The three Millennium problem solutions, RH, NSE, YME, and a Hilbert scale based quantum geometrodynamics

Klaus Braun

PART II

3D-NSE and YME mass gap solutions in a distributional Hilbert scale frame enabling a quantum gravity theory

Klaus Braun
July 31, 2019
www.fuchs-braun.com

1. The $\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp}$ decomposition for a quantum space-time model

The following sections are about *„a theory, where the wave function provides a complete description of the physical reality, i.e. every element of the physical reality does have a counterpart in the physical theory (EiA2)*; the theory is claimed to be *(1) correct, and (2) the description given by the theory is complete*" (EiA2).

This section is about a common Hilbert space framework enabling variational methods for nonlinear operators (VaM) for the considered mathematical physics models. It overcomes the (claimed) common purely mathematical handicaps for problem adequate solutions in alignment with the *purpose* of physical models. From a *physical* modelling perspective it is about a replacement of Dirac's model of the „*density*" of an idealized point mass or point charge, which is called the Dirac or Delta „function". It is a distribution equal to zero everywhere except for zero, and whose integral over the entire line is equal to one. The Dirac model of the „*density*" of an idealized point mass is replaced by Plemelj's concept of a „*mass element*" (PlJ), with the essential consequence, that the regularity requirement for those distributions $\mathrm{d}\mu$ are independent from the space-dimension in opposite to the Dirac function:

the regularity of Dirac's model of the <u>point</u> <u>mass</u> <u>density</u> of an idealized point mass is $\delta \in H_{-n/2-\varepsilon}$ ($\varepsilon > 0$, n = space dimension), while for Plemelj's <u>mass element</u> definition it holds $d\mu \in H_{-1/2}$.

From a mathematical point of view this means that a Lebesgue integral is replaced by a Stieltjes integral. The corresponding $\mathrm{H}_{-1/2}$ quantum state model (alternatively to the standard $\mathrm{L}_2 = \mathrm{H}_0$ model) goes along with a corresponding quantum energy Hilbert space model $\mathrm{H}_{1/2}$. Its definition follows the same building principles as for the standard Laplace operator in a $\mathrm{L}_2 = \mathrm{H}_0$ framework with its corresponding Dirichlet (energy inner product) integral $\mathrm{D}(\mathrm{u},\mathrm{v}) = (\nabla u, \nabla v)_0 = (u,v)_1$.

The decompositions

$$\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp} = H_{1/2}^*, \; H_{1/2} = H_1 \otimes H_1^{\neg} = H_{-1/2}^*$$

distinguish between elementary particle states & energy with or w/o „observed/measured mass". The „*symmetry break down*" model to „generate/explain" physical „mass" is replaced by a „projection of a *self-adjoint operator onto the observation/measure space* H_0" (**). In other words, the matter particles (fermions) are the manifestations of the vacuum energy (bosons).

The corresponding mass/energy Hilbert space is given by the decomposition $H_{1/2} = H_1 \times H_1^{\neg}$ into the „fermions" space and the orthogonal „bosons" space, including a Hilbert space based model of the Higgs boson, as well as a Cauchy problem representation of the Einstein-Vacuum field equation with an initial „inflation-field" with regularity $g_{Inflation} \in H_1^{\neg}$ without singularities for $t \to 0$, avoiding current early universe state model singularities.

(*) (PlJ) I, §8: "*bisher war es ueblich fuer das Potential V(p) die Form* $V(u)(s) = \oint \gamma(s-t)u(t)dt$ *vorauszusetzen, wobei dann* $u(t)dt$ *die Massendichtigkeit der Belegung genannt wurde. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschraenkung, dass dadurch dem Potentials V(p) der groesste Teil seiner Leistungsfaehigkeit hinweg genommen wird.*" $V(u)(s) = \oint \gamma(s-t)du(t)$." (PlJ) p. 11: "*Vom Integral* $\oint \frac{\partial U}{\partial n} ds$ *auf einer nichtgeschlossenen Kurve ergibt sich aus der Gleichung (6) eine Eigenschaft von grosser Wichtigkeit. Das Integral hängt nämlich nur von den Endpunkten ab und nicht von der näheren Form der sie verbindenden Integrationskurve in der Weise, dass die Integrale alle gleich einander gleich sind, welche Integrationswege entsprechen, die durch stetige Deformation im Regularitätsgebiete auseinander hervorgehen. Sind also* p *und* q *zwei Punkte im Regularitätsgebiete und verbindet man sie durch irgendeine Kurve (die Tangenten hat), so ist* $\int_p^q \frac{\partial U}{\partial n} ds$*wohl definiert und hat einen von der näheren Form der Kurve nicht abhängigen Wert.… Das Integral zwischen zwei Punkten* p *und* q $\bar{U}(q) = -\int_p^q \frac{\partial U}{\partial n} ds$ *ist, weil von der Kurve unabhängig, eine wohl definierte Funktion der Grenzen* p *und* q *und soll in seiner Abhängigkeit von* q *mit* $\bar{U}$ *bezeichnet werden.*"

(**) *The mathematical „fluid/quantum" state Hilbert space* $H_{-1/2}$ *and its „components" might be identified with the names,* $H_{-1/2}\colon Plemelj, Schrödinger$, $(H_0, H_0^{\perp}) = ((Fourier, Lebesgue), Stieltjes)$*, while the corresponding physical „energy" Hilbert space* $H_{1/2}$ *and its „component"* $H_0^{\perp}$ *might be identified with the names* $H_{1/2}\colon Calderón, Schrödinger$ *and* $H_0^{\perp}\colon Bohm$*. From a philosophical point of view, the spaces* $H_{\pm 1/2}$ *might be identified with the names,* $H_{\pm 1/2}\colon Leibniz, Kant, \ldots Schrödinger$*, while the "* $\otimes$ *" might be interpreted as Kant´s „borderline" between physical and meta-physical (transcendendal) „world", with the <u>very strong</u> assumption, that the set of integers <u>and</u> real numbers (where each irrational number is already a universe by itself (!)) with the Cantor cardinality* $\aleph$ *resp.* $2^{\aleph}$ *(i.e.* $R = Q \otimes Q^{\perp}$*) are on this side of the transcendence border (!).*

The considered physical problem areas are about the generation and transport of elementary particles and their energy over time, going along with „observed/measured" actions of such energy transports (radiation problems). The handicaps of today´s physical models are about „inappropriate" physical solution behaviors for $t \to 0$ (e.g. big bang), as well as blow-up effects for existing global bounded solutions until a certain point in time ($t < T_{Blow-up}$), or no existing global bounded solution at all (e.g. 3D-NSE).

The singularity behavior and the blow-up effects are the result of the chosen Sobolev space framework governed by the corresponding Sobolev embedding theorems:

i) already the most simple, linear homogenous heat equation with non-regular initial value function $g \in H_0$ shows a singular solution behavior for $t \to 0$ in the form

$$\|z(t)\|_k^2 \le c t^{-(k-l)} \|g\|_l^2 \quad , \quad \int_0^T t^{-1/2} \|z'\|_{-1/2}^2 dt \le c \|g\|_0^2 \quad (*)$$

ii) the global boundedness of the solution of the *2D*-NSE is governed by the ODE $y'(t) = y^2(t)$, $y(0) = y_0$ with the solution $y(t) = y_0/(1 - t \cdot y_0)$ becoming infinite in finite time (blow-up effect)

iii) the 3D-NSE is governed by the ODE $y'(t) = y^3(t)$, $y(0) = y_0$, i.e. there is no global global boundedness at all (which is the 3D-NSE Millennium problem with the proposed solution in (BrK2)).

Additionally, in the standard H_0 framework the non-linear part of the „energy"-norm vanishes. Thisis a great thing from a mathematical perspective, avoiding sophisticated estimating techniques, but a doubtful thing from a physical modelling perspective, as this term is the critical one, while at the same point in time, jepordizing all attempts to extend the existing 2-D NSE problem solution technique to the 3D case (GiY). The alternatively proposed "fluid state" Hilbert space $\mathrm{H}_{-1/2}$ with corresponding alternative energy ("velocity") space $\mathrm{H}_{1/2}$ avoids the blow-up effect due to Ricci ODE estimates in the form $y'(t) \le c \cdot y^{1/2}(t)$ (**), while enabling at the same time an „energy" norm inequality (including contributions from the non-linear term), based a corresponding Sobolevskii estimate.

The newly proposed scale value $\alpha = -1/2$ fulfills also the requirement $0 < \alpha < n/2 + \varepsilon$. It therefore provides an alternative model to the Dirac (Delta) „function".

(*) From $z(x,t) = \sum z_\nu(t)\phi_\nu(x)$ *it follows* $\dot{z} - z'' = \sum(\dot{z}_\nu(t) + \lambda_\nu z_\nu(t))\phi_\nu(x) = 0$. *Therefore* $z_\nu(t) = z_\nu(0)e^{-\lambda_\nu t}$ *and* $z_\nu(0) = g_\nu = (g, \phi_\nu)$. *Putting* $C_{k,l}(t) := sup\left\{\lambda_\nu^{k-l} e^{-2\lambda_\nu t} \middle| \lambda_\nu \ge m > 0\right\}$ *it follows* $\|z(t)\|_k^2 = \sum \lambda_\nu^k z_\nu^2(t) = \sum \lambda_\nu^k e^{-2\lambda_\nu t} g_\nu \le C_{k.l}(t) \sum \lambda_\nu^l e^{-2\lambda_\nu t}$. *The conditions* $(k-l)\lambda^{k-l-1}e^{-2\lambda_\nu t} + \lambda^{k-l}(-2t)e^{-2\lambda_\nu t} = 0$ *resp.* $(k-l)\lambda^{k-l-1}e^{-2\lambda_\nu t} = 2t\lambda^{k-l}e^{-2\lambda_\nu t}$ *leads to (for the critical case* $k > l$ *)* $\lambda \approx t^{-1}$ *.*

For the orthogonal set $\{w_i, \lambda_i\}$ *of eigenpairs of the non-stationary Stokes operator*

$$\bar{A} := \dot{w} + Aw = f \ , \ \ w(0) = 0 \ , \ \ \tau \in [0,t]$$

one gets $w_i(\tau) = \int_0^\tau e^{-\lambda_i(\tau - s)} f_i(s) ds$. *By changing the order of integration it follows for* $\beta > -1$

$$\int_0^t \tau^\beta w_i^2(\tau) d\tau \le \int_0^t \left[\int_0^\tau e^{-\lambda_i(\tau-s)} ds\right]\left[\int_0^\tau s^\beta e^{-\lambda_i(\tau-s)} f_i^2(s) ds\right] d\tau$$

$$\le \lambda_i^{-1} \int_0^t s^\beta f_i^2(s) \left[\int_\tau^t e^{-\lambda_i(\tau-s)} d\tau\right] ds \le \lambda_i^{-2} \int_0^t s^\beta f_i^2(s) ds \ .$$

From this one gets $\left|\left|\left|t^{\beta/2} w(t)\right|\right|\right|_{\alpha+2}^2 \le c \left|\left|\left|t^{\beta/2} \bar{A} w(t)\right|\right|\right|_\alpha^2$, $\beta > -1$, *with* $|||v(t)|||_\alpha^2 := \int_0^t \|v(s)\|_\alpha^2 ds$, $\alpha \in R$.

(**) *Lemma of Gronwall (general form): Let* $a(t)$ *and* $b(t)$ *nonnegative functions in* $[0, A)$ *and* $0 < \delta < 1$ *. Suppose a nonnegative function* $y(t)$ *satisfies the differential inequality*

$$y'(t) + b(t) \le \alpha(t) y^\delta(t) \quad \text{on} \quad [0, A)$$
$$y(0) = y_0.$$

Then for $0 \le t < A$

$$y(t) + \int_0^t b(\tau) d\tau \le (2^{\delta/(1-\delta)} + 1) y_0 + 2^{\delta/(1-\delta)} \left[\int_0^t \alpha(\tau) d\tau\right]^{\delta/(1-\delta)}$$

A global bounded solution of a $H_{-1/2}$ based variational representation of the 3D nonlinear, non-stationary NSE

For this section we also refer to (BrK2), (BrK10), (BrK related papers).

The Stokes operator is a projector from $A: L_2 \to L_\sigma^2 := \{v | v \in L_2 \wedge div(v) = 0\}$. The Hilbert scale is built on the Stokes operator on $\Omega \subseteq R^n$ $(n \geq 2)$ in the form $A = \int_0^\infty \lambda dE_\lambda$. The Stokes operator enables the definition of a related Hilbert scale ($\alpha \in R$) with a corresponding norm $\|u\|_\alpha := \|A^{\alpha/2}u\|$ (**), enabled by the corresponding positive selfadjoint fractional powers ((SoH), IV15)

$$A^\alpha = \int_0^\infty \lambda^\alpha dE_\lambda \ \ , -1 \leq \alpha \leq 1$$

The corresponding Stokes semigroup family $\{S(t)\}$ is built on the everywhere bounded, positive selfadjoint operator

$$S(t) := e^{-tA} := \int_0^\infty e^{-t\lambda} dE_\lambda \,|\lambda \geq 0, t \geq 0.$$

Putting $B(u) := P(u, grad)u)$ in the NSE and assuming $Pu_0 = u_0$, the NSE initial-boundary equation is given by $\frac{du}{dt} + Au + Bu = Pf$, $u(0) = u_0$. Multiplying this homogeneous equation with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_\alpha + (Au, u)_\alpha + (Bu, u)_\alpha = 0, \ (u(0), v)_\alpha = (u_0, v)_\alpha \text{ for all } \ v \in H_{-1/2}$$

We note that the the pressure p in the variational representation

$$(Au, v)_{-\frac{1}{2}} \coloneqq (\nabla u, \nabla v)_{-\frac{1}{2}} + (\nabla p, v)_{-\frac{1}{2}} = (u, v)_{\frac{1}{2}} + (p, v)_0 \qquad \text{for all } \ v \in H_{-1/2}$$
$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2} \qquad .$$

can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

with (R_1, R_2, R_3) is the Riesz transform.

