(3.20) \quad \frac{2\sqrt{c_0 c_1}}{\pi} \left(D + (1+\epsilon)(Q+1) \left(\varpi_\epsilon + \frac{1}{2} \log^+ \frac{D}{\frac{Q+1}{2}} \right) \right) \\ + 3c_1 \left(2 + \frac{(1+\epsilon)}{\epsilon} \log \frac{D}{\frac{Q+1}{2}} \right) \frac{x}{Q} + \frac{35c_0 c_2}{6\pi^2} q,$$

where we are bounding

$$(3.21) \quad \frac{5c_0 c_2}{6\pi^2} = \frac{5c_0}{6\pi^2} \frac{3\pi}{5\sqrt{c_0}} \left(1 + \sqrt{\frac{13}{3}} \right) = \frac{\sqrt{c_0}}{2\pi} \left(1 + \sqrt{\frac{13}{3}} \right) \leq \frac{2\sqrt{c_0 c_1}}{\pi} \cdot \frac{1}{4} \left(1 + \sqrt{\frac{13}{3}} \right)$$

and defining

$$(3.22) \quad \varpi_\epsilon = \sqrt{3+2\epsilon} + \left(\frac{1}{4} \left(1 + \sqrt{\frac{13}{3}} \right) - 1 \right) \frac{1}{2(1+\epsilon)}.$$

(Note that $\varpi_\epsilon < \sqrt{3}$ for $\epsilon < 0.1741$). A quick check against (3.16) shows that (3.20) is valid also when $D \leq Q/2$, even when $Q+1$ is replaced by $\min(Q+1, 2D)$. We bound Q from above by $x/|\delta|q$ and $\log^+ D/((Q+1)/2)$ by $\log^+ 2D/(x/|\delta|q+1)$, and obtain the result.

Case (b): $|\delta|$ *small*: $|\delta| \leq 1/2c_2$ or $D \leq Q_0/2$. Then $\min(c_2 x/q, D) \leq Q/2$. We start by bounding the first $q/2$ terms in (3.15) by (3.13) and Lemma 3.3:

$$(3.23) \quad \sum_{m \leq q/2} |T_m(\alpha)| \leq \sum_{m \leq q/2} \min \left(\frac{\frac{1}{2}|\eta'|_1}{|\sin(\pi m \alpha)|}, \frac{c_0 q/8x}{|\sin(\pi m \alpha)|^2} \right) \\ \leq \frac{|\eta'|_1}{\pi} q \max \left(2, \log \frac{c_0 e^3 q^2}{4\pi |\eta'|_1 x} \right).$$

If $q^2 < 2c_2 x$, we estimate the terms with $q/2 < m \leq c_2 x/q$ by Lemma 3.2, which is applicable because $\min(c_2 x/q, D) < Q/2$:

$$(3.24) \quad \sum_{\substack{\frac{q}{2} < m \leq D' \\ q \nmid m}} |T_m(\alpha)| \leq \sum_{j=1}^{\infty} \sum_{\substack{(j-\frac{1}{2})q < m \leq (j+\frac{1}{2})q \\ m \leq \min(\frac{c_2 x}{q}, D) \\ q \nmid m}} \min \left(\frac{x}{(j-\frac{1}{2})q} + \frac{|\eta'|_1}{2}, \frac{\frac{c_0}{4} \frac{(j+1/2)q}{x}}{(\sin \pi m \alpha)^2} \right) \\ \leq \frac{20}{3\pi^2} \frac{c_0 q^3}{4x} \sum_{1 \leq j \leq \frac{D'}{q} + \frac{1}{2}} \left(j + \frac{1}{2} \right) \leq \frac{20}{3\pi^2} \frac{c_0 q^3}{4x} \left(\frac{c_2 x D'}{2q^2} + \frac{3}{2} \left(\frac{c_2 x}{q^2} \right) + \frac{5}{8} \right) \\ \leq \frac{5c_0}{6\pi^2} \left(c_2 D' + 3c_2 q + \frac{5}{4} \frac{q^3}{x} \right) \leq \frac{5c_0 c_2}{6\pi^2} \left(D' + \frac{11}{2} q \right),$$

where we write $D' = \min(c_2x/q, D)$. If $c_2x/q \geq D$, we stop here. Assume that $c_2x/q < D$. Let $R = \max(c_2x/q, q/2)$. The terms we have already estimated are precisely those with $m \leq R$. We bound the terms $R < m \leq D$ by the second bound in Lemma 3.1:

$$(3.25) \quad \sum_{R < m \leq D} |T_m(\alpha)| \leq \sum_{j=0}^{\infty} \sum_{\substack{m > jq+R \\ m \leq \min((j+1)q+R, D)}} \min \left(\frac{c_1x}{jq+R}, \frac{\frac{c_0}{4} \frac{(j+1)q+R}{x}}{(\sin \pi m \alpha)^2} \right) \\ \leq \sum_{j=0}^{\lfloor \frac{1}{q}(D-R) \rfloor} \frac{3c_1x}{jq+R} + \frac{4q}{\pi} \sqrt{\frac{c_1c_0}{4} \left(1 + \frac{q}{jq+R} \right)}$$

(Note there is no need to use two successive approximations a/q , a'/q' as in case (a). We are also including all terms with m divisible by q , as we may, since $|T_m(\alpha)|$ is non-negative.) Now, much as before,

$$(3.26) \quad \sum_{j=0}^{\lfloor \frac{1}{q}(D-R) \rfloor} \frac{x}{jq+R} \leq \frac{x}{R} + \frac{x}{q} \int_R^D \frac{1}{t} dt \leq \min \left(\frac{q}{c_2}, \frac{2x}{q} \right) + \frac{x}{q} \log^+ \frac{D}{c_2x/q},$$

and

$$(3.27) \quad \sum_{j=0}^{\lfloor \frac{1}{q}(D-R) \rfloor} \sqrt{1 + \frac{q}{jq+R}} \leq \sqrt{1 + \frac{q}{R}} + \frac{1}{q} \int_R^D \sqrt{1 + \frac{q}{t}} dt \leq \sqrt{3} + \frac{D-R}{q} + \frac{1}{2} \log^+ \frac{D}{q/2}.$$

We sum with (3.23) and (3.24), and we obtain that (3.15) is at most

$$(3.28) \quad \frac{2\sqrt{c_0c_1}}{\pi} \left(\sqrt{3}q + D + \frac{q}{2} \log^+ \frac{D}{q/2} \right) + \left(3c_1 \log^+ \frac{D}{c_2x/q} \right) \frac{x}{q} \\ + 3c_1 \min \left(\frac{q}{c_2}, \frac{2x}{q} \right) + \frac{55c_0c_2}{12\pi^2} q + \frac{|\eta'|_1}{\pi} q \cdot \max \left(2, \log \frac{c_0e^3q^2}{4\pi|\eta'|_1x} \right),$$

where we are using the fact that $5c_0c_2/6\pi^2 < 2\sqrt{c_0c_1}/\pi$. A quick check against (3.24) shows that (because of the fact just stated) (3.28) is also valid when $c_2x/q \geq D$. \square

We will need a version of Lemma 3.4 with m and n restricted to the odd numbers. (We will barely be using the restriction of m , whereas the restriction on n is both (a) slightly harder to deal with, (b) something that can be turned to our advantage.)

Lemma 3.5. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ with $2\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $|\delta/x| \leq 1/qQ_0$, $q \leq Q_0$, $Q_0 \geq 16$. Let η be continuous, piecewise C^2 and compactly supported, with $|\eta|_1 = 1$ and $\eta'' \in L_1$. Let $c_0 \geq |\eta''|_\infty$.*

Let $1 \leq D \leq x$. Then, if $|\delta| \leq 1/2c_2$, where $c_2 = 6\pi/5\sqrt{c_0}$, the absolute value of

$$(3.29) \quad \sum_{\substack{m \leq D \\ m \text{ odd}}} \mu(m) \sum_{n \text{ odd}} e(\alpha mn) \eta \left(\frac{mn}{x} \right)$$

is at most

$$(3.30) \quad \frac{x}{2q} \min \left(1, \frac{c_0}{(\pi\delta)^2} \right) \left| \sum_{\substack{m \leq \frac{M}{q} \\ \gcd(m, 2q)=1}} \frac{\mu(m)}{m} \right| + O^* \left(\frac{c_0 q}{x} \left(\frac{1}{8} - \frac{1}{2\pi^2} \right) \left(\frac{D}{q} + 1 \right)^2 \right)$$

plus

$$(3.31) \quad \frac{2\sqrt{c_0 c_1}}{\pi} D + \frac{3c_1 x}{2q} \log^+ \frac{D}{c_2 x/q} + \frac{\sqrt{c_0 c_1}}{\pi} q \log^+ \frac{D}{q/2} \\ + \frac{2|\eta'|_1}{\pi} q \cdot \max \left(1, \log \frac{c_0 e^3 q^2}{4\pi |\eta'|_1 x} \right) + \left(\frac{2\sqrt{3c_0 c_1}}{\pi} + \frac{3c_1}{2c_2} + \frac{55c_0 c_2}{6\pi^2} \right) q,$$

where $c_1 = 1 + |\eta'|_1/(x/D)$ and $M \in [\min(Q_0/2, D), D]$. The same bound holds if $|\delta| \geq 1/2c_2$ but $D \leq Q_0/2$.

In general, if $|\delta| \geq 1/2c_2$, the absolute value of (3.8) is at most (3.30) plus

$$(3.32) \quad \frac{2\sqrt{c_0 c_1}}{\pi} \left(D + (1 + \epsilon) \min \left(\left\lfloor \frac{x}{|\delta|q} \right\rfloor + 1, 2D \right) \left(\sqrt{3 + 2\epsilon} + \frac{1}{2} \log^+ \frac{2D}{\frac{x}{|\delta|q}} \right) \right) \\ + \frac{3}{2} c_1 \left(2 + \frac{(1 + \epsilon)}{\epsilon} \log^+ \frac{2D}{\frac{x}{|\delta|q}} \right) \frac{x}{Q_0} + \frac{35c_0 c_2}{3\pi^2} q,$$

for $\epsilon \in (0, 1]$ arbitrary.

If q is even, the sum (3.30) can be replaced by 0.

Proof. The proof is almost exactly that of Lemma 3.4; we go over the differences. The parameters Q , Q' , a' , q' and M are defined just as before (with 2α wherever we had α).

Let us first consider $m \leq M$ odd and divisible by q . (Of course, this case arises only if q is odd.) For $n = 2r + 1$,

$$e(\alpha mn) = e(\alpha m(2r + 1)) = e(2\alpha r m) e(\alpha m) = e \left(\frac{\delta}{x} r m \right) e \left(\left(\frac{a}{2q} + \frac{\delta}{2x} + \frac{\kappa}{2} \right) m \right) \\ = e \left(\frac{\delta(2r + 1)}{2x} m \right) e \left(\frac{a + \kappa q}{2} \frac{m}{q} \right) = \kappa' e \left(\frac{\delta(2r + 1)}{2x} m \right),$$

where $\kappa \in \{0, 1\}$ and $\kappa' = e((a + \kappa q)/2) \in \{-1, 1\}$ are independent of m and n . Hence, by Poisson summation,

$$(3.33) \quad \sum_{n \text{ odd}} e(\alpha mn) \eta(mn/x) = \kappa' \sum_{n \text{ odd}} e((\delta m/2x)n) \eta(mn/x) \\ = \frac{\kappa'}{2} \left(\sum_n \widehat{f}(n) - \sum_n \widehat{f}(n + 1/2) \right),$$

where $f(u) = e((\delta m/2x)u) \eta((m/x)u)$. Now

$$\widehat{f}(t) = \frac{x}{m} \widehat{\eta} \left(\frac{x}{m} t - \frac{\delta}{2} \right).$$

Just as before, $|x/m| \geq 2|\delta q| \geq 2\delta$. Thus

$$\begin{aligned}
(3.34) \quad & \frac{1}{2} \left| \sum_n \widehat{f}(n) - \sum_n \widehat{f}(n+1/2) \right| \leq \frac{x}{m} \left(\frac{1}{2} \left| \widehat{\eta} \left(-\frac{\delta}{2} \right) \right| + \frac{1}{2} \sum_{n \neq 0} \left| \widehat{\eta} \left(\frac{x}{m} \frac{n}{2} - \frac{\delta}{2} \right) \right| \right) \\
& = \frac{x}{m} \left(\frac{1}{2} \left| \widehat{\eta} \left(-\frac{\delta}{2} \right) \right| + \frac{1}{2} \cdot O^* \left(\sum_{n \neq 0} \frac{1}{\left(\pi \left(\frac{nx}{m} - \delta \right) \right)^2} \right) \cdot \left| \widehat{\eta}' \right|_\infty \right) \\
& = \frac{x}{2m} \left| \widehat{\eta} \left(-\frac{\delta}{2} \right) \right| + \frac{m}{x} \frac{c_0}{2\pi^2} (\pi^2 - 4)x.
\end{aligned}$$

The contribution of the second term in the last line of (3.34) is

$$\sum_{\substack{m \leq M \\ m \text{ odd} \\ q|m}} \frac{m}{x} \frac{c_0}{2\pi^2} (\pi^2 - 4) = \frac{q}{x} \frac{c_0}{2\pi^2} (\pi^2 - 4) \cdot \sum_{\substack{m \leq M/q \\ m \text{ odd}}} m = \frac{qc_0}{x} \left(\frac{1}{8} - \frac{1}{2\pi^2} \right) \left(\frac{M}{q} + 1 \right)^2.$$

Hence, the absolute value of the sum of all terms with $m \leq M$ and $q|m$ is given by (3.30).

We define $T_{m,o}(\alpha)$ by

$$(3.35) \quad T_{m,o}(\alpha) = \sum_{n \text{ odd}} e(\alpha mn) \eta \left(\frac{mn}{x} \right).$$

Changing variables by $n = 2r + 1$, we see that

$$|T_{m,o}(\alpha)| = \left| \sum_r e(2\alpha \cdot mr) \eta(m(2r+1)/x) \right|.$$

Hence, instead of (3.13), we get that

$$(3.36) \quad |T_{m,o}(\alpha)| \leq \min \left(\frac{x}{2m} + \frac{1}{2} |\eta'|_1, \frac{\frac{1}{2} |\eta'|_1}{|\sin(2\pi m \alpha)|}, \frac{m}{x} \frac{c_0}{2} \frac{1}{(\sin 2\pi m \alpha)^2} \right).$$

We obtain (3.14), but with $T_{m,o}$ instead of T_m , $A = (x/2y_1)(1 + |\eta'|_1/(x/y_1))$ and $C = (c_0/2)(y_2/x)$, and so $c_1 = 1 + |\eta'|_1/(x/D)$.

The rest of the proof of Lemma 3.4 carries almost over word-by-word. (For the sake of simplicity, we do not really try to take advantage of the odd support of m here.) Since C has doubled, it would seem to make sense to reset the value of c_2 to be $c_2 = (3\pi/5\sqrt{2c_0})(1 + \sqrt{13/3})$; this would cause complications related to the fact that $5c_0c_2/3\pi^2$ would become larger than $2\sqrt{c_0}/\pi$, and so we set c_2 to the slightly smaller value $c_2 = 6\pi/5\sqrt{c_0}$ instead. This implies

$$(3.37) \quad \frac{5c_0c_2}{3\pi^2} = \frac{2\sqrt{c_0}}{\pi}.$$

The bound from (3.16) gets multiplied by 2 (but the value of c_2 has changed), the second line in (3.19) gets halved, (3.21) gets replaced by (3.37), the second term in the maximum in the second line of (3.23) gets doubled, the bound from (3.24) gets doubled, and the bound from (3.26) gets halved. \square

We will also need a version of Lemma 3.4 (or rather Lemma 3.5; we will decide to work with the restriction that n and m be odd) with a factor of $(\log n)$ within the inner sum.

Lemma 3.6. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ with $2\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $|\delta/x| \leq 1/qQ_0$, $q \leq Q_0$, $Q_0 \geq \max(16, 2\sqrt{x})$. Let η be continuous, piecewise C^2 and compactly supported, with $|\eta|_1 = 1$ and $\eta'' \in L_1$. Let $c_0 \geq |\widehat{\eta''}|_\infty$. Assume that, for any $\rho \geq \rho_0$, ρ_0 a constant, the function $\eta_{(\rho)}(t) = \log(\rho t)\eta(t)$ satisfies*

$$(3.38) \quad |\eta_{(\rho)}|_1 \leq \log(\rho)|\eta|_1, \quad |\eta'_{(\rho)}|_1 \leq \log(\rho)|\eta'|_1, \quad |\widehat{\eta''_{(\rho)}}|_\infty \leq c_0 \log(\rho)$$

Let $\sqrt{3} \leq D \leq \min(x/\rho_0, x/e)$. Then, if $|\delta| \leq 1/2c_2$, where $c_2 = 6\pi/5\sqrt{c_0}$, the absolute value of

$$(3.39) \quad \sum_{\substack{m \leq D \\ m \text{ odd}}} \mu(m) \sum_{\substack{n \\ n \text{ odd}}} (\log n) e(\alpha mn) \eta\left(\frac{mn}{x}\right)$$

is at most

$$(3.40) \quad \frac{x}{q} \min\left(1, \frac{c_0/\delta^2}{(2\pi)^2}\right) \left| \sum_{\substack{m \leq \frac{M}{q} \\ \gcd(m, q) = 1}} \frac{\mu(m)}{m} \log \frac{x}{mq} \right| + \frac{x}{q} |\widehat{\log \cdot \eta}(-\delta)| \left| \sum_{\substack{m \leq \frac{M}{q} \\ \gcd(m, q) = 1}} \frac{\mu(m)}{m} \right| \\ + O^*\left(c_0 \left(\frac{1}{2} - \frac{2}{\pi^2}\right) \left(\frac{D^2}{4qx} \log \frac{e^{1/2}x}{D} + \frac{1}{e}\right)\right)$$

plus

$$(3.41) \quad \frac{2\sqrt{c_0c_1}}{\pi} D \log \frac{ex}{D} + \frac{3c_1}{2} \frac{x}{q} \log^+ \frac{D}{c_2x/q} \log \frac{q}{c_2} \\ + \left(\frac{2|\eta'|_1}{\pi} \max\left(1, \log \frac{c_0e^3q^2}{4\pi|\eta'|_1x}\right) \log x + \frac{2\sqrt{c_0c_1}}{\pi} \left(\sqrt{3} + \frac{1}{2} \log^+ \frac{D}{q/2}\right) \log \frac{q}{c_2}\right) q \\ + \frac{3c_1}{2} \sqrt{\frac{2x}{c_2}} \log \frac{2x}{c_2} + \frac{20c_0c_2^{3/2}}{3\pi^2} \sqrt{2x} \log \frac{2\sqrt{ex}}{c_2}$$

for $c_1 = 1 + |\eta'|_1/(x/D)$. The same bound holds if $|\delta| \geq 1/2c_2$ but $D \leq Q_0/2$.

In general, if $|\delta| \geq 1/2c_2$, the absolute value of (3.39) is at most

$$(3.42) \quad \frac{2\sqrt{c_0c_1}}{\pi} D \log \frac{ex}{D} + \\ \frac{2\sqrt{c_0c_1}}{\pi} (1 + \epsilon) \left(\frac{x}{|\delta|q} + 1\right) \left(\sqrt{3 + 2\epsilon} \cdot \log^+ 2\sqrt{e}|\delta|q + \frac{1}{2} \log^+ \frac{2D}{|\delta|q} \log^+ 2|\delta|q\right) \\ + \left(\frac{3c_1}{4} \left(\frac{2}{\sqrt{5}} + \frac{1 + \epsilon}{2\epsilon} \log x\right) + \frac{40}{3} \sqrt{2c_0c_2^{3/2}}\right) \sqrt{x} \log x$$

for $\epsilon \in (0, 1]$.

Proof. Define Q , Q' , M , a' and q' as in the proof of Lemma 3.4. The same method of proof works as for Lemma 3.4; we go over the differences. When applying Poisson summation or (2.2), use $\eta_{(x/m)}(t) = (\log xt/m)\eta(t)$ instead of $\eta(t)$. Then use the bounds in (3.38) with $\rho = x/m$; in particular,

$$|\widehat{\eta''_{(x/m)}}|_\infty \leq c_0 \log \frac{x}{m}.$$

For $f(u) = e((\delta m/2x)u)(\log u)\eta((m/x)u)$,

$$\widehat{f}(t) = \frac{x}{m} \widehat{\eta_{(x/m)}} \left(\frac{x}{m} t - \frac{\delta}{2} \right)$$

and so

$$\begin{aligned} \frac{1}{2} \sum_n \left| \widehat{f}(n/2) \right| &\leq \frac{x}{m} \left(\frac{1}{2} \left| \widehat{\eta_{(x/m)}} \left(-\frac{\delta}{2} \right) \right| + \frac{1}{2} \sum_{n \neq 0} \left| \widehat{\eta} \left(\frac{x}{m} \frac{n}{2} - \frac{\delta}{2} \right) \right| \right) \\ &= \frac{1}{2} \frac{x}{m} \left(\widehat{\log \cdot \eta} \left(-\frac{\delta}{2} \right) + \log \left(\frac{x}{m} \right) \widehat{\eta} \left(-\frac{\delta}{2} \right) \right) + \frac{m}{x} \left(\log \frac{x}{m} \right) \frac{c_0}{2\pi^2} (\pi^2 - 4). \end{aligned}$$

The part of the main term involving $\log(x/m)$ becomes

$$\frac{x \widehat{\eta}(-\delta)}{2} \sum_{\substack{m \leq M \\ m \text{ odd} \\ q|m}} \frac{\mu(m)}{m} \log \left(\frac{x}{m} \right) = \frac{x \mu(q)}{q} \widehat{\eta}(-\delta) \cdot \sum_{\substack{m \leq M/q \\ \gcd(m, 2q)=1}} \frac{\mu(m)}{m} \log \left(\frac{x}{mq} \right)$$

for q odd. (We can see that this, like the rest of the main term, vanishes for m even.)

In the term in front of $\pi^2 - 4$, we find the sum

$$\sum_{\substack{m \leq M \\ m \text{ odd} \\ q|m}} \frac{m}{x} \log \left(\frac{x}{m} \right) \leq \frac{M}{x} \log \frac{x}{M} + \frac{q}{2} \int_0^{M/q} t \log \frac{x/q}{t} dt = \frac{M}{x} \log \frac{x}{M} + \frac{M^2}{4qx} \log \frac{e^{1/2} x}{M},$$

where we use the fact that $t \mapsto t \log(x/t)$ is increasing for $t \leq x/e$. By the same fact (and by $M \leq D$), $(M^2/q) \log(e^{1/2} x/M) \leq (D^2/q) \log(e^{1/2} x/D)$. It is also easy to see that $(M/x) \log(x/M) \leq 1/e$ (since $M \leq D \leq x$).

The basic estimate for the rest of the proof (replacing (3.13)) is

$$\begin{aligned} T_{m,o}(\alpha) &= \sum_{n \text{ odd}} e(\alpha mn)(\log n) \eta \left(\frac{mn}{x} \right) = \sum_{n \text{ odd}} e(\alpha mn) \eta_{(x/m)} \left(\frac{mn}{x} \right) \\ &= O^* \left(\min \left(\frac{x}{2m} |\eta_{(x/m)}|_1 + \frac{|\eta'_{(x/m)}|_1}{2}, \frac{\frac{1}{2} |\eta'_{(x/m)}|_1}{|\sin(2\pi m \alpha)|}, \frac{m}{x} \frac{\frac{1}{2} |\widehat{\eta''_{(x/m)}}|_\infty}{(\sin 2\pi m \alpha)^2} \right) \right) \\ &= O^* \left(\log \frac{x}{m} \cdot \min \left(\frac{x}{2m} + \frac{|\eta'|_1}{2}, \frac{\frac{1}{2} |\eta'|_1}{|\sin(2\pi m \alpha)|}, \frac{m}{x} \frac{c_0}{2} \frac{1}{(\sin 2\pi m \alpha)^2} \right) \right). \end{aligned}$$

We wish to bound

$$(3.43) \quad \sum_{\substack{m \leq M \\ q|m \\ m \text{ odd}}} |T_{m,o}(\alpha)| + \sum_{\frac{Q}{2} < m \leq D} |T_{m,o}(\alpha)|.$$

Just as in the proofs of Lemmas 3.4 and 3.5, we give two bounds, one valid for $|\delta|$ large ($|\delta| \geq 1/2c_2$) and the other for δ small ($|\delta| \leq 1/2c_2$). Again as in the proof of Lemma 3.5, we ignore the condition that m is odd in (3.15).

Consider the case of $|\delta|$ large first. Instead of (3.16), we have

$$(3.44) \quad \sum_{\substack{1 \leq m \leq M \\ q|m}} |T_m(\alpha)| \leq \frac{40}{3\pi^2} \frac{c_0 q^3}{2x} \sum_{0 \leq j \leq \frac{M}{q}} (j+1) \log \frac{x}{jq+1}.$$

Since

$$\begin{aligned}
\sum_{0 \leq j \leq \frac{M}{q}} (j+1) \log \frac{x}{jq+1} &\leq \log x + \frac{M}{q} \log \frac{x}{M} + \sum_{1 \leq j \leq \frac{M}{q}} \log \frac{x}{jq} + \sum_{1 \leq j \leq \frac{M}{q}-1} j \log \frac{x}{jq} \\
&\leq \log x + \frac{M}{q} \log \frac{x}{M} + \int_0^{\frac{M}{q}} \log \frac{x}{tq} dt + \int_1^{\frac{M}{q}} t \log \frac{x}{tq} dt \\
&\leq \log x + \left(\frac{2M}{q} + \frac{M^2}{2q^2} \right) \log \frac{e^{1/2}x}{M},
\end{aligned}$$

this means that

$$\begin{aligned}
(3.45) \quad \sum_{\substack{1 \leq m \leq M \\ q \nmid m}} |T_m(\alpha)| &\leq \frac{40}{3\pi^2} \frac{c_0 q^3}{4x} \left(\log x + \left(\frac{2M}{q} + \frac{M^2}{2q^2} \right) \log \frac{e^{1/2}x}{M} \right) \\
&\leq \frac{5c_0 c_2}{3\pi^2} M \log \frac{\sqrt{e}x}{M} + \frac{40}{3} \sqrt{2} c_0 c_2^{3/2} \sqrt{x} \log x,
\end{aligned}$$

where we are using the bounds $M \leq Q/2 \leq c_2 x/q$ and $q^2 \leq 2c_2 x$ (just as in (3.16)). Instead of (3.17), we have

$$\begin{aligned}
\sum_{j=0}^{\lfloor \frac{D-(Q+1)/2}{q'} \rfloor} \left(\log \frac{x}{jq' + \frac{Q+1}{2}} \right) \frac{x}{jq' + \frac{Q+1}{2}} &\leq \frac{x}{Q/2} \log \frac{2x}{Q} + \frac{x}{q'} \int_{\frac{Q+1}{2}}^D \log \frac{x}{t} \frac{dt}{t} \\
&\leq \frac{2x}{Q} \log \frac{2x}{Q} + \frac{x}{q'} \log \frac{2x}{Q} \log^+ \frac{2D}{Q};
\end{aligned}$$

recall that the coefficient in front of this sum will be halved by the condition that n is odd. Instead of (3.18), we obtain

$$\begin{aligned}
q' \sum_{j=0}^{\lfloor \frac{D-(Q+1)/2}{q'} \rfloor} \sqrt{1 + \frac{q'}{jq' + (Q+1)/2}} \left(\log \frac{x}{jq' + \frac{Q+1}{2}} \right) \\
\leq q' \sqrt{3+2\epsilon} \cdot \log \frac{2x}{Q+1} + \int_{\frac{Q+1}{2}}^D \left(1 + \frac{q'}{2t} \right) \left(\log \frac{x}{t} \right) dt \\
\leq q' \sqrt{3+2\epsilon} \cdot \log \frac{2x}{Q+1} + D \log \frac{ex}{D} - \frac{Q+1}{2} \log \frac{2ex}{Q+1} + \frac{q'}{2} \log \frac{2x}{Q+1} \log \frac{2D}{Q+1}.
\end{aligned}$$

(The bound $\int_a^b \log(x/t) dt/t \leq \log(x/a) \log(b/a)$ will be more practical than the exact expression for the integral.) Hence $\sum_{Q/2 < m \leq D} |T_m(\alpha)|$ is at most

$$\begin{aligned}
&\frac{2\sqrt{c_0 c_1}}{\pi} D \log \frac{ex}{D} \\
&+ \frac{2\sqrt{c_0 c_1}}{\pi} \left((1+\epsilon)\sqrt{3+2\epsilon} + \frac{(1+\epsilon)}{2} \log \frac{2D}{Q+1} \right) (Q+1) \log \frac{2x}{Q+1} \\
&- \frac{2\sqrt{c_0 c_1}}{\pi} \cdot \frac{Q+1}{2} \log \frac{2ex}{Q+1} + \frac{3c_1}{2} \left(\frac{2}{\sqrt{5}} + \frac{1+\epsilon}{\epsilon} \log^+ \frac{D}{Q/2} \right) \sqrt{x} \log \sqrt{x}.
\end{aligned}$$

Summing this to (3.45) (with $M = Q/2$), and using (3.21) and (3.22) as before, we obtain that (3.43) is at most

$$\begin{aligned} & \frac{2\sqrt{c_0c_1}}{\pi} D \log \frac{ex}{D} \\ & + \frac{2\sqrt{c_0c_1}}{\pi} (1+\epsilon)(Q+1) \left(\sqrt{3+2\epsilon} \log^+ \frac{2\sqrt{ex}}{Q+1} + \frac{1}{2} \log^+ \frac{2D}{Q+1} \log^+ \frac{2x}{Q+1} \right) \\ & + \frac{3c_1}{2} \left(\frac{2}{\sqrt{5}} + \frac{1+\epsilon}{\epsilon} \log^+ \frac{D}{Q/2} \right) \sqrt{x} \log \sqrt{x} + \frac{40}{3} \sqrt{2c_0c_2}^{3/2} \sqrt{x} \log x. \end{aligned}$$

Now we go over the case of $|\delta|$ small (or $D \leq Q_0/2$). Instead of (3.23), we have

$$(3.46) \quad \sum_{m \leq q/2} |T_{m,\circ}(\alpha)| \leq \frac{2|\eta'|_1}{\pi} q \max \left(1, \log \frac{c_0e^3q^2}{4\pi|\eta'|_1x} \right) \log x.$$

Suppose $q^2 < 2c_2x$. Instead of (3.24), we have

$$(3.47) \quad \begin{aligned} \sum_{\substack{q/2 < m \leq D' \\ q \nmid m}} |T_{m,\circ}(\alpha)| & \leq \frac{40}{3\pi^2} \frac{c_0q^3}{6x} \sum_{1 \leq j \leq \frac{D'}{q} + \frac{1}{2}} \left(j + \frac{1}{2} \right) \log \frac{x}{(j - \frac{1}{2})q} \\ & \leq \frac{10c_0q^3}{3\pi^2x} \left(\log \frac{2x}{q} + \frac{1}{q} \int_0^{D'} \log \frac{x}{t} dt + \frac{1}{q} \int_0^{D'} t \log \frac{x}{t} dt + \frac{D'}{q} \log \frac{x}{D'} \right) \\ & = \frac{10c_0q^3}{3\pi^2x} \left(\log \frac{2x}{q} + \left(\frac{2D'}{q} + \frac{(D')^2}{2q^2} \right) \log \frac{\sqrt{ex}}{D'} \right) \\ & \leq \frac{5c_0c_2}{3\pi^2} \left(4\sqrt{2c_2x} \log \frac{2x}{q} + 4\sqrt{2c_2x} \log \frac{\sqrt{ex}}{D'} + D' \log \frac{\sqrt{ex}}{D'} \right) \\ & \leq \frac{5c_0c_2}{3\pi^2} \left(D' \log \frac{\sqrt{ex}}{D'} + 4\sqrt{2c_2x} \log \frac{2\sqrt{e}}{c_2} x \right) \end{aligned}$$

where $D' = \min(c_2x/q, D)$. (We are using the bounds $q^3/x \leq (2c_2)^{3/2}$, $D'q^2/x \leq c_2q < c_2^{3/2}\sqrt{2x}$ and $D'q/x \leq c_2$.) Instead of (3.25), we have

$$\sum_{R < m \leq D} |T_{m,\circ}(\alpha)| \leq \sum_{j=0}^{\lfloor \frac{1}{q}(D-R) \rfloor} \left(\frac{3c_1}{jq+R} + \frac{4q}{\pi} \sqrt{\frac{c_1c_0}{4} \left(1 + \frac{q}{jq+R} \right)} \right) \log \frac{x}{jq+R},$$

where $R = \max(c_2x/q, q/2)$. We can simply reuse (3.26), multiplying it by $\log x/R$; we replace (3.27) by

$$\begin{aligned} q \sum_{j=0}^{\lfloor \frac{1}{q}(D-R) \rfloor} \sqrt{1 + \frac{q}{jq+R}} \log \frac{x}{jq+R} & \leq q \sqrt{1 + \frac{q}{R}} \log \frac{x}{R} + \int_R^D \sqrt{1 + \frac{q}{t}} \log \frac{x}{t} dt \\ & \leq \sqrt{3}q \log \frac{q}{c_2} + \left(D \log \frac{ex}{D} - R \log \frac{ex}{R} \right) + \frac{q}{2} \log \frac{q}{c_2} \log^+ \frac{D}{R}. \end{aligned}$$

We sum with (3.46) and (3.47), and obtain (3.42) as an upper bound for (3.43). \square

We will apply the following only for q relatively large.

