Kinematical/potential Fourier/Calderon waves/wavelets

Wavelets are proposed as appropriate analysis tool for the proposed single gravity and quantum field model, additionally to the standard Fourier wave analysis technique. Regarding all physical classical PDE model and their related variational representations in line with the Hamiltionian formalism) Fourier waves are related to the standard kinematical Hilbert space H(1); the corresponding variational theory and the related Ritz-Galerkin approximation procedure provide the mathematical framework for the well established (numerical) finite element methods. The physical notion "Fourier wave package" is related to the H(1)-complementary subspace of H(1/2). The corresponding tool set for analytical and approximation analysis are the (continuous) wavelets accompanied with the related approximation discrete wavelet methods.

There are at least two approaches to wavelet analysis, both are addressing the somehow contradiction by itself, that a function over the one-dimensional space R can be unfolded into a function over the two-dimensional half-plane (HoM1):

the first approach is the interpretation of the wavelet transform as a *time-frequency analysis* tool, where a one-dimensional information (a one-parameter family of purely oscillations) is somehow 'unfolded' into a two-dimensional time-frequency plane, i.e. a function over the real line is mapped into a function over the *time-frequency plane* that tells 'when' and which 'frequency' occurs. The (human perception) hearing process of a concert is somehow reflecting this kind of compromise (!!) between the (mathematically correct) (either) time localization or frequency localization on the one hand side, and the human perceived melodies, and hence music, just based on a received one-dimensional signal on the other hand side. The interpretation of the wavelet transform of a one-dimensional signal in this context is about a *time-frequency analysis* with the physical parameters "time" and "frequency" with constant relative bandwidth.

The second approach uses the wavelet analysis as a *mathematical microscope*. The idea is to look at the details that are added if one goes from a scale "a" to a scale " a-da", where "da" is infinitesimally small. This second approach is closely linked to approximation theory, e.g. in the context of the building of Calderon-Zygmund operators, based on the truncation of kernels (MeY). This *mathematical microscope tool* 'unfolds' a function over the one-dimensional space R into a function over the two-dimensional half-plane of "*positions*" and "*details*" (where is which detail generated?). This two-dimensional parameter space may also be called the *position-scale half-plane*. The interpretation of the wavelet transform in this context is about a *mathematical microscope* with the physical parameters "*position* (parameter a)", "*enlargement*" and "*optics* (wavelet function g)".

(LoA) remark 1.1.10: The second mathematical microscope approach enables a purely (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions f, g can be compared with each other by the corresponding "reproducing" ("duality") formula (see also (*) below), whereby

- the "bra(c)"-wavelet transform W(f) is inverted by the adjoint operator of the "(c)ket"-wavelet transform W(g) (given corresponding admissibility conditions are valid)

- the identity (*) provides also some additional degree of freedom in the way that in order to analyze a signal s(t) the wavelet f can be chosen properly according to the special situation of the underlying mathematical model. The prize to be paid is only later, when the "re-building" wavelet g needs to be built accordingly to enable the corresponding "synthesis"

- the Hilbert transform operator (which is valid for every Hilbert scale) is a "natural" partner of the wavelet transform operator, as it is skew-symmetric, rotation invariant and each Hilbert transformed "function" has vanishing constant Fourier term. The example in the context above is the Hilbert transform of the Gaussian/Maxwellian distribution function, the (odd) Dawson function, with the "polynomial degree" point of zero at +/- infinite.

Further details

The sine and cosine functions have unbounded support and they do not vanish at infinity. Their spectra are very local consisting of a finite sum of Dirac measures. Conversely, if one use approximations based on finite sum of Dirac measures the spectrum of the corresponding basis "functions" (which is basically the (cosine(x*s) + i * sine(x*s) function) does not vanishes at infinity in the frequency domain.

The wavelet concept is trying to overcome this issue, while basically looking for an orthogonal basis of a Hilbert space (e.g. L(2)=H(0) or H(-1/2)), constructed from a unique generation function g (the scaling function), via translation, dilation and linear combinations, whereby g can be localized in x (space variable) and s (Fourier variable). The admissibility condition for a wavelet governs the behavior of the wavelets in the neighborhood of the frequency zero. The (wavelet) admissibility condition is obviously related to the H(-1/2) Hilbert space norm in case of space dimension m=1.