Putting

$$y(t) := \begin{cases} \|u\|_{2\sigma}^2 & n = 2 \quad 0 < \sigma < 1/2 \\ \|u\|_1^2 & n = 3 \end{cases}$$

in case of $\alpha = 0$ one gets from the Sobolev estimates (GiY), (SoP)

$$\frac{1}{2}\frac{d}{dt}\|u\|_1^2 + \|u\|_2^2 \cong \frac{1}{2}\frac{d}{dt}\left\|A^{1/2}u\right\|^2 + \|Au\|^2 \leq c \begin{cases} \|u\|^2\left\|A^{1/2}u\right\|^4 \cong \|u\|_0^2\|u\|_1^4 & n = 2 \\ \left\|A^{1/2}u\right\|^6 \ \cong \ \|u\|_1^6 & n = 3 \end{cases} \quad .$$

The corresponding ODE inequality govering the global boundesness of the NSE solution is given in the form

$$y'(t) \leq c \cdot \begin{cases} \|u\|_1^2 \cdot y(t) & n = 2 \\ \|u\|_1^6 \cdot y^3(t) & n = 3 \end{cases}$$

For $n = 2$ this leads to a global boundedness estimate in the form $z'(t) \leq c \cdot \|u\|_1^2 \cdot z(t)$ resp. $z(t) \leq z(0) \cdot e^{c\int_0^t \|u\|_1^2(s)ds}$. For $n = 3$ there is no global bounded solution existing.

In case of $\alpha = -1/2$ one gets from Sobolevskii-estimates ((*), (GiY) lemma 3.2) the corresponding generalized "energy" inequality, given by

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu,u)_{-1/2}| \leq \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \leq c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \leq c\{\|u_0\|_{-1/2} + \|u_0\|_0^2\},$$

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$. For the norms $\|w\|_{q,\rho,T}^{\rho} := \int_0^T \|w(t)\|_{L_q}^{\rho} dt$ the (scaled) Serrin values are defined by

$$S(q,\rho) := \frac{n}{q} + \frac{2}{\rho}$$

The condition $S(q,\rho) \leq 1$ ensures convergent integrals. Uniqueness and regularity of NSE solutions are ensured, if $S(q,\rho) = n/2$. In case of space dimension $n = 3$, one knows, that for $q = 4$ and $\rho = 8$ the norm $|||w|||_{4.8.T} < \infty$ is bounded and, if a weak solution of the full linear case fulfills the Serrin condition $|||w|||_{4.8.T} < \infty$, then u is uniquely determined by the data f and u_0 (SoH).

On the other side, what is required from the NSE energy inequality, is

$$1 < S(q,\rho) < \frac{n}{2} = \frac{3}{2}$$

which leads to the Serrin gap problem of the 3-D non-linear, non-stationary NSE.

For later use we note that the counterpart of the (collision-free) NSE non-linear critical term in the Vlasov equation ((ChF) 7.2)

$$\frac{\partial}{\partial t}f + \boldsymbol{v} \cdot \nabla_x f + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial}{\partial v} f = 0 \ .$$

is given by the non-linear term $\mathrm{F}[\mathrm{f}] \cdot \nabla_{\mathrm{v}}\mathrm{f}$, whereby

$$F[f](t,x) := -\iint \nabla W(x-y) f(t,y,w) dw dy.$$

It is built under the assumptions, that the plasma is sufficiently hot (i.e. „plasma particle" collisions can be neglected) and, that the force F is entirely electromagnetic. The combined system with the related Vlasov-Poisson model

$$F = -\nabla W, \ -\Delta_x W = \rho, \quad W = \frac{1}{4\pi|x|} *_x \rho, \quad \rho(x,t) = \int_{R^n} f(x,v,t)dv$$

is called the Vlasov-Poisson-Boltzmann (VPB) system. The extension of the VPB system, where the Vlasov force F (or self-consistent force, or mean force …) is replaced by the Lorentz force determined by the electro-magnetic field created by the particles themselves, is described in (LiP1). The counterpart of the (NSE-) pressure p in the „Vlasov" case is the potential W, which is proposed to be replaced by its Riesz transform $\mathbf{W} = \mathrm{R}[\mathrm{W}]$. The counterpart of the Stokes operator with its intrinsic „reduced regularity" domain becomes the Vlasov operator $A^{(V)} := A_{Vlasov}$, defined by the linearized Vlasov equation (with the linearized term $(\nabla \mathrm{W} * \rho) \cdot \nabla_{\mathrm{v}} \mathrm{f}^0$) in an appropriate domain.

(*) (GiY) lemma 3.2.: For $0 \leq \delta < 1/2 + n \cdot (1 - 1/p)/2$ *it holds* $\left|A^{-\delta}P(u, grad)v\right|_p \leq M \cdot \left|A^{\theta}u\right|_p \cdot |A^{\rho}u|_p$ *with a constant* $M := M(\delta,\theta,\rho,p)$ *if* $\delta + \theta + \rho \geq n/2p + 1/2$, $\theta, \rho > 0, \theta + \rho > 1/2$. *Putting* $p := 2$, $\delta := 1/4$, $\theta := \rho := 1/2$ *fulfilling* $\theta + \rho \geq \frac{1}{4}(n+1) = 1$ *it follows*

$$\left\|A^{-\delta}P(u,grad)u\right\| \leq c\left\|A^{\theta}u\right\| \cdot \|A^{\rho}u\| = c\|u\|_{2\theta} \cdot \|u\|_{2\rho} = c\|u\|_1^2$$

resp.

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq |(Bu,u)_{-1/2}| \leq c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

The four Nature „forces" phenomena and corresponding self-adjoint operators

The proposed quantum/fluid state Hilbert space $\mathrm{H}_{-1/2}=H_0\otimes H_0^{\perp}$ resp. the corresponding quantum/fluid energy Hilbert space $\mathrm{H}_{1/2}=H_1\otimes H_1^{\perp}=H_{-1/2}^{*}$ comes along with a series of properties enabling an integrated mathematical framework, based on a common (space dimension independent) „mass element" concept. It enables an "one-energy" (field) concept and corresponding Partial Differential or Pseudo Differential equations specific manifestations/ forms of the considered "Nature forces". In other words, a „force" is the observed phenomenon of the considered physical situation, governed by the identical energy model concept. Depending from the considered physical problem area there are different candidates for corresponding self-adjoint operator definitions. We emphasize that an operator is only well-defined, if the mapping construction is combined with an approproriate domain:

(1) The Prandtl operator $\overline{\mathrm{P}}$, which is the double layer (hyper-singular integral) potential operator of the Neumann problem, fulfills the following properties ((LiI) Theorems 4.2.1, 4.2.2, 4.3.2):

i) the Prandtl operator $\overline{P}: H_r\rightarrow \widehat{H}_{r-1}$ is bounded for $0\le r\le 1$

ii) the Prandtl operator $\overline{P}: H_r\rightarrow \widehat{H}_{r-1}$ is Noetherian for $0<r<1$

iii) for $1/2\le r<1$, the exterior Neumann problem admits one and only one generalized solution.

Therefore, the Prandtl operator enables a combined (conservation of mass & (linear & angular) momentum balances) integral equations system, where the two momentum balances systems are modelled by corresponding momentum operator equations with corresponding domains according to $H_{1/2}=H_1\times H_1^{\neg}$.

(2) The Leray-Hopf projector is the matrix valued Fourier multiplier given by

$$P(\xi)=Id-\frac{\xi\otimes\xi}{|\xi|^2}=(\delta_{jk}-\frac{\xi_j\xi_k}{|\xi|^2})_{1\le j,k\le n}\quad , \; P=Id-R\otimes R=:Id-Q$$

resp.

$$P=Id-R\otimes R=:Id-Q=Id-\frac{D\otimes D}{D^2}Id-\Delta^{-1}(\nabla\times\nabla).$$

As the operator $Q:=R\otimes R=(R_jR_k)_{1\le j,k\le 1}=Q^2$ ($\mathrm{R_i}$ denote the Riesz operators) is an orthogonal projector, the Leray-Hopf operator is also an orthogonal projection, where the domain can be defined on each Hilbert scale. In (LeN1) an explicit expression fort he kernles of the Fourier multipliers of the corresponding Ossen operators are provided, which involves the incomplete gamma function and the confluent hypergeometric function of first kind.

(3) The collision operator of the Landau equation (see below) is given by

$$Q(f,f)=\frac{\partial}{\partial v_i}\left\{\int_{R^N}a_{ij}(v-w)\left[f(w)\frac{\partial f(v)}{\partial v_j}-f(v)\frac{\partial f(w)}{\partial w_j}\right]dw\right\}$$

with

$$a_{ij}(z)=\frac{a(z)}{|z|}\left\{\delta_{ij}-\frac{z_iz_j}{|z|^2}\right\}=\frac{a(z)}{|z|}P(z):=\frac{1-[1-a(z)]}{|z|}[Id-Q](z)\quad Q(z):=(R_iR_j)_{1\le i,j\le N}$$

and $\mathrm{a(z)}$ symmetric, non-negative and even in z and with an unknown function f corresponding at each time t to the density of particle at the point x with velocity v. It can be approximated by a linear Pseudo Differential Operator (PDO) of order zero with symbol

$$b_{ij}(z)=z\cdot a_{ij}(z)=\frac{z}{|z|}\left\{\delta_{ij}-\frac{z_iz_j}{|z|^2}\right\}=\frac{z}{|z|}P(z):=\frac{z}{|z|}[Id-Q](z)$$

whereby $a_{ij}(z)$ denotes the symbol of the Oseen kernel (LeN).

(4) With regards to the Maxwell equations the components of the electric and magnetic field forces $\underline{E}$, $\underline{H}$ build the 4-dimensional electromagnetic field force tensor $F_{ik} = (\underline{E}, \underline{H})$. The Maxwell stress tensor is given by $\sigma_{ik} = \frac{1}{4\pi}\left\{-E_i E_k - H_i H_k + \frac{1}{2}\delta_{ik}(E^2 + H^2)\right\}$.

(5) In (CoM) for the time-harmonic Maxwell equations (KiA), there is a coercive bilinear form (for the Sobolev space H_1) provided, containing tangential derivatives of the normal and tangential components of the field on the boundary, vanishing on the subspace H_1. Thus the variational formulations of „electric" or „magnetic" boundary value problems with homogeneous bondary conditions are not changed.

(6) With regards to "initial data for the Cauchy problem in general relativity" we refer to the corresponding lecture notes ((PoD) lecture 2) with the applied Weyl tensor representation

$$C_{iklm} = R_{iklm} - \frac{1}{2}R_{il}g_{km} + \frac{1}{2}R_{im}g_{kl} + \frac{1}{2}R_{kl}g_{im} - \frac{1}{2}R_{km}g_{il} + \frac{1}{6}R(g_{il}g_{km} - g_{im}g_{kl})$$

The reduced Einstein equations representation is given by $R^H_{\alpha,\alpha\beta} \coloneqq -\frac{1}{2}g^{\gamma,\gamma}g_{\alpha\beta,\gamma\delta} + Q(g,\partial g) = 0$.

The reduced Einstein-Vacuum equations are quasilinear hyperbolic equations in the form

$$\boldsymbol{g}^{\alpha\beta}\partial_\alpha\partial_\beta \boldsymbol{g}_{ik} = N_{ik}(\boldsymbol{g},\partial\boldsymbol{g}) \ , \ i,k = 0,\dots,3$$

where N is quadratic in the first derivatives of the metric.