Lemma 3.7. *Let $\alpha \in \mathbb{R}/\mathbb{Z}$ with $2\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $|\delta/x| \leq 1/qQ_0$, $q \leq Q_0$, $Q_0 \geq \max(2e, 2\sqrt{x})$. Let η be continuous, piecewise C^2 and compactly supported, with $|\eta|_1 = 1$ and $\eta'' \in L_1$. Let $c_0 \geq |\widehat{\eta''}|_\infty$. Let $c_2 = 6\pi/5\sqrt{c_0}$. Assume that $x \geq e^2c_2/2$.*

Let $U, V \geq 1$ satisfy $UV + (19/18)Q_0 \leq x/5.6$. Then, if $|\delta| \leq 1/2c_2$, the absolute value of

$$(3.48) \quad \left| \sum_{\substack{v \leq V \\ v \text{ odd}}} \Lambda(v) \sum_{\substack{u \leq U \\ u \text{ odd}}} \mu(u) \sum_{\substack{n \\ n \text{ odd}}} e(\alpha v u n) \eta(v u n / x) \right|$$

is at most

$$(3.49) \quad \frac{x}{2q} \min\left(1, \frac{c_0}{(\pi\delta)^2}\right) \log Vq + O^*\left(\frac{1}{4} - \frac{1}{\pi^2}\right) \cdot c_0 \left(\frac{D^2 \log V}{2qx} + \frac{3c_4 UV^2}{2x} + \frac{(U+1)^2 V}{2x} \log q\right)$$

plus

$$(3.50) \quad \frac{2\sqrt{c_0 c_1}}{\pi} \left(D \log \frac{D}{\sqrt{e}} + q \left(\sqrt{3} \log \frac{c_2 x}{q} + \frac{\log D}{2} \log^+ \frac{D}{q/2}\right)\right) + \frac{3c_1 x}{2q} \log D \log^+ \frac{D}{c_2 x/q} + \frac{2|\eta'|_1}{\pi} q \max\left(1, \log \frac{c_0 e^3 q^2}{4\pi|\eta'|_1 x}\right) \log \frac{q}{2} + \frac{3c_1}{2\sqrt{2c_2}} \sqrt{x} \log \frac{c_2 x}{2} + \frac{25c_0}{4\pi^2} (2c_2)^{3/2} \sqrt{x} \log x,$$

where $D = UV$ and $c_1 = 1 + |\eta'|_1/(2x/D)$ and $c_4 = 1.03884$. The same bound holds if $|\delta| \geq 1/2c_2$ but $D \leq Q_0/2$.

In general, if $|\delta| \geq 1/2c_2$, the absolute value of (3.48) is at most (3.49) plus

$$(3.51) \quad \frac{2\sqrt{c_0 c_1}}{\pi} D \log \frac{D}{e} + \frac{2\sqrt{c_0 c_1}}{\pi} (1 + \epsilon) \left(\frac{x}{|\delta|q} + 1\right) \left(\left(\sqrt{3 + 2\epsilon} - 1\right) \log \frac{\frac{x}{|\delta|q} + 1}{\sqrt{2}} + \frac{1}{2} \log D \log^+ \frac{e^2 D}{\frac{x}{|\delta|q}}\right) + \left(\frac{3c_1}{2} \left(\frac{1}{2} + \frac{3(1 + \epsilon)}{16\epsilon} \log x\right) + \frac{20c_0}{3\pi^2} (2c_2)^{3/2}\right) \sqrt{x} \log x$$

for $\epsilon \in (0, 1]$.

Proof. We proceed essentially as in Lemma 3.4 and Lemma 3.5. Let Q, q' and Q' be as in the proof of Lemma 3.5, that is, with 2α where Lemma 3.4 uses α .

Let $M = \min(UV, Q/2)$. We first consider the terms with $uv \leq M$, u and v odd, uv divisible by q . If q is even, there are no such terms. Assume q is odd. Then, by (3.33) and (3.34), the absolute value of the contribution of these terms is at most

$$(3.52) \quad \sum_{\substack{a \leq M \\ a \text{ odd} \\ q|a}} \left(\sum_{\substack{v|a \\ a/U \leq v \leq V}} \Lambda(v) \mu(a/v) \right) \left(\frac{x \widehat{\eta}(-\delta/2)}{2a} + O\left(\frac{a |\widehat{\eta''}|_\infty}{x} \frac{1}{2\pi^2} \cdot (\pi^2 - 4)\right) \right).$$

Now

$$\begin{aligned}
& \sum_{\substack{a \leq M \\ a \text{ odd} \\ q|a}} \sum_{\substack{v|a \\ a/U \leq v \leq V}} \frac{\Lambda(v)\mu(a/v)}{a} \\
&= \sum_{\substack{v \leq V \\ v \text{ odd} \\ \gcd(v,q)=1}} \frac{\Lambda(v)}{v} \sum_{\substack{u \leq \min(U, M/V) \\ u \text{ odd} \\ q|u}} \frac{\mu(u)}{u} + \sum_{\substack{p^\alpha \leq V \\ p \text{ odd} \\ p|q}} \frac{\Lambda(p^\alpha)}{p^\alpha} \sum_{\substack{u \leq \min(U, M/V) \\ u \text{ odd} \\ \frac{q}{\gcd(q, p^\alpha)} | u}} \frac{\mu(u)}{u} \\
&= \frac{\mu(q)}{q} \sum_{\substack{v \leq V \\ v \text{ odd} \\ \gcd(v,q)=1}} \frac{\Lambda(v)}{v} \sum_{\substack{u \leq \min(U/q, M/Vq) \\ \gcd(u, 2q)=1}} \frac{\mu(u)}{u} \\
&\quad + \frac{\mu\left(\frac{q}{\gcd(q, p^\alpha)}\right)}{q} \sum_{\substack{p^\alpha \leq V \\ p \text{ odd} \\ p|q}} \frac{\Lambda(p^\alpha)}{p^\alpha / \gcd(q, p^\alpha)} \sum_{\substack{u \leq \min\left(\frac{U}{q/\gcd(q, p^\alpha)}, \frac{M/V}{q/\gcd(q, p^\alpha)}\right) \\ u \text{ odd} \\ \gcd\left(u, \frac{q}{\gcd(q, p^\alpha)}\right)=1}} \frac{\mu(u)}{u} \\
&= \frac{1}{q} \cdot O^* \left(\sum_{\substack{v \leq V \\ \gcd(v, 2q)=1}} \frac{\Lambda(v)}{v} + \sum_{\substack{p^\alpha \leq V \\ p \text{ odd} \\ p|q}} \frac{\log p}{p^\alpha / \gcd(q, p^\alpha)} \right),
\end{aligned}$$

where we are using (2.7) to bound the sums on u by 1. We notice that

$$\begin{aligned}
\sum_{\substack{p^\alpha \leq V \\ p \text{ odd} \\ p|q}} \frac{\log p}{p^\alpha / \gcd(q, p^\alpha)} &\leq \sum_{\substack{p \text{ odd} \\ p|q}} (\log p) \left(v_p(q) + \sum_{\substack{\alpha > v_p(q) \\ p^\alpha \leq V}} \frac{1}{p^{\alpha - v_p(q)}} \right) \\
&\leq \log q + \sum_{\substack{p \text{ odd} \\ p|q}} (\log p) \sum_{\substack{\beta > 0 \\ p^\beta \leq \frac{V}{p^{v_p(q)}}}} \frac{\log p}{p^\beta} \leq \log q + \sum_{\substack{v \leq V \\ v \text{ odd} \\ \gcd(v, q)=1}} \frac{\Lambda(v)}{v},
\end{aligned}$$

and so

$$\begin{aligned}
\sum_{\substack{a \leq M \\ a \text{ odd} \\ q|a}} \sum_{\substack{v|a \\ a/U \leq v \leq V}} \frac{\Lambda(v)\mu(a/v)}{a} &= \frac{1}{q} \cdot O^* \left(\log q + \sum_{\substack{v \leq V \\ \gcd(v, 2)=1}} \frac{\Lambda(v)}{v} \right) \\
&= \frac{1}{q} \cdot O^*(\log q + \log V)
\end{aligned}$$

by (2.12). The absolute value of the sum of the terms with $\widehat{\eta}(-\delta/2)$ in (3.52) is thus at most

$$\frac{x \widehat{\eta}(-\delta/2)}{q} (\log q + \log V) \leq \frac{x}{2q} \min\left(1, \frac{c_0}{(\pi\delta)^2}\right) \log Vq,$$

where we are bounding $\widehat{\eta}(-\delta/2)$ by (2.1).

The other terms in (3.52) contribute at most

$$(3.53) \quad (\pi^2 - 4) \frac{|\widehat{\eta}'|_\infty}{2\pi^2} \frac{1}{x} \sum_{\substack{u \leq U \\ uv \text{ odd} \\ uv \leq M, q|uv \\ u \text{ sq-free}}} \sum_{v \leq V} \Lambda(v) uv.$$

For any R , $\sum_{u \leq R, u \text{ odd}, q|u} \leq R^2/4q + 3R/4$. Using the estimates (2.12), (2.15) and (2.16), we obtain that the double sum in (3.53) is at most

$$(3.54) \quad \begin{aligned} & \sum_{\substack{v \leq V \\ \gcd(v, 2q)=1}} \Lambda(v)v \sum_{\substack{u \leq \min(U, M/v) \\ u \text{ odd} \\ q|u}} u + \sum_{\substack{p^\alpha \leq V \\ p \text{ odd} \\ p|q}} (\log p)p^\alpha \sum_{\substack{u \leq U \\ u \text{ odd} \\ \frac{q}{\gcd(q, p^\alpha)} | u}} u \\ & \leq \sum_{\substack{v \leq V \\ \gcd(v, 2q)=1}} \Lambda(v)v \cdot \left(\frac{(M/v)^2}{4q} + \frac{3M}{4v} \right) + \sum_{\substack{p^\alpha \leq V \\ p \text{ odd} \\ p|q}} (\log p)p^\alpha \cdot \frac{(U+1)^2}{4} \\ & \leq \frac{M^2 \log V}{4q} + \frac{3c_4}{4} MV + \frac{(U+1)^2}{4} V \log q, \end{aligned}$$

where $c_4 = 1.03884$.

From this point onwards, we use the easy bound

$$\left| \sum_{\substack{v|a \\ a/U \leq v \leq V}} \Lambda(v) \mu(a/v) \right| \leq \log a.$$

What we must bound now is

$$(3.55) \quad \sum_{\substack{m \leq UV \\ m \text{ odd} \\ q \nmid m \text{ or } m > M}} (\log m) \sum_{n \text{ odd}} e(\alpha mn) \eta(mn/x).$$

The inner sum is the same as the sum $T_{m, \circ}(\alpha)$ in (3.35); we will be using the bound (3.36). Much as before, we will be able to ignore the condition that m is odd.

Let $D = UV$. What remains to do is similar to what we did in the proof of Lemma 3.4 (or Lemma 3.5).

Case (a). δ large: $|\delta| \geq 1/2c_2$. Instead of (3.16), we have

$$\sum_{\substack{1 \leq m \leq M \\ q \nmid m}} (\log m) |T_{m, \circ}(\alpha)| \leq \frac{40}{3\pi^2} \frac{c_0 q^3}{4x} \sum_{0 \leq j \leq \frac{M}{q}} (j+1) \log(j+1)q,$$

and, since $M \leq \min(c_2x/q, D)$, $q \leq \sqrt{2c_2x}$ (just as in the proof of Lemma 3.4) and

$$\begin{aligned} \sum_{0 \leq j \leq \frac{M}{q}} (j+1) \log(j+1)q &\leq \frac{M}{q} \log M + \left(\frac{M}{q} + 1\right) \log(M+1) + \frac{1}{q^2} \int_0^M t \log t \, dt \\ &\leq \left(2\frac{M}{q} + 1\right) \log x + \frac{M^2}{2q^2} \log \frac{M}{\sqrt{e}}, \end{aligned}$$

we conclude that

$$(3.56) \quad \sum_{\substack{1 \leq m \leq M \\ q|m}} |T_{m,\circ}(\alpha)| \leq \frac{5c_0c_2}{3\pi^2} M \log \frac{M}{\sqrt{e}} + \frac{20c_0}{3\pi^2} (2c_2)^{3/2} \sqrt{x} \log x.$$

Instead of (3.17), we have

$$\begin{aligned} \sum_{j=0}^{\lfloor \frac{D-(Q+1)/2}{q'} \rfloor} \frac{x}{jq' + \frac{Q+1}{2}} \log \left(jq' + \frac{Q+1}{2} \right) &\leq \frac{x}{\frac{Q+1}{2}} \log \frac{Q+1}{2} + \frac{x}{q'} \int_{\frac{Q+1}{2}}^D \frac{\log t}{t} dt \\ &\leq \frac{2x}{Q} \log \frac{Q}{2} + \frac{(1+\epsilon)x}{2\epsilon Q} \left((\log D)^2 - \left(\log \frac{Q}{2} \right)^2 \right). \end{aligned}$$

Instead of (3.18), we estimate

$$\begin{aligned} q' \sum_{j=0}^{\lfloor \frac{D-\frac{Q+1}{2}}{q'} \rfloor} \left(\log \left(\frac{Q+1}{2} + jq' \right) \right) \sqrt{1 + \frac{q'}{jq' + \frac{Q+1}{2}}} \\ \leq q' \left(\log D + (\sqrt{3+2\epsilon} - 1) \log \frac{Q+1}{2} \right) + \int_{\frac{Q+1}{2}}^D \log t \, dt + \int_{\frac{Q+1}{2}}^D \frac{q' \log t}{2t} dt \\ \leq q' \left(\log D + (\sqrt{3+2\epsilon} - 1) \log \frac{Q+1}{2} \right) + \left(D \log \frac{D}{e} - \frac{Q+1}{2} \log \frac{Q+1}{2e} \right) \\ + \frac{q'}{2} \log D \log^+ \frac{D}{\frac{Q+1}{2}}. \end{aligned}$$

We conclude that, when $D \geq Q/2$, the sum $\sum_{Q/2 < m \leq D} (\log m) |T_m(\alpha)|$ is at most

$$\begin{aligned} \frac{2\sqrt{c_0c_1}}{\pi} \left(D \log \frac{D}{e} + (Q+1) \left((1+\epsilon)(\sqrt{3+2\epsilon} - 1) \log \frac{Q+1}{2} - \frac{1}{2} \log \frac{Q+1}{2e} \right) \right) \\ + \frac{\sqrt{c_0c_1}}{\pi} (Q+1)(1+\epsilon) \log D \log^+ \frac{e^2 D}{\frac{Q+1}{2}} \\ + \frac{3c_1}{2} \left(\frac{2x}{Q} \log \frac{Q}{2} + \frac{(1+\epsilon)x}{2\epsilon Q} \left((\log D)^2 - \left(\log \frac{Q}{2} \right)^2 \right) \right). \end{aligned}$$

We must now add this to (3.56). Since

$$(1+\epsilon)(\sqrt{3+2\epsilon} - 1) \log \sqrt{2} - \frac{1}{2} \log 2e + \frac{1 + \sqrt{13/3}}{2} \log 2\sqrt{e} > 0$$

and $Q \geq 2\sqrt{x}$, we conclude that (3.55) is at most

$$(3.57) \quad \begin{aligned} & \frac{2\sqrt{c_0c_1}}{\pi} D \log \frac{D}{e} \\ & + \frac{2\sqrt{c_0c_1}}{\pi} (1+\epsilon)(Q+1) \left((\sqrt{3+2\epsilon}-1) \log \frac{Q+1}{\sqrt{2}} + \frac{1}{2} \log D \log^+ \frac{e^2 D}{\frac{Q+1}{2}} \right) \\ & + \left(\frac{3c_1}{2} \left(\frac{1}{2} + \frac{3(1+\epsilon)}{16\epsilon} \log x \right) + \frac{20c_0}{3\pi^2} (2c_2)^{3/2} \right) \sqrt{x} \log x. \end{aligned}$$

Case (b). δ small: $|\delta| \leq 1/2c_2$ or $D \leq Q/2$. The analogue of (3.23) is a bound of

$$\leq \frac{2|\eta'|_1}{\pi} q \max \left(1, \log \frac{c_0 e^3 q^2}{4\pi |\eta'|_1 x} \right) \log \frac{q}{2}$$

for the terms with $m \leq q/2$. If $q^2 < 2c_2x$, then, much as in (3.24), we have

$$(3.58) \quad \begin{aligned} \sum_{\substack{\frac{q}{2} < m \leq D' \\ q \nmid m}} |T_{m,\circ}(\alpha)| (\log m) & \leq \frac{10}{\pi^2} \frac{c_0 q^3}{3x} \sum_{1 \leq j \leq \frac{D'}{q} + \frac{1}{2}} \left(j + \frac{1}{2} \right) \log(j+1/2)q \\ & \leq \frac{10}{\pi^2} \frac{c_0 q}{3x} \int_q^{D'+\frac{3}{2}q} x \log x \, dx. \end{aligned}$$

Since

$$\begin{aligned} \int_q^{D'+\frac{3}{2}q} x \log x \, dx & = \frac{1}{2} \left(D' + \frac{3}{2}q \right)^2 \log \frac{D'+\frac{3}{2}q}{\sqrt{e}} - \frac{1}{2} q^2 \log \frac{q}{\sqrt{e}} \\ & = \left(\frac{1}{2} D'^2 + \frac{3}{2} D'q \right) \left(\log \frac{D'}{\sqrt{e}} + \frac{3}{2} \frac{q}{D'} \right) + \frac{9}{8} q^2 \log \frac{D'+\frac{3}{2}q}{\sqrt{e}} - \frac{1}{2} q^2 \log \frac{q}{\sqrt{e}} \\ & = \frac{1}{2} D'^2 \log \frac{D'}{\sqrt{e}} + \frac{3}{2} D'q \log D' + \frac{9}{8} q^2 \left(\frac{2}{9} + \frac{3}{2} + \log \left(D' + \frac{19}{18}q \right) \right), \end{aligned}$$

where $D' = \min(c_2x/q, D)$, and since the assumption $(UV + (19/18)Q_0) \leq x/5.6$ implies that $(2/9 + 3/2 + \log(D' + (19/18)q)) \leq x$, we conclude that

$$(3.59) \quad \begin{aligned} & \sum_{\substack{\frac{q}{2} < m \leq D' \\ q \nmid m}} |T_{m,\circ}(\alpha)| (\log m) \\ & \leq \frac{5c_0c_2}{3\pi^2} D' \log \frac{D'}{\sqrt{e}} + \frac{10c_0}{3\pi^2} \left(\frac{3}{4} (2c_2)^{3/2} \sqrt{x} \log x + \frac{9}{8} (2c_2)^{3/2} \sqrt{x} \log x \right) \\ & \leq \frac{5c_0c_2}{3\pi^2} D' \log \frac{D'}{\sqrt{e}} + \frac{25c_0}{4\pi^2} (2c_2)^{3/2} \sqrt{x} \log x. \end{aligned}$$

Let $R = \max(c_2x/q, q/2)$. We bound the terms $R < m \leq D$ as in (3.25), with a factor of $\log(jq + R)$ inside the sum. The analogues of (3.26) and (3.27) are

$$(3.60) \quad \begin{aligned} & \sum_{j=0}^{\lfloor \frac{1}{q}(D-R) \rfloor} \frac{x}{jq+R} \log(jq+R) \leq \frac{x}{R} \log R + \frac{x}{q} \int_R^D \frac{\log t}{t} dt \\ & \leq \sqrt{\frac{2x}{c_2}} \log \sqrt{\frac{c_2x}{2}} + \frac{x}{q} \log D \log^+ \frac{D}{R}, \end{aligned}$$

where we use the assumption that $x \geq e^2 c/2$, and

$$(3.61) \quad \sum_{j=0}^{\lfloor \frac{1}{q}(D-R) \rfloor} \log(jq + R) \sqrt{1 + \frac{q}{jq + R}} \leq \sqrt{3} \log R \\ + \frac{1}{q} \left(D \log \frac{D}{e} - R \log \frac{R}{e} \right) + \frac{1}{2} \log D \log \frac{D}{R}$$

(or 0 if $D < R$). We sum with (3.59) and the terms with $m \leq q/2$, and obtain, for $D' = c_2 x/q = R$,

$$\frac{2\sqrt{c_0 c_1}}{\pi} \left(D \log \frac{D}{\sqrt{e}} + q \left(\sqrt{3} \log \frac{c_2 x}{q} + \frac{\log D}{2} \log^+ \frac{D}{q/2} \right) \right) \\ + \frac{3c_1}{2} \frac{x}{q} \log D \log^+ \frac{D}{c_2 x/q} + \frac{2|\eta'|_1}{\pi} q \max \left(1, \log \frac{c_0 e^3 q^2}{4\pi |\eta'|_1 x} \right) \log \frac{q}{2} \\ + \frac{3c_1}{2\sqrt{2}c_2} \sqrt{x} \log \frac{c_2 x}{2} + \frac{25c_0}{4\pi^2} (2c_2)^{3/2} \sqrt{x} \log x,$$

which, it is easy to check, is also valid even if $D' = D$ (in which case (3.60) and (3.61) do not appear) or $R = q/2$ (in which case (3.59) does not appear). \square

4. TYPE II

We must now consider the sum

$$(4.1) \quad S_{II} = \sum_{\substack{m > U \\ \gcd(m, v) = 1}} \left(\sum_{\substack{d > U \\ d|m}} \mu(d) \right) \sum_{\substack{n > V \\ \gcd(n, v) = 1}} \Lambda(n) e(\alpha mn) \eta(mn/x).$$

Here the main improvements over classical treatments are as follows:

- (1) obtaining cancellation in the term

$$\sum_{\substack{d > U \\ d|m}} \mu(d)$$

leading to a gain of a factor of \log ;

- (2) using a large sieve for primes, getting rid of a further \log ;
- (3) exploiting, via a non-conventional application of the principle of the large sieve (Lemma 4.3), the fact that α is in the tail of an interval (when that is the case).

Some of the techniques developed for (1) should be applicable to other instances of Vaughan's identity in the literature.

It is technically helpful to express η as the (multiplicative) convolution of two functions of compact support – preferably the same function:

$$(4.2) \quad \eta(x) = \int_0^\infty \eta_1(t) \eta_1(x/t) \frac{dt}{t}.$$

For the smoothing function $\eta(t) = \eta_2(t) = 4 \max(\log 2 - |\log 2t|, 0)$, (4.2) holds with

$$(4.3) \quad \eta_1(t) = \begin{cases} 2 & \text{if } t \in (1/2, 1] \\ 0 & \text{otherwise.} \end{cases}$$

We will work with $\eta_1(t)$ as in (4.3) for convenience, yet what follows should carry over to other (non-negative) choices of η_1 .

By (4.2), the sum (4.1) equals
(4.4)

$$\begin{aligned} & 4 \int_0^\infty \sum_{\substack{m>U \\ \gcd(m,v)=1}} \left(\sum_{\substack{d>U \\ d|m}} \mu(d) \right) \sum_{\substack{n>V \\ \gcd(n,v)=1}} \Lambda(n) e(\alpha mn) \eta_1(t) \eta_1\left(\frac{mn/x}{t}\right) \frac{dt}{t} \\ &= 4 \int_V^{x/U} \sum_{\substack{\max(\frac{x}{2W}, U) < m \leq \frac{x}{W} \\ \gcd(m,v)=1}} \left(\sum_{\substack{d>U \\ d|m}} \mu(d) \right) \sum_{\substack{\max(V, \frac{W}{2}) < n \leq W \\ \gcd(n,v)=1}} \Lambda(n) e(\alpha mn) \frac{dW}{W} \end{aligned}$$

by the substitution $t = (m/x)W$. (We can assume $V \leq W \leq x/U$ because otherwise one of the sums in (4.5) is empty.)

We separate n prime and n non-prime. By Cauchy-Schwarz, the expression within the integral in (4.4) is then at most $\sqrt{S_1(U, W) \cdot S_2(U, V, W)} + \sqrt{S_1(U, W) \cdot S_3(W)}$, where

$$(4.5) \quad \begin{aligned} S_1(U, W) &= \sum_{\substack{\max(\frac{x}{2W}, U) < m \leq \frac{x}{W} \\ \gcd(m,v)=1}} \left(\sum_{\substack{d>U \\ d|m}} \mu(d) \right)^2, \\ S_2(U, V, W) &= \sum_{\substack{\max(\frac{x}{2W}, U) < m \leq \frac{x}{W} \\ \gcd(m,v)=1}} \left| \sum_{\substack{\max(V, \frac{W}{2}) < p \leq W \\ \gcd(p,v)=1}} (\log p) e(\alpha mp) \right|^2 \end{aligned}$$

and

$$(4.6) \quad \begin{aligned} S_3(W) &= \sum_{\substack{\frac{x}{2W} < m \leq \frac{x}{W} \\ \gcd(m,v)=1}} \left| \sum_{\substack{n \leq W \\ n \text{ non-prime}}} \Lambda(n) \right|^2 \\ &= \sum_{\substack{\frac{x}{2W} < m \leq \frac{x}{W} \\ \gcd(m,v)=1}} \left(1.42620W^{1/2} \right)^2 \leq 1.0171x + 2.0341W \end{aligned}$$

(by [RS62, Thm. 13]). We will assume $V \leq w$; thus the condition $\gcd(p, v) = 1$ will be fulfilled automatically and can be removed.

The contribution of $S_3(W)$ will be negligible. We must bound $S_1(U, W)$ and $S_2(U, V, W)$ from above.

4.1. The sum S_1 : cancellation. We shall bound

$$S_1(U, W) = \sum_{\substack{\max(U, x/2W) < m \leq x/W \\ \gcd(m,v)=1}} \left(\sum_{\substack{d>U \\ d|m}} \mu(d) \right)^2.$$

There will be what is perhaps a surprising amount of cancellation: the expression within the sum will be bounded by a constant on average.