We note that the hypothesis that a function g has compact support is essential to become a wavelet. Otherwise, it can be shown that there are infinitely supported solutions of the corresponding scaling equation. For instance, the Hilbert transform of the function g satisfies the scaling recursion whenever g does.

We further note the two fundamental examples of universal scaling functions (scaling functions for every rank), the sinc and the Haar scaling functions, which are Fourier transforms of each other.

The wavelet transform W(g)(v) of a function v with respect to a wavelet function g is an isometric mapping, whereby the corresponding adjoint operator is given by the inverse wavelet transform on its range. Let u,v denote two elements of a Hilbert space with inner product (u,v), let ((*,*)) denote the inner product of the Hilbert space H(-1/2). Let further f,g denote two wavelets with bounded inner product ((f,g)) and let (((*,*))) denote the inner product of the corresponding wavelet transforms W(f)(u), W(g)(v) with respect to the underlying Haar measure. Then (up to a constant) it holds

(*)   (((W(f)(u),W(g)(v)))) = ((f,g)) * (u,v) .

This identity (in combination with the below) enables a combined wave-wavelet ((H(0),H(-1)) concept for analysis of the H(-1/2) = H(0) * H(0)(ortho) framework, whereby in this specific case it holds (u,v):=((u,v)).

In (PaR) the wavelet transform for a class of distributions is provided, whereby the corresponding inversion formula is established by interpreting convergence in a weak distributional sense. In the context of above we note that log2(sin(x/2)) (with its corresponding 1st and 2nd derivatives, the cot(x) and the 1/(sin(x)*sin(x)) functions) is a L(2) function fulfilling the admissibility condition.

The Gaussian function stands out since it minimizes the Heisenberg uncertainty principle (DaS). The corresponding windowed Fourier (integral) transform is e.g. applied in quantum physics, where it is used for defining and investigating coherent states. It is related to the Weyl-Heisenberg group, while the corresponding wavelet (integral) transform is related to the affine group. In other words, from a group theory perspective windowed Fourier transforms and wavelet transforms are identical.

The wavelet mother function, which is directly connected to the Gaussian function (which is not a wavelet) is the Mexican hat function. It is basically the second derivative of the Gaussian function. In (DaS) a new interpretation of the Mexican hat function is provided: it can be interpreted as a minimizing function of an uncertainty principle, in case its rotation invariant form "A" has a certain form/representation.

The affine-linear group (where each element of that group has two components, while the Weyl-Heisenberg group has three components) of unitary operators equipped with the Haar measure is locally compact, i.e. the group multiplication and the inverse operation of the group are continuous mappings. For local compact groups there is an orthogonality relationship valid, which provides the common group theoretical denominator of windowed Fourier and wavelet transforms (GrA).

References

(ChK) Chakraborty K., Kanemitsu S., Long L. H., Quadratic reciprocity and Riemann's "non-differentiable" function, Research in Number Theory, Springer Open Journal, (2015) 1-14

(DaS) Dahlke S., Maass P., The Affine Uncertainty Principle in One and Two Dimensions, Computers Math. Applic. Vol. 30, No. 3-6, (1995), pp. 293-305

(FeP) Federbush P., Navier and Stokes Meet the Wavelet, Commun. Math. Phys. 155, 219-248 (1993)

(GrA) Grossmann A., Morlet J., Paul T., Transforms associated to square integrable group representations I: General results, J. Math. Phys. 26, (1985) pp. 2473-2479

(HoM) Holschneider M., Tchamitchian P., Pointwise analysis of Riemann's "non-differentiable" function, Invest. Math. 105, (1991) pp. 157-175

(HoM1) Holschneider M., Wavelets, An Analysis Tool, Oxford Science Publications, Clarendon Press, Oxford, 1995

(LoA) Louis A. K., Maaß P., Rieder A., Wavelets, Theorie und Anwendungen, B. G. Teubner Verlag, Stuttgart, 1998

(MeY) Meyer Y., Coifman R., Wavelets, Calderon-Zygmund and multilinear operators, Cambridge studies in advanced mathematics 48, Cambridge University Press, 1996

(OsK) Oskolkov K. I., Chakhkiev M. A., On Riemann "Nondifferentiable" Equation and Schrödinger Equation, Proceedings of the Steklov Institute of Mathematics, 2010, Vol. 269, pp. 186-196

(PaR) Pathak R. S., Singh A., Distributional Wavelet Transform, Proc. Natl. Acad. Sci., India, Sect. A. Phys. Sci. 86(2), (2016) 273-277