(7) The Riesz transforms (the n-dimensional generalization of the Hilbert transform) are special Calderón-Zygmund (Pseudo Differential, convolution) operators with symbols $m(\omega) \in C^\infty(R^n - \{0\})$, where $m(\mu\omega) = m(\omega)$, $\mu > 0$, where the mean of $m(\omega)$ on the unit sphere is zero and where it holds $m(\omega) = \frac{\omega_j}{|\omega|}$. They arise when study the Neumann problem in upper half-plane. The Riesz transforms

$$R_k u = -ic_n p.v.\int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y)dy \ \text{(with} \ \ c_n \coloneqq \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} \text{)}$$

commutes with translations and homothesis, having nice properties relative to rotation. Especially the latter one play a key role in the concepts of the proposed concept of „rotating differentials" with respect to the rotation group $SO(n)$:

> let $m \coloneqq m(x) \coloneqq (m_1(x), \dots m_n(x))$ be the vector of the Mikhlin multipliers of the Riesz operators and $\rho = \rho_{ik} \in SO(n)$, then it holds $m(\rho(x)) = \rho(m(x))$ (i.e. $m_j(\rho(x)) = \sum \rho_{jk} m_k(x)$), because of

$$m(\rho(x)) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(x\rho^{-1}(y)) + log\left|\frac{1}{x\rho^{-1}(y)}\right|)\frac{y}{|y|}d\sigma(y) = c_n \int_{S^{n-1}} (\frac{\pi i}{2} sign(xy) + log\left|\frac{1}{xy}\right|)\frac{y}{|y|}d\sigma(y) \ .$$

The Riesz operators are related to the Caldéron- Zygmund operators $T(f) = S * F$ with a distribution S defined by a homogeneous function of degree zero, satisfying a kind of average mean zero condition on the unit sphere with its underlying rotation invariant probability measure (MeY). The search for conditions of minimal regularity in the context of the „pointwise multiplication" operator A is about an analysis of the commutator $[T, A]$. This leads to the „Caldéron operator"

$$(\Lambda u)(x) = (\sum_{k=1}^n R_k D_k u)(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^n p.v.\int_{-\infty}^{\infty} \sum_{k=1}^n \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy = -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v.\int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta\Lambda^{-1})u(x)$$

with symbol $|\nu|$ and its inverse operator ((EsG) (3.15), (3.17), (3.35))

$$(\Lambda^{-1}u)(x) = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v.\int_{-\infty}^{\infty} \frac{u(y)}{|x-y|^{n-1}} dy \ , \ n \geq 2 \ .$$

In dimension 1, this is about $\Lambda = DH$ where H denotes the Hilbert transform and D the Schrödinger momentum operator in the form $P \coloneqq D = -i\frac{d}{dx}$ ((MeY) p. 5). The Schrödinger momentum operator in dimension n, and its related Hamiltonin operator is given by $P \coloneqq -i\hbar\nabla = \frac{\hbar}{i}\nabla$ resp. $H \coloneqq -\frac{\hbar^2}{2m}\Delta = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2$.

The operator concerned with the time-harmonic Maxwell equation and the radiation problem is the D'Alembert operator related to the wave equation:

$$\Box u := \ddot{u} - \Delta u \quad .$$

The electrodynamic in the special relativity theory is described by the four-vector formalism of the space-time given by the equation $\Box \vec{A} = \frac{4\pi}{c}\vec{j}$, with the four-vector potential $\vec{A}$, where its curvature determines the electric and magnetic field forces, and j denotes the four-current-density.

The Einstein operator is given by

$$G = R_{ik} - R\frac{g_{ik}}{2}$$

with the corresponding gravity field equations

$$G = -\kappa T_{ik}$$

with the corresponding motion equations

$$\frac{d}{d\tau}\left(g_{\mu,\nu}\frac{dx^{\mu}}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial x^{\alpha}}{\partial\tau}\frac{\partial x^{\beta}}{\partial\tau}$$

for the path $x^{\mu} = x^{\mu}(t)$ of a particle. The change from the Newton model is about a change from the potential equation to the Einstein equation

$$-\Delta\Phi = -4\pi k\rho \qquad \rightarrow \qquad G = -\kappa T_{ik}$$

and a change from the motion equations

$$\frac{d^2\vec{x}}{dt^2} = -grad\Phi \qquad \rightarrow \qquad \frac{d}{d\tau}\left(g_{\mu,\nu}\frac{dx^{\mu}}{d\tau}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\frac{\partial x^{\alpha}}{\partial\tau}\frac{\partial x^{\beta}}{\partial\tau}.$$

Instead of one potential equation we now have 10 equations with 10 potentials Φ_{ik}; instead of a linear operator, we now have a non-linear operator, i.e. the gravity potential is no longer the sum of single gravitation potentials. Additionally there is a circle structure, i.e. the potentials are a functions of the T_{ik} ($\Phi_{ik} = f(T_{ik})$), while the space-time structure is a function of the potentials ($f(\Phi_{ik})$). The matter, as described by the energy-momentum tensor T_{ik}, reflecting the principles of energy and momentum conservation, generates a curvature of the space-time and particles move along of geodesics (*).

(*) (RoC) 1.1.3: The physical meaning of general relativity (GR): GR is the discovery that spacetime and the gravitational field are the same entity. What we call „spacetime" is itself a physical object, in many respects similar to the electromagnetic field. We can say that GR is the discovery hat there is no spacetime at all. What Newton called „space", and Minkowski called „spacetime", is unmasked: it is nothing but a dynamic object – the gravitational field – in a regime in which we neglect its dynamics. …., the universe is not made up of fields on spacetime; it is made up of fields on fields.

The Bel-Robinson tensor to prove the nonlinear gravitational stability of the Minkowski space-time

The Einstein field equations are proposed to be re-formulated as a weak (!) least action minimization problem by correspondingly defined variational equations representation with initial value functions, in line with the alternatively proposed radiation model:

The building principles for an appropriately defined variational representation is about finding a the way,

(1) how "*Space-time geometry "tells" mass-energy how to move*", can be obtained by those representation and that the multiple tests (observed phenomena) of the geometrical structure and of the geodesic equation of motion

(2) "*where mass-energy "tells" space-time geometry how to curve*") is modelled (as a kind of symmetry break down) as approximation solution in the compactly embedded sub-spaces H_0 resp. H_1 of $\mathrm{H}_{-1/2}$ resp. $\mathrm{H}_{1/2}$.

The Maxwell field strength tensor is constructed from the exterior derivative of the Maxwell vector potential $(\varphi, \vec{A})$, which is a 1-form (a gauge field). The standard exterior derivative is proposed to be replaced by the corresponding Plemelj concept.

The energy-momentum tensor of an electromagnetic field for the solutions of the Maxwell equations plays precisely the same role as the Bel-Robinson tensor (that is a four tensor quadratic in a Weyl tensorfield W fulfilling $*(*W) = -W$)

$$Q_{\alpha\beta\gamma\delta} = \frac{1}{2}(W_{\alpha\mu\beta\rho}W_{\gamma\delta}^{\mu\rho} + * W_{\alpha\mu\beta\rho} * W_{\gamma\delta}^{\mu\rho})$$

for the solutions of the Bianchi equations *(*)*.

In (ChD1) the Bel-Robinson tensor is applied to prove the nonlinear gravitational stability of the Minkowski space-time *(**)*. The ideas around the Bianchi equation of an E-V space-time (the electric-magnetic decomposition, null decomposition of a Weyl field, null-structure equation in space-time) are at the heart of the analysis in (ChD1).

(*) (KLS1) 3.2: „The primary example of the solution of the Bianchi equations is the Riemann curvature tensor of an Einstein vacuum space-time „The Bianchi equations look complicated. This is obvious formally, but it becomes even apparent if we decompose the Weyl field W into its „electric" and „magnetic" parts. …. The two covariant symmetric traceless tensor fields $E = i_{(T,T)}W$ and $H = i_{(T,T)} * W$, tangent to the hypersurface determines completely the Weyl tensor field. The corresponding Bianchi equations for this decomposition are given by the following Maxwell-type equations:

$$\Phi^{-1}\partial_t E + curl H = \rho(E,H),\ \Phi^{-1}\partial_t H - curl E = \sigma(E,H)$$
$$div E = k \therefore H\ ,\ div H = -k \therefore E$$

The explicit expressions of $\rho(E,H)$ and $\sigma(E,H)$ can be found in (ChD), p. 146. The strong formal analogy with the Maxwell equations goes even further. In fact, the Bianchi equations possess a tensor analogous to the energy-momentum tensor, the Bel-Robinson tensor

$$Q_{\alpha\beta\gamma\delta} = \tfrac{1}{2}(W_{\alpha\mu\beta\rho}W_{\gamma\delta}^{\mu\rho} + * W_{\alpha\mu\beta\rho} * W_{\gamma\delta}^{\mu\rho})\ ."$$

(**) (ChD1): The problem of stability of the Minkowski space-time is closely related to that of characterizing the space-time solution of the E-V equations, which are globally asymptotically flat – as defined in physics literature, space-times that becomes flat as we approach infinity in any direction. Despite the central importance that such space-times have in General Relativity as corresponding to isolated systems, it is not at all settled how to define them correctly, consistent with the field equations. … The present state of understanding was set by Penrose (PeR2), (PeR3), who formulized the idea of asymptotics flatness by adding a boundary at infinity attached through a smooth conformal compactification. However, it remains questionable whether there exists any nontrivial solution of the field equations that satisfies the Penrose requirements. Indeed, his regularity assumptions translate into fall-off conditions of a curvature that may be too stringent and thus may fail to be satisfied by any solution that would allow gravitational waves. Moreover, the picture given by conformal compactification fails to address the crucial issue of the relationship between conditions in the past and behavior in the future. …. We believe that a real understanding of assymptotically flat spaces can only be accomplished by constructing them from initial data and studying their asymptotiv behavior. In addition, only such a construction can address the crucial issue of the relationship between conditions in the past and behavior in the future, an issue that the conformal compactification leaves entirely open. …

In the least precise version our main result asserts the following:

Theorem 1.0.1 (First version of Main Theorem, 2nd & 3rd versions pp. 17 & 298) Any strongly asymptotically flat initial data set that satisfies, in addition, a global smallness assumption, leads to a unique, globally hyperbolic, smooth, and geodesically complete solution of the E-V equations. Moreover, this development is globally asymptotically flat, by which we mean that its Riemann curvature tensor approaches zero on any causial or spacelike geodesic, as the corresponding affine parameter tends to infinity."

A proper Cauchy problem formulation of the Einstein-Vacuum equations $\boldsymbol{R}_{\alpha\beta}(\boldsymbol{g})=0$, where $\boldsymbol{g}$ is an unknown four dimensional Lorentz metric and $\boldsymbol{R}_{\alpha\beta}$ is its Ricci curvature tensor, is about finding a metric g on $\sum_0$ coinciding with the Riemannian metric g_{ij} and that the tensor k_{ij} is the second fundamental form of the hypersurface $\sum_0 = t = 0$. The latter property can be expressed as follows. Let T denote the unit vector field normal to the level hypersurfaces of the time foliation $\sum_t$. Then $k_{ij} = -\frac{1}{2} L_T g_{ij} \rceil \sum_0$, where L_T denotes the Lie derivative in the direction of the vector field T (KlS). The Einstein field equations are overdetermined, i.e. from a mathematical point of view they are not well defined. As a „physical problem" consequence, there is the so-called „*gauge freedom*" of the Einstein field equations, allowing a special choice of gauge to resolve given ambiguities, e.g. special wave coordinates $x^\alpha, \alpha = 0, \dots$ with $\nabla g_{\alpha\beta}(0) \in H^{s-1}(\sum_0)$ and $\dot{g}_{\alpha\beta}(0) \in H^{s-1}(\sum_0)$ for $s \geq 2 + \varepsilon$ (KlS).

As the causal structure of an arbitrary Einstein space-time can have undesirable pathologies. In (ChD1) the existence of a Cauchy hypersurface is postulated, which is a hypersurface with the property that any causal curve intersects it at precisely one point. Such space-times allow the existence of a globally defined differentiable function t (called time function) whose gradient $\boldsymbol{D}t$ (whereby $\boldsymbol{D}$ denotes the covariant differentiation) is timelike everywhere. The foliation given by its level surfaces is called a t-foliation. Topologically, a space-time foliated by the level surfaces of a time function is diffeomorphic to a product manifold $\sum \times \boldsymbol{D}$, where $\sum$ is a 3-dimensional manifold.

With respect to the proposed decomposition of the newly proposed „fermions" Hilbert space we note that the condition that gravity is always be attractive (HaS) or not is given by the energy-momentum tensor inequalities

$T_{ab} = V^a V^b \geq \frac{1}{2} V_a V^b T$ resp. $T_{ab} = V^a V^b < \frac{1}{2} V_a V^b T$ for any time-like vector V^a,

With respect to the regularity requirements for the standard theory we note that in (HaS) it is adopted that space-time consists only of points at which the metric is Lorentzian and suitable differentiable (say C^2). The proposed Hilbert space framework enables also the building of hyperboloids with corresponding hyperbolic and conical regions ((VaM) and below), to build a Hilbert space framework, overcoming the the current handicaps of the solutions in (ChD1) $(*)$:

(*) (ChD1): The **main difficulties** one encounters in the proof of our result are (1) The **problem of coordinates**, and (2) **The strongly nonlinear hyperbolic features of the Einstein equations**.

. (1) The problem of coordinates is the first major difficulty one has to overcome when trying to solve the Cauchy problem for the Einstein equations. In short, one is faced with the following dilemma: … coordinates seem to be necessary even to allow the formulation of well-posed Cauchy problems and a proof of a local in time existence result. Nevertheless, as the particular case of wave coordinates illustrates, the coordinates may lead, in the large, to problems of their own. ….