4.1.1. *Reduction to a sum with μ .* We can write

$$(4.7) \quad \sum_{\substack{\max(U, x/2W) < m \leq x/W \\ \gcd(m, v) = 1}} \left(\sum_{\substack{d > U \\ d|m}} \mu(d) \right)^2 = \sum_{\substack{\frac{x}{2W} < m \leq \frac{x}{W} \\ \gcd(m, v) = 1}} \sum_{d_1, d_2 | m} \mu(d_1 > U) \mu(d_2 > U) \\ = \sum_{\substack{r_1 < x/WU \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \sum_{\substack{r_2 < x/WU \\ \gcd(l, r_1 r_2) = 1 \\ r_1 l, r_2 l > U \\ \gcd(l, v) = 1}} \sum_l \mu(r_1 l) \mu(r_2 l) \sum_{\substack{\frac{x}{2W} < m \leq \frac{x}{W} \\ r_1 r_2 l | m \\ \gcd(m, v) = 1}} 1,$$

where we write $d_1 = r_1 l$, $d_2 = r_2 l$, $l = \gcd(d_1, d_2)$. (The inequality $r_1 < x/WU$ comes from $r_1 r_2 l | m$, $m \leq x/W$, $r_2 l > U$; $r_2 < x/WU$ is proven in the same way.) Now (4.7) equals

$$(4.8) \quad \sum_{\substack{s < \frac{x}{WU} \\ \gcd(s, v) = 1}} \sum_{\substack{r_1 < \frac{x}{WUs} \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \sum_{r_2 < \frac{x}{WUs}} \mu(r_1) \mu(r_2) \sum_{\substack{\max\left(\frac{U}{\min(r_1, r_2)}, \frac{x/W}{2r_1 r_2 s}\right) < l \leq \frac{x/W}{r_1 r_2 s} \\ \gcd(l, r_1 r_2) = 1, (\mu(l))^2 = 1 \\ \gcd(l, v) = 1}} 1,$$

where we have set $s = m/(r_1 r_2 l)$.

Lemma 4.1. *Let $z, y > 0$. Then*

$$(4.9) \quad \sum_{\substack{r_1 < y \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \sum_{r_2 < y} \mu(r_1) \mu(r_2) \sum_{\substack{\min\left(\frac{z/y}{\min(r_1, r_2)}, \frac{z}{2r_1 r_2}\right) < l \leq \frac{z}{r_1 r_2} \\ \gcd(l, r_1 r_2) = 1, (\mu(l))^2 = 1 \\ \gcd(l, v) = 1}} 1$$

equals

$$(4.10) \quad \frac{6z}{\pi^2} \frac{v}{\sigma(v)} \sum_{\substack{r_1 < y \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \sum_{r_2 < y} \frac{\mu(r_1) \mu(r_2)}{\sigma(r_1) \sigma(r_2)} \left(1 - \max\left(\frac{1}{2}, \frac{r_1}{y}, \frac{r_2}{y}\right) \right) \\ + O^* \left(5.08 \zeta\left(\frac{3}{2}\right)^2 y \sqrt{z} \cdot \prod_{p|v} \left(1 + \frac{1}{\sqrt{p}}\right) \left(1 - \frac{1}{p^{3/2}}\right)^2 \right).$$

If $v = 2$, the error term in (4.10) can be replaced by

$$(4.11) \quad O^* \left(1.27 \zeta\left(\frac{3}{2}\right)^2 y \sqrt{z} \cdot \left(1 + \frac{1}{\sqrt{2}}\right) \left(1 - \frac{1}{2^{3/2}}\right)^2 \right).$$

Proof. By Möbius inversion, (4.9) equals

$$(4.12) \quad \sum_{\substack{r_1 < y \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \sum_{r_2 < y} \mu(r_1) \mu(r_2) \quad \sum_{\substack{l \leq \frac{z}{r_1 r_2} \\ l > \min\left(\frac{z/y}{\min(r_1, r_2)}, \frac{z}{2r_1 r_2}\right) \\ \gcd(l, v) = 1}} \sum_{\substack{d_1 | r_1, d_2 | r_2 \\ d_1 d_2 | l}} \mu(d_1) \mu(d_2) \\ \sum_{\substack{d_3 | v \\ d_3 | l}} \mu(d_3) \quad \sum_{\substack{m^2 | l \\ \gcd(m, r_1 r_2 v) = 1}} \mu(m).$$

We can change the order of summation of r_i and d_i by defining $s_i = r_i/d_i$, and we can also use the obvious fact that the number of integers in an interval $(a, b]$ divisible by d is $(b - a)/d + O^*(1)$. Thus (4.12) equals

$$(4.13) \quad \sum_{\substack{d_1, d_2 < y \\ \gcd(d_1, d_2) = 1 \\ \gcd(d_1 d_2, v) = 1}} \mu(d_1) \mu(d_2) \quad \sum_{\substack{s_1 < y/d_1 \\ s_2 < y/d_2 \\ \gcd(d_1 s_1, d_2 s_2) = 1 \\ \gcd(s_1 s_2, v) = 1}} \mu(d_1 s_1) \mu(d_2 s_2) \\ \sum_{d_3 | v} \mu(d_3) \quad \sum_{\substack{m \leq \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}} \\ \gcd(m, d_1 s_1 d_2 s_2 v) = 1}} \frac{\mu(m)}{d_1 d_2 d_3 m^2} \frac{z}{s_1 d_1 s_2 d_2} \left(1 - \max\left(\frac{1}{2}, \frac{s_1 d_1}{y}, \frac{s_2 d_2}{y}\right)\right)$$

plus

$$(4.14) \quad O^* \left(\sum_{\substack{d_1, d_2 < y \\ \gcd(d_1 d_2, v) = 1}} \sum_{\substack{s_1 < y/d_1 \\ s_2 < y/d_2 \\ \gcd(s_1 s_2, v) = 1}} \sum_{\substack{d_3 | v \\ \gcd(d_3, v) = 1}} \sum_{\substack{m \leq \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}} \\ m \text{ sq-free}}} 1 \right).$$

If we complete the innermost sum in (4.13) by removing the condition $m \leq \sqrt{z/(d_1^2 s_1 d_2^2 s_2)}$, we obtain (reintroducing the variables $r_i = d_i s_i$)

$$(4.15) \quad z \cdot \sum_{\substack{r_1, r_2 < y \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \frac{\mu(r_1) \mu(r_2)}{r_1 r_2} \left(1 - \max\left(\frac{1}{2}, \frac{r_1}{y}, \frac{r_2}{y}\right)\right) \\ \sum_{\substack{d_1 | r_1 \\ d_2 | r_2}} \sum_{d_3 | v} \sum_{\substack{m \\ \gcd(m, r_1 r_2 v) = 1}} \frac{\mu(d_1) \mu(d_2) \mu(m) \mu(d_3)}{d_1 d_2 d_3 m^2}$$

times z . Now (4.15) equals

$$\begin{aligned} & \sum_{\substack{r_1, r_2 < y \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \frac{\mu(r_1)\mu(r_2)z}{r_1 r_2} \left(1 - \max\left(\frac{1}{2}, \frac{r_1}{y}, \frac{r_2}{y}\right)\right) \prod_{p|r_1 r_2 v} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \nmid r_1 r_2 \\ p \nmid v}} \left(1 - \frac{1}{p^2}\right) \\ &= \frac{6z}{\pi^2} \frac{v}{\sigma(v)} \sum_{\substack{r_1, r_2 < y \\ \gcd(r_1, r_2) = 1 \\ \gcd(r_1 r_2, v) = 1}} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} \left(1 - \max\left(\frac{1}{2}, \frac{r_1}{y}, \frac{r_2}{y}\right)\right), \end{aligned}$$

i.e., the main term in (4.10). It remains to estimate the terms used to complete the sum; their total is, by definition, given exactly by (4.13) with the inequality $m \leq \sqrt{z/(d_1^2 s d_2^2 s_2 d_3)}$ changed to $m > \sqrt{z/(d_1^2 s d_2^2 s_2 d_3)}$. This is a total of size at most

$$(4.16) \quad \frac{1}{2} \sum_{\substack{d_1, d_2 < y \\ \gcd(d_1 d_2, v) = 1}} \sum_{\substack{s_1 < y/d_1 \\ s_2 < y/d_2 \\ \gcd(s_1 s_2, v) = 1}} \sum_{d_3 | v} \sum_{\substack{m > \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}} \\ m \text{ sq-free}}} \frac{1}{d_1 d_2 d_3 m^2} \frac{z}{s_1 d_1 s_2 d_2}.$$

Adding this to (4.14), we obtain, as our total error term,

$$(4.17) \quad \sum_{\substack{d_1, d_2 < y \\ \gcd(d_1 d_2, v) = 1}} \sum_{\substack{s_1 < y/d_1 \\ s_2 < y/d_2 \\ \gcd(s_1 s_2, v) = 1}} \sum_{d_3 | v} f\left(\sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}}\right),$$

where

$$f(x) := \sum_{\substack{m \leq x \\ m \text{ sq-free}}} 1 + \frac{1}{2} \sum_{\substack{m > x \\ m \text{ sq-free}}} \frac{x^2}{m^2}.$$

It is easy to see that $f(x)/x$ has a local maximum exactly when x is a square-free (positive) integer. We can hence check that

$$f(x) \leq \frac{1}{2} \left(2 + 2 \left(\frac{\zeta(2)}{\zeta(4)} - 1.25\right)\right) x = 1.26981 \dots x$$

for all $x \geq 0$ by checking all integers smaller than a constant and using $\{m : m \text{ sq-free}\} \subset \{m : 4 \nmid m\}$ and $1.5 \cdot (3/4) < 1.26981$ to bound f from below for x larger than a constant. Therefore, (4.17) is at most

$$\begin{aligned} 1.27 & \sum_{\substack{d_1, d_2 < y \\ \gcd(d_1 d_2, v) = 1}} \sum_{\substack{s_1 < y/d_1 \\ s_2 < y/d_2 \\ \gcd(s_1 s_2, v) = 1}} \sum_{d_3 | v} \sqrt{\frac{z}{d_1^2 s_1 d_2^2 s_2 d_3}} \\ &= 1.27 \sqrt{z} \prod_{p|v} \left(1 + \frac{1}{\sqrt{p}}\right) \cdot \left(\sum_{\substack{d < y \\ \gcd(d, v) = 1}} \sum_{\substack{s < y/d \\ \gcd(s, v) = 1}} \frac{1}{d\sqrt{s}} \right)^2. \end{aligned}$$

We can bound the double sum simply by

$$\sum_{\substack{d < y \\ \gcd(d,v)=1}} \sum_{s < y/d} \frac{1}{\sqrt{sd}} \leq 2 \sum_{d < y} \frac{\sqrt{y/d}}{d} \leq 2\sqrt{y} \cdot \zeta\left(\frac{3}{2}\right) \prod_{p|v} \left(1 - \frac{1}{p^{3/2}}\right).$$

Alternatively, if $v = 2$, we bound

$$\sum_{\substack{s < y/d \\ \gcd(s,v)=1}} \frac{1}{\sqrt{s}} = \sum_{\substack{s < y/d \\ s \text{ odd}}} \frac{1}{\sqrt{s}} \leq 1 + \frac{1}{2} \int_1^{y/d} \frac{1}{\sqrt{s}} ds = \sqrt{y/d}$$

and thus

$$\sum_{\substack{d < y \\ \gcd(d,v)=1}} \sum_{\substack{s < y/d \\ \gcd(s,v)=1}} \frac{1}{\sqrt{sd}} \leq \sum_{\substack{d < y \\ \gcd(d,2)=1}} \frac{\sqrt{y/d}}{d} \leq \sqrt{y} \left(1 - \frac{1}{2^{3/2}}\right) \zeta\left(\frac{3}{2}\right).$$

□

Applying Lemma 4.1 with $y = S/s$ and $z = x/Ws$, where $S = x/WU$, we obtain that (4.8) equals

$$(4.18) \quad \frac{6x}{\pi^2 W} \frac{v}{\sigma(v)} \sum_{\substack{s < S \\ \gcd(s,v)=1}} \frac{1}{s} \sum_{\substack{r_1 < S/s \\ \gcd(r_1,r_2)=1 \\ \gcd(r_1 r_2, v)=1}} \sum_{r_2 < S/s} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} \left(1 - \max\left(\frac{1}{2}, \frac{r_1}{S/s}, \frac{r_2}{S/s}\right)\right) \\ + O^* \left(5.04 \zeta\left(\frac{3}{2}\right)^3 S \sqrt{\frac{x}{W}} \prod_{p|v} \left(1 + \frac{1}{\sqrt{p}}\right) \left(1 - \frac{1}{p^{3/2}}\right)^3 \right),$$

with 5.04 replaced by 1.27 if $v = 2$. The main term in (4.18) can be written as

$$(4.19) \quad \frac{6x}{\pi^2 W} \frac{v}{\sigma(v)} \sum_{\substack{s \leq S \\ \gcd(s,v)=1}} \frac{1}{s} \int_{1/2}^1 \sum_{\substack{r_1 \leq \frac{uS}{s} \\ \gcd(r_1,r_2)=1 \\ \gcd(r_1 r_2, v)=1}} \sum_{r_2 \leq \frac{uS}{s}} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} du.$$

From now on, we will focus on the cases $v = 1$ and $v = 2$ for simplicity. (Higher values of v do not seem to be really profitable in the last analysis.)

4.1.2. Explicit bounds for a sum with μ . We must estimate the expression within parentheses in (4.19). It is not too hard to show that it tends to 0; the first part of the proof of Lemma 4.2 will reduce this to the fact that $\sum_n \mu(n)/n = 0$. Obtaining good bounds is a more delicate matter. For our purposes, we will need the expression to converge to 0 at least as fast as $1/(\log)^2$, with a good constant in front. For this task, the bound (2.8) on $\sum_{n \leq x} \mu(n)/n$ is enough.

Lemma 4.2. *Let*

$$g_v(x) := \sum_{\substack{r_1 \leq x \\ \gcd(r_1,r_2)=1 \\ \gcd(r_1 r_2, v)=1}} \sum_{r_2 \leq x} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)},$$

where $v = 1$ or $v = 2$. Then

$$|g_1(x)| \leq \begin{cases} 1/x & \text{if } 33 \leq x \leq 10^6, \\ \frac{1}{x}(111.536 + 55.768 \log x) & \text{if } 10^6 \leq x < 10^{10}, \\ \frac{0.0044325}{(\log x)^2} + \frac{0.1079}{\sqrt{x}} & \text{if } x \geq 10^{10}, \end{cases}$$

$$|g_2(x)| \leq \begin{cases} 2.1/x & \text{if } 33 \leq x \leq 10^6, \\ \frac{1}{x}(1634.34 + 817.168 \log x) & \text{if } 10^6 \leq x < 10^{10}, \\ \frac{0.038128}{(\log x)^2} + \frac{0.2046}{\sqrt{x}} & \text{if } x \geq 10^{10}. \end{cases}$$

The proof involves what may be called a version of Rankin's trick, using Dirichlet series and the behavior of $\zeta(s)$ near $s = 1$. The statements for $x \leq 10^6$ are proven by direct computation.⁴

Proof. Clearly

$$\begin{aligned} g(x) &= \sum_{\substack{r_1 \leq x \\ \gcd(r_1, r_2, v)=1}} \sum_{\substack{r_2 \leq x \\ \gcd(r_1, r_2, v)=1}} \left(\sum_{d | \gcd(r_1, r_2)} \mu(d) \right) \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} \\ &= \sum_{\substack{d \leq x \\ \gcd(d, v)=1}} \mu(d) \sum_{\substack{r_1 \leq x \\ \gcd(r_1, r_2, v)=1}} \sum_{\substack{r_2 \leq x \\ \gcd(r_1, r_2, v)=1}} \frac{\mu(r_1)\mu(r_2)}{\sigma(r_1)\sigma(r_2)} \\ (4.20) \quad &= \sum_{\substack{d \leq x \\ \gcd(d, v)=1}} \frac{\mu(d)}{(\sigma(d))^2} \sum_{\substack{u_1 \leq x/d \\ \gcd(u_1, dv)=1}} \sum_{\substack{u_2 \leq x/d \\ \gcd(u_2, dv)=1}} \frac{\mu(u_1)\mu(u_2)}{\sigma(u_1)\sigma(u_2)} \\ &= \sum_{\substack{d \leq x \\ \gcd(d, v)=1}} \frac{\mu(d)}{(\sigma(d))^2} \left(\sum_{\substack{r \leq x/d \\ \gcd(r, dv)=1}} \frac{\mu(r)}{\sigma(r)} \right)^2. \end{aligned}$$

Moreover,

$$\begin{aligned} \sum_{\substack{r \leq x/d \\ \gcd(r, dv)=1}} \frac{\mu(r)}{\sigma(r)} &= \sum_{\substack{r \leq x/d \\ \gcd(r, dv)=1}} \frac{\mu(r)}{r} \sum_{d' | r} \prod_{p | d'} \left(\frac{p}{p+1} - 1 \right) \\ &= \sum_{\substack{d' \leq x/d \\ \mu(d')^2=1 \\ \gcd(d', dv)=1}} \left(\prod_{p | d'} \frac{-1}{p+1} \right) \sum_{\substack{r \leq x/d \\ \gcd(r, dv)=1 \\ d' | r}} \frac{\mu(r)}{r} \\ &= \sum_{\substack{d' \leq x/d \\ \mu(d')^2=1 \\ \gcd(d', dv)=1}} \frac{1}{d' \sigma(d')} \sum_{\substack{r \leq x/dd' \\ \gcd(r, dd'v)=1}} \frac{\mu(r)}{r} \end{aligned}$$

⁴Using D. Platt's implementation [Pla11] of double-precision interval arithmetic. (In fact, one gets $2.0895071/x$ instead of $2.1/x$.)

and

$$\sum_{\substack{r \leq x/dd' \\ \gcd(r, dd'v)=1}} \frac{\mu(r)}{r} = \sum_{\substack{d'' \leq x/dd' \\ d'' | (dd'v)^\infty}} \frac{1}{d''} \sum_{r \leq x/dd'd''} \frac{\mu(r)}{r}.$$

Hence
(4.21)

$$|g(x)| \leq \sum_{\substack{d \leq x \\ \gcd(d, v)=1}} \frac{(\mu(d))^2}{(\sigma(d))^2} \left(\sum_{\substack{d' \leq x/d \\ \mu(d')^2=1 \\ \gcd(d', dv)=1}} \frac{1}{d' \sigma(d')} \sum_{\substack{d'' \leq x/dd' \\ d'' | (dd'v)^\infty}} \frac{1}{d''} f(x/dd'd'') \right)^2,$$

where $f(t) = \left| \sum_{r \leq t} \mu(r)/r \right|$.

We intend to bound the function $f(t)$ by a linear combination of terms of the form $t^{-\delta}$, $\delta \in [0, 1/2)$. Thus it makes sense now to estimate $F_v(s_1, s_2, x)$, defined to be the quantity

$$\sum_{\gcd(d, v)=1} \frac{(\mu(d))^2}{(\sigma(d))^2} \left(\sum_{\substack{d'_1 \\ \gcd(d'_1, dv)=1}} \frac{\mu(d'_1)^2}{d'_1 \sigma(d'_1)} \sum_{\substack{d''_1 | (dd'_1 v)^\infty}} \frac{1}{d''_1} \cdot (dd'_1 d''_1)^{1-s_1} \right) \left(\sum_{\substack{d'_2 \\ \gcd(d'_2, dv)=1}} \frac{\mu(d'_2)^2}{d'_2 \sigma(d'_2)} \sum_{\substack{d''_2 | (dd'_2 v)^\infty}} \frac{1}{d''_2} \cdot (dd'_2 d''_2)^{1-s_2} \right).$$

for $s_1, s_2 \in [1/2, 1]$. This is equal to

$$\sum_{\gcd(d, v)=1} \frac{\mu(d)^2}{d^{s_1+s_2}} \prod_{p|d} \frac{1}{(1+p^{-1})^2 (1-p^{-s_1}) \prod_{p|v} \frac{1}{(1-p^{-s_1})(1-p^{-s_2})} (1-p^{-s_2})} \cdot \left(\sum_{\substack{d' \\ \gcd(d', dv)=1}} \frac{\mu(d')^2}{(d')^{s_1+1}} \prod_{p'|d'} \frac{1}{(1+p'^{-1})(1-p'^{-s_1})} \right) \cdot \left(\sum_{\substack{d' \\ \gcd(d', dv)=1}} \frac{\mu(d')^2}{(d')^{s_2+1}} \prod_{p'|d'} \frac{1}{(1+p'^{-1})(1-p'^{-s_2})} \right),$$

which in turn can easily be seen to equal

$$(4.22) \quad \prod_{p|v} \left(1 + \frac{p^{-s_1} p^{-s_2}}{(1-p^{-s_1} + p^{-1})(1-p^{-s_2} + p^{-1})} \right) \prod_{p|v} \frac{1}{(1-p^{-s_1})(1-p^{-s_2})} \cdot \prod_{p|v} \left(1 + \frac{p^{-1} p^{-s_1}}{(1+p^{-1})(1-p^{-s_1})} \right) \cdot \prod_{p|v} \left(1 + \frac{p^{-1} p^{-s_2}}{(1+p^{-1})(1-p^{-s_2})} \right)$$

Now, for any $0 < x \leq y \leq x^{1/2} < 1$,

$$(1+x-y)(1-xy)(1-xy^2) - (1+x)(1-y)(1-x^3) = (x-y)(y^2-x)(xy-x-1)x \leq 0,$$

and so

$$(4.23) \quad 1 + \frac{xy}{(1+x)(1-y)} = \frac{(1+x-y)(1-xy)(1-xy^2)}{(1+x)(1-y)(1-xy)(1-xy^2)} \leq \frac{(1-x^3)}{(1-xy)(1-xy^2)}.$$

For any $x \leq y_1, y_2 < 1$ with $y_1^2 \leq x, y_2^2 \leq x$,

$$(4.24) \quad 1 + \frac{y_1 y_2}{(1-y_1+x)(1-y_2+x)} \leq \frac{(1-x^3)^2(1-x^4)}{(1-y_1 y_2)(1-y_1 y_2^2)(1-y_1^2 y_2)}.$$

This can be checked as follows: multiplying by the denominators and changing variables to $x, s = y_1 + y_2$ and $r = y_1 y_2$, we obtain an inequality where the left side, quadratic on s with positive leading coefficient, must be less than or equal to the right side, which is linear on s . The left side minus the right side can be maximal for given x, r only when s is maximal or minimal. This happens when $y_1 = y_2$ or when either $y_i = \sqrt{x}$ or $y_i = x$ for at least one of $i = 1, 2$. In each of these cases, we have reduced (4.24) to an inequality in two variables that can be proven automatically⁵ by a quantifier-elimination program; the author has used QEPCAD [HB11] to do this.

Hence $F_v(s_1, s_2, x)$ is at most

$$(4.25) \quad \prod_{p|v} \frac{(1-p^{-3})^2(1-p^{-4})}{(1-p^{-s_1-s_2})(1-p^{-2s_1-s_2})(1-p^{-s_1-2s_2})} \cdot \prod_{p|v} \frac{1}{(1-p^{-s_1})(1-p^{-s_2})} \\ \cdot \prod_{p|v} \frac{1-p^{-3}}{(1+p^{-s_1-1})(1+p^{-2s_1-1})} \prod_{p|v} \frac{1-p^{-3}}{(1+p^{-s_2-1})(1+p^{-2s_2-1})} \\ = C_{v,s_1,s_2} \cdot \frac{\zeta(s_1+1)\zeta(s_2+1)\zeta(2s_1+1)\zeta(2s_2+1)}{\zeta(3)^4\zeta(4)(\zeta(s_1+s_2)\zeta(2s_1+s_2)\zeta(s_1+2s_2))^{-1}},$$

where

$$C_{v,s_1,s_2} = \begin{cases} 1 & \text{if } v = 1, \\ \frac{(1-2^{-s_1-2s_2})(1+2^{-s_1-1})(1+2^{-2s_1-1})(1+2^{-s_2-1})(1+2^{-2s_2-1})}{(1-2^{-s_1+s_2})^{-1}(1-2^{-2s_1-s_2})^{-1}(1-2^{-s_1})(1-2^{-s_2})(1-2^{-3})^4(1-2^{-4})} & \text{if } v = 2. \end{cases}$$

For $1 \leq t \leq x$, (2.8) and (2.11) imply

$$(4.26) \quad f(t) \leq \begin{cases} \sqrt{\frac{2}{t}} & \text{if } x \leq 10^{10} \\ \sqrt{\frac{2}{t}} + \frac{0.03}{\log x} \left(\frac{x}{t}\right)^{\frac{\log \log 10^{10}}{\log x - \log 10^{10}}} & \text{if } x > 10^{10}, \end{cases}$$

where we are using the fact that $\log x$ is convex-down. Note that, again by convexity,

$$\frac{\log \log x - \log \log 10^{10}}{\log x - \log 10^{10}} < (\log t)'|_{t=\log 10^{10}} = \frac{1}{\log 10^{10}} = 0.0434294 \dots$$

Obviously, $\sqrt{2/t}$ in (4.26) can be replaced by $(2/t)^{1/2-\epsilon}$ for any $\epsilon \geq 0$.

⁵In practice, the case $y_i = \sqrt{x}$ leads to a polynomial of high degree, and quantifier elimination increases sharply in complexity as the degree increases; a stronger inequality of lower degree (with $(1-3x^3)$ instead of $(1-x^3)^2(1-x^4)$) was given to QEPCAD to prove in this case.

By (4.21) and (4.26),

$$|g_v(x)| \leq \left(\frac{2}{x}\right)^{1-2\epsilon} F_v(1/2 + \epsilon, 1/2 + \epsilon, x)$$

for $x \leq 10^{10}$. We set $\epsilon = 1/\log x$ and obtain from (4.25) that

$$(4.27) \quad \begin{aligned} F_v(1/2 + \epsilon, 1/2 + \epsilon, x) &\leq C_{v, \frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon} \frac{\zeta(1+2\epsilon)\zeta(3/2)^4\zeta(2)^2}{\zeta(3)^4\zeta(4)} \\ &\leq 55.768 \cdot C_{v, \frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon} \cdot \left(1 + \frac{\log x}{2}\right), \end{aligned}$$

where we use the easy bound $\zeta(s) < 1 + 1/(s-1)$ obtained by

$$\sum n^s < 1 + \int_1^\infty t^s dt.$$

(For sharper bounds, see [BR02].) Now

$$C_{2, \frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon} \leq \frac{(1 - 2^{-3/2-\epsilon})^2(1 + 2^{-3/2})^2(1 + 2^{-2})^2(1 - 2^{-1-2\epsilon})}{(1 - 2^{-1/2})^2(1 - 2^{-3})^4(1 - 2^{-4})} \leq 14.652983,$$

whereas $C_{1, \frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon} = 1$. (We are assuming $x \geq 10^6$, and so $\epsilon \leq 1/(\log 10^6)$.) Hence

$$|g_v(x)| \leq \begin{cases} \frac{1}{x}(111.536 + 55.768 \log x) & \text{if } v = 1, \\ \frac{1}{x}(1634.34 + 817.168 \log x) & \text{if } v = 2. \end{cases}$$

for $10^6 \leq x < 10^{10}$.

For general x , we must use the second bound in (4.26). Define $c = 1/(\log 10^{10})$. We see that, if $x > 10^{10}$,

$$\begin{aligned} |g_v(x)| &\leq \frac{0.03^2}{(\log x)^2} F_1(1-c, 1-c) \cdot C_{v, 1-c, 1-c} \\ &\quad + 2 \cdot \frac{\sqrt{2} \cdot 0.03}{\sqrt{x} \log x} F(1-c, 1/2) \cdot C_{v, 1-c, 1/2} \\ &\quad + \frac{1}{x}(111.536 + 55.768 \log x) \cdot C_{v, \frac{1}{2} + \epsilon, \frac{1}{2} + \epsilon}. \end{aligned}$$

For $v = 1$, this gives

$$\begin{aligned} |g_1(x)| &\leq \frac{0.0044325}{(\log x)^2} + \frac{2.1626}{\sqrt{x} \log x} + \frac{1}{x}(111.536 + 55.768 \log x) \\ &\leq \frac{0.0044325}{(\log x)^2} + \frac{0.1079}{\sqrt{x}}; \end{aligned}$$

for $v = 2$, we obtain

$$\begin{aligned} |g_2(x)| &\leq \frac{0.038128}{(\log x)^2} + \frac{25.607}{\sqrt{x} \log x} + \frac{1}{x}(1634.34 + 817.168 \log x) \\ &\leq \frac{0.038128}{(\log x)^2} + \frac{0.2046}{\sqrt{x}}. \end{aligned}$$

□

4.1.3. *Estimating the triple sum.* We will now be able to bound the triple sum in (4.19), viz.,

$$(4.28) \quad \sum_{\substack{s \leq S \\ \gcd(s,v)=1}} \frac{1}{s} \int_{1/2}^1 g_v(uS/s) du,$$

where g_v is as in Lemma 4.2.

As we will soon see, Lemma 4.2 that (4.28) is bounded by a constant (essentially because the integral $\int_0^{1/2} 1/t(\log t)^2$ converges). We must give as good a constant as we can, since it will affect the largest term in the final result.

Clearly $g_v(R) = g_v(\lfloor R \rfloor)$. The contribution of each $g_v(m)$, $1 \leq m \leq S$, to (4.28) is exactly $g_v(m)$ times

$$(4.29) \quad \begin{aligned} & \sum_{\substack{\frac{S}{m+1} < s \leq \frac{S}{m} \\ \gcd(s,v)=1}} \frac{1}{s} \int_{ms/S}^1 du + \sum_{\substack{\frac{S}{2m} < s \leq \frac{S}{m+1} \\ \gcd(s,v)=1}} \frac{1}{s} \int_{ms/S}^{(m+1)s/S} du + \sum_{\substack{\frac{S}{2(m+1)} < s \leq \frac{S}{2m} \\ \gcd(s,v)=1}} \frac{1}{s} \int_{1/2}^{(m+1)s/S} du \\ &= \sum_{\substack{\frac{S}{m+1} < s \leq \frac{S}{m} \\ \gcd(s,v)=1}} \left(\frac{1}{s} - \frac{m}{S} \right) + \sum_{\substack{\frac{S}{2m} < s \leq \frac{S}{m+1} \\ \gcd(s,v)=1}} \frac{1}{S} + \sum_{\substack{\frac{S}{2(m+1)} < s \leq \frac{S}{2m} \\ \gcd(s,v)=1}} \left(\frac{m+1}{S} - \frac{1}{2s} \right). \end{aligned}$$

Write $f(t) = 1/S$ for $S/2m < t \leq S/(m+1)$, $f(t) = 0$ for $t > S/m$ or $t < S/2(m+1)$, $f(t) = 1/t - m/S$ for $S/(m+1) < t \leq S/m$ and $f(t) = (m+1)/S - 1/2t$ for $S/2(m+1) < t \leq S/2m$; then (4.29) equals $\sum_{n: \gcd(n,v)=1} f(n)$. By Euler-Maclaurin (second order),

$$(4.30) \quad \begin{aligned} \sum_n f(n) &= \int_{-\infty}^{\infty} f(x) - \frac{1}{2} B_2(\{x\}) f''(x) dx = \int_{-\infty}^{\infty} f(x) + O^* \left(\frac{1}{12} |f''(x)| \right) dx \\ &= \int_{-\infty}^{\infty} f(x) dx + \frac{1}{6} \cdot O^* \left(\left| f' \left(\frac{3}{2m} \right) \right| + \left| f' \left(\frac{s}{m+1} \right) \right| \right) \\ &= \frac{1}{2} \log \left(1 + \frac{1}{m} \right) + \frac{1}{6} \cdot O^* \left(\left(\frac{2m}{s} \right)^2 + \left(\frac{m+1}{s} \right)^2 \right). \end{aligned}$$

Similarly,

$$\begin{aligned} \sum_{n \text{ odd}} f(n) &= \int_{-\infty}^{\infty} f(2x+1) - \frac{1}{2} B_2(\{x\}) \frac{d^2 f(2x+1)}{dx^2} dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx - 2 \int_{-\infty}^{\infty} \frac{1}{2} B_2 \left(\left\{ \frac{x-1}{2} \right\} \right) f''(x) dx \\ &= \frac{1}{2} \int_{-\infty}^{\infty} f(x) dx + \frac{1}{6} \int_{-\infty}^{\infty} O^* (|f''(x)|) dx \\ &= \frac{1}{4} \log \left(1 + \frac{1}{m} \right) + \frac{1}{3} \cdot O^* \left(\left(\frac{2m}{s} \right)^2 + \left(\frac{m+1}{s} \right)^2 \right). \end{aligned}$$

We use these expressions for $m \leq C_0$, where $C_0 \geq 33$ is a constant to be computed later; they will give us the main term. For $m > C_0$, we use the bounds on $|g(m)|$ that Lemma 4.2 gives us.

(Starting now and for the rest of the paper, we will focus on the cases $v = 1$, $v = 2$ when giving explicit computational estimates. All of our procedures would allow higher values of v as well, but, as will become clear much later, the gains from higher values of v are offset by losses and complications elsewhere.)

Let us estimate (4.28). Let

$$c_{v,0} = \begin{cases} 1/6 & \text{if } v = 1, \\ 1/3 & \text{if } v = 2, \end{cases} \quad c_{v,1} = \begin{cases} 1 & \text{if } v = 1, \\ 2.5 & \text{if } v = 2, \end{cases}$$

$$c_{v,2} = \begin{cases} 55.768\dots & \text{if } v = 1, \\ 817.168\dots & \text{if } v = 2, \end{cases} \quad c_{v,3} = \begin{cases} 111.536\dots & \text{if } v = 1, \\ 1634.34\dots & \text{if } v = 2, \end{cases}$$

$$c_{v,4} = \begin{cases} 0.0044325\dots & \text{if } v = 1, \\ 0.038128\dots & \text{if } v = 2, \end{cases} \quad c_{v,5} = \begin{cases} 0.1079\dots & \text{if } v = 1, \\ 0.2046\dots & \text{if } v = 2. \end{cases}$$

Then (4.28) equals

$$\begin{aligned} & \sum_{m \leq C_0} g_v(m) \cdot \left(\frac{\phi(v)}{2v} \log \left(1 + \frac{1}{m} \right) + O^* \left(c_{v,0} \frac{5m^2 + 2m + 1}{S^2} \right) \right) \\ & + \sum_{S/10^6 \leq s < S/C_0} \frac{1}{s} \int_{1/2}^1 O^* \left(\frac{c_{v,1}}{uS/s} \right) du \\ & + \sum_{S/10^{10} \leq s < S/10^6} \frac{1}{s} \int_{1/2}^1 O^* \left(\frac{c_{v,2} \log(uS/s) + c_{v,3}}{uS/s} \right) du \\ & + \sum_{s < S/10^{10}} \frac{1}{s} \int_{1/2}^1 O^* \left(\frac{c_{v,4}}{(\log uS/s)^2} + \frac{c_{v,5}}{\sqrt{uS/s}} \right) du, \end{aligned}$$

which is

$$\begin{aligned} & \sum_{m \leq C_0} g_v(m) \cdot \frac{\phi(v)}{2v} \log \left(1 + \frac{1}{m} \right) + \sum_{m \leq C_0} |g(m)| \cdot O^* \left(c_{v,0} \frac{5m^2 + 2m + 1}{S^2} \right) \\ & + O^* \left(c_{v,1} \frac{\log 2}{C_0} + \frac{\log 2}{10^6} (c_{v,3} + c_{v,2}(1 + \log 10^6)) + \frac{2 - \sqrt{2}}{10^{10/2}} c_{v,5} \right) \\ & + O^* \left(\sum_{s < S/10^{10}} \frac{c_{v,4}/2}{s(\log S/2s)^2} \right) \end{aligned}$$

for $S \geq (C_0 + 1)$. Note that $\sum_{s < S/10^{10}} \frac{1}{s(\log S/2s)^2} = \int_0^{2/10^{10}} \frac{1}{t(\log t)^2} dt$.

Now

$$\frac{c_{v,4}}{2} \int_0^{2/10^{10}} \frac{1}{t(\log t)^2} dt = \frac{c_{v,4}/2}{\log(10^{10}/2)} = \begin{cases} 0.00009923\dots & \text{if } v = 1 \\ 0.000853636\dots & \text{if } v = 2. \end{cases}$$

and

$$\frac{\log 2}{10^6} (c_{v,3} + c_{v,2}(1 + \log 10^6)) + \frac{2 - \sqrt{2}}{10^5} c_{v,5} = \begin{cases} 0.0006506\dots & \text{if } v = 1 \\ 0.009525\dots & \text{if } v = 2. \end{cases}$$

For $C_0 = 10000$,

$$\frac{\phi(v)}{v} \frac{1}{2} \sum_{m \leq C_0} g_v(m) \cdot \log \left(1 + \frac{1}{m} \right) = \begin{cases} 0.362482\dots & \text{if } v = 1, \\ 0.360576\dots & \text{if } v = 2, \end{cases}$$

$$c_{v,0} \sum_{m \leq C_0} |g_v(m)|(5m^2 + 2m + 1) \leq \begin{cases} 6204066.5\dots & \text{if } v = 1, \\ 15911340.1\dots & \text{if } v = 2, \end{cases}$$

and

$$c_{v,1} \cdot (\log 2)/C_0 = \begin{cases} 0.00006931\dots & \text{if } v = 1, \\ 0.00017328\dots & \text{if } v = 2. \end{cases}$$

Thus, for $S \geq 100000$,

$$(4.31) \quad \sum_{\substack{s \leq S \\ \gcd(s,v)=1}} \frac{1}{s} \int_{1/2}^1 g_v(uS/s) du \leq \begin{cases} 0.36393 & \text{if } v = 1, \\ 0.37273 & \text{if } v = 2. \end{cases}$$

For $S < 100000$, we proceed as above, but using the exact expression (4.29) instead of (4.30). Note (4.29) is of the form $f_{s,m,1}(S) + f_{s,m,2}(S)/S$, where both $f_{s,m,1}(S)$ and $f_{s,m,2}(S)$ depend only on $\lfloor S \rfloor$ (and on s and m). Summing over $m \leq S$, we obtain a bound of the form

$$\sum_{\substack{s \leq S \\ \gcd(s,v)=1}} \frac{1}{s} \int_{1/2}^1 g_v(uS/s) du \leq G_v(S)$$

with

$$G_v(S) = K_{v,1}(|S|) + K_{v,2}(|S|)/S,$$

where $K_{v,1}(n)$ and $K_{v,2}(n)$ can be computed explicitly for each integer n . (For example, $G_v(S) = 1 - 1/S$ for $1 \leq S < 2$ and $G_v(S) = 0$ for $S < 1$.)

It is easy to check numerically that this implies that (4.31) holds not just for $S \geq 100000$ but also for $40 \leq S < 100000$ (if $v = 1$) or $16 \leq S < 100000$ (if $v = 2$). Using the fact that $G_v(S)$ is non-negative, we can compare $\int_1^T G_v(S) dS/S$ with $\log(T + 1/N)$ for each $T \in [2, 40] \cap \frac{1}{N}\mathbb{Z}$ (N a large integer) to show, again numerically, that

$$(4.32) \quad \int_1^T G_v(S) \frac{dS}{S} \leq \begin{cases} 0.3698 \log T & \text{if } v = 1, \\ 0.37273 \log T & \text{if } v = 2. \end{cases}$$

(We use $N = 100000$ for $v = 1$; already $N = 10$ gives us the answer above for $v = 2$. Indeed, computations suggest the better bound 0.358 instead of 0.37273; we are committed to using 0.37273 because of (4.31).)

Multiplying by $6v/\pi^2\sigma(v)$, we conclude that

$$(4.33) \quad S_1(U, W) = \frac{x}{W} \cdot H_1 \left(\frac{x}{WU} \right) + O^* \left(5.08\zeta(3/2)^3 \frac{x^{3/2}}{W^{3/2}U} \right)$$

if $v = 1$,

$$(4.34) \quad S_1(U, W) = \frac{x}{W} \cdot H_2 \left(\frac{x}{WU} \right) + O^* \left(1.27\zeta(3/2)^3 \frac{x^{3/2}}{W^{3/2}U} \right)$$

if $v = 2$, where

$$(4.35) \quad H_1(S) = \begin{cases} \frac{6}{\pi^2} G_1(S) & \text{if } 1 \leq S < 40, \\ 0.22125 & \text{if } S \geq 40, \end{cases} \quad H_2(s) = \begin{cases} \frac{4}{\pi^2} G_2(S) & \text{if } 1 \leq S < 16, \\ 0.15107 & \text{if } S \geq 16. \end{cases}$$

Hence (by (4.32))

$$(4.36) \quad \int_1^T H_v(S) \frac{dS}{S} \leq \begin{cases} 0.22482 \log T & \text{if } v = 1, \\ 0.15107 \log T & \text{if } v = 2; \end{cases}$$

moreover, $H_1(S) \leq 3/\pi^2$, $H_2(S) \leq 2/\pi^2$ for all S .

* * *

Note. There is another way to obtain cancellation on μ , applicable when $(x/W) > Uq$ (as is unfortunately never the case in our main application). For this alternative to be taken, one must either apply Cauchy-Schwarz on n rather than m (resulting in exponential sums over m) or lump together all m near each other and in the same congruence class modulo q before applying Cauchy-Schwarz on m (one can indeed do this if δ is small). We could then write

$$\sum_{\substack{m \sim W \\ m \equiv r \pmod q}} \sum_{\substack{d|m \\ d > U}} \mu(d) = - \sum_{\substack{m \sim W \\ m \equiv r \pmod q}} \sum_{\substack{d|m \\ d \leq U}} \mu(d) = - \sum_{d \leq U} \mu(d) (W/qd + O(1))$$

and obtain cancellation on d . If $Uq \geq (x/W)$, however, the error term dominates.

4.2. The sum S_2 : the large sieve, primes and tails. We must now bound

$$(4.37) \quad S_2(U', W', W) = \sum_{\substack{U' < m \leq \frac{x}{W'} \\ \gcd(m, v) = 1}} \left| \sum_{W' < p \leq W} (\log p) e(\alpha m p) \right|^2.$$

for $U' = \max(U, x/2W)$, $W' = \max(V, W/2)$. (The condition $\gcd(p, v) = 1$ will be fulfilled automatically by the assumption $V > v$.)

From a modern perspective, this is clearly a case for a large sieve. It is also clear that we ought to try to apply a large sieve for sequences of prime support. What is subtler here is how to do things well for very large q (i.e., x/q small). This is in some sense a dual problem to that of q small, but it poses additional complications; for example, it is not obvious how to take advantage of prime support for very large q .

As in type I, we avoid this entire issue by forbidding q large and then taking advantage of the error term δ/x in the approximation $\alpha = \frac{a}{q} + \frac{\delta}{x}$. This is one of the main innovations here. Note this alternative method will allow us to take advantage of prime support.

A key situation to study is that of frequencies α_i clustering around given rationals a/q while nevertheless keeping at a certain small distance from each other.

Lemma 4.3. *Let $q \geq 1$. Let $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}/\mathbb{Z}$ be of the form $\alpha_i = a_i/q + v_i$, $0 \leq a_i < q$, where the elements $v_i \in \mathbb{R}$ all lie in an interval of length $v > 0$, and where $a_i = a_j$ implies $|v_i - v_j| > \nu > 0$. Assume $\nu + v \leq 1/q$. Then, for any*

$W, W' \geq 1, W' \geq W/2,$

$$(4.38) \quad \sum_{i=1}^k \left| \sum_{W' < p \leq W} (\log p) e(\alpha_i p) \right|^2 \leq \min \left(1, \frac{2q}{\phi(q)} \frac{1}{\log((q(\nu+v))^{-1})} \right) \cdot (W - W' + \nu^{-1}) \sum_{W' < p \leq W} (\log p)^2.$$

Proof. For any distinct i, j , the angles α_i, α_j are separated by at least ν (if $a_i = a_j$) or at least $1/q - |v_i - v_j| \geq 1/q - v \geq \nu$ (if $a_i \neq a_j$). Hence we can apply the large sieve (in the optimal $N + \delta^{-1} - 1$ form due to Selberg [Sel91] and Montgomery-Vaughan [MV74]) and obtain the bound in (4.38) with 1 instead of $\min(1, \dots)$ immediately.

We can also apply Montgomery's inequality ([Mon68], [Hux72]; see the expositions in [Mon71, pp. 27–29] and [IK04, §7.4]). This gives us that the left side of (4.38) is at most

$$(4.39) \quad \left(\sum_{\substack{r \leq R \\ \gcd(r,q)=1}} \frac{(\mu(r))^2}{\phi(r)} \right)^{-1} \sum_{\substack{r \leq R \\ \gcd(r,q)=1}} \sum_{\substack{a' \bmod r \\ \gcd(a',r)=1}} \sum_{i=1}^k \left| \sum_{W' < p \leq W} (\log p) e((\alpha_i + a'/r)p) \right|^2$$

If we add all possible fractions of the form $a'/r, r \leq R, \gcd(r, q) = 1$, to the fractions a_i/q , we obtain fractions that are separated by at least $1/qR^2$. If $\nu + v \geq 1/qR^2$, then the resulting angles $\alpha_i + a'/r$ are still separated by at least ν . Thus we can apply the large sieve to (4.39); setting $R = 1/\sqrt{(\nu+v)q}$, we see that we gain a factor of

$$(4.40) \quad \sum_{\substack{r \leq R \\ \gcd(r,q)=1}} \frac{(\mu(r))^2}{\phi(r)} \geq \frac{\phi(q)}{q} \sum_{r \leq R} \frac{(\mu(r))^2}{\phi(r)} \geq \frac{\phi(q)}{q} \sum_{d \leq R} \frac{1}{d} \geq \frac{\phi(q)}{2q} \log((q(\nu+v))^{-1}),$$

since $\sum_{d \leq R} 1/d \geq \log(R)$ for all $R \geq 1$ (integer or not). \square

Let us first give a bound on sums of the type of $S_2(U, V, W)$ using prime support but not the error terms (or Lemma 4.3).

Lemma 4.4. *Let $W \geq 1, W' \geq W/2$. Let $\alpha = a/q + O^*(1/qQ), q \leq Q$. Then*

$$(4.41) \quad \sum_{A_0 < m \leq A_1} \left| \sum_{W' < p \leq W} (\log p) e(\alpha m p) \right|^2 \leq \left[\frac{A_1 - A_0}{\min(q, \lceil Q/2 \rceil)} \right] \cdot (W - W' + 2q) \sum_{W' < p \leq W} (\log p)^2.$$

If $q < W/2$ and $Q \geq 3.5W$, the following bound also holds:

$$(4.42) \quad \sum_{A_0 < m \leq A_1} \left| \sum_{W' < p \leq W} (\log p) e(\alpha m p) \right|^2 \leq \left[\frac{A_1 - A_0}{q} \right] \cdot \frac{q}{\phi(q)} \frac{W}{\log(W/2q)} \cdot \sum_{W' < p \leq W} (\log p)^2.$$

If $A_1 - A_0 \leq \varrho q$ and $q \leq \rho Q$, $\varrho, \rho \in [0, 1]$, the following bound also holds:

$$(4.43) \quad \sum_{A_0 < m \leq A_1} \left| \sum_{W' < p \leq W} (\log p) e(\alpha m p) \right|^2 \leq (W - W' + q/(1 - \varrho\rho)) \sum_{W' < p \leq W} (\log p)^2.$$

The inequality (4.42) can be stronger than (4.42) only when $q < W/7.2638\dots$ (if q is odd) or $q < W/92.514\dots$ (if q is even).

Proof. Let $k = \min(q, \lceil Q/2 \rceil) \geq \lceil q/2 \rceil$. We split $(A_0, A_1]$ into $\lceil (A_1 - A_0)/k \rceil$ blocks of at most k consecutive integers $m_0 + 1, m_0 + 2, \dots$. For m, m' in such a block, αm and $\alpha m'$ are separated by a distance of at least

$$|\{(a/q)(m - m')\}| - O^*(k/qQ) = 1/q - O^*(1/2q) \geq 1/2q.$$

By the large sieve

$$(4.44) \quad \sum_{a=1}^q \left| \sum_{W' < p \leq W} (\log p) e(\alpha(m_0 + a)p) \right|^2 \leq ((W - W') + 2q) \sum_{W' < p \leq W} (\log p)^2.$$

We obtain (4.41) by summing over all $\lceil (A_1 - A_0)/k \rceil$ blocks.

If $A_1 - A_0 \leq |\varrho q|$ and $q \leq \rho Q$, $\varrho, \rho \in [0, 1]$, we obtain (4.43) simply by applying the large sieve without splitting the interval $A_0 < m \leq A_1$.

Let us now prove (4.42). We will use Montgomery's inequality, followed by Montgomery and Vaughan's large sieve with weights. An angle $a/q + a'_1/r_1$ is separated from other angles $a'/q + a'_2/r_2$ ($r_1, r_2 \leq R$, $\gcd(a_i, r_i) = 1$) by at least $1/qr_1R$, rather than just $1/qR^2$. We will choose R so that $qR^2 < Q$; this implies $1/Q < 1/qR^2 \leq 1/qr_1R$.

By Montgomery's inequality [IK04, Lemma 7.15], applied (for each $1 \leq a \leq q$) to $S(\alpha) = \sum_n a_n e(\alpha n)$ with $a_n = \log(n) e(\alpha(m_0 + a)n)$ if n is prime and $a_n = 0$ otherwise,

$$(4.45) \quad \frac{1}{\phi(r)} \left| \sum_{W' < p \leq W} (\log p) e(\alpha(m_0 + a)p) \right|^2 \leq \sum_{\substack{a' \bmod r \\ \gcd(a', r) = 1}} \left| \sum_{W' < p \leq W} (\log p) e\left(\left(\alpha(m_0 + a) + \frac{a'}{r}\right)p\right) \right|^2.$$

for each square-free $r \leq W'$. We multiply both sides of (4.45) by $(W/2 + (3/2)(1/qrR - 1/Q)^{-1})^{-1}$ and sum over all $a = 0, 1, \dots, q-1$ and all square-free $r \leq R$ coprime to q ; we will later make sure that $R \leq W'$. We obtain that

$$(4.46) \quad \sum_{\substack{r \leq R \\ \gcd(r, q) = 1}} \left(\frac{W}{2} + \frac{3}{2} \left(\frac{1}{qrR} - \frac{1}{Q} \right)^{-1} \right)^{-1} \frac{\mu(r)^2}{\phi(r)} \cdot \sum_{a=1}^q \left| \sum_{W' < p \leq W} (\log p) e(\alpha(m_0 + a)p) \right|^2$$

is at most

$$(4.47) \quad \sum_{\substack{r \leq R \\ \gcd(r,q)=1 \\ r \text{ sq-free}}} \left(\frac{W}{2} + \frac{3}{2} \left(\frac{1}{qrR} - \frac{1}{Q} \right)^{-1} \right)^{-1} \\ \sum_{a=1}^q \sum_{\substack{a' \bmod r \\ \gcd(a',r)=1}} \left| \sum_{W' < p \leq W} (\log p) e \left(\left(\alpha(m_0 + a) + \frac{a'}{r} \right) p \right) \right|^2$$

We now apply the large sieve with weights [MV73, (1.6)], recalling that each angle $\alpha(m_0 + a) + a'/r$ is separated from the others by at least $1/qrR - 1/Q$; we obtain that (4.47) is at most $\sum_{W' < p \leq W} (\log p)^2$. It remains to estimate the sum in the first line of (4.46). (We are following here a procedure analogous to that used in [MV73] to prove the Brun-Titchmarsh theorem.)

Assume first that $q \leq W/13.5$. Set

$$(4.48) \quad R = \left(\sigma \frac{W}{q} \right)^{1/2},$$

where $\sigma = 1/2e^{2 \cdot 0.25068} = 0.30285\dots$. It is clear that $qR^2 < Q$, $q < W'$ and $R \geq 2$. Moreover, for $r \leq R$,

$$\frac{1}{Q} \leq \frac{1}{3.5W} \leq \frac{\sigma}{3.5} \frac{1}{\sigma W} = \frac{\sigma}{3.5} \frac{1}{qR^2} \leq \frac{\sigma/3.5}{qrR}.$$

Hence

$$\begin{aligned} \frac{W}{2} + \frac{3}{2} \left(\frac{1}{qrR} - \frac{1}{Q} \right)^{-1} &\leq \frac{W}{2} + \frac{3}{2} \frac{qrR}{1 - \sigma/3.5} = \frac{W}{2} + \frac{3r}{2(1 - \frac{\sigma}{3.5})} \cdot 2\sigma \frac{W}{2} \\ &= \frac{W}{2} \left(1 + \frac{3\sigma}{1 - \sigma/3.5} \frac{rW}{R} \right) < \frac{W}{2} \left(1 + \frac{rW}{R} \right) \end{aligned}$$

and so

$$\begin{aligned} \sum_{\substack{r \leq R \\ \gcd(r,q)=1}} \left(\frac{W}{2} + \frac{3}{2} \left(\frac{1}{qrR} - \frac{1}{Q} \right)^{-1} \right)^{-1} \frac{\mu(r)^2}{\phi(r)} \\ \geq \frac{2}{W} \sum_{\substack{r \leq R \\ \gcd(r,q)=1}} (1 + rR^{-1})^{-1} \frac{\mu(r)^2}{\phi(r)} \geq \frac{2}{W} \frac{\phi(q)}{q} \sum_{r \leq R} (1 + rR^{-1})^{-1} \frac{\mu(r)^2}{\phi(r)}. \end{aligned}$$

For $R \geq 2$,

$$\sum_{r \leq R} (1 + rR^{-1})^{-1} \frac{\mu(r)^2}{\phi(r)} > \log R + 0.25068;$$

this is true for $R \geq 100$ by [MV73, Lemma 8] and easily verifiable numerically for $2 \leq R < 100$. (It suffices to verify this for R integer with $r < R$ instead of $r \leq R$, as that is the worst case.)

Now

$$\log R = \frac{1}{2} \left(\log \frac{W}{2q} + \log 2\sigma \right) = \frac{1}{2} \log \frac{W}{2q} - 0.25068.$$

Hence

$$\sum_{r \leq R} (1 + rR^{-1})^{-1} \frac{\mu(r)^2}{\phi(r)} > \frac{1}{2} \log \frac{W}{2q}$$

and the statement follows.

Now consider the case $q > W/13.5$. If q is even, then, in this range, inequality (4.41) is always better than (4.42), and so we are done. Assume, then, that $W/13.5 < q \leq W/2$ and q is odd. We set $R = 2$; clearly $qR^2 < W \leq Q$ and $q < W/2 \leq W'$, and so this choice of R is valid. It remains to check that

$$\frac{1}{\frac{W}{2} + \frac{3}{2} \left(\frac{1}{2q} - \frac{1}{Q}\right)^{-1}} + \frac{1}{\frac{W}{2} + \frac{3}{2} \left(\frac{1}{4q} - \frac{1}{Q}\right)^{-1}} \geq \frac{1}{W} \log \frac{W}{2q}.$$

This follows because

$$\frac{1}{\frac{1}{2} + \frac{3}{2} \left(\frac{t}{2} - \frac{1}{3.5}\right)^{-1}} + \frac{1}{\frac{1}{2} + \frac{3}{2} \left(\frac{t}{4} - \frac{1}{3.5}\right)^{-1}} \geq \log \frac{t}{2}$$

for all $2 \leq t \leq 13.5$. □

We need a version of Lemma 4.4 with m restricted to the odd numbers.

Lemma 4.5. *Let $W \geq 1$, $W' \geq W/2$. Let $2\alpha = a/q + O^*(1/qQ)$, $q \leq Q$. Then*

$$(4.49) \quad \sum_{\substack{A_0 < m \leq A_1 \\ m \text{ odd}}} \left| \sum_{W' < p \leq W} (\log p) e(\alpha mp) \right|^2 \leq \left[\frac{A_1 - A_0}{\min(2q, Q)} \right] \cdot (W - W' + 2q) \sum_{W' < p \leq W} (\log p)^2.$$

If $q < W/2$ and $Q \geq 3.5W$, the following bound also holds:

$$(4.50) \quad \sum_{\substack{A_0 < m \leq A_1 \\ m \text{ odd}}} \left| \sum_{W' < p \leq W} (\log p) e(\alpha mp) \right|^2 \leq \left[\frac{A_1 - A_0}{2q} \right] \cdot \frac{q}{\phi(q)} \frac{W}{\log(W/2q)} \cdot \sum_{W' < p \leq W} (\log p)^2.$$

If $A_1 - A_0 \leq 2\rho q$ and $q \leq \rho Q$, $\rho, \rho \in [0, 1]$, the following bound also holds:

$$(4.51) \quad \sum_{A_0 < m \leq A_1} \left| \sum_{W' < p \leq W} (\log p) e(\alpha mp) \right|^2 \leq (W - W' + q/(1 - \rho)) \sum_{W' < p \leq W} (\log p)^2.$$

Proof. We follow the proof of Lemma 4.4, noting the differences. Let $k = \min(q, \lceil Q/2 \rceil) \geq \lceil q/2 \rceil$, just as before. We split $(A_0, A_1]$ into $\lceil (A_1 - A_0)/k \rceil$ blocks of at most $2k$ consecutive integers; any such block contains at most k odd numbers. For odd m, m' in such a block, αm and $\alpha m'$ are separated by a distance of

$$|\{\alpha(m - m')\}| = \left| \left\{ 2\alpha \frac{m - m'}{2} \right\} \right| = |\{(a/q)k\}| - O^*(k/qQ) \geq 1/2q.$$

We obtain (4.49) and (4.51) just as we obtained (4.41) and (4.43) before. To obtain (4.50), proceed again as before, noting that the angles we are working with can be labelled as $\alpha(m_0 + 2a)$, $0 \leq a < q$. \square

The idea now (for large δ) is that, if δ is not negligible, then, as m increases, am loops around the circle \mathbb{R}/\mathbb{Z} roughly repeats itself every q steps – but with a slight displacement. This displacement gives rise to a configuration to which Lemma 4.3 is applicable.

Proposition 4.6. *Let $x \geq W \geq 1$, $W' \geq W/2$, $U' \geq x/2W$. Let $Q \geq 3.5W$. Let $2\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $|\delta/x| \leq 1/qQ$, $q \leq Q$. Let $S_2(U', W', W)$ be as in (4.37) with $v = 2$.*

For $q \leq \rho Q$, where $\rho \in [0, 1]$,

$$(4.52) \quad S_2(U', W', W) \leq \left(\max(1, 2\rho) \left(\frac{x}{8q} + \frac{x}{2W} \right) + \frac{W}{2} + 2q \right) \cdot \sum_{W' < p \leq W} (\log p)^2$$

If $q < W/2$,

$$(4.53) \quad S_2(U', W', W) \leq \left(\frac{x}{4\phi(q)} \frac{1}{\log(W/2q)} + \frac{q}{\phi(q)} \frac{W}{\log(W/2q)} \right) \cdot \sum_{W' < p \leq W} (\log p)^2.$$

If $W > x/4q$, the following bound also holds:

$$(4.54) \quad S_2(U', W', W) \leq \left(\frac{W}{2} + \frac{q}{1 - x/4Wq} \right) \sum_{W' < p \leq W} (\log p)^2.$$

If $\delta \neq 0$ and $x/4W + q \leq x/|\delta|q$,

$$(4.55) \quad S_2(U', W', W) \leq \min \left(1, \frac{2q/\phi(q)}{\log \left(\frac{x}{|\delta|q} \left(q + \frac{x}{4W} \right)^{-1} \right)} \right) \cdot \left(\frac{x}{|\delta|q} + \frac{W}{2} \right) \sum_{W' < p \leq W} (\log p)^2.$$

Lastly, if $\delta \neq 0$ and $q \leq \rho Q$, where $\rho \in [0, 1]$,

$$(4.56) \quad S_2(U', W', W) \leq \left(\frac{x}{|\delta|q} + \frac{W}{2} + \frac{x}{8(1-\rho)Q} + \frac{x}{4(1-\rho)W} \right) \sum_{W' < p \leq W} (\log p)^2.$$

The trivial bound would be in the order of

$$S_2(U', W', W) = (x/2 \log x) \sum_{W' < p \leq W} (\log p)^2.$$

In practice, (4.54) gets applied when $W \geq x/q$.