. (2) The other major obstacle in the study of the Einstein equations consists in their hyperbolic and strongly nonlinear character. The only powerful analytic tool we have in the study of nonlinear hyperbolic equations in the physical space-time dimension are the energy estimates**. Yet the classical energy estimates are limited to proving results that are local in time**. The difficulty has to do with the fact that, in order to control the higher energy norms of the solutions, one has to control the integral in time of their bounds in uniform norm. …new techniques were developped, based on modified energy estimates and the invariance property of corresponding linear equations, which were applied to prove global or long-term existence results for nonlinear wave equations… one uses the Killing and conformal Killing vectorfields generated by the conformal group of the Minkowski space-time to define a global energy norm that is invariant relative to the linear evolution. The precise asymptotic behavior, including the uniform bounds previously mentioned, are then an immediate consequence of a global version of the Sobolev inequalities (KlS2), (KlS3). ….The relevant linearized equations for the E-V equations are the Bianchi equations in Minkowski space-time. … Its complete asymptotic properties are analyzed by using only energy estimates and the conformal invariance properties of the equations. …To derive a global existence result, however, one also needs to investigate the structure of the nonlinear terms. It is well known that arbitrary quadratic nonlinear perturbances of the scalar wave, even when derivable from a Lagrangean, could lead to formation of singularities unless a certain structural condition, which we have called the null condition, is satisfied. It turns out that the appropriate, tensorial version of this structural condition is satisfied by the Einstein equations. **Roughly speaking, one could say that the troublesome nonlinear terms, which could have led to formation of singularities, are in fact excluded due to the covariance and algebraic properties of the Einstein equations**. … These basic algebraic properties of the Einstein equations, which allow us to prove the global existence result, are **in sharp contrast with the nonlinear hyperbolic equations of classical continuum mechanics**. Indeed, the equations of nonlinear elasticity (JoF) and compressible fluids (SiT), in four space-time dimensions, form singularities even for arbitrary small initial conditions."

Singularities in the general relativity theory
This is about a quote of the abstract of (TrH):

„Regular solutions of EINSTEIN's equations mean very different things. In the case of the empty-space equations, $R_{ik}=0$, such solutions must be metrics $g_{ik}(x^l)$ without additional singular „field sources" (Einstein's Particle problem"). However the „phenomemological matter" is defined by thr EINSTEIN equations $R_{ik}-\frac{1}{2}g_{ik}R=-\mu T_{ik}$ itselves. Therefore if 10 regular functions $g_{ik}(x^l)$ are given (which the inequalities of LORENTZ-signature fulfill) then these g_{ik} define 10 functions $T_{ik}(x^l)$ without singularities. But, the matter-tensor T_{ik} must fulfil the to inequalitites $T\geq 0, T_0^0\geq \frac{1}{2}T$ only and therefore the EINSTEIN-equations with „phenomenological matter" must mean two inequalities $R\geq 0, R_0^0\leq 0$, which are incompatible with permanently regular metric with LORENTZ-signature, generally."

Penrose's speculation about time (1989)

(PeR4) p.443-4: *„I suggest that we may actually be going badly wrong when we apply the usual physical rules for time when we consider consciousness! . . . My guess is that there is something illusory here. . .and the time of our perceptions does not 'really' flow in quite the linear forward-moving way that we perceive it to flow (whatever that might mean!). The temporal ordering that we 'appear' to perceive is, I am claiming, something that we impose upon our perceptions in order to make sense of them in relation to the uniform forward time-progression of an external physical reality*."

In (RoC) 2.4.4 the meanings of time are considered.

A new type of cosmological solutions of the gravity field equations
This is about a the first section of the paper from K. Gödel (GöK):

All cosmological solutions with non-vanishing density of matter known at present have the common property that, in a certain sense, they contain an „absolute" time coordinate, owing to the fact that there exists a one-parametric system of three-spaces everywhere orthogonal on the world lines of matter. It is easily seen that the non-existence of such a system of three-spaces is equivalent with a rotation of matter relatively to the compass of inertia. In this paper I am proposing a solution (with a cosmological term $\neq 0$) which exhibits such a rotation. This solution, or rather the four-dimensional space S which it defines, hast he further properties: (1) – (9), e.g.

(1) S is homogeneous

(2) …. so that any two world lines of matter are equidistant

(3) S has rotational symmetry

(4) … That is, a positive direction of time can consistently be introduced in the whole solution

(5) It is not possible to assign a time coordinate to each space-time point in such a way that the coordinate always increases, if one moves in a positive time-like direction; …

(6) … it is theoretically possible in these worlds to travel into the past, or otherwise influence the past

(7) There exist no three-spaces which are everywhere space-like and intersect each world line of matter in one point

(8) … an absolute time does not exist, even if it is not required to agree in direction with the times of all possible observers (where absolute means: definable without reference to individual objects, such as e.g. a particular galactic system).

(9) Matter everywhere rotates relatively to the compass of inertia with the angular velocity $2\sqrt{\pi\mu\rho}$, where ρ is the mean density of matter and μ Newton's gravitational constant.

The shift of the Hilbert scale value from $\alpha = 0$ to $\alpha = -1/2$ closes the above "*Serrin gap*". It might be seen as a purely mathematical trick to overcome a physical model. It is just the other way around, which should become clear, when looking at physical problems areas, like

- the physical "*mass gap*" problem in the YME (*)
- the mathematical eigenvalue and eigenfunction solutions of the number operator of the harmonic quantum oscillator starting with index $n = 0$, not as physically required with $n = 1$
- (the number operator is the product of the generation and annihilation operators)
- the missing „*how?*" interaction of attractive and/or repulsive fermions
  e.g. to distinguish between unperturbed cold and hot plasma
- the missing Huygens' principle in any quantum (gravity) model
- the (by Einstein claimed) identity „ *heavy mass = inertial mass* "
- the dynamic space-time variable in the Einstein equations
- the covariant elliptic vs. hyperbolic type equations in the Riemannian vs. Einstein geometry
- Einstein's quadratic mean energy formula of an electric oscillator with a given frequency based on Planck's radiation law (**)
- the missing „*initial value inflation field*" model for a well-posed Cauchy Einstein-Vacuum field equations problem
  the hyperbolic structure and the strongly nonlinear charachter of the Einstein equation, where classical energy estimates cannot be applied to prove global or long-term existing solutions (***).

We mention that the Brownian motion is given by the Gaussian function and that the white noise can be defined as the derivative of a Brownian motion, i.e. a Brownian motion is obtained as the integral of a white noise signal $dB(t)$. It does not exist in the ordinary case: all derivatives of the Brownian motion are generalized functions on the same space.

(*) The classical Yang-Mills theory is a generalization of the Maxwell theory of electromagnetism where the chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles /gluons. However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the mass gap. Another aspect of confinement is asymptotic freedom which makes it conceivable that quantum Yang-Mills theory exists without restriction to low energy scales. The problem is to establish rigorously the existence of the quantum Yang-Mills theory and a mass gap.

(**) $\langle \varepsilon^2 \rangle = \left(h\omega\rho + \frac{c^3}{8\pi v^2}\rho^2 \right) v dv$; $\langle \varepsilon \rangle = \frac{1}{2}\hbar\omega + \frac{\hbar\omega}{e^{\frac{\hbar\omega}{kT}} - 1}$ with the divergent zero point energy $\varepsilon = \frac{1}{2}\hbar\omega$ for $T = 0$

(***) (ChD): The simplest solution of the Einstein-Vacuum (E-V) equations is the Minkowski space-time R^{3+1} that is the space R^4 together with a given Einstein metric, and a canonical coordinate system (x^0, x^1, x^2, x^3) such that $\langle \partial_\alpha, \partial_\beta \rangle = \quad {}_{\alpha\beta},\ \alpha, \beta = 0,1,2,3.$ At the present time it is not known whether there are, apart from the Minkowski space, any smooth, geodesically complete solution, which becomes flat at infinity on any given spacelike direction. Any attempt to simplify the problem significantly by looking or solutions with additional symmetries fails as a consequence of the well-known results of Lichnerowicz for static solutions and Birkhoff for spherically symmetric solutions. According to Lichnerowicz, a static solution that is geodesically complete and flat at infinity on any spacelike hypersurface must be flat. The Birkhoff theorem asserts that all spherically symmetric solutions of the E-V equations are static. Thus, disregarding the Schwarzschild solution, which is not geodesically complete, the only such solution that becomes flat at spacelike infinity is the Minkowski space-time.

The problem of stability of the Minkowski space-time is closely related to that of characterizing the space-time solution of the E-V equations, which are globally asymptotically flat – as defined in physics literature, space-times that becomes flat as we approach infinity in any direction. Despite the central importance that such space-times have in General Relativity as corresponding to isolated systems, it is not at all settled how **to define them correctly, consistent with the field equations**. ... The present state of understanding was set by Penrose (PeR2), (PeR3), who formulized the idea of asymptotics flatness by adding a boundary at infinity attached through a smooth *conformal compactification*. However, it remains questionable whether there exists any nontrivial solution of the field equations that satisfies the Penrose requirements. Indeed, his regularity assumptions translate into fall-off conditions of a curvature that may be too stringent and thus may fail to be satisfied by any solution that would allow gravitational waves. Moreover, the picture given by conformal compactification fails to address the crucial issue of the relationship between conditions in the past and behavior in the future.

Mathematical cornerstones for an integrated Hilbert space/scale theory

Mathematical cornerstones for an integrated Hilbert space/scale theory (also addressing the „hidden variables in quantum theory" concept of D. Bohm (BoD)) are

- Pseudo Differential Operators
- approximation theory in Hilbert scale ((NiJ), (NiJ1))
- the theory of spaces with an indefinite metric defining manifold, which represents a hyperboloid with corresponding hyperbolic and conical regions
- the wavelet theory (FaM), (HoM)
- methods to solve (nonlinear) complementary extremal problems
- undistorted spherical travelling waves characterizing a space-time frame with dimension $n = m + 1 = 4$ (*).

The proposed Hilbert space framework (with its relationship to the Zeta function theory and the Hilbert-Polya/ Berry-Keating conjecture) enables a combined usage of spectral theory, variational methods for non-linear operators (VaM), Galerkin-Ritz approximation theory (VeW), and tools like Pseudo-Differential operators ((EsG), (LoA), (PeB)), degenerated hypergeometric functions (GrI), Hilbert (resp. Riesz) transform(s) and wavelets (HoM).

The link between PDO and the Galerkin-Ritz approximation theory is given by the Garding inequality and the concept of hypoellipticity ((AzA), (GaL), (PeB)). The norms of the Hilbert scale H_α can be enriched with an additional norm enjoying an "exponential decay" behavior. Each Hilbert space norm with a<0 is governed by the norm of the Hilbert space H_0 and this "exp-decay" norm. This property is proposed to be applied in the context of the decomposition of the Hilbert space $\mathrm{H}_{-1/2} = H_0 \otimes H_0^\perp$ ((BrK), (BrK1), (BrK3), (BrK7)).

The today's standard quantum state resp. energy spaces are $H_0 = L_2$ resp. H_1, i.e. those Hilbert spaces are compactly embedded subspaces of the proposed new ones.

The Hilbert space in (BaB) (in the context of a RH criterion) is about of all sequences $a = \{a_n | n \in N\}$ of complex numbers such that

$$\sum_{n=1}^{\infty} \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \leq \omega_n \leq \frac{c_2}{n^2}$$

which is isomorph to the Hilbert space $H_{-1} \cong l_2^{-1}$. Let $\gamma := \{11,1,1,......\}$ then it holds

$$\|\gamma\|_{-1}^2 = \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e. $\gamma \in l_2^{-1} \cong H_{-1}$. For the Zeta function on the critical line it holds $\Xi \in \mathrm{H}_{-1}(-\infty, \infty)$, i.e. there exists a Zeta function representation a Hermitian operator in a weak $\mathrm{H}_{-1/2}$ −sense in the form $\bar{\mathrm{H}}[\bar{\mathrm{u}}] = \Xi \in \mathrm{H}_{-1/2}$ given by

$$\bar{u} \in H_{1/2} : \ (\bar{u}, v)_{1/2} = (\Xi, v)_{-1/2} \ \ \forall v \in H_{1/2}.$$

The Berry-Keating conjecture puts the zeros of the Zeta function (on the critical line, if the RH is true) in relationship to the (energy level) eigenvalues associated with the classical Hermitian operator $H(x,p) = x \cdot p \sim x \cdot \frac{d}{dx}$, where x denotes the position coordinate and p the conjugate momentum. The Friedrichs extension of the variational representation of the Zeta function (on the critical) with L_2-test space indicated a $H_{-1/2}$ quantum state space with related $H_{1/2}$ energy space. In a weak $\mathrm{H}_{-1/2}$ representation Einstein's quadratic mean energy formula of an electric oscillator is given by

$$\langle \varepsilon^2 \rangle_{-1/2} = \left(h\omega\rho + \frac{c^3}{8\pi\nu^2} \rho^2 \right) d\nu .$$

An integrated Hilbert space model enabling a Nonstandard Model of Elementary Particles (NMEP)

For this section we also refer to ((BrK), (BrK1), (BrK8)). Applying the physical quantum (fluid) Hilbert (state) space $\mathrm{H}_{-1/2}$ to the 3-D non-linear, non-stationary NSE enables a well posed variational representation of the NSE with appropriate valid energy inequality, closing the Serrin gap problem.