Proof. Let us first prove statements (4.53) and (4.52), which do not involve δ . Assume first $q \leq W/2$. Then, by (4.50) with $A_0 = U'$, $A_1 = x/W$,

$$S_2(U', W', W) \leq \left(\frac{x/W - U'}{2q} + 1 \right) \frac{q}{\phi(q)} \frac{W}{\log(W/2q)} \sum_{W' < p \leq W} (\log p)^2.$$

Clearly $(x/W - U')W \leq (x/2W) \cdot W = x/2$. Thus (4.53) holds.

Assume now that $q \leq \rho Q$. Apply (4.49) with $A_0 = U'$, $A_1 = x/W$. Then

$$S_2(U', W', W) \leq \left(\frac{x/W - U'}{q \cdot \min(2, \rho^{-1})} + 1 \right) (W - W' + 2q) \sum_{W' < p \leq W} (\log p)^2.$$

Now

$$\begin{aligned}
& \left(\frac{x/W - U'}{q \cdot \min(2, \rho^{-1})} + 1 \right) \cdot (W - W' + 2q) \\
& \leq \left(\frac{x}{W} - U' \right) \frac{W - W'}{q \min(2, \rho^{-1})} + \max(1, 2\rho) \left(\frac{x}{W} - U' \right) + W/2 + 2q \\
& \leq \frac{x/4}{q \min(2, \rho^{-1})} + \max(1, 2\rho) \frac{x}{2W} + W/2 + 2q.
\end{aligned}$$

This implies (4.52).

If $W > x/4q$, apply (4.43) with $\varrho = x/4Wq$, $\rho = 1$. This yields (4.54).

Assume now that $\delta \neq 0$ and $x/4W + q \leq x/|\delta q|$. Let $Q' = x/|\delta q|$. For any m_1, m_2 with $x/2W < m_1, m_2 \leq x/W$, we have $|m_1 - m_2| \leq x/2W \leq 2(Q' - q)$, and so

$$(4.57) \quad \left| \frac{m_1 - m_2}{2} \cdot \delta/x + q\delta/x \right| \leq Q'|\delta|/x = \frac{1}{q}.$$

The conditions of Lemma 4.3 are thus fulfilled with $\nu = (x/4W) \cdot |\delta|/x$ and $\nu = |\delta q|/x$. We obtain that $S_2(U', W', W)$ is at most

$$\min \left(1, \frac{2q}{\phi(q)} \frac{1}{\log((q(\nu + \nu))^{-1})} \right) (W - W' + \nu^{-1}) \sum_{W' < p \leq W} (\log p)^2.$$

Here $W - W' + \nu^{-1} = W - W' + x/|q\delta| \leq W/2 + x/|q\delta|$ and

$$(q(\nu + \nu))^{-1} = \left(q \frac{|\delta|}{x} \right)^{-1} \left(q + \frac{x}{4W} \right)^{-1}.$$

Lastly, assume $\delta \neq 0$ and $q \leq \rho Q$. We let $Q' = x/|\delta q| \geq Q$ again, and we split the range $U' < m \leq x/W$ into intervals of length $2(Q' - q)$, so that (4.57) still holds within each interval. We apply Lemma 4.3 with $\nu = (Q' - q) \cdot |\delta|/x$ and $\nu = |\delta q|/x$. We obtain that $S_2(U', W', W)$ is at most

$$\left(1 + \frac{x/W - U'}{2(Q' - q)} \right) (W - W' + \nu^{-1}) \sum_{W' < p \leq W} (\log p)^2.$$

Here $W - W' + \nu^{-1} \leq W/2 + x/q|\delta|$ as before. Moreover,

$$\begin{aligned}
\left(\frac{W}{2} + \frac{x}{q|\delta|} \right) \left(1 + \frac{x/W - U'}{2(Q' - q)} \right) & \leq \left(\frac{W}{2} + Q' \right) \left(1 + \frac{x/2W}{2(1 - \rho)Q'} \right) \\
& \leq \frac{W}{2} + Q' + \frac{x}{8(1 - \rho)Q'} + \frac{x}{4W(1 - \rho)} \\
& \leq \frac{x}{|\delta q|} + \frac{W}{2} + \frac{x}{8(1 - \rho)Q} + \frac{x}{4(1 - \rho)W}.
\end{aligned}$$

Hence (4.56) holds. \square

5. TOTALS

Let x be given. We will choose U, V, W later; assume from the start that $2 \cdot 10^6 \leq V < x/4$ and $UV \leq x$. Starting in section 5.2, we will also assume that $x \geq x_0 = 10^{25}$.

Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be given. We choose an approximation $2\alpha = a/q + \delta/x$, $\gcd(a, q) = 1$, $q \leq Q$, $|\delta/x| \leq 1/qQ$. We assume $Q \geq \max(16, 2\sqrt{x})$ and

$Q \geq \max(2U, x/U)$. Let $S_{I,1}$, $S_{I,2}$, S_{II} , S_0 be as in (2.21), with the smoothing function $\eta = \eta_2$ as in (1.4).

The term S_0 is 0 because $V < x/4$ and η_2 is supported on $[-1/4, 1]$. We set $v = 2$.

5.1. Contributions of different types.

5.1.1. *Type I terms: $S_{I,1}$.* The term $S_{I,1}$ can be handled directly by Lemma 3.6, with $\rho_0 = 4$ and $D = U$. (Condition (3.38) is valid thanks to (2.6).) Since $U \leq Q/2$, the contribution of $S_{I,1}$ gets bounded by (3.40) and (3.41): the absolute value of $S_{I,1}$ is at most

$$(5.1) \quad \begin{aligned} & \frac{x}{q} \min\left(1, \frac{c_0/\delta^2}{(2\pi)^2}\right) \left| \sum_{\substack{m \leq \frac{U}{q} \\ \gcd(m,q)=1}} \frac{\mu(m)}{m} \log \frac{x}{mq} \right| + \frac{x}{q} |\widehat{\log \cdot \eta}(-\delta)| \left| \sum_{\substack{m \leq \frac{U}{q} \\ \gcd(m,q)=1}} \frac{\mu(m)}{m} \right| \\ & + \frac{2\sqrt{c_0 c_1}}{\pi} \left(U \log \frac{ex}{U} + \sqrt{3}q \log \frac{q}{c_2} + \frac{q}{2} \log \frac{q}{c_2} \log^+ \frac{2U}{q} \right) + \frac{3c_1}{2} \frac{x}{q} \log \frac{q}{c_2} \log^+ \frac{U}{\frac{c_2 x}{q}} \\ & + \frac{3c_1}{2} \sqrt{\frac{2x}{c_2}} \log \frac{2x}{c_2} + \left(\frac{c_0}{2} - \frac{2c_0}{\pi^2} \right) \left(\frac{U^2}{4qx} \log \frac{e^{1/2}x}{U} + \frac{1}{e} \right) \\ & + \frac{2|\eta'|_1}{\pi} q \max\left(1, \log \frac{c_0 e^3 q^2}{4\pi |\eta'|_1 x}\right) \log x, \end{aligned}$$

where $c_0 = 31.521$ (by Lemma A.5), $c_1 = 1.0000028 > 1 + (8 \log 2)/V \geq 1 + (8 \log 2)/(x/U)$, $c_2 = 6\pi/5\sqrt{c_0} = 0.67147\dots$. By (2.1), (A.17) and Lemma A.6,

$$|\widehat{\log \cdot \eta}(-\delta)| \leq \min\left(2 - \log 4, \frac{24 \log 2}{\pi^2 \delta^2}\right).$$

By (2.7), (2.9) and (2.10), the first line of (5.1) is at most

$$\begin{aligned} & \frac{x}{q} \min\left(1, \frac{c'_0}{\delta^2}\right) \left(\min\left(\frac{4}{5} \frac{q/\phi(q)}{\log^+ \frac{U}{q^2}}, 1\right) \log \frac{x}{U} + 1.00303 \frac{q}{\phi(q)} \right) \\ & + \frac{x}{q} \min\left(2 - \log 4, \frac{c''_0}{\delta^2}\right) \min\left(\frac{4}{5} \frac{q/\phi(q)}{\log^+ \frac{U}{q^2}}, 1\right), \end{aligned}$$

where $c'_0 = 0.798437 > c_0/(2\pi)^2$, $c''_0 = 1.685532$. Clearly $c''_0/c_0 > 1 > 2 - \log 4$.

Taking derivatives, we see that $t \mapsto (t/2) \log(t/c_2) \log^+ 2U/t$ takes its maximum (for $t \in [1, 2U]$) when $\log(t/c_2) \log^+ 2U/t = \log t/c_2 - \log^+ 2U/t$; since $t \rightarrow \log t/c_2 - \log^+ 2U/t$ is increasing on $[1, 2U]$, we conclude that

$$\frac{q}{2} \log \frac{q}{c_2} \log^+ \frac{2U}{q} \leq U \log \frac{2U}{c_2}.$$

Similarly, $t \mapsto t \log(x/t) \log^+(U/t)$ takes its maximum at a point $t \in [0, U]$ for which $\log(x/t) \log^+(U/t) = \log(x/t) + \log^+(U/t)$, and so

$$\frac{x}{q} \log \frac{q}{c_2} \log^+ \frac{U}{\frac{c_2 x}{q}} \leq \frac{U}{c_2} (\log x + \log U).$$

We conclude that

$$(5.2) \quad \begin{aligned} |S_{I,1}| &\leq \frac{x}{q} \min\left(1, \frac{c'_0}{\delta^2}\right) \left(\min\left(\frac{4q/\phi(q)}{5 \log^+ \frac{U}{q^2}}, 1\right) \left(\log \frac{x}{U} + c_{3,I}\right) + c_{4,I} \frac{q}{\phi(q)} \right) \\ &\quad + \left(c_{7,I} \log \frac{q}{c_2} + c_{8,I} \log x \max\left(1, \log \frac{c_{11,I} q^2}{x}\right) \right) q + c_{10,I} \frac{U^2}{4qx} \log \frac{e^{1/2} x}{U} \\ &\quad + \left(c_{5,I} \log \frac{2U}{c_2} + c_{6,I} \log xU \right) U + c_{9,I} \sqrt{x} \log \frac{2x}{c_2} + \frac{c_{10,I}}{e}, \end{aligned}$$

where c_2 and c'_0 are as above, $c_{3,I} = 2.11104 > c''_0/c'_0$, $c_{4,I} = 1.00303$, $c_{5,I} = 3.57422 > 2\sqrt{c_0 c_1}/\pi$, $c_{6,I} = 2.23389 > 3c_1/2c_2$, $c_{7,I} = 6.19072 > 2\sqrt{3c_0 c_1}/\pi$, $c_{8,I} = 3.53017 > 2(8 \log 2)/\pi$, $c_{9,I} = 2.58877 > 3\sqrt{2}c_1/2\sqrt{c_2}$, $c_{10,I} = 9.37301 > c_0(1/2 - 2/\pi^2)$ and $c_{11,I} = 9.0857 > c_0 e^3/(4\pi \cdot 8 \log 2)$.

5.1.2. *Type I terms: $S_{I,2}$. The case $q \leq Q/V$.*

If $q \leq Q/V$, then, for $v \leq V$,

$$2v\alpha = \frac{va}{q} + O^*\left(\frac{v}{Qq}\right) = \frac{va}{q} + O^*\left(\frac{1}{q^2}\right),$$

and so va/q is a valid approximation to $2v\alpha$. (Here we are using v to label an integer variable bounded above by $v \leq V$; we no longer need v to label the quantity in (2.22), since that has been set equal to the constant 2.) Moreover, for $Q_v = Q/v$, we see that $2v\alpha = (va/q) + O^*(1/qQ_v)$. If $\alpha = a/q + \delta/x$, then $v\alpha = va/q + \delta/(x/v)$. Now

$$(5.3) \quad S_{I,2} = \sum_{\substack{v \leq V \\ v \text{ odd}}} \Lambda(v) \sum_{\substack{m \leq U \\ m \text{ odd}}} \mu(m) \sum_{\substack{n \\ n \text{ odd}}} e((v\alpha) \cdot mn) \eta(mn/(x/v)).$$

We can thus estimate $S_{I,2}$ by applying Lemma 3.5 to each inner double sum in (5.3). We obtain that, if $|\delta| \leq 1/2c_2$, where $c_2 = 6\pi/5\sqrt{c_0}$ and $c_0 = 31.521$, then $|S_{I,2}|$ is at most

$$(5.4) \quad \sum_{v \leq V} \Lambda(v) \left(\frac{x/v}{2q_v} \min\left(1, \frac{c_0}{(\pi\delta)^2}\right) \left| \sum_{\substack{m \leq M_v/q \\ \gcd(m, 2q)=1}} \frac{\mu(m)}{m} \right| + \frac{c_{10,I} q}{4x/v} \left(\frac{U}{q_v} + 1\right)^2 \right)$$

plus

$$(5.5) \quad \begin{aligned} &\sum_{v \leq V} \Lambda(v) \left(\frac{2\sqrt{c_0 c_1}}{\pi} U + \frac{3c_1}{2} \frac{x}{vq_v} \log^+ \frac{U}{\frac{c_2 x}{vq_v}} + \frac{\sqrt{c_0 c_1}}{\pi} q_v \log^+ \frac{U}{q_v/2} \right) \\ &+ \sum_{v \leq V} \Lambda(v) \left(c_{8,I} \max\left(\log \frac{c_{11,I} q_v^2}{x/v}, 1\right) q_v + \left(\frac{2\sqrt{3c_0 c_1}}{\pi} + \frac{3c_1}{2c_2} + \frac{55c_0 c_2}{6\pi^2} \right) q_v \right), \end{aligned}$$

where $q_v = q/\gcd(q, v)$, $M_v \in [\min(Q/2v, U), U]$ and $c_1 = 1.0000028$; if $|\delta| \geq 1/2c_2$, then $|S_{I,2}|$ is at most (5.4) plus

$$(5.6) \quad \sum_{v \leq V} \Lambda(v) \left(\frac{\sqrt{c_0 c_1}}{\pi/2} U + \frac{3c_1}{2} \left(2 + \frac{(1+\epsilon)}{\epsilon} \log^+ \frac{2U}{\frac{x/v}{|\delta|q_v}} \right) \frac{x/v}{Q/v} + \frac{35c_0 c_2}{3\pi^2} q_v \right) \\ + \sum_{v \leq V} \Lambda(v) \frac{\sqrt{c_0 c_1}}{\pi/2} (1+\epsilon) \min \left(\left\lfloor \frac{x/v}{|\delta|q_v} \right\rfloor + 1, 2U \right) \left(\sqrt{3+2\epsilon} + \frac{\log^+ \frac{2U}{\left\lfloor \frac{x/v}{|\delta|q_v} \right\rfloor + 1}}{2} \right)$$

Write $S_V = \sum_{v \leq V} \Lambda(v)/(vq_v)$. By (2.12),

$$(5.7) \quad S_V \leq \sum_{v \leq V} \frac{\Lambda(v)}{vq} + \sum_{\substack{v \leq V \\ \gcd(v,q) > 1}} \frac{\Lambda(v)}{v} \left(\frac{\gcd(q,v)}{q} - \frac{1}{q} \right) \\ \leq \frac{\log V}{q} + \frac{1}{q} \sum_{p|q} (\log p) \left(v_p(q) + \sum_{\substack{\alpha \geq 1 \\ p^{\alpha+v_p(q)} \leq V}} \frac{1}{p^\alpha} - \sum_{\substack{\alpha \geq 1 \\ p^\alpha \leq V}} \frac{1}{p^\alpha} \right) \\ \leq \frac{\log V}{q} + \frac{1}{q} \sum_{p|q} (\log p) v_p(q) = \frac{\log Vq}{q}.$$

This helps us to estimate (5.4). We could also use this to estimate the second term in the first line of (5.5), but, for that purpose, it will actually be wiser to use the simpler bound

$$(5.8) \quad \sum_{v \leq V} \Lambda(v) \frac{x}{vq_v} \log^+ \frac{U}{\frac{c_2 x}{vq_v}} \leq \sum_{v \leq V} \Lambda(v) \frac{U/c_2}{e} \leq \frac{1.0004}{ec_2} UV$$

(by (2.14) and the fact that $t \log^+ A/t$ takes its maximum at $t = A/e$).

We bound the sum over m in (5.4) by (2.7) and (2.9). To bound the terms involving $(U/q_v + 1)^2$, we use

$$\sum_{v \leq V} \Lambda(v)v \leq 0.5004V^2 \quad (\text{by (2.17)}), \\ \sum_{v \leq V} \Lambda(v)v \gcd(v,q)^j \leq \sum_{v \leq V} \Lambda(v)v + V \sum_{\substack{v \leq V \\ \gcd(v,q) \neq 1}} \Lambda(v) \gcd(v,q)^j, \\ \sum_{\substack{v \leq V \\ \gcd(v,q) \neq 1}} \Lambda(v) \gcd(v,q) \leq \sum_{p|q} (\log p) \sum_{1 \leq \alpha \leq \log_p V} p^{v_p(q)} \leq \sum_{p|q} (\log p) \frac{\log V}{\log p} p^{v_p(q)} \\ \leq (\log V) \sum_{p|q} p^{v_p(q)} \leq q \log V$$

and

$$\begin{aligned} \sum_{\substack{v \leq V \\ \gcd(v, q) \neq 1}} \Lambda(v) \gcd(v, q)^2 &\leq \sum_{p|q} (\log p) \sum_{1 \leq \alpha \leq \log_p V} p^{v_p(q) + \alpha} \\ &\leq \sum_{p|q} (\log p) \cdot 2p^{v_p(q)} \cdot p^{\log_p V} \leq 2qV \log q. \end{aligned}$$

Using (2.14) and (5.7) as well, we conclude that (5.4) is at most

$$\begin{aligned} \frac{x}{2q} \min \left(1, \frac{c_0}{(\pi\delta)^2} \right) \min \left(\frac{4}{5} \frac{q/\phi(q)}{\log^+ \frac{\min(Q/2V, U)}{2q}}, 1 \right) \log Vq \\ + \frac{c_{10, I}}{4x} \left(0.5004V^2q \left(\frac{U}{q} + 1 \right)^2 + 2UVq \log V + 2U^2V \log V \right). \end{aligned}$$

Assume $Q \leq 2UV/e$. Using (2.14), (5.8), (2.18) and the inequality $vq \leq Vq \leq Q$ (which implies $q/2 \leq U/e$), we see that (5.5) is at most

$$\begin{aligned} 1.0004 \left(\left(\frac{2\sqrt{c_0c_1}}{\pi} + \frac{3c_1}{2ec_2} \right) UV + \frac{\sqrt{c_0c_1}}{\pi} Q \log \frac{U}{q/2} \right) \\ + \left(c_{5, I_2} \max \left(\log \frac{c_{11, I} q^2}{x}, 2 \right) + c_{6, I_2} \right) Q, \end{aligned}$$

where $c_{5, I_2} = 3.53312 > 1.0004 \cdot c_{8, I}$ and

$$c_{6, I_2} = \frac{2\sqrt{3c_0c_1}}{\pi} + \frac{3c_1}{2c_2} + \frac{55c_0c_2}{6\pi^2}.$$

The expressions in (5.6) get estimated similarly. In particular,

$$\begin{aligned} \sum_{v \leq V} \Lambda(v) \min \left(\left\lfloor \frac{x/v}{|\delta|q_v} \right\rfloor + 1, 2U \right) \cdot \frac{1}{2} \log^+ \frac{2U}{\left\lfloor \frac{x/v}{|\delta|q_v} \right\rfloor + 1} \\ \leq \sum_{v \leq V} \Lambda(v) \max_{t > 0} t \log^+ \frac{U}{t} \leq \sum_{v \leq V} \Lambda(v) \frac{U}{e} = \frac{1.0004}{e} UV, \end{aligned}$$

but

$$\begin{aligned} \sum_{v \leq V} \Lambda(v) \min \left(\left\lfloor \frac{x/v}{|\delta|q_v} \right\rfloor + 1, 2U \right) &\leq \sum_{v \leq \frac{x}{2U|\delta|q}} \Lambda(v) \cdot 2U \\ &+ \sum_{\substack{\frac{x}{2U|\delta|q} < v \leq V \\ \gcd(v, q) = 1}} \Lambda(v) \frac{x/|\delta|}{vq} + \sum_{v \leq V} \Lambda(v) + \sum_{\substack{v \leq V \\ \gcd(v, q) \neq 1}} \Lambda(v) \frac{x/|\delta|}{v} \left(\frac{1}{q_v} - \frac{1}{q} \right) \\ &\leq 1.03883 \frac{x}{|\delta|q} + \frac{x}{|\delta|q} \max \left(\log V - \log \frac{x}{2U|\delta|q} + \log \frac{3}{\sqrt{2}}, 0 \right) \\ &+ V + \frac{x}{|\delta|} \frac{1}{q} \sum_{p|q} (\log p) v_p(q) \\ &\leq \frac{x}{|\delta|q} \left(1.03883 + \log q + \log^+ \frac{6UV|\delta|q}{\sqrt{2}x} \right) + 1.0004V \end{aligned}$$

by (2.12), (2.13), (2.14) and (2.15); we are proceeding much as in (5.7).

If $|\delta| \leq 1/2c_2$, then, assuming $Q \leq 2UV/e$, we conclude that $|S_{I,2}|$ is at most

$$(5.9) \quad \frac{x}{2\phi(q)} \min\left(1, \frac{c_0}{(\pi\delta)^2}\right) \min\left(\frac{4/5}{\log^+ \frac{Q}{4Vq^2}}, 1\right) \log Vq \\ + c_{8,I_2} \frac{x}{q} \left(\frac{UV}{x}\right)^2 \left(1 + \frac{q}{U}\right)^2 + \frac{c_{10,I}}{2} \left(\frac{UV}{x}q \log V + \frac{U^2V}{x} \log V\right)$$

plus

$$(5.10) \quad (c_{4,I_2} + c_{9,I_2})UV + (c_{10,I_2} \log \frac{U}{q} + c_{5,I_2} \max\left(\log \frac{c_{11,I}q^2}{x}, 2\right) + c_{12,I_2}) \cdot Q,$$

where

$$\begin{aligned} c_{4,I_2} &= 3.57422 > 2\sqrt{c_0c_1}/\pi, \\ c_{5,I_2} &= 3.53312 > 1.0004 \cdot c_{8,I}, \\ c_{8,I_2} &= 1.17257 > \frac{c_{10,I}}{4} \cdot 0.5004, \\ c_{9,I_2} &= 0.82214 > 3c_1 \cdot 1.0004/2ec_2, \\ c_{10,I_2} &= 1.78783 > 1.0004\sqrt{c_0c_1}/\pi, \\ c_{12,I_2} &= 28.26771 > c_{6,I_2} + c_{10,I_2} \log 2. \end{aligned}$$

If $|\delta| \geq 1/2c_2$, then $|S_{I,2}|$ is at most (5.9) plus

$$(5.11) \quad (c_{4,I_2} + (1 + \epsilon)c_{13,I_2})UV + c_\epsilon \left(c_{14,I_2} \left(\log q + \log^+ \frac{6UV|\delta|q}{\sqrt{2x}}\right) + c_{15,I_2}\right) \frac{x}{|\delta|q} \\ + c_{16,I_2} \left(2 + \frac{1 + \epsilon}{\epsilon} \log^+ \frac{2UV|\delta|q}{x}\right) \frac{x}{Q/V} + c_{17,I_2}Q + c_\epsilon \cdot c_{18,I_2}V,$$

where

$$\begin{aligned} c_{13,I_2} &= 1.31541 > \frac{2\sqrt{c_0c_1}}{\pi} \cdot \frac{1.0004}{e}, \\ c_{14,I_2} &= 3.57422 > \frac{2\sqrt{c_0c_1}}{\pi}, \\ c_{15,I_2} &= 3.71301 > \frac{2\sqrt{c_0c_1}}{\pi} \cdot 1.03883, \\ c_{16,I_2} &= 1.50061 > 1.0004 \cdot 3c_1/2 \\ c_{17,I_2} &= 25.0295 > 1.0004 \cdot \frac{35c_0c_2}{3\pi^2}, \\ c_{18,I_2} &= 3.57565 > \frac{2\sqrt{c_0c_1}}{\pi} \cdot 1.0004, \end{aligned}$$

and $c_\epsilon = (1 + \epsilon)\sqrt{3 + 2\epsilon}$. We recall that $c_2 = 6\pi/5\sqrt{c_0} = 0.67147\dots$. We will choose $\epsilon \in (0, 1)$ later.

The case $q > Q/V$. We use Lemma 3.7 in this case.

5.1.3. *Type II terms.* As we showed in (4.1)–(4.6), S_{II} (given in (4.1)) is at most

$$(5.12) \quad 4 \int_V^{x/U} \sqrt{S_1(U, W) \cdot S_2(U, V, W)} \frac{dW}{W} + 4 \int_V^{x/U} \sqrt{S_1(U, W) \cdot S_3(W)} \frac{dW}{W},$$

where S_1 , S_2 and S_3 are as in (4.5) and (4.6). We bounded S_1 in (4.33) and (4.34), S_2 in Prop. 4.6 and S_3 in (4.6).

We first recall our estimate for S_1 . In the whole range $[V, x/U]$ for W , we know from (4.33) and (4.34) that $S_1(U, W)$ is at most

$$(5.13) \quad \frac{2}{\pi^2} \frac{x}{W} + \kappa_0 \zeta(3/2)^3 \frac{x}{W} \sqrt{\frac{x/WU}{U}},$$

where

$$\kappa_0 = 1.27.$$

(We recall we are working with $v = 2$.)

We have better estimates for the constant in front in some parts of the range; in what is usually the main part, (4.34) and (4.36) give us a constant of 0.15107 instead of $2/\pi^2$. Note that $1.27\zeta(3/2)^3 = 22.6417\dots$. We should choose U, V so that the first term dominates. For the while being, assume only

$$(5.14) \quad U \geq 5 \cdot 10^5 \frac{x}{VU};$$

then (5.13) gives

$$(5.15) \quad S_1(U, W) \leq \kappa_1 \frac{x}{W},$$

where

$$\kappa_1 = \frac{2}{\pi^2} + \frac{22.6418}{\sqrt{10^6/2}} \leq 0.2347.$$

This will suffice for our cruder estimates.

The second integral in (5.12) is now easy to bound. By (4.6),

$$S_3(W) \leq 1.0171x + 2.0341W \leq 1.0172x,$$

since $W \leq x/U \leq x/5 \cdot 10^5$. Hence

$$\begin{aligned} 4 \int_V^{x/U} \sqrt{S_1(U, W) \cdot S_3(W)} \frac{dW}{W} &\leq 4 \int_V^{x/U} \sqrt{\kappa_{w,1} \frac{x}{W} \cdot 1.0172x} \frac{dW}{W} \\ &\leq \kappa_9 \frac{x}{\sqrt{V}}, \end{aligned}$$

where

$$\kappa_9 = 8 \cdot \sqrt{1.0172 \cdot \kappa_1} \leq 3.9086.$$

(We are using the easy bound $\sqrt{a+b+c} \leq \sqrt{a} + \sqrt{b} + \sqrt{c}$.)

Let us now examine S_2 , which was bounded in Prop. 4.6. Recall $W' = \max(V, W/2)$, $U' = \max(U, x/2W)$. Since $W' \geq W/2$ and $W \geq V \geq 117$, we can always bound

$$(5.16) \quad \sum_{W' < p \leq W} (\log p)^2 \leq \frac{1}{2} W (\log W).$$

by (2.19).

Bounding S_2 for δ arbitrary. We set

$$W_0 = \min(\max(2\theta q, V), x/U),$$

where $\theta \geq e$ is a parameter that will be set later.

For $V \leq W < W_0$, we use the bound (4.52):

$$\begin{aligned} S_2(U', W', W) &\leq \left(\max(1, 2\rho) \left(\frac{x}{8q} + \frac{x}{2W} \right) + \frac{W}{2} + 2q \right) \cdot \frac{1}{2} W (\log W) \\ &\leq \max\left(\frac{1}{2}, \rho\right) \left(\frac{W}{8q} + \frac{1}{2} \right) x \log W + \frac{W^2 \log W}{4} + qW \log W, \end{aligned}$$

where $\rho = q/Q$.

If $W_0 > V$, the contribution of the terms with $V \leq W < W_0$ to (5.12) is (by 5.15) bounded by

$$\begin{aligned}
& 4 \int_V^{W_0} \sqrt{\kappa_1 \frac{x}{W} \left(\frac{\rho_0}{4} \left(\frac{W}{4q} + 1 \right) x \log W + \frac{W^2 \log W}{4} + qW \log W \right)} \frac{dW}{W} \\
& \leq \frac{\kappa_2}{2} \sqrt{\rho_0 x} \int_V^{W_0} \frac{\sqrt{\log W}}{W^{3/2}} dW + \frac{\kappa_2}{2} \sqrt{x} \int_V^{W_0} \frac{\sqrt{\log W}}{W^{1/2}} dW \\
(5.17) \quad & + \kappa_2 \sqrt{\frac{\rho_0 x^2}{16q} + qx} \int_V^{W_0} \frac{\sqrt{\log W}}{W} dW \\
& \leq \left(\kappa_2 \sqrt{\rho_0} \frac{x}{\sqrt{V}} + \kappa_2 \sqrt{xW_0} \right) \sqrt{\log W_0} \\
& + \frac{2\kappa_2}{3} \sqrt{\frac{\rho_0 x^2}{16q} + qx} \left((\log W_0)^{3/2} - (\log V)^{3/2} \right),
\end{aligned}$$

where $\rho_0 = \max(1, 2\rho)$ and

$$\kappa_2 = 4\sqrt{\kappa_1} \leq 1.93768.$$

We now examine the terms with $W \geq W_0$. (If $\theta q > x/U$, then $W_0 = U/x$, the contribution of the case is nil, and the computations below can be ignored.)

We use (4.53):

$$S_2(U', W', W) \leq \left(\frac{x}{4\phi(q) \log(W/2q)} + \frac{q}{\phi(q) \log(W/2q)} \frac{W}{2} \right) \cdot \frac{1}{2} W \log W.$$

By $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, we can take out the $q/\phi(q) \cdot W/\log(W/2q)$ term and estimate its contribution on its own; it is at most

$$\begin{aligned}
(5.18) \quad & 4 \int_{W_0}^{x/U} \sqrt{\kappa_1 \frac{x}{W} \cdot \frac{q}{\phi(q)} \cdot \frac{1}{2} W^2 \frac{\log W}{\log W/2q}} \frac{dW}{W} \\
& = \frac{\kappa_2}{\sqrt{2}} \sqrt{\frac{q}{\phi(q)}} \int_{W_0}^{x/U} \sqrt{\frac{x \log W}{W \log W/2q}} dW \\
& \leq \frac{\kappa_2}{\sqrt{2}} \sqrt{\frac{qx}{\phi(q)}} \int_{W_0}^{x/U} \frac{1}{\sqrt{W}} \left(1 + \sqrt{\frac{\log 2q}{\log W/2q}} \right) dW
\end{aligned}$$

Now

$$\int_{W_0}^{x/U} \frac{1}{\sqrt{W}} \sqrt{\frac{\log 2q}{\log W/2q}} dW \leq \sqrt{2q \log 2q} \int_{\max(\theta, V/2q)}^{x/2Uq} \frac{1}{\sqrt{t \log t}} dt.$$

We bound this last integral somewhat crudely: for $T \geq e$,

$$(5.19) \quad \int_e^T \frac{1}{\sqrt{t \log t}} dt \leq 2.3 \sqrt{\frac{T}{\log T}}$$

(by numerical work for $e \leq T \leq T_0$ and by comparison of derivatives for $T > T_0$, where $T_0 = e^{(1-2/2.3)^{-1}} = 2135.94\dots$). Since $\theta \geq e$, this gives us that

$$\begin{aligned} & \int_{W_0}^{x/U} \frac{1}{\sqrt{W}} \left(1 + \sqrt{\frac{\log 2q}{\log W/2q}} \right) dW \\ & \leq 2\sqrt{\frac{x}{U}} + 2.3\sqrt{2q \log 2q} \cdot \sqrt{\frac{x/2Uq}{\log x/2Uq}}, \end{aligned}$$

and so (5.18) is at most

$$\sqrt{2}\kappa_2 \sqrt{\frac{q}{\phi(q)}} \left(1 + 1.15\sqrt{\frac{\log 2q}{\log x/2Uq}} \right) \frac{x}{\sqrt{U}}.$$

We are left with what will usually be the main term, viz.,

$$(5.20) \quad 4 \int_{W_0}^{x/U} \sqrt{S_1(U, W)} \cdot \left(\frac{x}{8\phi(q)} \frac{\log W}{\log W/2q} \right) W \frac{dW}{W},$$

which, by (4.34), is at most $x/\sqrt{\phi(q)}$ times the integral of

$$\frac{1}{W} \sqrt{\left(2H_2\left(\frac{x}{WU}\right) + \frac{\kappa_4}{2} \sqrt{\frac{x/WU}{U}} \right) \frac{\log W}{\log W/2q}}$$

for W going from W_0 to x/U , where H_2 is as in (4.35) and

$$\kappa_4 = 4\kappa_0\zeta(3/2)^3 \leq 90.5671.$$

By the arithmetic/geometric mean inequality, the integrand is at most $1/W$ times

$$(5.21) \quad \frac{\beta + \beta^{-1} \cdot 2H_2(x/WU)}{2} + \frac{\beta^{-1} \kappa_4}{2} \sqrt{\frac{x/WU}{U}} + \frac{\beta}{2} \frac{\log 2q}{\log W/2q}$$

for any $\beta > 0$. We will choose β later.

The first summand in (5.21) gives what we can think of as the main or worst term in the whole paper; let us compute it first. The integral is

$$(5.22) \quad \begin{aligned} \int_{W_0}^{x/U} \frac{\beta + \beta^{-1} \cdot 2H_2(x/WU)}{2} \frac{dW}{W} &= \int_1^{x/ UW_0} \frac{\beta + \beta^{-1} \cdot 2H_2(s)}{2} \frac{ds}{s} \\ &\leq \left(\frac{\beta}{2} + \frac{\kappa_6}{4\beta} \right) \log \frac{x}{UW_0} \end{aligned}$$

by (4.36), where

$$\kappa_6 = 0.60428.$$

Thus the main term is simply

$$(5.23) \quad \left(\frac{\beta}{2} + \frac{\kappa_6}{4\beta} \right) \frac{x}{\sqrt{\phi(q)}} \log \frac{x}{UW_0}.$$

The integral of the second summand is at most

$$\beta^{-1} \cdot \frac{\kappa_4}{4} \frac{\sqrt{x}}{U} \int_V^{x/U} \frac{dW}{W^{3/2}} \leq \beta^{-1} \cdot \frac{\kappa_4}{2} \sqrt{\frac{x/UV}{U}}.$$

By (5.14), this is at most

$$\frac{\beta^{-1}}{\sqrt{2}} \cdot 10^{-3} \cdot \kappa_4 \leq \beta^{-1} \kappa_7/2,$$

where

$$\kappa_7 = \frac{\sqrt{2}\kappa_4}{1000} \leq 0.1281.$$

Thus the contribution of the second summand is at most

$$\frac{\beta^{-1}\kappa_7}{2} \cdot \frac{x}{\sqrt{\phi(q)}}.$$

The integral of the third summand in (5.21) is

$$(5.24) \quad \frac{\beta}{2} \int_{W_0}^{x/U} \frac{\log 2q}{\log W/2q} \frac{dW}{W}.$$

If $V < 2\theta q \leq x/U$, this is

$$\begin{aligned} \frac{\beta}{2} \int_{2\theta q}^{x/U} \frac{\log 2q}{\log W/2q} \frac{dW}{W} &= \frac{\beta}{2} \log 2q \cdot \int_{\theta}^{x/2Uq} \frac{1}{\log t} \frac{dt}{t} \\ &= \frac{\beta}{2} \log 2q \cdot \left(\log \log \frac{x}{2Uq} - \log \log \theta \right). \end{aligned}$$

If $2\theta q > x/U$, the integral is over an empty range and its contribution is hence 0.

If $2\theta q \leq V$, (5.24) is

$$\begin{aligned} \frac{\beta}{2} \int_V^{x/U} \frac{\log 2q}{\log W/2q} \frac{dW}{W} &= \frac{\beta \log 2q}{2} \int_{V/2q}^{x/2Uq} \frac{1}{\log t} \frac{dt}{t} \\ &= \frac{\beta \log 2q}{2} \cdot \left(\log \log \frac{x}{2Uq} - \log \log V/2q \right) \\ &= \frac{\beta \log 2q}{2} \cdot \log \left(1 + \frac{\log x/UV}{\log V/2q} \right). \end{aligned}$$

(Of course, $\log(1 + (\log x/UV)/(\log V/2q)) \leq (\log x/UV)/(\log V/2q)$; this is smaller than $(\log x/UV)/\log 2q$ when $V/2q > 2q$.)

The total bound for (5.20) is thus

$$(5.25) \quad \frac{x}{\sqrt{\phi(q)}} \cdot \left(\beta \cdot \left(\frac{1}{2} \log \frac{x}{UW_0} + \frac{\Phi}{2} \right) + \beta^{-1} \left(\frac{1}{4} \kappa_6 \log \frac{x}{UW_0} + \frac{\kappa_7}{2} \right) \right),$$

where

$$(5.26) \quad \Phi = \begin{cases} \log 2q \left(\log \log \frac{x}{2Uq} - \log \log \theta \right) & \text{if } V/2\theta < q < x/(2\theta U). \\ \log 2q \log \left(1 + \frac{\log x/UV}{\log V/2q} \right) & \text{if } q \leq V/2\theta. \end{cases}$$

Choosing β optimally, we obtain that (5.20) is at most

$$(5.27) \quad \frac{x}{\sqrt{2\phi(q)}} \sqrt{\left(\log \frac{x}{UW_0} + \Phi \right) \left(\kappa_6 \log \frac{x}{UW_0} + 2\kappa_7 \right)},$$

where Φ is as in (5.26).

Bounding S_2 for $|\delta| \geq 8$. Let us see how much a non-zero δ can help us. It makes sense to apply (4.55) only when $|\delta| \geq 4$; otherwise (4.53) is almost certainly better. Now, by definition, $|\delta|/x \leq 1/qQ$, and so $|\delta| \geq 8$ can happen only when $q \leq x/8Q$.

With this in mind, let us apply (4.55). Note first that

$$\begin{aligned} \frac{x}{|\delta q|} \left(q + \frac{x}{4W} \right)^{-1} &\geq \frac{1/|\delta q|}{\frac{x}{4} + \frac{1}{4W}} \geq \frac{4/|\delta q|}{\frac{1}{2Q} + \frac{1}{W}} \\ &\geq \frac{4W}{|\delta|q} \cdot \frac{1}{1 + \frac{W}{2Q}} \geq \frac{4W}{|\delta|q} \cdot \frac{1}{1 + \frac{x/U}{2Q}}. \end{aligned}$$

This is at least $2 \min(2Q, W)/|\delta q|$. Thus we may apply (4.55)–(4.56) when $|\delta q| \leq 2 \min(2Q, W)$. Since $Q \geq x/U$, we know that $\min(2Q, W) = W$ for all $W \leq x/U$, and so it is enough to assume that $|\delta q| \leq 2W$.

Recalling also (5.16), we see that (4.55) gives us
(5.28)

$$S_2(U', W', W) \leq \min \left(1, \frac{2q/\phi(q)}{\log \left(\frac{4W}{|\delta|q} \cdot \frac{1}{1 + \frac{x/U}{2Q}} \right)} \right) \left(\frac{x}{|\delta q|} + \frac{W}{2} \right) \cdot \frac{1}{2} W (\log W).$$

Similarly to before, we define $W_0 = \max(V, \theta|\delta q|)$, where $\theta \geq 1$ will be set later. For $W \geq W_0$, we certainly have $|\delta q| \leq 2W$. Hence the part of (5.12) coming from the range $W_0 \leq W < x/U$ is

$$\begin{aligned} &4 \int_{W_0}^{x/U} \sqrt{S_1(U, W) \cdot S_2(U, V, W)} \frac{dW}{W} \\ (5.29) \quad &\leq 4 \sqrt{\frac{q}{\phi(q)}} \int_{W_0}^{x/U} \sqrt{S_1(U, W) \cdot \frac{\log W}{\log \left(\frac{4W}{|\delta|q} \cdot \frac{1}{1 + \frac{x/U}{2Q}} \right)} \left(\frac{Wx}{|\delta q|} + \frac{W^2}{2} \right) \frac{dW}{W}}. \end{aligned}$$

By (4.34), the contribution of the term $Wx/|\delta q|$ to (5.29) is at most

$$\frac{4x}{\sqrt{|\delta|\phi(q)}} \int_{W_0}^{x/U} \sqrt{\left(H_2 \left(\frac{x}{WU} \right) + \frac{\kappa_4}{4} \sqrt{\frac{x/WU}{U}} \right) \frac{\log W}{\log \left(\frac{4W}{|\delta|q} \cdot \frac{1}{1 + \frac{x/U}{2Q}} \right)} \frac{dW}{W}}$$

Note that $1 + (x/U)/2Q \leq 3/2$. Proceeding as in (5.20)–(5.27), we obtain that this is at most

$$\frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\left(\log \frac{x}{UW_0} + \Phi \right) \left(\kappa_6 \log \frac{x}{UW_0} + 2\kappa_7 \right)},$$

where

$$(5.30) \quad \Phi = \begin{cases} \log \frac{(1+\epsilon_1)|\delta q|}{4} \log \left(1 + \frac{\log x/UV}{\log 4V/|\delta|(1+\epsilon_1)q} \right) & \text{if } |\delta q| \leq V/\theta, \\ \log \frac{3|\delta q|}{8} \left(\log \log \frac{8x}{3U|\delta q|} - \log \log \frac{8\theta}{3} \right) & \text{if } V/\theta < |\delta q| \leq x/\theta U, \end{cases}$$

where $\epsilon_1 = x/2UQ$. This is what we think of as the main term.

By (5.15), the contribution of the term $W^2/2$ to (5.29) is at most

$$(5.31) \quad 4 \sqrt{\frac{q}{\phi(q)}} \int_V^{x/U} \sqrt{\frac{\kappa_1}{2} x \frac{dW}{\sqrt{W}}} \cdot \max_{V \leq W \leq \frac{x}{\theta}} \sqrt{\frac{\log W}{\max \left(\log \frac{2W}{|\delta q|}, 2 \right)}}.$$

Since $t \rightarrow (\log t)/(\log t/c)$ is decreasing for $t > c$, (5.31) is at most

$$4\sqrt{2\kappa_1} \sqrt{\frac{q}{\phi(q)}} \frac{x}{\sqrt{U}} \sqrt{\frac{\log W_0}{\max\left(\log \frac{8W_0}{3|\delta q|}, \frac{8}{3}\right)}}.$$

If $W_0 > V$, we also have to consider the range $V \leq W < W_0$. The part of (5.12) coming from this is

$$4 \int_V^{\theta|\delta q|} \sqrt{S_1(U, W) \cdot (\log W) \left(\frac{Wx}{2|\delta q|} + \frac{W^2}{4} + \frac{Wx}{16(1-\rho)Q} + \frac{x}{8(1-\rho)} \right) \frac{dW}{W}}.$$

We have already counted the contribution of $W^2/4$ in the above. The terms $Wx/2|\delta q|$ and $Wx/(16(1-\rho)Q)$ contribute at most

$$\begin{aligned} & 4\sqrt{\kappa_1} \int_V^{\theta|\delta q|} \sqrt{\frac{x}{W} \cdot (\log W) W \left(\frac{x}{2|\delta q|} + \frac{x}{16(1-\rho)Q} \right) \frac{dW}{W}} \\ &= 4\sqrt{\kappa_1} x \left(\frac{1}{\sqrt{2|\delta q|}} + \frac{1}{4\sqrt{(1-\rho)Q}} \right) \int_V^{\theta|\delta q|} \sqrt{\log W} \frac{dW}{W} \\ &\leq \frac{2\kappa_2}{3} x \left(\frac{1}{\sqrt{2|\delta q|}} + \frac{1}{4\sqrt{(1-\rho)Q}} \right) \left((\log \theta|\delta q|)^{3/2} - (\log V)^{3/2} \right). \end{aligned}$$

The term $x/8(1-\rho)$ contributes

$$\begin{aligned} \sqrt{\kappa_1} x \int_V^{\theta|\delta q|} \sqrt{\frac{\log W}{W(1-\rho)}} \frac{dW}{W} &\leq \frac{\sqrt{\kappa_1} x}{\sqrt{1-\rho}} \int_V^\infty \frac{\sqrt{\log W}}{W^{3/2}} dW \\ &\leq \frac{\kappa_2 x}{2\sqrt{(1-\rho)V}} (\sqrt{\log V} + \sqrt{1/\log V}), \end{aligned}$$

where we use the estimate

$$\begin{aligned} \int_V^\infty \frac{\sqrt{\log W}}{W^{3/2}} dW &= \frac{1}{\sqrt{V}} \int_1^\infty \frac{\sqrt{\log u + \log V}}{u^{3/2}} du \\ &\leq \frac{1}{\sqrt{V}} \int_1^\infty \frac{\sqrt{\log V}}{u^{3/2}} du + \frac{1}{\sqrt{V}} \int_1^\infty \frac{1}{2\sqrt{\log V}} \frac{\log u}{u^{3/2}} du \\ &= 2\frac{\sqrt{\log V}}{\sqrt{V}} + \frac{1}{2\sqrt{V}\log V} \cdot 4 \leq \frac{2}{\sqrt{V}} (\sqrt{\log V} + \sqrt{1/\log V}). \end{aligned}$$

It is time to collect all type II terms. Let us start with the case of general δ . We will set $\theta \geq e$ later. If $q \leq V/2\theta$, then $|S_{II}|$ is at most (5.32)

$$\begin{aligned} & \frac{x}{\sqrt{2\phi(q)}} \cdot \sqrt{\left(\log \frac{x}{UV} + \log 2q \log \left(1 + \frac{\log x/UV}{\log V/2q} \right) \right) \left(\kappa_6 \log \frac{x}{UV} + 2\kappa_7 \right)} \\ &+ \sqrt{2}\kappa_2 \sqrt{\frac{q}{\phi(q)}} \left(1 + 1.15 \sqrt{\frac{\log 2q}{\log x/2Uq}} \right) \frac{x}{\sqrt{U}} + \kappa_9 \frac{x}{\sqrt{V}}. \end{aligned}$$

If $V/2\theta < q \leq x/2\theta U$, then $|S_{II}|$ is at most

$$(5.33) \quad \begin{aligned} & \frac{x}{\sqrt{2\phi(q)}} \cdot \sqrt{\left(\log \frac{x}{U \cdot 2\theta q} + \log 2q \log \frac{\log x/2Uq}{\log \theta}\right) \left(\kappa_6 \log \frac{x}{U \cdot 2\theta q} + 2\kappa_7\right)} \\ & + \sqrt{2}\kappa_2 \sqrt{\frac{q}{\phi(q)}} \left(1 + 1.15 \sqrt{\frac{\log 2q}{\log x/2Uq}}\right) \frac{x}{\sqrt{U}} + (\kappa_2 \sqrt{\log 2\theta q} + \kappa_9) \frac{x}{\sqrt{V}} \\ & + \frac{\kappa_2}{6} \left((\log 2\theta q)^{3/2} - (\log V)^{3/2}\right) \frac{x}{\sqrt{q}} \\ & + \kappa_2 \left(\sqrt{2\theta \cdot \log 2\theta q} + \frac{2}{3}((\log 2\theta q)^{3/2} - (\log V)^{3/2})\right) \sqrt{qx}, \end{aligned}$$

where we use the fact that $Q \geq x/U$ (implying that $\rho_0 = \max(1, 2q/Q)$ equals 1 for $q \leq x/2U$). Finally, if $q > x/2\theta U$,

$$(5.34) \quad \begin{aligned} |S_{II}| & \leq (\kappa_2 \sqrt{2 \log x/U} + \kappa_9) \frac{x}{\sqrt{V}} + \kappa_2 \sqrt{\log x/U} \frac{x}{\sqrt{U}} \\ & + \frac{2\kappa_2}{3} ((\log x/U)^{3/2} - (\log V)^{3/2}) \left(\frac{x}{2\sqrt{2q}} + \sqrt{qx}\right). \end{aligned}$$

Now let us examine the alternative bounds for $|\delta| \geq 8$. If $|\delta q| \leq V/\theta$, then $|S_{II}|$ is at most

$$(5.35) \quad \begin{aligned} & \frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\log \frac{x}{UV} + \log \frac{|\delta q|(1+\epsilon_1)}{4} \log \left(1 + \frac{\log x/UV}{\log \frac{4V}{|\delta|(1+\epsilon_1)q}}\right)} \\ & \cdot \sqrt{\kappa_6 \log \frac{x}{UV} + 2\kappa_7} \\ & + \kappa_2 \sqrt{\frac{2q}{\phi(q)}} \cdot \sqrt{\frac{\log V}{\log 2V/|\delta q|}} \cdot \frac{x}{\sqrt{U}} + \kappa_9 \frac{x}{\sqrt{V}}, \end{aligned}$$

where $\epsilon_1 = x/2UQ$. If $|\delta q| > V/\theta$, then $|S_{II}|$ is at most

$$(5.36) \quad \begin{aligned} & \frac{2x}{\sqrt{|\delta|\phi(q)}} \sqrt{\left(\log \frac{x}{U \cdot \theta|\delta q|} + \log \frac{3|\delta q|}{8} \log \frac{\log \frac{8x}{3U|\delta q|}}{\log 8\theta/3}\right) \left(\kappa_6 \log \frac{x}{U \cdot \theta|\delta q|} + 2\kappa_7\right)} \\ & + \frac{2\kappa_2}{3} \left(\frac{x}{\sqrt{2|\delta q|}} + \frac{x}{4\sqrt{Q-q}}\right) \left((\log \theta|\delta q|)^{3/2} - (\log V)^{3/2}\right) \\ & + \left(\frac{\kappa_2}{\sqrt{2(1-\rho)}} \left(\sqrt{\log V} + \sqrt{1/\log V}\right) + \kappa_9\right) \frac{x}{\sqrt{V}} \\ & + \kappa_2 \sqrt{\frac{q}{\phi(q)}} \cdot \sqrt{\log \theta|\delta q|} \cdot \frac{x}{\sqrt{U}}, \end{aligned}$$

where $\rho = q/Q$. (Note that $|\delta| \leq x/Qq$ implies $\rho \leq x/4Q^2$, and so ρ will be very small and $Q - q$ will be very close to Q .)

5.2. Adjusting parameters. Calculations. We must bound the exponential sum $\sum_n \Lambda(n)e(\alpha n)\eta(n/x)$. By (2.20), it is enough to sum the bounds we obtained in §5.1. We will now see how it will be best to set U , V and other parameters.

Usually, the largest terms will be

$$(5.37) \quad C_0 UV,$$

where

$$(5.38) \quad C_0 = \begin{cases} c_{4,I_2} + c_{9,I_2} = 4.39636 & \text{if } |\delta| \leq 1/2c_2 \sim 0.74463, \\ c_{4,I_2} + (1 + \epsilon)c_{13,I_2} = 4.88963 + 1.31541\epsilon & \text{if } |\delta| > 1/2c_2 \end{cases}$$

(from (5.10) and (5.11), type I; $\epsilon \in (0, 1)$ will be set later) and

$$(5.39) \quad \frac{x}{\sqrt{\delta_0 \phi(q)}} \sqrt{\log \frac{x}{UV} + (\log \delta_0 (1 + \epsilon_1) q) \log \left(1 + \frac{\log \frac{x}{UV}}{\log \frac{V}{\delta_0 (1 + \epsilon_1) q}} \right)} \sqrt{\kappa_6 \log \frac{x}{UV} + 2\kappa_7}$$

(from (5.32) and (5.35), type II; here $\delta_0 = \max(2, |\delta|/4)$, while $\epsilon_1 = x/2UQ$ for $|\delta| > 8$ and $\epsilon_1 = 0$ for $|\delta| < 8$.)

We set $UV = \varkappa x / \sqrt{q\delta_0}$; we must choose $\varkappa > 0$.

Let us first optimise \varkappa in the case $|\delta| \leq 4$, so that $\delta_0 = 2$ and $\epsilon_1 = 0$. For the purpose of choosing \varkappa , we replace $\sqrt{\phi(q)}$ by \sqrt{q}/C_1 , where $C_1 = 2.3536 \sim 510510/\phi(510510)$, and also replace V by q^2/c , c a constant. We use the approximation

$$\begin{aligned} \log \left(1 + \frac{\log \frac{x}{UV}}{\log \frac{V}{|2q|}} \right) &= \log \left(1 + \frac{\log(\sqrt{2q}/\varkappa)}{\log(q/2c)} \right) = \log \left(\frac{3}{2} + \frac{\log 2\sqrt{c}/\varkappa}{\log q/2c} \right) \\ &\sim \log \frac{3}{2} + \frac{2 \log 2\sqrt{c}/\varkappa}{3 \log q/2c}. \end{aligned}$$

What we must minimize, then, is

$$(5.40) \quad \begin{aligned} &\frac{C_0 \varkappa}{\sqrt{2q}} + \frac{C_1}{\sqrt{2q}} \sqrt{\left(\log \frac{\sqrt{2q}}{\varkappa} + \log 2q \left(\log \frac{3}{2} + \frac{2 \log \frac{2\sqrt{c}}{\varkappa}}{3 \log \frac{q}{2c}} \right) \right) \left(\kappa_6 \log \frac{\sqrt{2q}}{\varkappa} + 2\kappa_7 \right)} \\ &\leq \frac{C_0 \varkappa}{\sqrt{2q}} + \frac{C_1}{2\sqrt{q}} \frac{\sqrt{\kappa_6}}{\sqrt{\kappa'_1}} \sqrt{\kappa'_1 \log q - \left(\frac{5}{3} + \frac{2 \log 4c}{3 \log \frac{q}{2c}} \right) \log \varkappa + \kappa'_2} \\ &\quad \cdot \sqrt{\kappa'_1 \log q - 2\kappa'_1 \log \varkappa + \frac{4\kappa'_1 \kappa_7}{\kappa_6} + \kappa'_1 \log 2} \\ &\leq \frac{C_0}{\sqrt{2q}} \left(\varkappa + \kappa'_4 \left(\kappa'_1 \log q - \left(\left(\frac{5}{6} + \kappa'_1 \right) + \frac{1 \log 4c}{3 \log \frac{q}{2c}} \right) \log \varkappa + \kappa'_3 \right) \right), \end{aligned}$$

where

$$\begin{aligned} \kappa'_1 &= \frac{1}{2} + \log \frac{3}{2}, \quad \kappa'_2 = \log \sqrt{2} + \log 2 \log \frac{3}{2} + \frac{\log 4c \log 2q}{3 \log q/2c}, \\ \kappa'_3 &= \frac{1}{2} \left(\kappa'_2 + \frac{4\kappa'_1 \kappa_7}{\kappa_6} + \kappa'_1 \log 2 \right) = \frac{\log 4c}{6} + \frac{(\log 4c)^2}{6 \log \frac{q}{2c}} + \kappa'_5, \\ \kappa'_4 &= \frac{C_1}{C_0} \sqrt{\frac{\kappa_6}{2\kappa'_1}} \sim \begin{cases} 0.30925 & \text{if } |\delta| \leq 4 \\ \frac{0.27805}{1+0.26902\epsilon} & \text{if } |\delta| > 4, \end{cases} \\ \kappa'_5 &= \frac{1}{2} (\log \sqrt{2} + \log 2 \log \frac{3}{2} + \frac{4\kappa'_1 \kappa_7}{\kappa_6} + \kappa'_1 \log 2) \sim 1.01152. \end{aligned}$$

Taking derivatives, we see that the minimum is attained when

$$(5.41) \quad \varkappa = \left(\frac{5}{6} + \kappa'_1 + \frac{1 \log 4c}{3 \log \frac{q}{2c}} \right) \kappa'_4 \sim \left(1.7388 + \frac{\log 4c}{3 \log \frac{q}{2c}} \right) \cdot 0.30925$$

provided that $|\delta| \leq 4$. (What we obtain for $|\delta| > 4$ is essentially the same, only with $\log \delta_0 q = \log |\delta|q/4$ instead of $\log q$, and $0.27805/(1 + 0.26902\epsilon)$ in place of 0.30925 .) For $q = 5 \cdot 10^5$, $c = 2.5$ and $|\delta| \leq 4$ (typical values in the most delicate range), we get that \varkappa should be $0.55834\dots$, and the last line of (5.40) is then $0.02204\dots$; for $q = 10^6$, $c = 10$, $|\delta| \leq 4$, we get that \varkappa should be $0.57286\dots$, and the last line of (5.40) is then $0.01656\dots$. If $|\delta| > 4$, $|\delta|q = 5 \cdot 10^5$, $c = 2.5$ and $\epsilon = 0.2$ (say), then $\varkappa = 0.47637\dots$, and the last line of (5.40) is $0.02243\dots$; if $|\delta| > 4$, $|\delta|q = 10^6$, $c = 10$ and $\epsilon = 0.2$, then $\varkappa = 0.48877\dots$, and the last line of (5.40) is $0.01684\dots$.

(A back-of-the-envelope calculation suggests that choosing $w = 1$ instead of $w = 2$ would have given bounds worse by about 15 percent.)

We make the choices

$$\varkappa = 1/2, \quad \text{and so} \quad UV = \frac{1}{2\sqrt{q\delta_0}}$$

for the sake of simplicity. (Unsurprisingly, (5.40) changes very slowly around its minimum.)

Now we must decide how to choose U , V and Q , given our choice of UV . We will actually make two sets of choices. First, we will use the $S_{I,2}$ estimates for $q \leq Q/V$ to treat all α of the form $\alpha = a/q + O^*(1/qQ)$, $q \leq y$. (Here y is a parameter satisfying $y \leq Q/V$.) The remaining α then get treated with the (coarser) $S_{I,2}$ estimate for $q > Q/V$, with Q reset to a lower value (call it Q'). If α was not treated in the first go (so that it must be dealt with the coarser estimate) then $\alpha = a'/q' + \delta'/x$, where either $q' > y$ or $\delta'q' > x/Q$. (Otherwise, $\alpha = a'/q' + O^*(1/q'Q)$ would be a valid estimate with $q' \leq y$.)

The value of Q' is set to be smaller than Q both because this is helpful (it diminishes error terms that would be large for large q) and because this is now harmless (since we are no longer assuming that $q \leq Q/V$).