The Standard Model of Elementary Particles (SMEP) is concered with gauges theory and variational principles. Each of the observed Nature „force" phenomena are related to a specific gauge group. The model does not provide any explanation where the related elementary „particles" are coming from (or have been generated out of „first mover" resp. out of mass-less photons) during the inflation phase of current big bang assumption and why their mass have their specific values.

The standard energy Hilbert space H_1 enables a differentiation of "elementary particles" with and w/o mass (modelled by the orthogonal decomposition of the Hilbert spaces $\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. $H_{1/2} = H_1 \otimes H_1^{\perp}$). The Hilbert space H_1 is proposed to be interpreted as *„ferminons mass/energy"* space; $H_1^{\perp}$ is proposed to be interpreted as the orthogonal *„bosons energy"* space. Both together build the newly proposed quantum energy space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$. The sub-space $H_1^{\perp}$ may be interpreted as zero point energy space containing "wave package" resp. „eigen-differential" „elements".

The decomposition of the quantum state space $H_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. the quantum energy space $H_{1/2} = H_1 \otimes H_1^{\perp}$ goes along with the Fourier wave resp. the Calderón wavelet tool (*). While the Fourier waves enable an analysis of the test space H_0, wavelets enable an alternative analyis tool for a specific densely embedded subspace of H_0, as the (wavelet) admissibility condition for a $\psi \in H_0$ is a weak one, as for each $\psi, \hat{\psi} \in H_0$: it holds $\|\psi_\varepsilon - \psi\|_{L_2}^2 \rightarrow 0$ for

$$\hat{\psi}_\varepsilon := \begin{cases} \hat{\psi}(\omega), & |\omega| \geq \varepsilon \\ 0, & else \end{cases}$$

In (FaM1) a review is provided for wavelet transforms and their applications to MHD and plasma turbulence.

Applying the physical quantum (fluid) Hilbert (state) space $\mathrm{H}_{-1/2}$ to a correspondingly defined variational representation of the Maxwell equations enables a quantum field model, which overcomes the current mass gap problem of the YME.

The macroscopic and microscopic state of quanta relate to corresponding frequencies of its vibrations. The corresponding action variables of the system ((HeW) II.1.c) define the related kinematical (physical) and thermodynamical concept of "*time*", as described in ((RoC), (SmL)), (RoC1), section 13) (**), (***).

(*) (HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify

$$b \;\leftrightarrow\; \text{position}; \quad (a\Delta)^{-1} \;\leftrightarrow\; \text{enlargement}; \; g \;\leftrightarrow \text{optics."}$$

(**) (PeR): „one of the deepest mysteries of our universe is the puzzle of whence it came."

(***) (RoC1), section 13: "Our interaction with the world is partial, which is why we see it in blurred way. To this blurring is added quantum indeterminacy. The ignorance that follows from this determines the existence of a particular variable - thermal time - and of an entropy that quantifies our uncertainty. Perhaps we belong to a particular subset of the world that interacts with the rest of it in such a way that this entropy is lower in one direction of our thermal time."

The wavelet tool to govern the alternative $H_1^{\perp}$ ground state energy (Hilbert) space

The proposed quantum state Hilbert space $\mathrm{H}_{-1/2}$ goes along with a replacement of of Dirac „function" model $\delta \in \mathrm{H}_{-n/2-\varepsilon}$ ($\varepsilon > 0$, $\mathrm{n} =$ space dimension) of the <u>point</u> <u>mass</u> <u>density</u> <u>of an</u> <u>idealized point mass</u> by Plemelj's <u>mass element</u> definition. The decomposition of the quantum state space $H_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. the quantum energy space $H_{1/2} = H_1 \otimes H_1^{\perp}$ goes along with the Fourier wave resp. the Calderón wavelet tool. The admissibility condition for wavelets goes along with its dual space $H_{-1/2}^{*} = \mathrm{H}_{1/2}$. The admissibility condition for the definition of a wavelet ensures the validity of the inverse wavelet transform, which is valid for all Hilbert scale values. A L_2 – based Fourier wave analysis is the baseline for statistical analysis, as well as for PDE and PDO theory. Therefore, the decomposition of the quantum state space $H_0 \otimes H_0^{\perp}$ resp. the quantum energy space $H_1 \otimes H_1^{\perp}$ is very much related to the „hidden variables in quantum theory" concept of D. Bohm (BoD). We further note that for a convenient choice of the two wavelet functions the Gibbs phenomenon disappears and that the Hilbert transform of a wavelet is again a wavelet.

While the Fourier waves enable an analysis of the test space H_0, wavelets enable an alternative analyis tool for a specific densely embedded subspace of H_0, as the (wavelet) admissibility condition for a $\psi \in H_0$ is a weak one, as for each $\psi, \hat{\psi} \in H_0$: it holds $\|\psi_\varepsilon - \psi\|_{L_2}^2 \to 0$ for

$$\hat{\psi}_\varepsilon := \begin{cases} \hat{\psi}(\omega), & |\omega| \geq \varepsilon \\ 0, & else \end{cases}$$

A L_2 – based Fourier wave analysis is the baseline for statistical analysis, as well as for PDE and PDO theory.

The wavelet duality relationship provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "*microscope observations*" of <u>*two*</u> wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula (***). The prize to be paid is about additional efforts, when re-building the reconstruction wavelet.

The concept of an *optical function* (***) is an essential tool in the strategy to overcome technical difficulties to overcome the problems of „coordinates", and the „strongly nonlinear hyperbolic features of the Einstein equations" for a global stability of the Minkowski space (*). It is basically about appropriately modified Killing and conformal Killing vectorfields in the definition of the basic norm (**).

(*) (ChD1) p. 12-15: The **main difficulties** one encounters in the proof of our result are (1) The **problem of coordinates**, and (2) **The strongly nonlinear hyperbolic features of the Einstein equations**.

(**) (ChD1) p. 15-16: It is applied to govern the mass term that appears in the Schwarzschild part of an strongly asymptotically flat initial data set, which has the long-range effect of changing the asymptotic position of the null geodesic cones relative to the maximal foliation. They are expected to diverge logarithmically from their corresponding position in flat space-time. In adition to this, their asymptotic shear differs drastically from that in the Minkowski space-time. The difference reflects the presence of gravitational radiation in any nontrivial perturbance of the Minkowski space-time. To take this effect into account an optical function is constructed, whose level surfaces are outgoing null hypersurfaces related by a translation at infinity.

(***) (HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify

$b \leftrightarrow$ position; $(a\Delta)^{-1} \leftrightarrow$ enlargement; $g \leftrightarrow$ optics. "

A wavelet transform is defined by

$$\widehat{W_\vartheta[f]}(a,\omega) := (2\pi|a|)^{\frac{1}{2}} c_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-a\omega)\hat{f}(\omega).$$

For $\varphi, \vartheta \in L_2(R)$, $f_1, f_2 \in L_2(R)$ from the so-called admissibilty condition, given by

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\bar{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

one gets the duality relationship

$$(W_\vartheta\, f_1, W_\varphi^* f_2)_{L_2\left(R^2, \frac{dadb}{a^2}\right)} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$$ i.e. $W_\varphi^* W_\vartheta\ [f] = c_{\vartheta\varphi} f$ in a L_2 –sense.

It is the counterpart of the Fourier inverse with respect to the wavelet theory proven by Calderón´s reproducing formula (MeY)

$$f = \int_0^\infty \varphi_a * \varphi_a * f \frac{da}{a}$$

with

$$\varphi_a(x) = a^{-n}\varphi(\tfrac{x}{a}),\ \int_0^\infty \frac{|\hat{\varphi}(a\omega)|^2}{a} da = 1.$$

The simplest possible wavelet is the Haar wavelet

$$\psi(x) := \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x \le 1 \\ 0 & else \end{cases}$$

with its corresponding Fourier transform

$$\hat{\psi}(\omega) = \frac{i}{\sqrt{2\pi}} e^{-\frac{i\omega}{2}} sinc\left(\frac{\omega}{4}\right) \cdot \sin\left(\frac{\omega}{4}\right) = \frac{i}{\sqrt{2\pi}} e^{-\frac{i\omega}{2}} \frac{\omega}{4} \cdot \left[\frac{\sin(\omega/4)}{\omega/4}\right]^2 \ (\hat{\psi}(0) = 0).$$

The Hilbert transform of the Haar scaling function decays only as $1/|x|$, while the Hilbert transform of the Haar wavelet

$$H[\psi](x) = \frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{|x-y|>\varepsilon} \frac{t\psi(t)dt}{x(x-t)}$$

(which is again a wavelet) shows a better decay of $1/|x|^2$ (ChK1).

For the continuous wavelet transform on R^n the admissibility condition is given by

$$\int_{R^n} \frac{|\hat{\varphi}(\omega)|^2}{|\omega|^n} d\omega < \infty$$

The extension of the Hilbert transform to higher dimensions leads to the Riesz operators. In (StE), (StE1) their relationship to the concpet of conjugate harmonic functions in several variables are considered. The corresponding analogy for the Cauchy-Riemann differential equations to space dimension 3 is provided in (RuC). In (ArN) the Riesz transforms on spheres are explored.

In (DaS) families of wavelets are constructed that minimize an uncertainty relation associated with square integrable representations of some canonical groups. Especially, there is a new interpretation of the Mexican hat function provided.

In (PeM) representations of solutions of the wave equation based on relativistic wavelets are provided.

The question of the appropriate common set of postulates to derive a unified general relativity and quantum mechanics theory

The GRT is built on Riemann´s mathematical concept of „manifolds"; we emphasis, that the mathematical model of the GRT even requires „*differentiable*" manifolds, whereby only *continuous* manifolds are required by physical GRT modelling aspects, w/o taking into account any appropriate quantum theoretical modelling requirements. Therefore, challenging the „continuity" concept, taking into account also its relationship to the quantum theory Hilbert space framework H_a and the related Sobolev embedding theorem, leads to the proposed replacement of the Dirac function concept by an alternative $H_{-1/2}$ −quantum state Hilbert space, which is independently defined and applicable from any space dimension n.

The Sobolev embedding theorem states, that $\mathrm{H_k}$ is a sub-space of C^0 (continuous functions) for $k > n/2$. In other words, there is no concept of "continuous velocity/momentum" in the proposed Hilbert space framework, i.e. there is no Frechet differential existing ((VaM) 3.3). This refers to one of the several proposals, which have been made to drop some of the common sense notions about the universe ((KaM) 1.1), which is about continuity, i.e. space-time must be granular. The size of these grains would provide a natural cutoff for the Feynman integrals, allowing to have a finite S-matrix.

The concept of „continuity" was one element of the list in (KuM) („continuity", „causality", unitary", locality", „point particle") to be challenged to get rid off, in order to enable a consistent GRT and quantum theory (*).

The proposed variation Hilbert space frame is built on the space-time frame with dimension $n = m + 1 = 4$. Therefore the Huygens' Principle (which is also valid for the initial value problem of the wave equation) is valid for all considered "wave" PDE, overcoming e.g. the $n > 10$ requirement of the string theory. At the same time, the characteristics roles of a space-time dimension = 4 is also underlined by the specific role of undistorted spherical travelling waves (**).

In (KiA1) relatively undistorted wave solutions of the form $u = gf(\theta)$, of the wave equation in three space variables are considered, where $\theta = \theta(x, y, z, t)$ and $g = g(x, y, z, t)$ are the phase and the amplitude, and f is an arbitrary wave form function of a single variable. The plane and spherical waves are explicite solutions of this problem. In those cases the phase does not determine the amplitude uniquely. In the plane wave, g can be multiplied by an arbitrary harmonic function, and in the spherical wave, g can be multiplied by any arbitrary function harmonic on the sphere. Choosing the delta function as f one obtain a diffusionsless solution of the two-dimensional wave equation, which still not contradict to the Huygens principle.

(*) (KaM): "Because general relativity and quantum mechanics can be derived from a small set of postulates, one or more of these postulates must be wrong. The key must be to drop one or more of these assumptions about Nature on which we have constructed general relativity and quantum mechanics. Over the years several proposals have been made to drop some of our commonsense notions about the universe: continuity, causality, unitarity, locality, point particles."

(**) (CoR) p. 763: „relatively undistorted spherical waves relate to the problem of transmitting with perfect fidelity signals in all directions. All we can do here is to formulate a conjecture which will be given some support in article 3:
Courant-Hilbert conjecture: „Families of sherical waves for arbitrary time-like lines exist only in the case of two or four variables, and then only if the differential equation is equivalent to the wave equation.

p. 765: A proof of this conjecture would show that four-dimensional physical space-time world of classical physics enjoys an essential distinction.

Altogether, the question of Huygens' principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem (see §5), a problem which is still completely open."