5.2.1. *First choice of parameters: $q \leq y$.* The largest items affected strongly by our choices at this point are

$$(5.42) \quad \begin{aligned} c_{16,I_2} \left(2 + \frac{1+\epsilon}{\epsilon} \log^+ \frac{2UV|\delta|q}{x} \right) \frac{x}{Q/V} + c_{17,I_2} Q & \quad (\text{from } S_{I,2}, |\delta| > 1/2c_2), \\ \left(c_{10,I_2} \log \frac{U}{q} + 2c_{5,I_2} + c_{12,I_2} \right) Q & \quad (\text{from } S_{I,2}, |\delta| \leq 1/2c_2), \end{aligned}$$

and

$$(5.43) \quad \kappa_2 \sqrt{\frac{2q}{\phi(q)}} \left(1 + 1.15 \sqrt{\frac{\log 2q}{\log x/2Uq}} \right) \frac{x}{\sqrt{U}} + \kappa_9 \frac{x}{\sqrt{V}} \quad (\text{from } S_{II}).$$

In addition, we have a relatively mild but important dependence on V in the main term (5.39). We must also respect the condition $q \leq Q/V$, the lower bound on U given by (5.14) and the assumptions made at the beginning of section 5 (e.g. $Q \geq x/U$, $V \geq 2 \cdot 10^6$). Recall that $UV = x/\sqrt{q\delta}$.

We set

$$Q = \frac{x}{8y},$$

since we will then have not just $q \leq y$ but also $q|\delta| \leq x/Q = 8y$, and so $q\delta_0 \leq 4y$. We want $q \leq Q/V$ to be true whenever $q \leq y$; this means that

$$q \leq \frac{Q}{V} = \frac{QU}{UV} = \frac{QU}{x/2\sqrt{q\delta_0}} = \frac{U\sqrt{q\delta_0}}{4y}$$

must be true when $q \leq y$, and so it is enough to set $U = 4y^2/\sqrt{q\delta_0}$. The following choices make sense: we will work with the parameters

$$(5.44) \quad \begin{aligned} y &= \frac{x^{1/3}}{6}, & Q &= \frac{x}{8y} = \frac{3}{4}x^{2/3}, & x/UV &= 2\sqrt{q\delta_0} \leq 2\sqrt{2y}, \\ U &= \frac{4y^2}{\sqrt{q\delta_0}} = \frac{x^{2/3}}{9\sqrt{q\delta_0}}, & V &= \frac{x}{(x/UV) \cdot U} = \frac{x}{8y^2} = \frac{9x^{1/3}}{2}, \end{aligned}$$

where, as before, $\delta_0 = \max(2, |\delta|/4)$. Thus $\epsilon_1 \leq x/2UQ \leq 2\sqrt{6}/x^{1/6}$. Assuming

$$(5.45) \quad x \geq 2.16 \cdot 10^{20},$$

we obtain that $U/(x/UV) \geq (x^{3/2}/9\sqrt{q\delta_0})/(2\sqrt{q\delta_0}) = x^{2/3}/18q\delta_0 \geq x^{1/3}/6 \geq 5 \cdot 10^5$, and so (5.14) holds. We also get that $\epsilon_1 \leq 0.002$.

Since $V = x/8y^2 = (9/2)x^{1/3}$, (5.45) also implies that $V \geq 2 \cdot 10^6$ (in fact, $V \geq 27 \cdot 10^6$). It is easy to check that

$$(5.46) \quad V < x/4, \quad UV \leq x, \quad Q \geq \sqrt{ex}, \quad Q \geq \max(U, x/U),$$

as stated at the beginning of section 5. Let $\theta = (3/2)^3 = 27/8$. Then

$$(5.47) \quad \begin{aligned} \frac{V}{2\theta q} &= \frac{x/8y^2}{2\theta q} \geq \frac{x}{16\theta y^3} = \frac{x}{54y^3} = 4 > 1, \\ \frac{V}{\theta|\delta q|} &= \frac{x/8y^2}{8\theta y} \geq \frac{x}{64\theta y^3} = \frac{x}{216y^3} = 1. \end{aligned}$$

The first type I bound is

$$(5.48) \quad \begin{aligned} |S_{I,1}| &\leq \frac{x}{q} \min\left(1, \frac{c'_0}{\delta^2}\right) \left(\min\left(\frac{\frac{4}{5}\frac{q}{\phi(q)}}{\log^+ \frac{x^{2/3}}{9q^{\frac{5}{2}}\delta_0^{\frac{1}{2}}}}, 1\right) \left(\log 9x^{\frac{1}{3}}\sqrt{q\delta_0} + c_{3,I}\right) + \frac{c_{4,I}q}{\phi(q)} \right) \\ &\quad + \left(c_{7,I} \log \frac{y}{c_2} + c_{8,I} \log x\right) y + \frac{c_{10,I}x^{1/3}}{3^4 2^2 q^{3/2} \delta_0^{\frac{1}{2}}} (\log 9x^{1/3} \sqrt{eq\delta_0}) \\ &\quad + \left(c_{5,I} \log \frac{2x^{2/3}}{9c_2\sqrt{q\delta_0}} + c_{6,I} \log \frac{x^{5/3}}{9\sqrt{q\delta_0}}\right) \frac{x^{2/3}}{9\sqrt{q\delta_0}} + c_{9,I} \sqrt{x} \log \frac{2x}{c_2} + \frac{c_{10,I}}{e}, \end{aligned}$$

where the constants are as in §5.1.1. The function $x \rightarrow (\log cx)/(\log x/R)$, $c, R \geq 1$, attains its maximum on $[R', \infty]$, $R' > R$, at $x = R'$. Hence, for $q\delta_0$ fixed,

$$(5.49) \quad \min\left(\frac{4/5}{\log^+ \frac{4x^{2/3}}{9(\delta_0 q)^{\frac{5}{2}}}}, 1\right) \left(\log 9x^{\frac{1}{3}}\sqrt{q\delta_0} + c_{3,I}\right)$$

attains its maximum at $x = (27/8)e^{6/5}(q\delta_0)^{15/4}$, and so

$$\begin{aligned} & \min \left(\frac{4/5}{\log^+ \frac{4x^{2/3}}{9(\delta_0 q)^{5/2}}}, 1 \right) \left(\log 9x^{1/3} \sqrt{q\delta_0} + c_{3,I} \right) + c_{4,I} \\ & \leq \log \frac{27}{2} e^{2/5} (\delta_0 q)^{7/4} + c_{3,I} + c_{4,I} \leq \frac{7}{4} \log \delta_0 q + 6.11676. \end{aligned}$$

Examining the other terms in (5.48) and using (5.45), we conclude that

$$\begin{aligned} (5.50) \quad |S_{I,1}| & \leq \frac{x}{q} \min \left(1, \frac{c'_0}{\delta^2} \right) \cdot \min \left(\frac{q}{\phi(q)} \left(\frac{7}{4} \log \delta_0 q + 6.11676 \right), \frac{1}{2} \log x + 5.65787 \right) \\ & \quad + \frac{x^{2/3}}{\sqrt{q\delta_0}} (0.67845 \log x - 1.20818) + 0.0507x^{2/3}, \end{aligned}$$

where we are using (5.45) to simplify the smaller error terms. (The bound $(1/2) \log x + 5.65787$ comes from a trivial bound on (5.49).) We recall that $c'_0 = 0.798437 > c_0/(2\pi)^2$.

Let us now consider $S_{I,2}$. The terms that appear both for $|\delta|$ small and $|\delta|$ large are given in (5.9). The second line in (5.9) equals

$$\begin{aligned} c_{8,I_2} & \left(\frac{x}{4q^2\delta_0} + \frac{2UV^2}{x} + \frac{qV^2}{x} \right) + \frac{c_{10,I_2}}{2} \left(\frac{q}{2\sqrt{q\delta_0}} + \frac{x^{2/3}}{18q\delta_0} \right) \log \frac{9x^{1/3}}{2} \\ & \leq c_{8,I_2} \left(\frac{x}{4q^2\delta_0} + \frac{9x^{1/3}}{2\sqrt{2}} + \frac{27}{8} \right) + \frac{c_{10,I_2}}{2} \left(\frac{y^{1/6}}{2^{3/2}} + \frac{x^{2/3}}{18q\delta_0} \right) \left(\frac{1}{3} \log x + \log \frac{9}{2} \right) \\ & \leq 0.29315 \frac{x}{q^2\delta_0} + (0.00828 \log x + 0.03735) \frac{x^{2/3}}{\sqrt{q\delta_0}} + 0.00153\sqrt{x}, \end{aligned}$$

where we are using (5.45) to simplify. Now

$$(5.51) \quad \min \left(\frac{4/5}{\log^+ \frac{Q}{4Vq^2}}, 1 \right) \log Vq = \min \left(\frac{4/5}{\log^+ \frac{y}{4q^2}}, 1 \right) \log \frac{9x^{1/3}q}{2}$$

can be bounded trivially by $\log(9x^{1/3}q/2) \leq (2/3) \log x + \log 3/4$. We can also bound (5.51) as we bounded (5.49) before, namely, by fixing q and finding the maximum for x variable. In this way, we obtain that (5.51) is maximal for $y = 4e^{4/5}q^2$; since, by definition, $x^{1/3}/6 = y$, (5.51) then equals

$$\log \frac{9(6 \cdot 4e^{4/5}q^2)q}{2} = 3 \log q + \log 108 + \frac{4}{5} \leq 3 \log q + 5.48214.$$

If $|\delta| \leq 1/2c_2$, we must consider (5.10). This is at most

$$\begin{aligned} & (c_{4,I_2} + c_{9,I_2}) \frac{x}{2\sqrt{q\delta_0}} + (c_{10,I_2} \log \frac{x^{2/3}}{9q^{3/2}\sqrt{\delta_0}} + 2c_{5,I_2} + c_{12,I_2}) \cdot \frac{3}{4} x^{2/3} \\ & \leq \frac{2.19818x}{\sqrt{q\delta_0}} + (0.89392 \log x + 23.0896)x^{2/3}. \end{aligned}$$

If $|\delta| > 1/2c_2$, we must consider (5.11) instead. For $\epsilon = 0.07$, that is at most

$$\begin{aligned} & (c_{4,I_2} + (1 + \epsilon)c_{13,I_2}) \frac{x}{2\sqrt{q\delta_0}} + (3.30386 \log \delta q^3 + 16.4137) \frac{x}{|\delta|q} \\ & + (68.8137 \log |\delta|q + 36.7795)x^{2/3} + 29.7467x^{1/3} \\ & = 2.49086 \frac{x}{\sqrt{q\delta_0}} + (3.30386 \log \delta q^3 + 16.4137) \frac{x}{|\delta|q} + (22.9379 \log x + 56.576)x^{2/3}. \end{aligned}$$

Hence

$$\begin{aligned} (5.52) \quad |S_{I,2}| & \leq 2.49086 \frac{x}{\sqrt{q\delta_0}} \\ & + x \cdot \min \left(1, \frac{4c'_0}{\delta^2} \right) \min \left(\frac{\frac{3}{2} \log q + 2.74107}{\phi(q)}, \frac{\frac{1}{3} \log x + \frac{1}{2} \log \frac{3}{4}}{q} \right) \\ & + 0.29315 \frac{x}{q^2\delta_0} + (22.9462 \log x + 56.6134)x^{2/3} \end{aligned}$$

plus a term $(3.30386 \log \delta q^2 + 16.4137) \cdot (x/|\delta|q)$ that appears if and only if $|\delta| \geq 1/2c_2$.

For type II, we have to consider two cases: (a) $|\delta| < 8$, and (b) $|\delta| \geq 8$. Consider first $|\delta| < 8$. Then $\delta_0 = 2$. Recall that $\theta = 27/8$. We have $q \leq V/2\theta$ and $|\delta q| \leq V/\theta$ thanks to (5.47). We apply (5.32), and obtain that, for $|\delta| < 8$,

$$\begin{aligned} (5.53) \quad |S_{II}| & \leq \frac{x}{\sqrt{2\phi(q)}} \cdot \sqrt{\frac{1}{2} \log 4q\delta_0 + \log 2q \log \left(1 + \frac{\frac{1}{2} \log 4q\delta_0}{\log \frac{V}{2q}} \right)} \\ & \cdot \sqrt{0.30214 \log 4q\delta_0 + 0.2562} \\ & + 8.22088 \sqrt{\frac{q}{\phi(q)}} \left(1 + 1.15 \sqrt{\frac{\log 2q}{\log \frac{9x^{1/3}\sqrt{\delta_0}}{2\sqrt{q}}}} \right) (q\delta_0)^{1/4} x^{2/3} + 1.84251x^{5/6} \\ & \leq \frac{x}{\sqrt{2\phi(q)}} \cdot \sqrt{C_{x,2q} \log 2q + \frac{\log q}{2}} \cdot \sqrt{0.30214 \log 2q + 0.67506} \\ & + 16.404 \sqrt{\frac{q}{\phi(q)}} x^{3/4} + 1.84251x^{5/6} \end{aligned}$$

where we define

$$C_{x,t} := \log \left(1 + \frac{\log 4t}{2 \log \frac{9x^{1/3}}{2.004t}} \right)$$

for $0 < t < 9x^{1/3}/2$. (We have 2.004 here instead of 2 because we want a constant $\geq 2(1 + \epsilon_1)$ in later occurrences of $C_{x,t}$, for reasons that will soon become clear.)

For purposes of later comparison, we remark that $16.404 \leq 1.5785x^{3/4-4/5}$ for $x \geq 2.16 \cdot 10^{20}$.

Consider now case (b), namely, $|\delta| \geq 8$. Then $\delta_0 = |\delta|/4$. By (5.47), $|\delta q| \leq V/\theta$. Hence, (5.35) gives us that

$$\begin{aligned}
(5.54) \quad |S_{II}| &\leq \frac{2x}{\sqrt{|\delta|\phi(q)}} \cdot \sqrt{\frac{1}{2} \log |\delta q| + \log \frac{|\delta q|(1+\epsilon_1)}{4} \log \left(1 + \frac{\log |\delta| q}{2 \log \frac{18x^{1/3}}{|\delta|(1+\epsilon_1)q}}\right)} \\
&\quad \cdot \sqrt{0.30214 \log |\delta| q + 0.2562} \\
&\quad + 8.22088 \sqrt{\frac{q}{\phi(q)}} \sqrt{\frac{\log \frac{9x^{1/3}}{2}}{\log \frac{12x^{1/3}}{|\delta q|}}} \cdot (q\delta_0)^{1/4} x^{2/3} + 1.84251x^{5/6} \\
&\leq \frac{x}{\sqrt{\delta_0\phi(q)}} \sqrt{C_{x,\delta_0q} \log \delta_0(1+\epsilon_1)q + \frac{\log 4\delta_0q}{2}} \sqrt{0.30214 \log \delta_0q + 0.67506} \\
&\quad + 1.68038 \sqrt{\frac{q}{\phi(q)}} x^{4/5} + 1.84251x^{5/6},
\end{aligned}$$

since

$$8.22088 \sqrt{\frac{\log \frac{9x^{1/3}}{2}}{\log \frac{12x^{1/3}}{|\delta q|}}} \cdot (q\delta_0)^{1/4} \leq 8.22088 \sqrt{\frac{\log \frac{9x^{1/3}}{2}}{\log 9}} \cdot (x^{1/3}/3)^{1/4} \leq 1.68038x^{4/5-2/3}$$

for $x \geq 2.16 \cdot 10^{20}$. Clearly

$$\log \delta_0(1+\epsilon_1)q \leq \log \delta_0q + \log(1+\epsilon_1) \leq \log \delta_0q + \epsilon_1.$$

Now note the fact ([RS62, Thm. 15]) that $q/\phi(q) < F(q)$, where

$$(5.55) \quad F(q) = e^\gamma \log \log q + \frac{2.50637}{\log \log q}.$$

Moreover, $q/\phi(q) \leq 3$ for $q < 30$. Since $F(30) > 3$ and $F(t)$ is increasing for $t \geq 30$, we conclude that, for any q and for any $r \geq \max(q, 30)$, $q/\phi(q) < F(r)$. In particular, $q/\phi(q) \leq F(y) = F(x^{1/3}/6)$ (since, by (5.45), $x \geq 180^3$). It is easy to check that $x \rightarrow \sqrt{F(x^{1/3}/6)}x^{4/5-5/6}$ is decreasing for $x \geq 180^3$. Using (5.45), we conclude that $1.67718\sqrt{q/\phi(q)}x^{4/5} \leq 0.83574x^{5/6}$. This allows us to simplify the last lines of (5.53) and (5.54).

It is time to sum up $S_{I,1}$, $S_{I,2}$ and S_{II} . The main terms come from the first lines of (5.53) and (5.54) and the first term of (5.52). Lesser-order terms can be dealt with roughly: we bound $\min(1, c'_0/\delta^2)$ and $\min(1, 4c'_0/\delta^2)$ from above by $2/\delta_0$ (somewhat brutally) and $1/q^2\delta_0$ by $1/q\delta_0$ (again, coarsely). For $|\delta| \geq 1/2c_2$,

$$\frac{1}{|\delta|} \leq \frac{4c_2}{\delta_0}, \quad \frac{\log |\delta|}{|\delta|} \leq \frac{2}{e \log 2} \cdot \frac{\log \delta_0}{\delta_0};$$

we use this to bound the term in the comment after (5.52). The terms inversely proportional to q , $\phi(q)$ or q^2 thus add up to at most

$$\begin{aligned}
& \frac{2x}{\delta_0} \cdot \min \left(\frac{\frac{7}{4} \log \delta_0 q + 6.11676}{\phi(q)}, \frac{\frac{1}{2} \log x + 5.65787}{q} \right) \\
& + \frac{2x}{\delta_0} \cdot \min \left(\frac{\frac{3}{2} \log q + 2.74107}{\phi(q)}, \frac{\frac{1}{3} \log x + \frac{1}{2} \log \frac{3}{4}}{q} \right) \\
& + 0.29315 \frac{x}{q\delta_0} + \frac{4c_2 x}{q\delta_0} (3.30386 \log q^2 + 16.4137) + \frac{2x}{(e \log 2)q\delta_0} \cdot 3.30386 \log \delta_0 \\
& \leq \frac{2x}{\delta_0} \min \left(\frac{\log \delta_0^{7/4} q^{13/4} + 8.858}{\phi(q)}, \frac{\frac{5}{6} \log x + 5.515}{q} \right) \\
& + \frac{2x}{\delta_0 q} (8.874 \log q + 1.7535 \log \delta_0 + 22.19) \\
& \leq \frac{2x}{\delta_0} \left(\min \left(\frac{\log \delta_0^{7/4} q^{13/4} + 8.858}{\phi(q)}, \frac{\frac{5}{6} \log x + 5.515}{q} \right) + \frac{\log q^{\frac{80}{9}} \delta_0^{\frac{16}{9}} + 22.19}{q} \right).
\end{aligned}$$

As for the other terms – we use (5.45) to bound $x^{2/3}$ and $x^{2/3} \log x$ by a small constant times $x^{5/6}$. We bound $x^{2/3}/\sqrt{q\delta_0}$ by $x^{2/3}/\sqrt{2}$ (in (5.50)).

The sums $S_{0,\infty}$ and $S_{0,w}$ in (2.23) are 0 (by (5.45)). We conclude that, for $q \leq y = x^{1/3}/6$, $x \geq 2.16 \cdot 10^{20}$ and $\eta = \eta_2$ as in (1.4),

$$\begin{aligned}
(5.56) \quad & |S_\eta(x, \alpha)| \leq |S_{I,1}| + |S_{I,2}| + |S_{II}| \\
& \leq \frac{x}{\sqrt{\phi(q)\delta_0}} \sqrt{C_{x,\delta_0 q} (\log \delta_0 q + 0.002) + \frac{\log 4\delta_0 q}{2} \sqrt{0.30214 \log \delta_0 q + 0.67506}} \\
& + \frac{2.49086x}{\sqrt{q\delta_0}} + \frac{2x}{\delta_0} \min \left(\frac{\log \delta_0^{7/4} q^{13/4} + \frac{80}{9}}{\phi(q)}, \frac{\frac{5}{6} \log x + \frac{50}{9}}{q} \right) + \frac{2x \log q^{\frac{80}{9}} \delta_0^{\frac{16}{9}} + \frac{111}{5}}{\delta_0 q} \\
& + 3.14624x^{5/6},
\end{aligned}$$

where

$$(5.57) \quad \delta_0 = \max(2, |\delta|/4), \quad C_{x,t} = \log \left(1 + \frac{\log 4t}{2 \log \frac{9x^{1/3}}{2.004t}} \right).$$

Since $C_{x,t}$ is an increasing function as a function of t (for x fixed and $t \leq 9x^{1/3}/2.004$) and $\delta_0 q \leq 2y$, we see that $C_{x,t} \leq C_{x,2y}$. It is clear that $x \mapsto C_{x,t}$ (fixed t) is decreasing function of x . For $x = 2.16 \cdot 10^{20}$, $C_{x,2y} = 1.39942 \dots$. Also, compare the value $C_{3.1 \cdot 10^{28}, 2 \cdot 10^6} = 0.64020 \dots$ given by (5.57) to the value of $1.196 \dots - 0.5 = 0.696 \dots$ for $C_{3.1 \cdot 10^{28}, 2 \cdot 10^6}$ in a previous version [Helb] of the present paper. (The largest gains are elsewhere.)

5.2.2. Second choice of parameters. If, with the original choice of parameters, we obtained $q > y = x^{1/3}/6$, we now reset our parameters (Q , U and V). Recall that, while the value of q may now change (due to the change in Q), we will be able to assume that either $q > y$ or $|\delta q| > x/(x/8y) = 8y$.

We want $U/(x/UV) \geq 5 \cdot 10^5$ (this is (5.14)). We also want UV small. With this in mind, we let

$$V = \frac{x^{1/3}}{3}, \quad U = 500\sqrt{6}x^{1/3}, \quad Q = \frac{x}{U} = \frac{x^{2/3}}{500\sqrt{6}}.$$

Then (5.14) holds (as an equality). Since we are assuming (5.45), we have $V \geq 2 \cdot 10^6$. It is easy to check that (5.45) also implies that $U < \sqrt{x}$ and $Q > \sqrt{ex}$, and so the inequalities in (5.46) hold.

Write $2\alpha = a/q + \delta/x$ for the new approximation; we must have either $q > y$ or $|\delta| > 8y/q$, since otherwise a/q would already be a valid approximation under the first choice of parameters. Thus, either (a) $q > y$, or both (b1) $|\delta| > 8$ and (b2) $|\delta|q > 8y$. Since now $V = 2y$, we have $q > V/2\theta$ in case (a) and $|\delta q| > V/\theta$ in case (b) for any $\theta \geq 1$. We set $\theta = e^2$.

By (5.2),

$$\begin{aligned} |S_{I,1}| &\leq \frac{x}{q} \min\left(1, \frac{c'_0}{\delta^2}\right) \left(\log x^{2/3} - \log 500\sqrt{6} + c_{3,I} + c_{4,I} \frac{q}{\phi(q)}\right) \\ &\quad + \left(c_{7,I} \log \frac{Q}{c_2} + c_{8,I} \log x \log c_{11,I} \frac{Q^2}{x}\right) Q + c_{10,I} \frac{U^2}{4x} \log \frac{e^{1/2} x^{2/3}}{500\sqrt{6}} + \frac{c_{10,I}}{e} \\ &\quad + \left(c_{5,I} \log \frac{1000\sqrt{6}x^{1/3}}{c_2} + c_{6,I} \log 500\sqrt{6}x^{4/3}\right) \cdot 500\sqrt{6}x^{1/3} + c_{9,I} \sqrt{x} \log \frac{2x}{c_2} \\ &\leq \frac{x}{q} \min\left(1, \frac{c'_0}{\delta^2}\right) \left(\frac{2}{3} \log x - 4.99944 + 1.00303 \frac{q}{\phi(q)}\right) + \frac{1.063}{10000} x^{2/3} (\log x)^2, \end{aligned}$$

where we are bound $\log c_{11,I} Q^2/x$ by $\log x^{1/3}$. Just as before, we use the assumption (5.45) when we have to bound a lower-order term (such as $x^{1/2} \log x$) by a multiple of a higher-order term (such as $x^{2/3} (\log x)^2$).

We have $q/\phi(q) \leq F(Q)$ (where F is as in (5.55)) and we can check that

$$1.00303F(Q) \leq 0.0327 \log x + 4.99944$$

for all $x \geq 10^6$. We have either $q > y$ or $q|\delta| > 8y$; if $q|\delta| > 8y$ but $q \leq y$, then $|\delta| \geq 8$, and so $c'_0/\delta^2 q < 1/8|\delta|q < 1/64y < 1/y$. Hence

$$\begin{aligned} |S_{I,1}| &\leq 4.1962x^{2/3} \log x + 0.090843x^{2/3} + 0.001063x^{2/3} (\log x)^2 \\ &\leq 4.1982x^{2/3} \log x + 0.001063x^{2/3} (\log x)^2. \end{aligned}$$

We bound $|S_{I,2}|$ using Lemma 3.7. First we bound (3.49): this is at most

$$\begin{aligned} &\frac{x}{2q} \min\left(1, \frac{4c'_0}{\delta^2}\right) \log \frac{x^{1/3} q}{3} \\ &+ c_0 \left(\frac{1}{4} - \frac{1}{\pi^2}\right) \left(\frac{(UV)^2 \log \frac{x^{1/3}}{3}}{2x} + \frac{3c_4}{2} \frac{500\sqrt{6}}{9} + \frac{(500\sqrt{6}x^{1/3} + 1)^2 x^{1/3}}{3x}\right), \end{aligned}$$

where $c_4 = 1.03884$. We bound the second line of this using (5.45). As for the first line, we have either $q \geq y$ (and so the first line is at most $(x/2y)(\log x^{1/3} y/3)$) or $q < y$ and $4c'_0/\delta^2 q < 1/16y < 1/y$ (and so the same bound applies). Hence (3.49) is at most

$$\frac{3}{2} x^{2/3} \left(\frac{2}{3} \log x - \log 9\right) + 0.02017 x^{2/3} \log x.$$

Now we bound (3.50), which comes up when $|\delta| \leq 1/2c_2$, where $c_2 = 6\pi/5\sqrt{c_0}$, $c_0 = 31.521$ (and so $c_2 = 0.6714769\dots$). Since $1/2c_2 < 8$, it follows that $q > y$ (the alternative $q \leq y$, $|\delta q| > 2y$ is impossible). Then (3.50) is at most

$$(5.58) \quad \begin{aligned} & \frac{2\sqrt{c_0c_1}}{\pi} \left(UV \log \frac{UV}{\sqrt{e}} + Q \left(\sqrt{3} \log \frac{c_2x}{Q} + \frac{\log UV}{2} \log \frac{UV}{Q/2} \right) \right) \\ & + \frac{3c_1}{2} \frac{x}{y} \log UV \log \frac{UV}{c_2x/y} + \frac{16 \log 2}{\pi} Q \log \frac{c_0e^3Q^2}{4\pi \cdot 8 \log 2 \cdot x} \log \frac{Q}{2} \\ & + \frac{3c_1}{2\sqrt{2c_2}} \sqrt{x} \log \frac{c_2x}{2} + \frac{25c_0}{4\pi^2} (3c_2)^{1/2} \sqrt{x} \log x, \end{aligned}$$

where $c_1 = 1.0000028 > 1 + (8 \log 2)/V$. Here $\log(c_0e^3Q^2/(4\pi \cdot 8 \log 2 \cdot x)) \log Q/2$ is at most $\log x^{1/3} \log x^{2/3}$. Using this and (5.45), we get that (5.58) is at most

$$\begin{aligned} & 1177.617x^{2/3} \log x + 0.0006406x^{2/3}(\log x)^2 + 29.5949x^{1/2} \log x \\ & \leq 1177.64x^{2/3} \log x + 0.0006406x^{2/3}(\log x)^2. \end{aligned}$$

If $|\delta| > 1/2c_2$, then we know that $|\delta q| > \max(y/2c_2, 2y) = y/2c_2$. Thus (3.51) (with $\epsilon = 0.01$) is at most

$$\begin{aligned} & \frac{2\sqrt{c_0c_1}}{\pi} UV \log \frac{UV}{\sqrt{e}} \\ & + \frac{2.02\sqrt{c_0c_1}}{\pi} \left(\frac{x}{y/2c_2} + 1 \right) \left((\sqrt{3.02} - 1) \log \frac{\frac{x}{y/2c_2} + 1}{\sqrt{2}} + \frac{1}{2} \log UV \log \frac{e^2UV}{\frac{x}{y/2c_2}} \right) \\ & + \left(\frac{3c_1}{2} \left(\frac{1}{2} + \frac{3.03}{0.16} \log x \right) + \frac{20c_0}{3\pi^2} (2c_2)^{3/2} \right) \sqrt{x} \log x. \end{aligned}$$

Again by (5.45), this simplifies to

$$\leq 1212.591x^{2/3} \log x + 29.131x^{1/2} \log x \leq 1213.15x^{2/3}(\log x)^2.$$

Hence, in total and for any $|\delta|$,

$$|S_{I,2}| \leq 1213.15x^{2/3}(\log x) + 0.0006406x^{2/3}(\log x)^2.$$

Now we must estimate S_{II} . As we said before, either (a) $q > y/4$, or both (b1) $|\delta| > 8$ and (b2) $|\delta q| > 8y$. Recall that $\theta = e^2$. In case (a), we use (5.33), and obtain that, if $y/4 < q \leq x/2e^2U$, $|S_{II}|$ is at most

$$(5.59) \quad \begin{aligned} & \frac{x\sqrt{F(q)}}{\sqrt{2q}} \sqrt{\left(\log \frac{x}{U \cdot 2e^2q} + \log 2q \log \frac{\log x/(2Uq)}{\log e^2} \right) \left(\kappa_6 \log \frac{x}{U \cdot 2e^2q} + 2\kappa_7 \right)} \\ & + \sqrt{2}\kappa_2 \sqrt{F\left(\frac{x}{2e^2U}\right)} \left(1 + 1.15\sqrt{\frac{\log x/e^2U}{2}} \right) \frac{x}{\sqrt{U}} + (\kappa_2 \sqrt{\log x/U} + \kappa_9) \frac{x}{\sqrt{V}} \\ & + \frac{\kappa_2}{6} \left((\log(e^2y/2))^{3/2} - (\log y)^{3/2} \right) \frac{x}{\sqrt{y}} \\ & + \kappa_2 \left(\sqrt{2e^2 \cdot \log x/U} + \frac{2}{3}((\log x/U)^{3/2} - (\log V)^{3/2}) \right) \frac{x}{\sqrt{2e^2U}}, \end{aligned}$$

where F is as in (5.55). It is easy to check that $q \rightarrow (\log 2q)(\log \log q)/q$ is decreasing for $q \geq y$ (indeed for $q \geq 9$), and so the first line of (5.59) is minimal for $q = y$. Asymptotically, the largest term in (5.59) comes from the last line (of order $x^{5/6}(\log x)^{3/2}$), even if the first line is larger in practice (while being of order

$x^{5/6}(\log x) \log \log x$). The ratio of (5.59) (for $q = y = x^{1/3}/6$) to $x^{5/6}(\log x)^{3/2}$ is descending for $x \geq x_0 = 2.16 \cdot 10^{20}$; its value at $x = x_0$ gives

$$(5.60) \quad |S_{II}| \leq 0.272652x^{5/6}(\log x)^{3/2}$$

in case (a), for $q \leq x/2e^2U$.