The NMEP in a Minkowski space and an alternative „Cycles of Time" concept (*)

The proposed energy Hilbert space is

$$H_{1/2} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}$$

in a Minkowski space-time framework. The corresponding elements of the dual Hilbert space $\mathrm{H}_{-1/2}$ model space-time states, while their related "space-time energy" elements are governed by its "dual" (wavelets) elements in $\mathrm{H}_{1/2}$. In combination with a revisited radiation problem representation the model is proposed as alternative „time model" to the concept proposed on (PeR) (*). The decomposition $H_1^{(-)} \otimes H_1^{(+)}$ is about a model for orthogonal repulsive and attractive elementary particles subspaces of H_1 (see also section 2a below):

The standard energy Hilbert space H_1 is proposed to be interpreted as *„ferminons mass/energy"* space; $H_1^{\perp}$ is proposed to be interpreted as the orthogonal *„bosons energy"* space. Both together build the newly proposed quantum energy space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$, whereby the Hilbert (sub-) space H_1 is the model of the physical (fermions) reality of the overall quantum-mechanical description (EiA2). The model meets the statement about physical meaning of general relativity on (RoC) (**), but in a newly Hilbert scale (back stage) framework.

The Friedrichs extension of the Laplacian operator is a selfadjoint, bounded operator B with domain H_1. Thus, the operator B induces a decomposition of H into the direct sum of two subspaces, enabling the definition of a potential and a corresponding „grad" potential operator. Then a potential criterion defines a manifold, which represents a hyperboloid in the Hilbert space H_1 with corresponding hyperbolic and conical regions ((VaM) 11.2). The direct sum of the corresponding two subspaces of $H = H_1$ are proposed as a model to define a decomposition of the „fermions" space H_1 into

$$H_1 = H_1^{repulsive} \otimes H_1^{attrac} =: H_1^{(-)} \otimes H_1^{(+)}.$$

whereby the potential criterion defines repulsive resp. attractive elementary mass particles. Then the corresponding proposed quantum energy Hilbert space (including attractive gravitons $\in H_1^{(+)}$) is given by

$$H_{1/2} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}.$$

The $H_{1/2}$ space as first cohomology is fundamental to explain the properties of period mapping on the universal Teichmüller space (***).

In (KrR1) a formal differentiability condition is given characterizing conformal mappings in R^4. The approach applies variable orthogonal sets, so-called moving frames.

In GR a frame field (also called a tetrad field) is a set of four (one time-like and three space-like) orthogonal vector fields, defined on a Lorentz manifold. All tensorial quantities defined on the manifold can be expressed by the frame field and ist dual coframe field. The related gravitational field e is a *one-form* $e^I(x) = e^I_\mu(x)dx^\mu$ with values in Minkowski space ((RoC) p. 33) (****).

(*) (PeR): „The two key ideas underlying this novel proposal are penetrating analysis of the Second Law of thermodynamics – according to which the „randdomness" oft he workd is continually increasing – and a thorough examination of the light-cone geometry of space-time. Penrose is able to combine these two central themes to show how the expected ultimate fate of our accelating, expanding universe can actually be reinterpreted s the „big bang" of a new universe.
(**) (RoC) p. 9: …., the Universe is not made up of fields on spacetime; it is made up of fields on fields.
(***) (NaS): „The Hilbert space $H_{1/2}$ can be interpreted as the first cohomology space with real coefficients oft he „universal Riemann surface" – namely the unit disk – in a Hodge-theoretic sense."
(****) (RoC) p. 34: I call „gravitational field" the tetrad field rather than Einstein's metric field. There are three reasons for this (1) the standard model cannot be written in terms of g because ferminos require the tetrad formalism (2) the tetrad field e is nowadays more utilized than g in quantum gravity, and (3) I think that e represents the gravitational fields in a more conceptually clean way than g (see section 2.2.3). The relation with the metric formalism is given in Section 2.1.5.

A tetrad field e determines uniquely a torsion-free spin connection $\omega = \omega[e]$. Ist compatibility condition with e ((RoC) (2.6)) and the Einstein equations ((RoC) (2.11)) are the field equations of GR in the absence of other fields. They are the Euler-Lagrange equations of the action $S[e,\omega]$ ((RoC) (2.12)). Replacing ω with $\omega[e]$ leads to the second order action formalism $S[e]$ ((RoC) (2.16)). The two Lagrange formalisms are not equivalent in the presence of fermions (*).

The Lagrange formalism is related to the concept of „force", while the Hamiltonian formalism is related to the concept of „energy". Both formalisms are equivalent *only* (!) in case the Legendre (contact) transform can be applied. Our proposed „alternative energy (Hilbert space) concept" goes along with reduced regularity assumptions of the concerned operators (similar to the regularity reduction when moving from standard potential function („*mass density*") definition to Plemelj's „*mass element*" concept ($\sim C^1 \rightarrow C^0$)), (PlJ).

The Hamiltonian formalism puts then the spot on the Noether theorem concering the invariance of an Hamiltonian operator under an infinitesimal transform.

The physical interpretation of a no longer valid Lagrange formalism is similar to the thermal time hypothesis in (RoC) (**), based on the statement, that „*In Nature, there is no preferred physical time variable t*":

It is about „a force is a statistical „measured" phenomenon of the mathematical model of the considered physical problem", only, and not any a priori hypothetical „Nature force" that drives the system to a preferred statistical state. We claim that an all-in-one Hamiltonian formalism of the several Lagrange formalisms, e.g. $S_{Einst}[e,\omega]$, $S_{Matter}[e,A]$, $S_{YM}[e,A]$ ((RoC), (2.12), (2.29), (2.30)) unifies the „*multiple lagrangian and hamiltonian world of the SMEP*" ((RoC) p. 38, 290) to a „*single hamiltonian world of a NMEP*".

The symmetry group $SU(2)$ is the next higher symmetry group to the Maxwell equation related symmetry group $U(1)$, which is diffeomorph to the unit circle. $SU(2)$ is the group of quaternions of absolute value one, which is diffeomorph to the 3-sphere. $SU(3)$ is the standard (symmetry) model for the strong nuclear „QCD force" interaction between three particles (red, blue, green quarks). The quaternions symmetry group $SU(2)$ governs the interaction between two particles. Therefore, the newly proposed elementary quantum energy space $\mathrm{H}_1 = H_1^{(-)} \otimes H_1^{(+)}$ (repulsive and attractive elementary mass particles) are governed by the symmetry group $SU(2)xSU(2)$ (two repulsive and two attractive interacting elementary particles, each). The attractive EP related symmetry group replaces the SMEP-$SU(3)$ symmetry group with its $2^3 = 8$ possible pairwise (colorful) Gluons quarks interaction types, modelling the (QCD-) „strong nuclear force", while at the same time, including also the graviton (with velocity beyond the light velocity border).

In (WeD) a way to find Haar measure on $SU(2)$ is provided.

The operator concerned with the time-harmonic Maxwell equation and the radiation problem is the D'Alembert operator related to the wave equation. In the special relativity theory the electrodynamic is described by the four-vector formalism of the space-time given by the equation $\Box \vec{A} = \frac{4\pi}{c}\vec{J}$, with the four-vector potential $\vec{A}$, where its curvature determines the electric and magnetic field forces, and j denotes the four-current-density.

(*) (RoC) p. 36: the formalism in (2.12) where e and ω (the spin connection, which is also a one-form with values in the Lie algebra of the Lorentz group $SO(3,1)$) are independent is called the first-order formalism. The two formalism are not equivalent in the presence of fermions; we do not know which one is physically correct, because the effect of gravity on single fermions is hard to measure.

(**) (RoC) p. 143: **The thermal time hypothesis**: In Nature, there is no preferred physical time variable t. There are no equilibrium states ρ_0 preferred a priori. Rather, all variables are equivalent: we can find the system in an arbitrary state ρ; if the system is in a state ρ, then a preferred variable is singled out by the state of the system. This variable is what we call time. …. In other words, it is the statistical state that determines which variable is physical time, and not any a priori hypothetical „flow" that drives the system to a preferred statistical state.

An alternative weak $\mathrm{H}_{-1/2}$ based radiation representation with time-asymmetric solution

The proposed quantum/fluid state Hilbert space $\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. the corresponding quantum/fluid energy Hilbert space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp} = H_{-1/2}^{*}$ enables the definition of a well-posed not time-symmetric radiation problem in the 4-dimensional Minkowski space based on a not time-symmetric initial value condition $\in H_{-1/2}$ (with its relationship to the „wavelet-space"), triggering a „symmetry break down" for $t > 0$. The proposed quantum state Hilbert space $\mathrm{H}_{-1/2}$ goes along with a replacement of of Dirac „function" model $\delta \in \mathrm{H}_{-n/2-\varepsilon}$ ($\varepsilon > 0$, n = space dimension) of the <u>point</u> <u>mass</u> <u>density</u> <u>of an idealized point</u> <u>mass</u> by Plemelj's <u>mass element</u> definition. This indicates also to revisit „*the comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem*" (CoR). In order to also anticipate the time-symmetry issue we propose an alternative Green function defining distribution equation, where the Dirac „function" is replaced by a proper wavelet function.

The current standard radiation model is about an inhomogeneous wave equation with homogeneous initial conditions ((CoR) p. 696:

$$\square u := \ddot{u} - \Delta u = g(t)\delta(x,y,z) \quad , \quad u(x,0) = \dot{u}(x,0) = 0$$

building a radiation problem solution by a limiting process from solutions of nonhomogeneous equations (**). The Green function of the D'Alembert operator is the solution $S(t,x)$ of the distribution equation

$$\square u = \ddot{u} - \Delta u = -\delta(t)\delta^3(x,y,z).$$

Its Fourier function representation $\hat{S}(t,x)$ is defined by the equation

$$(\omega^2 - |k|^2)\hat{S}(\omega,k) = 1.$$

The corresponding integral representation of the Green function is then given by ($m = 3$)

$$S(t,x) = \int\int e^{-i\omega t + ikx} \frac{1}{\omega^2 - |k|^2} \frac{dk}{(2\pi)^m} \frac{d\omega}{2\pi} \; .$$

We note that the poles of the function

$$g(\omega,k) := \frac{1}{k^2 - (\frac{\omega}{c})^2}$$

correspond to the dispersion relations for the electromagnetic waves in a vacuum given by $\omega^2 = c^2 k^2$.

In the context of the Vlasov's formula for the plasma dielectric for the longitudinal oscillators (and the corresponding non-linear Landau damping problem (2c)) we note the integral ((ShF) p. 392)

$$W\left((\frac{\omega}{k})^2\right) = 2\int_{-\infty}^{\infty} \frac{F_0{}'(v^2)dv}{v^2 - (\frac{\omega}{k})^2} \quad .$$

(*) (CoR) p. 765: „Huygens' principle stipulates that the solution at a point does not depend on the totality of initial data within the conoid of dependence but only on data on the characteristic rays through that point … It is proven, that for the wave equation in 3,5,7,… space dimensions, and for equivalent equations, the Huygens' principle is valid. For differential equations of second order with variable coefficients Hadamard's conjecture states that the same theorem holds even if the coefficients are not constant. Examples to the contrary show that this conjecture cannot be completely true in this form, although it is highly plausible that somehow it is essentially correct. … Altogehter, the question of Huygens's principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem**, a problem which is still completely open**." Altogether, the question of Huygens' principle for second order equations should be considered in the light of the much more comprehensive problem of the exact domain of dependence and influence for any hyperbolic problem (see §5), a problem which is still completely open."

The solution of the initial value problem by Fourier transform is given by

$$u(x,t)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int e^{ix\omega}\frac{\sin(c|\omega|t)}{c|\omega|}\hat{g}(\omega)d\omega$$

whereby $\frac{\sin(c|\omega|t)}{c|\omega|}$ satisfy the Garding's hyperbolicity condition.

The newly proposed Green function $S(t,\boldsymbol{x})$ of the D'Alembert operator with newly range $\mathrm{H}_{-1/2}$ is then defined by

$$(\Box[S],v)_{-1/2}=(\psi,v)_{-1/2}\quad \forall v\in H_{-1/2}.$$

Let $g=g_1+g_1^{\perp}\in H_{1/2}=H_1\otimes H_1^{\perp}$ be the quantum energy of a corresponding quantum state $h=h_1+h_1^{\perp}\in H_{-1/2}=H_0\otimes H_0^{\perp}$. Then the corresponding alternative representation of the newly proposed radiation problem is given by

$$(\Box u,v)_{-\frac{1}{2}}=(h_1^{\perp},v)_{-\frac{1}{2}}=(g_1^{\perp},v)_{\frac{1}{2}}\qquad \forall v\in H_{-1/2}\,,\,t\geq 0$$
$$(u,v)_0=(\dot{u},v)_0=0\,,\qquad \forall v\in H_0\qquad t=0$$

with the corresponding energy equality

$$\frac{1}{2}\frac{d}{dt}\left\{\|\dot{u}\|_{-\frac{1}{2}}^2(t)+\|u\|_{\frac{1}{2}}^2(t)\right\}=(h_1^{\perp},\dot{u})_{-\frac{1}{2}}\leq\|h_1^{\perp}\|_{-\frac{1}{2}}(t)\|\dot{u}\|_{-\frac{1}{2}}(t)\leq\frac{1}{2}\|h_1^{\perp}\|_{-\frac{1}{2}}^2(t)+\frac{1}{2}\|\dot{u}\|_{-\frac{1}{2}}^2(t)\,,\quad \|u\|_0^2(0)=\|\dot{u}\|_0^2(0)=0.$$

The lemma of Gronwall leads to the related a priori energy estimate in the form

$$\|\dot{u}\|_{-\frac{1}{2}}^2(t)+\|u\|_{\frac{1}{2}}^2(t)\leq\frac{1}{2}e^t\cdot\int_0^t\|h_1^{\perp}\|_{-\frac{1}{2}}^2(\tau)d\tau=\frac{1}{2}e^t\cdot\int_0^t\|g_1^{\perp}\|_{\frac{1}{2}}^2(\tau)d\tau.$$

In (AhJ) the „scattering trinity" is considered for the Helmholtz equation with radiation condition as a model for exterior scattering problem. The trinity is about the null-field method, modified Green functions (for the Dirichlet and Neumann problem) and the corresponding reproducing kernel (the difference between the modifed Dirichlet/ Neumann Green functions).

The revisited exterior integral relation ((AhJ) (5.2))

$$\int_{\partial B}\frac{\partial K(x,y)}{\partial n(y)}u(y)dS_y=\int K(x,y)\frac{\partial u(y)}{\partial n(y)}dS_y$$

with its related „trinity" partners leads to a Hilbert space framework in line with the proposed „quantum state Hilbert space $H_{-1/2}=H_0\otimes H_0^{\perp}$, resp. the corresponding energy Hilbert space Hilbert space $H_{1/2}=H_1\otimes H_1^{\perp}$, where the spectrum of H_1 are governed by Fourier waves, while the spectrum of $H_1^{\perp}$ are governed by Calderón wavelets.

Asymtotic Behavior of Some Evolution Systems

For this section we refer to (BrH), where a semi-group $S_t : C \rightarrow C$ of nonlinear contractions on a closed convex subset of a Hilbert space H is considered. In general $S_t[u]$ does not converge to a limit as $t \rightarrow \infty$.

The central tool in (BrH) is about ergodic mean

$$\sigma_t := \frac{1}{t}\int_0^t S_\tau[u]\,d\tau \ ,\ u \in C$$

and the Cesaro means

$$\sigma_n := \frac{1}{n}(u + Tu + \cdots + T^{n-1}u)$$

of a contraction T in H having at least one fixed point. In general σ_t and σ_n does not converge strongly, but σ_t converges weakly as $t \rightarrow \infty$ to a limit σ, and σ_n converges weakly as $n \rightarrow \infty$ to a fixed point of T.

Assuming that T is an odd contraction on H, i.e. $S_t[-u] = -S_t[u]$, σ_n converges strongly as $n \rightarrow \infty$ to a fixed point of T.

The above results are also valid with the more averaging process

$$\int_0^t S_\tau[u]a_n(\tau)\,d\tau$$

with $a_n \in L_1(0,\infty)$, $a_n \geq 0$, $\int_0^\infty a_n(\tau)\,d\tau = 1$, and $\int_0^\infty |da_n(\tau)|\,d\tau \rightarrow 0$ for $n \rightarrow \infty$.

Stability theory of differential equations in the context of upper bounds on the norm of a solution of the equation

$$\frac{dz}{dx} = F(x,z)$$

are e.g. considered in (BeR). In (LaC1) upper and lower bounds are provided.

2. The $H_1 = H_1^{repulsive} \otimes H_1^{attrac}$ decomposition to model repulsive and attractive elementary mass particles

The Standard Model of Elementary Particles (SMEP) is concered with gauges theory and variational principles. Each of the observed Nature „force" phenomena are related to a specific gauge group. The model does not provide any explanation where the related elementary „particles" are coming from (or have been generated out of „first mover" resp. out of mass-less photons) during the inflation phase of current big bang assumption and why their mass have their specific values.

The standard energy Hilbert space H_1 is proposed to be interpreted as *„ferminons mass/energy"* space; $H_1^{\perp}$ is proposed to be interpreted as the orthogonal *„bosons energy"* space. Both together build the newly proposed quantum energy space $\mathrm{H}_{1/2} = H_1 \otimes H_1^{\perp}$, whereby the Hilbert (sub-) space H_1 is the model of the physical (fermions) reality of the overall quantum-mechanical description (EiA2). It may be appropriate to revisit

- the spinors and space-time concept and the physical interpretation of the underlying variables in the context of the „rotating repulsive and attractive fermions" interpretation of the newly proposed quantum energy space (PeR5)
- Einstein's idealized experiment considering the motion of a single electron moving in a field of force with a given potential (*).

A selfadjoint operator B defined on all of the Hilbert space H (e.g. $H = H_1$ and B the Friedrichs extension of the Laplacian operator) is bounded. Thus, the operator B induces a decomposition of H into the direct sum of the subspaces, enabling the definition of a potential and a corresponding „grad" potential operator. Then a potential criterion defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions ((VaM) 11.2). The direct sum of the corresponding two subspaces of $H = H_1$ are proposed as a model to define a decomposition of the „fermions" space H_1 into

$$H_1 = H_1^{repulsive} \otimes H_1^{attrac} =: H_1^{(-)} \otimes H_1^{(+)}.$$

The potential criterion defines repulsive resp. attractive elementary mass particles. Then the corresponding proposed quantum energy Hilbert space (including attractive gravitons $\in H_1^{(+)}$) is given by

$$H_{1/2} = H_1^{(-)} \otimes H_1^{(+)} \otimes H_1^{\perp}.$$

(*) ((HeW) p. 36, englisch version): „critique of the corpuscular theory": „The motion and spreading of probability packets has been studied by various authors, … A simple consideration of Ehrenfest's may be mentioned, … considering the motion of a single electron moving in a field of force whose potential is $V(q)$. … If there were no spreading at all, it would be possible to make a Fourier analysis of the probability density into which only integral multiples of the fundamental frequency of the orbit enter. As a matter of fact, however, the „overtones" of quantum theory are not exactly integral multiples of this fundamental frequency. The time in which the phase of the quantum theoretical overtones will be qualitatively the same as the time required for the spreading of the wave packet. Let J be the action variabe of classical theory, then this time will be $t = \frac{1}{h\frac{\partial v}{\partial J}}$ and the number of revolutions performed in this time is $N \sim \frac{1}{h\frac{\partial v}{\partial J}}$. In the special case of the harmonic oscillator, N becomes infinite – the wave packet remains small for all times. In general, however, N will be of the order of magnitude of the quantum number n ".

In relation to these considerations, one other idealized experiment (due to Einstein) may be considered. We imagine a phton which is represented by a wave packet built up out of Maxwell waves. (For a single photon the configuration space has only three dimensions; the Schrödinger equation of a photon can thus be regarded as formally identical with the Maxwell equations.) It will thus have a certain spatial extension and also a certain range of frequency. By reflection at a definite probability for finding the photon either in one part or in the other part oft he divided wave packet. After sufficient time the two parts will be sparated by any distance disired; now if any experiemnt yields the result that the photon is, say, in the reflected part of the packet, then the probability of finding the photon in the other part of the packet immediately becomes zero. The experiment at the position of the reflected packet thus exerts a kind of action (reduction of the wave packet) at the distant point occupied by the transmitted packet, and one sees that this action is propagated with velocity greater than that of light. However, it is also obvious that this kind of action can never be utilized for transmission of signals so that it is not in conflict with the postulates of the theory of relativity".

The theory of Hilbert spaces with an indefinite metric is provided in e.g. ((DrM), (AzT), (DrM), (VaM)). Following the investigations of Pontrjagin and Iohvidov on linear operators in a Hilbert space with an indefinite inner product, M. G. Krein proved the Pontrjagin-Iohvidov-Krein theorem (FaK).

In case of a Hilbert space H, this is about a decomposition of H into an orthonal sum of two spaces H^1 and H^2 with corresponding projection operators P^1 and P^2 (see also the problem of S. L. Sobolev concerning Hermitean operators in spaces with indefinite metric, (VaM) IV). We note, that for a vector space H, the empty set, the space H, and any linear subspace of H are convex cones.

For x being an element of H this is about a defined "potential" ((VaM) (11.1))

$$\varphi(x) := \left((x)\right)^2 = \|P^1 x\|^2 - \|P^2 x\|^2$$

and a corresponding "grad" potential operator $\boldsymbol{W}(x)$, given by

$$\boldsymbol{W}(x) = \frac{1}{2}\mathrm{grad}\varphi(x) := P^1(x) - P^2(x) \quad , \quad \text{(VaM) (11.4).}$$

The potential criterion $\varphi(x) = c > 0$ defines a manifold, which represents a hyperboloid in the Hilbert space H with corresponding hyperbolic and conical regions. It provides a model for „symmetry break down" phenomena by choosing $P^1 := P$, $P^2 := I - P$ for the orthogonal projections $P: H_{-1/2} \to H_0$, $P: H_{1/2} \to H_1$, leading to the decompositions $\mathrm{H}_{-1/2} = H_0 \otimes H_0^{\perp}$, $H_{1/2} = H_1 \otimes H_0^{\perp}$.

The tool set for an appropropriate generalization of the above "grad" definition in case of non-linear problems is about the (homogeneous, not always linear in h) Gateaux differential (or weak differential) $\boldsymbol{VF}(x, h)$ of a functional $\boldsymbol{F}$ at a point x in the direction h ((VaM) §3)).

If there exists an operator A with $D(A) = H_1$, $R(A) = H_0$ and $\|x\|_1 = \|Ax\|_0$, whereby the operator A is positive definite, self-adjoint and A^{-1} is compact, the corresponding eigenvalue problem $A\varphi_i = \sigma_i \varphi_i$ has infinite solutions $\{\sigma_{i,}\varphi_i\}$ with $\sigma_i \to \infty$ and $(\varphi_i, \varphi_k) = \delta_{i,k}$. For each element $x \in H_1 = A^{-1}H_0$ it holds the representation

$$x = \sum_{i=1}^{\infty} (x, \varphi_i)\, \varphi_i .$$

Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x, y)_\alpha := \sum_i^{\infty} \lambda_i^{\alpha} (x, \phi_i)\, (y, \phi_i) = \sum_i^{\infty} \lambda_i^{\alpha} x_i y_i \ .$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form $e^{-\sqrt{\lambda_i} t}$ given by

$$(x, y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i} t} (x, \phi_i)(y, \phi_i) \quad , \quad \|x\|_{(t)}^2 := (x, x)_{(t)}^2$$

The approximation "quality" of the proposed $\mathrm{H}_{-1/2}$ –quantum state Hilbert space with respect to the „observable space" norm of H_0 is governed by the inequality

$$\|x\|_{-1/2}^2 \le \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \delta \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\delta} x_i^2 .$$

The estimate is valid for all $\alpha > 0$ in the form $\|x\|_{-\alpha}^2 \le \delta^{2\alpha} \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2$, which follows from the inequality $\lambda^{-\alpha} \le \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$, being valid for any $t, \delta, \alpha > 0$ and $\lambda \ge 1$. For a related approximation theory we refer to (BrK8), (NiJ), NiJ1).

Applying the mathematical wavelet (microscopic view) tool is then about an analysis of a quantum state $x = x_0 + x_0^{\perp} \in H_0 \otimes H_0^{\perp}$. Putting $\sigma := \|x_0^{\perp}\|_{-1/2}^2$ the approximation "quality" of a quantum state with respect to the „observable space" norm of H_0 is governed by the inequality

$$\|x\|_{-1/2}^2 \le \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\sigma} x_i^2 .$$

The standard model of a momentum operator is a Partial Differential Operator of order 1 with corresponding domain C^1 (or, in case of a Hilbert space based variational representation the corresponding Sobolev space governed by the Sobolev embedding theorem $H_k \in C^0$). It is proposed to be replaced by the Calderón-Zygmund integrodifferential operator with symbol $|\nu|$ ([EsG] (3.15), (3.17), (3.35)), defined by

$$(\Lambda u)(x) = (\sum_{k=1}^{n} R_k D_k u)(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \sum_{k=1}^{n} p.v. \int_{-\infty}^{\infty} \sum_{k=1}^{n} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy \, (\Lambda u)(x)$$

$$= -\frac{\Gamma(\frac{n-1}{2})}{2\pi^{\frac{n+1}{2}}} p.v. \int_{-\infty}^{\infty} \frac{\Delta_y u(y)}{|x-y|^{n-1}} dy = -(\Delta \Lambda^{-1}) u(x)$$

resp.

$$\Lambda^{-1} u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}} \int_{-\infty}^{\infty} \frac{u(y)dy}{|x-y|^{n-1}} \, , \quad n \geq 2 \, ,$$

whereby $R_k$denotes the Riesz operators

$$R_k u = -i \frac{\Gamma(\frac{n+1}{2})}{\pi^{(n+1)/2}} p.v. \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} u(y) dy.$$

Non-linear minimization problems can be analyzed as saddle point problems on convex manifolds in the following form (VeW):

$$(*) \quad J(u): a(u,u) - F(u) \rightarrow min \;, \quad u - u_0 \in U.$$

Let $a(\cdot,\cdot): \; V \times V \rightarrow R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V \; and F(\cdot): V \rightarrow R$ a functional with the following properties:

i) $F(\cdot): V \rightarrow R$ is convex on the linear manifold $u_0 + U$, i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \leq (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

ii) $F(u) \geq \alpha$ for every $u \in u_0 + U$

iii) $F(\cdot): V \rightarrow R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \rightarrow R$ with

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = F_u(v).$$

Then the minimum problem $(*)$ is equivalent to the variational equation

$$a(u,\phi) + F_u(\phi) = 0 \quad \text{for every} \quad \phi \in U$$

and admits only an unique solution.