If $x/2e^2U < q \leq Q$, we use (5.34). In this range, $x/2\sqrt{2q} + \sqrt{qx}$ adopts its maximum at $q = Q$ (because $x/2\sqrt{2q}$ for $q = x/2e^2U$ is smaller than \sqrt{qx} for $q = Q$, by (5.45)). A brief calculation starting from (5.34) then gives that

$$|S_{II}| \leq 0.10198x^{5/6}(\log x)^{3/2},$$

where we use (5.45) yet again to simplify.

Finally, let us treat case (b), that is, $|\delta| > 8$ and $|\delta|q > 8y$; we can also assume $q \leq y$, as otherwise we are in case (a), which has already been treated. Since $|\delta/x| \leq x/Q$, we know that $|\delta q| \leq x/Q = U$. From (5.36), we obtain that $|S_{II}|$ is at most

$$\begin{aligned} & \frac{2x\sqrt{F(y)}}{\sqrt{8y}} \sqrt{\left(\log \frac{x}{U \cdot e^2 \cdot 8y} + \log 3y \log \frac{\log x/3Uy}{\log 8e^2/3}\right) \left(\kappa_6 \log \frac{x}{U \cdot e^2 \cdot 2y} + 2\kappa_7\right)} \\ & + \frac{2\kappa_2}{3} \left(\frac{x}{\sqrt{16y}}((\log 8e^2y)^{3/2} - (\log y)^{3/2}) + \frac{x/4}{\sqrt{Q-y}}((\log e^2U)^{3/2} - (\log y)^{3/2})\right) \\ & + \left(\frac{\kappa_2}{\sqrt{2(1-y/Q)}}(\sqrt{\log V} + \sqrt{1/\log V}) + \kappa_9\right) \frac{x}{\sqrt{V}} \\ & + \kappa_2 \sqrt{2F(y)} \cdot \sqrt{\frac{\log e^2U}{\log 8e^2/3}} \cdot \frac{x}{\sqrt{U}}, \end{aligned}$$

We take the maximum of the ratio of this to $x^{5/6}(\log x)^{3/2}$, and obtain

$$|S_{II}| \leq 0.24956x^{5/6}(\log x)^{3/2}.$$

Thus (5.60) gives the worst case.

We now take totals, and obtain

$$(5.61) \quad \begin{aligned} S_\eta(x, \alpha) & \leq |S_{I,1}| + |S_{I,2}| + |S_{II}| \\ & \leq (4.1982 + 1213.15)x^{2/3} \log x + (0.001063 + 0.0006406)x^{2/3}(\log x)^2 \\ & \quad + 0.272652x^{5/6}(\log x)^{3/2} \\ & \leq 0.27266x^{5/6}(\log x)^{3/2} + 1217.35x^{2/3} \log x, \end{aligned}$$

where we use (5.45) yet again.

5.3. Conclusion.

Proof of main theorem. We have shown that $|S_\eta(\alpha, x)|$ is at most (5.56) for $q \leq x^{1/3}/6$ and at most (5.61) for $q > x^{1/3}/6$. It remains to simplify (5.56) slightly. Let

$$\rho = \frac{C_{x_1, 2q_0}(\log 2q_0 + 0.002) + \frac{\log 8q_0}{2}}{0.30214 \log 2q_0 + 0.67506} = 3.61407\dots,$$

where $x_1 = 10^{25}$, $q_0 = 2 \cdot 10^5$. (We will be optimizing matters for $x = x_1$, $\delta_0 q = 2q_0$, with very slight losses in nearby ranges.) By the geometric mean/arithmetical

mean inequality,

$$\sqrt{C_{x_1, \delta_0 q}(\log \delta_0 q + 0.002) + \frac{\log 4\delta_0 q}{2}} \sqrt{0.30214 \log \delta_0 q + 0.67506}.$$

is at most

$$\begin{aligned} & \frac{1}{2} \left(\frac{1}{\sqrt{\rho}} \left(C_{x_1, \delta_0 q}(\log \delta_0 q + 0.002) + \frac{\log 4\delta_0 q}{2} \right) + \sqrt{\rho}(0.30214 \log \delta_0 q + 0.67506) \right) \\ & \leq \frac{C_{x, \delta_0 q}}{2\sqrt{\rho}}(\log \delta_0 q + 0.002) + \left(\frac{1}{4\sqrt{\rho}} + \frac{\sqrt{\rho} \cdot 0.30214}{2} \right) \log \delta_0 q \\ & \quad + \frac{1}{2} \left(\frac{\log 2}{\sqrt{\rho}} + \frac{\sqrt{\rho}}{2} \cdot 0.67506 \right) \\ & \leq 0.27125 \log \left(1 + \frac{\log 4t}{2 \log \frac{9x^{1/3}}{2.004t}} \right) (\log \delta_0 q + 0.002) + 0.4141 \log \delta_0 q + 0.49911. \end{aligned}$$

Now, for $x \geq x_0 = 2.16 \cdot 10^{20}$,

$$\frac{C_{x,t}}{\log t} \leq \frac{C_{x_0,t}}{\log t} \leq 0.08659$$

for $t \leq 10^6$, and

$$\frac{C_{x,t}}{\log t} \leq \frac{C_{6t^3,t}}{\log t} \leq \frac{1}{\log t} \log \left(1 + \frac{\log 4t}{2 \log \frac{27}{1.002}} \right) \leq 0.08659$$

if $10^6 < t \leq x^{1/3}/6$. Hence

$$0.27125 \cdot C_{x, \delta_0 q} \cdot 0.002 \leq 0.000047 \log \delta_0 q.$$

We conclude that, for $q \leq x^{1/3}/6$,

$$\begin{aligned} |S_\eta(\alpha, x)| & \leq \frac{R_{x, \delta_0 q} \log \delta_0 q + 0.49911}{\sqrt{\phi(q)} \delta_0} \cdot x + \frac{2.491x}{\sqrt{q\delta_0}} \\ & \quad + \frac{2x}{\delta_0} \min \left(\frac{\log \delta_0^{\frac{7}{4}} q^{\frac{13}{4}} + \frac{80}{9}}{\phi(q)}, \frac{\frac{5}{6} \log x + \frac{50}{9}}{q} \right) + \frac{2x \log q^{\frac{80}{9}} \delta_0^{\frac{16}{9}} + \frac{111}{5}}{q} + 3.2x^{5/6}, \end{aligned}$$

where

$$R_{x,t} = 0.27125 \log \left(1 + \frac{\log 4t}{2 \log \frac{9x^{1/3}}{2.004t}} \right) + 0.41415.$$

□

APPENDIX A. NORMS OF FOURIER TRANSFORMS

Our aim here is to give upper bounds on $|\widehat{\eta}_2'|_\infty$, where η_2 is as in (1.4). We will do considerably better than the trivial bound $|\widehat{\eta}''|_\infty \leq |\eta''|_1$.

Lemma A.1. *For every $t \in \mathbb{R}$,*

$$(A.1) \quad |4e(-t/4) - 4e(-t/2) + e(-t)| \leq 7.87052.$$

We will describe an extremely simple, but rigorous, procedure to find the maximum. Since $|g(t)|^2$ is C^2 (in fact smooth), there are several more efficient and equally rigorous algorithms – for starters, the bisection method with error bounded in terms of $|(|g|^2)''|_\infty$.

Proof. Let

$$(A.2) \quad g(t) = 4e(-t/4) - 4e(-t/2) + e(-t).$$

For $a \leq t \leq b$,

$$(A.3) \quad g(t) = g(a) + \frac{t-a}{b-a}(g(b) - g(a)) + \frac{1}{8}(b-a)^2 \cdot O^*\left(\max_{v \in [a,b]} |g''(v)|\right).$$

(This formula, in all likelihood well-known, is easy to derive. First, we can assume without loss of generality that $a = 0$, $b = 1$ and $g(a) = g(b) = 0$. Dividing by g by $g(t)$, we see that we can also assume that $g(t)$ is real (and in fact 1). We can also assume that g is real-valued, in that it will be enough to prove (A.3) for the real-valued function $\Re g$, as this will give us the bound $g(t) = \Re g(t) \leq (1/8) \max_v |(\Re g)''(v)| \leq \max_v |g''(v)|$ that we wish for. Lastly, we can assume (by symmetry) that $0 \leq t \leq 1/2$, and that g has a local maximum or minimum at t . Writing $M = \max_{u \in [0,1]} |g''(u)|$, we then have:

$$\begin{aligned} g(t) &= \int_0^t g'(v)dv = \int_0^t \int_t^v g''(u)dudv = O^*\left(\int_0^t \left| \int_t^v Mdu \right| dv\right) \\ &= O^*\left(\int_0^t (v-t)Mdv\right) = O^*\left(\frac{1}{2}t^2M\right) = O^*\left(\frac{1}{8}M\right), \end{aligned}$$

as desired.)

We obtain immediately from (A.3) that

$$(A.4) \quad \max_{t \in [a,b]} |g(t)| \leq \max(|g(a)|, |g(b)|) + \frac{1}{8}(b-a)^2 \cdot \max_{v \in [a,b]} |g''(v)|.$$

For any $v \in \mathbb{R}$,

$$(A.5) \quad |g''(v)| \leq \left(\frac{\pi}{2}\right)^2 \cdot 4 + \pi^2 \cdot 4 + (2\pi)^2 = 9\pi^2.$$

Clearly $g(t)$ depends only on $t \bmod 4\pi$. Hence, by (A.4) and (A.5), to estimate $\max_{t \in \mathbb{R}} |g(t)|$ with an error of at most ϵ , it is enough to subdivide $[0, 4\pi]$ into intervals of length $\leq \sqrt{8\epsilon/9\pi^2}$ each. We set $\epsilon = 10^{-6}$ and compute. \square

Lemma A.2. *Let $\eta_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$ be as in (1.4). Then*

$$(A.6) \quad |\widehat{\eta_2''}|_\infty \leq 31.521.$$

This should be compared with $|\eta_2''|_1 = 48$.

Proof. We can write

$$(A.7) \quad \eta_2''(x) = 4(4\delta_{1/4}(x) - 4\delta_{1/2}(x) + \delta_1(x)) + f(x),$$

where δ_{x_0} is the point measure at x_0 of mass 1 (Dirac delta function) and

$$f(x) = \begin{cases} 0 & \text{if } x < 1/4 \text{ or } x \geq 1, \\ -4x^{-2} & \text{if } 1/4 \leq x < 1/2, \\ 4x^{-2} & \text{if } 1/2 \leq x < 1. \end{cases}$$

Thus $\widehat{\eta_2''}(t) = 4g(t) + \widehat{f}(t)$, where g is as in (A.2). It is easy to see that $|f'|_1 = 2 \max_x f(x) - 2 \min_x f(x) = 160$. Therefore,

$$(A.8) \quad \left| \widehat{f}(t) \right| = \left| \widehat{f}'(t)/(2\pi it) \right| \leq \frac{|f'|_1}{2\pi|t|} = \frac{80}{\pi|t|}.$$

Since $31.521 - 4 \cdot 7.87052 = 0.03892$, we conclude that (A.6) follows from Lemma A.1 and (A.8) for $|t| \geq 655 > 80/(\pi \cdot 0.03892)$.

It remains to check the range $t \in (-655, 655)$; since $4g(-t) + \widehat{f}(-t)$ is the complex conjugate of $4g(t) + \widehat{f}(t)$, it suffices to consider t non-negative. We use (A.4) (with $4g + \widehat{f}$ instead of g) and obtain that, to estimate $\max_{t \in \mathbb{R}} |4g + \widehat{f}(t)|$ with an error of at most ϵ , it is enough to subdivide $[0, 655)$ into intervals of length $\leq \sqrt{2\epsilon/|(4g + \widehat{f})''|_\infty}$ each and check $|4g + \widehat{f}(t)|$ at the endpoints. Now, for every $t \in \mathbb{R}$,

$$\left| \left(\widehat{f} \right)''(t) \right| = \left| (-2\pi i)^2 x^2 \widehat{f}(t) \right| = (2\pi)^2 \cdot O^*(|x^2 f|_1) = 12\pi^2.$$

By this and (A.5), $|(4g + \widehat{f})''|_\infty \leq 48\pi^2$. Thus, intervals of length δ_1 give an error term of size at most $24\pi^2 \delta_1^2$. We choose $\delta_1 = 0.001$ and obtain an error term less than 0.000237 for this stage.

To evaluate $\widehat{f}(t)$ (and hence $4g(t) + \widehat{f}(t)$) at a point, we use Simpson's rule on subdivisions of the intervals $[1/4, 1/2]$, $[1/2, 1]$ into $200 \cdot \max(1, \lfloor \sqrt{|t|} \rfloor)$ sub-intervals each.⁶ The largest value of $\widehat{f}(t)$ we find is $31.52065\dots$, with an error term of at most $4.5 \cdot 10^{-5}$. \square

Lemma A.3. *Let $\eta_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$ be as in (1.4). Let $\eta_y(t) = \log(yt)\eta_2(t)$, where $y \geq 4$. Then*

$$(A.9) \quad |\eta'_y|_1 < (\log y)|\eta'_2|_1.$$

This was sketched in [Helb, (2.4)].

Proof. Recall that $\text{supp}(\eta_2) = (1/4, 1)$. For $t \in (1/4, 1/2)$,

$$\eta'_y(t) = (4 \log(yt) \log 4t)' = \frac{4 \log 4t}{t} + \frac{4 \log yt}{t} \geq \frac{8 \log 4t}{t} > 0,$$

whereas, for $t \in (1/2, 1)$,

$$\eta'_y(t) = (-4 \log(yt) \log t)' = -\frac{4 \log yt}{t} - \frac{4 \log t}{t} = -\frac{4 \log yt^2}{t} < 0,$$

where we are using the fact that $y \geq 4$. Hence $\eta_y(t)$ is increasing on $(1/4, 1/2)$ and decreasing on $(1/2, 1)$; it is also continuous at $t = 1/2$. Hence $|\eta'_y|_1 = 2|\eta_y(1/2)|$. We are done by

$$2|\eta_y(1/2)| = 2 \log \frac{y}{2} \cdot \eta_2(1/2) = \log \frac{y}{2} \cdot 8 \log 2 < \log y \cdot 8 \log 2 = (\log y)|\eta'_2|_1.$$

\square

Lemma A.4. *Let $y \geq 4$. Let $g(t) = 4e(-t/4) - 4e(-t/2) + e(-t)$ and $k(t) = 2e(-t/4) - e(-t/2)$. Then, for every $t \in \mathbb{R}$,*

$$(A.10) \quad |g(t) \cdot \log y - k(t) \cdot 4 \log 2| \leq 7.87052 \log y.$$

Proof. By Lemma A.1, $|g(t)| \leq 7.87052$. Since $y \geq 4$, $k(t) \cdot (4 \log 2)/\log y \leq 6$. For any complex numbers z_1, z_2 with $|z_1|, |z_2| \leq \ell$, we can have $|z_1 - z_2| > \ell$ only if $|\arg(z_1/z_2)| > \pi/3$. It is easy to check that, for all $t \in [-2, 2]$,

$$\left| \arg \left(\frac{g(t) \cdot \log y}{4 \log 2 \cdot k(t)} \right) \right| = \left| \arg \left(\frac{g(t)}{k(t)} \right) \right| < 0.7 < \frac{\pi}{3}.$$

⁶The author's code uses D. Platt's implementation [Pla11] of double-precision interval arithmetic (based on Lambov's [Lam08] ideas).

(It is possible to bound maxima rigorously as in (A.4).) Hence (A.10) holds. \square

Lemma A.5. *Let $\eta_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$ be as in (1.4). Let $\eta_{(y)}(t) = (\log yt)\eta_2(t)$, where $y \geq 4$. Then*

$$(A.11) \quad |\widehat{\eta_{(y)}''}|_\infty < 31.521 \cdot \log y.$$

Proof. Clearly

$$\begin{aligned} \eta_{(y)}''(x) &= \eta_2''(x)(\log y) + \left((\log x)\eta_2''(x) + \frac{2}{x}\eta_2'(x) - \frac{1}{x^2}\eta_2(x) \right) \\ &= \eta_2''(x)(\log y) + 4(\log x)(4\delta_{1/4}(x) - 4\delta_{1/2}(x) + \delta_1(x)) + h(x), \end{aligned}$$

where

$$h(x) = \begin{cases} 0 & \text{if } x < 1/4 \text{ or } x > 1, \\ \frac{4}{x^2}(2 - 2\log 2x) & \text{if } 1/4 \leq x < 1/2, \\ \frac{4}{x^2}(-2 + 2\log x) & \text{if } 1/2 \leq x < 1. \end{cases}$$

(Here we are using the expression (A.7) for $\eta_2''(x)$.) Hence

$$(A.12) \quad \widehat{\eta_{(y)}''}(t) = (4g(t) + \widehat{f}(t))(\log y) + (-16\log 2 \cdot k(t) + \widehat{h}(t)),$$

where $k(t) = 2e(-t/4) - e(-t/2)$. Just as in the proof of Lemma A.2,

$$(A.13) \quad |\widehat{f}(t)| \leq \frac{|f'|_1}{2\pi|t|} \leq \frac{80}{\pi|t|}, \quad |\widehat{h}(t)| \leq \frac{160(1 + \log 2)}{\pi|t|}.$$

Again as before, this implies that (A.11) holds for

$$|t| \geq \frac{1}{\pi \cdot 0.03892} \left(80 + \frac{160(1 + \log 2)}{(\log 4)} \right) = 2252.51.$$

Note also that it is enough to check (A.11) for $t \geq 0$, by symmetry. Our remaining task is to prove (A.11) for $0 \leq t \leq 2252.21$.

Let $I = [0.3, 2252.21] \setminus [3.25, 3.65]$. For $t \in I$, we will have

$$(A.14) \quad \arg \left(\frac{4g(t) + \widehat{f}(t)}{-16\log 2 \cdot k(t) + \widehat{h}(t)} \right) \subset \left(-\frac{\pi}{3}, \frac{\pi}{3} \right).$$

(This is actually true for $0 \leq t \leq 0.3$ as well, but we will use a different strategy in that range in order to better control error terms.) Consequently, by Lemma A.2 and $\log y \geq \log 4$,

$$\begin{aligned} |\widehat{\eta_{(y)}''}(t)| &< \max(|4g(t) + \widehat{f}(t)| \cdot (\log y), |16\log 2 \cdot k(t) - \widehat{h}(t)|) \\ &< \max(31.521(\log y), |48\log 2 + 25|) = 31.521 \log y, \end{aligned}$$

where we bound $\widehat{h}(t)$ by (A.13) and by a numerical computation of the maximum of $|\widehat{h}(t)|$ for $0 \leq t \leq 4$ as in the proof of Lemma A.2.

It remains to check (A.14). Here, as in the proof of Lemma A.4, the allowable error is relatively large (the expression on the left of (A.14) is actually contained in $(-1, 1)$ for $t \in I$). We decide to evaluate the argument in (A.14) at all $t \in 0.005\mathbb{Z} \cap I$, computing $\widehat{f}(t)$ and $\widehat{h}(t)$ by numerical integration (Simpson's rule) with a subdivision of $[-1/4, 1]$ into 5000 intervals. Proceeding as in the proof of Lemma A.1, we see that the sampling induces an error of at most

$$(A.15) \quad \frac{1}{2}0.005^2 \max_{v \in I} (4|g''(v)| + |(\widehat{f})''(t)|) \leq \frac{0.0001}{8}48\pi^2 < 0.00593$$

in the evaluation of $4g(t) + \widehat{f}(t)$, and an error of at most

$$(A.16) \quad \begin{aligned} & \frac{1}{2} 0.005^2 \max_{v \in I} (16 \log 2 \cdot |k''(v)| + |(\widehat{h})''(t)|) \\ & \leq \frac{0.0001}{8} (16 \log 2 \cdot 6\pi^2 + 24\pi^2 \cdot (2 - \log 2)) < 0.0121 \end{aligned}$$

in the evaluation of $16 \log 2 \cdot |k''(v)| + |(\widehat{h})''(t)|$.

Running the numerical evaluation just described for $t \in I$, the estimates for the left side of (A.14) at the sample points are at most 0.99134 in absolute value; the absolute values of the estimates for $4g(t) + \widehat{f}(t)$ are all at least 2.7783, and the absolute values of the estimates for $|-16 \log 2 \cdot \log k(t) + \widehat{h}(t)|$ are all at least 2.1166. Numerical integration by Simpson's rule gives errors bounded by 0.17575 percent. Hence the absolute value of the left side of (A.14) is at most

$$\begin{aligned} & 0.99134 + \arcsin \left(\frac{0.00593}{2.7783} + 0.0017575 \right) + \arcsin \left(\frac{0.0121}{2.1166} + 0.0017575 \right) \\ & \leq 1.00271 < \frac{\pi}{3} \end{aligned}$$

for $t \in I$.

Lastly, for $t \in [0, 0.3] \cup [3.25, 3.65]$, a numerical computation (samples at $0.001\mathbb{Z}$; interpolation as in Lemma A.2; integrals computed by Simpson's rule with a subdivision into 1000 intervals) gives

$$\max_{t \in [0, 0.3] \cup [3.25, 3.65]} \left(|(4g(t) + \widehat{f}(t))| + \frac{|-16 \log 2 \cdot k(t) + \widehat{h}(t)|}{\log 4} \right) < 29.08,$$

and so $\max_{t \in [0, 0.3] \cup [3.25, 3.65]} |\widehat{\eta''(y)}|_\infty < 29.1 \log y < 31.521 \log y$. \square

An easy integral gives us that the function $\log \cdot \eta_2$ satisfies

$$(A.17) \quad |\log \cdot \eta_2|_1 = 2 - \log 4$$

The following function will appear only in a lower-order term; thus, an ℓ_1 estimate will do.

Lemma A.6. *Let $\eta_2 : \mathbb{R}^+ \rightarrow \mathbb{R}$ be as in (1.4). Then*

$$(A.18) \quad |(\log \cdot \eta_2)''|_1 = 96 \log 2.$$

Proof. The function $\log \cdot \eta(t)$ is 0 for $t \notin [1/4, 1]$, is increasing and negative for $t \in (1/4, 1/2)$ and is decreasing and positive for $t \in (1/2, 1)$. Hence

$$\begin{aligned} |(\log \cdot \eta_2)''|_\infty &= 2 \left((\log \cdot \eta_2)' \left(\frac{1}{2} \right) - (\log \cdot \eta_2)' \left(\frac{1}{4} \right) \right) \\ &= 2(16 \log 2 - (-32 \log 2)) = 96 \log 2. \end{aligned}$$

\square

REFERENCES

- [AS64] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.
- [BR02] G. Bastien and M. Rogalski. Convexité, complète monotonie et inégalités sur les fonctions zêta et gamma, sur les fonctions des opérateurs de Baskakov et sur des fonctions arithmétiques. *Canad. J. Math.*, 54(5):916–944, 2002.

- [But11] Y. Buttkewitz. Exponential sums over primes and the prime twin problem. *Acta Math. Hungar.*, 131(1-2):46–58, 2011.
- [CW89] J. R. Chen and T. Z. Wang. On the Goldbach problem. *Acta Math. Sinica*, 32(5):702–718, 1989.
- [Dab96] H. Daboussi. Effective estimates of exponential sums over primes. In *Analytic number theory, Vol. 1 (Allerton Park, IL, 1995)*, volume 138 of *Progr. Math.*, pages 231–244. Birkhäuser Boston, Boston, MA, 1996.
- [DEtRZ97] J.-M. Deshouillers, G. Effinger, H. te Riele, and D. Zinoviev. A complete Vinogradov 3-primes theorem under the Riemann hypothesis. *Electron. Res. Announc. Amer. Math. Soc.*, 3:99–104, 1997.
- [DR01] H. Daboussi and J. Rivat. Explicit upper bounds for exponential sums over primes. *Math. Comp.*, 70(233):431–447 (electronic), 2001.
- [EM95] M. El Marraki. Fonction sommatoire de la fonction de Möbius. III. Majorations asymptotiques effectives fortes. *J. Théor. Nombres Bordeaux*, 7(2):407–433, 1995.
- [EM96] M. El Marraki. Majorations de la fonction sommatoire de la fonction $\frac{\mu(n)}{n}$. Univ. Bordeaux 1, preprint (96-8), 1996.
- [GR96] A. Granville and O. Ramaré. Explicit bounds on exponential sums and the scarcity of squarefree binomial coefficients. *Mathematika*, 43(1):73–107, 1996.
- [HB85] D. R. Heath-Brown. The ternary Goldbach problem. *Rev. Mat. Iberoamericana*, 1(1):45–59, 1985.
- [HB11] H. Hong and Ch. W. Brown. QEPCAD B – Quantifier elimination by partial cylindrical algebraic decomposition, May 2011. version 1.62.
- [Hela] H. A. Helfgott. Major arcs for Goldbach’s problem. Preprint.
- [Helb] H. A. Helfgott. Minor arcs for Goldbach’s problem. Preprint. Available as [arXiv:1205.5252](https://arxiv.org/abs/1205.5252).
- [HL23] G. H. Hardy and J. E. Littlewood. Some problems of ‘Partitio numerorum’; III: On the expression of a number as a sum of primes. *Acta Math.*, 44(1):1–70, 1923.
- [Hux72] M. N. Huxley. Irregularity in sifted sequences. *J. Number Theory*, 4:437–454, 1972.
- [IK04] H. Iwaniec and E. Kowalski. *Analytic number theory*, volume 53 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2004.
- [Lam08] B. Lambov. Interval arithmetic using SSE-2. In *Reliable Implementation of Real Number Algorithms: Theory and Practice. International Seminar Dagstuhl Castle, Germany, January 8-13, 2006*, volume 5045 of *Lecture Notes in Computer Science*, pages 102–113. Springer, Berlin, 2008.
- [LW02] M.-Ch. Liu and T. Wang. On the Vinogradov bound in the three primes Goldbach conjecture. *Acta Arith.*, 105(2):133–175, 2002.
- [Mon68] H. L. Montgomery. A note on the large sieve. *J. London Math. Soc.*, 43:93–98, 1968.
- [Mon71] H. L. Montgomery. *Topics in multiplicative number theory*. Lecture Notes in Mathematics, Vol. 227. Springer-Verlag, Berlin, 1971.
- [MV73] H. L. Montgomery and R. C. Vaughan. The large sieve. *Mathematika*, 20:119–134, 1973.
- [MV74] H. L. Montgomery and R. C. Vaughan. Hilbert’s inequality. *J. London Math. Soc. (2)*, 8:73–82, 1974.
- [Pla11] D. Platt. *Computing degree 1 L-functions rigorously*. PhD thesis, Bristol University, 2011.
- [Rama] O. Ramaré. Explicit estimates on several summatory functions involving the Moebius function. Preprint.
- [Ramb] O. Ramaré. Explicit estimates on the summatory functions of the moebius function with coprimality restrictions. Preprint.
- [Ramc] O. Ramaré. From explicit estimates for the primes to explicit estimates for the Moebius function. Preprint.
- [Ramd] O. Ramaré. A sharp bilinear form decomposition for primes and moebius function. Preprint. To appear in *Acta. Math. Sinica*.
- [Ram10] O. Ramaré. On Bombieri’s asymptotic sieve. *J. Number Theory*, 130(5):1155–1189, 2010.
- [RR96] O. Ramaré and R. Rumely. Primes in arithmetic progressions. *Math. Comp.*, 65(213):397–425, 1996.

- [RS62] J. B. Rosser and L. Schoenfeld. Approximate formulas for some functions of prime numbers. *Illinois J. Math.*, 6:64–94, 1962.
- [RS75] J. B. Rosser and L. Schoenfeld. Sharper bounds for the Chebyshev functions $\theta(x)$ and $\psi(x)$. *Math. Comp.*, 29:243–269, 1975. Collection of articles dedicated to Derrick Henry Lehmer on the occasion of his seventieth birthday.
- [Sel91] A. Selberg. Lectures on sieves. In *Collected papers, vol. II*, pages 66–247. Springer Berlin, 1991.
- [Tao] T. Tao. Every odd number greater than 1 is the sum of at most five primes. Preprint. Available as [arXiv:1201.6656](https://arxiv.org/abs/1201.6656).
- [Vau77] R.-C. Vaughan. Sommes trigonométriques sur les nombres premiers. *C. R. Acad. Sci. Paris Sér. A-B*, 285(16):A981–A983, 1977.
- [Vin37] I. M. Vinogradov. Representation of an odd number as a sum of three primes. *Dokl. Akad. Nauk. SSR*, 15:291–294, 1937.
- [Vin04] I. M. Vinogradov. *The method of trigonometrical sums in the theory of numbers*. Dover Publications Inc., Mineola, NY, 2004. Translated from the Russian, revised and annotated by K. F. Roth and Anne Davenport, Reprint of the 1954 translation.

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