In case the sub-space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \rightarrow R$ is continuous with respect to convergence in the energy norm, then there exists a solution. We note that the energy functional is even strongly convex in whole V.

The proposed „energy" Hilbert space $H_{1/2}$ enables e.g. the method of Noble ((VeW) 6.2.4), (ArA) 4.2), which is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a "Hamiltonian" function $W(\cdot,\cdot)$ which combines the pair of underlying operator equations (based on the "Gateaux derivative" concept)

$$Tu = \frac{\partial W(\acute{u},u)}{\partial \acute{u}} \;, \; T^*\acute{u} = \frac{\partial W(\acute{u},u)}{\partial u} \quad u \in E = H_{1/2} \;, \; \acute{u} \in \acute{E} = H_{-1/2}.$$

With respect to the Bianchi identities we emphasis that for the inner product $(u,v)_{-1/2}$ of $H_{-1/2}$ the following relationships hold true:

$$(div(u), v)_{-\frac{1}{2}} \sim (u, \nabla v)_{-\frac{1}{2}} \sim (u, v)_0.$$

The decomposition of the quantum state space $H_0 \otimes H_0^\perp$ resp. the quantum energy space $H_1 \otimes H_1^\perp$ is very much related to the „hidden variables in quantum theory" concept of D. Bohm (BoD) with the notions of implicate and explicate order:

((BoD), A2): „*It is important to emphasize, however, that mathematics and physics are not being regarded here as separate but mutually related structures (so that, for example, one could be said to apply mathematics to physics as paint is applied to wood). Rather, it is being suggested that mathematics and physics are to be considered as aspects of a single undivided whole*".

The implicate or enfolded order is about „a new notion of order, that may be appropriate to a universe of unbroken wholeness. In the enfolded order, space and time are no longer the dominant factors determining the relationships of dependence or independence of different elements."

((BoD), A3): „*Implicate order is generally to be described not in terms of simple geometric transformations, such as translations, rotations, and dilations, but rather in terms of a different kind of operations*. … *What happens in the broader context of implicate order we shall call a metamorphosis. … An example of such a metamorphosis metamorphosis M is determed by the Green's function relating amplitudes at the illuminated structure to those at the photographic plate*".

In our case this relates to the closed sub-spaces $H_0^\perp$ and $H_1^\perp$.

Rather, an entirely different sort of basic connection of elements is possible, from which our ordinary notions of space and time, along with those of separately existent material particles, are abstracted as forms derived from the deeper order. These ordinary notions in fact appear in what is called the explicate or unfolded order, which is a special and distinguished form contained within the general totality of all the implicate orders… Explicate order arises primarily as a certain aspect of sense perception and of experience with the content of such sense perception. It may be added that, in physics, explicate order generally reveals itself in the sensibly observable results of functioning of an instrument. … „*What is common to the functioning of instruments generally used in physical research is that the sensibly perceptible content is ultimately describable in terms of a Euclidean system of order and measure, i.e., one that can adequately be understood in terms of ordinary Euclidean geometry. … The general transformations are considered to be the essential determining features of a geometry in a Euclidean space of three dimensions; those are displacement operators, rotation operators and the dilatation operator* ((BoD) A.2).

In our case this is related to the Hilbert spaces H_0 and H_1. The Euclidean systems of order and measure are strongly related to the Archimedian principle.

„*Of course, in the quantum theory, the algebraic terms are interpreted as standing for 'physical observables' to which they correspond. However, in the approach that is being suggested here, such terms are not to be regarded as standing for anything in particular. … This means, of course, that we do not regard terms like 'particle', 'charge', 'mass', 'position', 'momentum', etc., as having primary relevance in the algebraic language. Rather, at best, they will have to come out as high-level abstractions.*" ((BoD) A4).

3. The $H_{-1/2}$ Hilbert space and a new ground quantum state $H_0^{\perp}$ & ground quantum energy $H_1^{\perp}$ model

For this section we also refer to (BrK), (BrK1), (BrK3). In quantum mechanics, a boson is a "particle" that follows the Bose-Einstein statistics ("photon gases"). A characteristic of bosons is that their statistics do not restrict the number of them that occupy the same quantum state. All bosons can be brought into the energetically lowest quantum state, where they show the same "collective" behavior. Unlike bosons, two identical fermions cannot occupy the same quantum state. Fermions follow the Fermi statistics (e.g. (AnJ)). With respect to the above extended domain of the Schrödinger momentum operator, we propose to identify H_0 as quantum state space for the fermions (which is compactly embedded into $H_{-1/2}$), and $H_0^{\perp}$ as quantum state space for the bosons.

The "fermions quantum state" Hilbert space H_0 is dense in $H_{-1/2}$ with respect to the $H_{-1/2}$ –norm, while the (orthogonal) "bosons quantum state" Hilbert space $H_0^{\perp}$ is a closed subspace of $H_{-1/2}$, resp. the "mass/energy fermions" Hilbert space H_1 is dense in $H_{1/2}$ with respect to the $H_{1/2}$ –norm, while the "energy bosons" Hilbert space is a closed subspace of $H_{1/2}$.

The concept of "vacuons" (i.e. the vacuum expectation values of scalar fields) in the context of "spontaneous" breakdown of symmetry (HiP) then corresponds to the orthogonal projection $H_{1/2} \rightarrow H_1$.

Dirac´s point mass density concept leads to the „*generating/detecti*ng" of the „positron" with opposite charge than the „electron" in the context of building a relativistic Schrödinger equation. It is being followed by a sequence of other elementary particles for other phenomena with spin and flavor. This conception is complementary to Schrödinger's view of the (elementary „particles") world. All of those EP are beyond Kant´s transcendence border to theoretical metaphysics, while the „*matter-mind*" relationship is still an open question. Schrödinger's thoughts about appropriate answers are about the "perception process between "subconscious" and "awareness" of human mind", which lead him to the notion of "Differential" (ScE1). The central differentiator between „real" and „hyper-real" numbers from a mathematical point of view is the Archimedean axiom *(*)* (see also (EhP), (RoJ), (WoW)).

The Schrödinger (differentiation) operator is not bounded with respect to the norm of L_2, i.e. only on a dense subspace of L_2 a corresponding spectral representation of this operator can be defined. The not vanishing constant Fourier term of the baseline Hermite polynomial (which is the Gaussian function) leads to mathematical challenges with respect to the creation and annihilation operators of the related Hamiltonian operator of the quantum oscillator model. The Hilbert transform of a function f has always vanishing constant Fourier terms. As a consequence, the Hilbert-transformed Schrödinger operator form with extended domain $\mathrm{H}_{-1/2}$ is bounded (with respect to the norm of L_2) leading to a bounded Hermitian operator with corresponding spectral form representation.

Based on the newly defined common Hilbert space domain spectral theory can be applied, while

- the (physical) test space keeps the same, i.e. $L_2 = H_0$
- the current domains of the considered operators are extended to enable a (convergent) energy norm $\|x\|_{1/2}$ and a corresponding weak variation representation of the considered operator equations with respect to the inner product $(x, y)_{-1/2}$.

(*) *The mathematical counterpart of Leibniz's transcendental concept of a "differential = monad" corresponds to Robinson's hyper-real number (ideal points). From a mathematical point of view the extension from the field of „real numbers" to „hyper-real numbers" is built on exactly the same mathematical axioms (i.e. it is also an ordered field), „just" the Archimedean axiom is missing. An ordered field A is said to be Archimedean if for all $x, y \in A$, where $0 < x < y$ there is an $n \in N$ such that $nx > y$. In other words, any „distance" y can be (over-estimating) measured by a multile of a given normed length x.*

The corresponding notions from variation theory are “energy norm” and “operator norm” with correspondingly defined minimization problems (“energy” resp. “action” minimization problems). The corresponding eigenvalue problem of an operator T is then related to the inner product $(Tx,x)_{-1/2}$.

With respect to the newly proposed Pseudo-differential and Fourier multiplier operators with extended fractional Hilbert scale domain we note the following:

- the Maxwell equations are represented by differential equations or integral equations. Both representations are considered as equivalent, i.e. the different corresponding domains regularity is neglected
- The Maxwell equations are considered as equivalent to the corresponding wave equations for the electromagnetic potentials, if the Lorentz condition $\nabla\vec{A}-\frac{1}{c}\frac{\partial\varphi}{\partial t}=0$ is fulfilled, which comes along with higher mathematical regularity requirements for those potentials than justified by the physical problem
- the Lagrange (“force”) and the Hamiltonian (“energy”) formalisms are considered as equivalent. The mathematical proof is based on the Legendre transform, i.e. the equivalence is only valid if the assumptions of the Legendre transform are fulfilled.

In the cases above, corresponding (mathematical) regularity assumptions are required to enable those propositions. A restriction of the domain regularity of the considered operators leads to no longer well-defined classical differential equations resp. to no longer valid Lagrange formalism. In other words, the provided consistent model in the distributional framework represents the mathematical/transcendental view of the considered physical world, while the corresponding classical solutions of the several differential equations are mathematical approximations to those physical models. This concept also overcomes the “physical interpretation” challenge of the “Neumann PDE” representation of the pressure p in the NSE model.

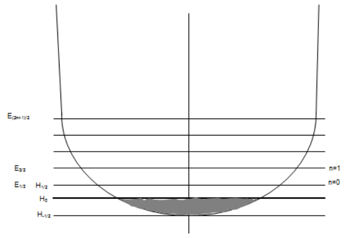
A successfully applied least action principle (being interpreted as a maxime of Kant's reflective judgment) results into appropriate consistent mathematical-physical models, those models can be declared as law of natures. The above is related to the three "forces of nature" as modelled by the SMEP. The nature of those elementary particles and the way they move, is described by quantum mechanics, but quantum mechanics cannot deal with the curvature of space-time. Space-time are manifestations of a physical field, the gravitational field. At the same time, physical fields have quantum character: granular, probabilistic, manifesting through interactions. The to be defined common mathematical solution framework needs to provide a quantum state of a gravitational field, i.e. a quantum state of space. The crucial difference between the photons characterized by the Maxwell equations (the quanta of the electromagnetic field) and the to be defined quanta of gravity is, that photons exists in space, whereas the quanta of gravity constitute space themselves ((RoC2) p. 148).

The proposed mathematical framework provides a common baseline to integrate quantum mechanics & thermodynamics with gravity & thermodynamics. From a physical model problem perspective this is about a common mathematical framework for black body radiation ((BrK4) remark 2.6, Note O55, O71, O72) and black hole radiation ((RoC3) p. 56, 60 ff)). The thermodynamics is the common physical theory denominator with the Planck concept of zero point energy of the harmonic quantum oscillator (BrK), (BrK1), and the Boltzmann entropy concept. An integrated model needs to combine the underlying Bose-Einstein and the Dirac-Fermi statistic.

In the context of the newly proposed “energy-space” $\mathrm{H}_{1/2}=H_1\otimes H_1^{\perp}=H_{-1/2}^{*}$ we also refer to the Bose-Einstein condensation, where below the critical temperature T_C BEC “normal gas” particles coexist in equilibrium with “condensed” particles. Unlike a liquid droplet in a gas, here the “condensed” particles are not separated in space $(H_{-1/2}=H_0\otimes H_0^{\perp}=H_{1/2}^{*})$ from normal particles. Instead they are separated in momentum space. The condensed particles all occupy a single quantum state of zero momentum, while normal particles all have finite momentum.

The wave-mechanical vibrations correspond to the motion of particles of a gas resp. the eigenvalues and eigen-functions of the harmonic quantum oscillator. The alternatively proposed $\mathrm{H}_{1/2}$ energy space is claimed to enable Schrödinger's "purely quantum wave" vision, which is about half-odd integers, rather than integers quantum numbers. As a consequence the corresponding eigenvalue and eigenfunction solutions of the number operator (i.e. the product of generation and annihilation operators) start with index $n=1$, not already with $n=0$. The quantum energy for the quatum state with index $n=0$ is modelled by the closed sub-space $H_1^{\perp}$ of the energy Hilbert space $H_{1/2}=H_1 \otimes H_1^{\perp}$. The eigenvalue and eigenfunction solutions of the number operator (i.e. the product of generation and annihilation operators) starting with index $n=1$ are modelled by the densely embedded Hilbert space H_1 of $H_{1/2}$. The change from the standard energy space H_1 to $\mathrm{H}_{1/2}$ also anticipates the linkages of the crystal lattices to the Heisenberg uncertainty relation; such a missing proper linkage was the reason, why Schrödinger not adopted his „half-odd integers" idea, continuing to take for the quantum number n_s the integers, beginning with $n=0$ ((ScE) p. 51).

With respect to the ladder operators of the harmonic quantum oscillator the proposed alternative quantum state and related energy Hilbert scales can be visualized by



Hilbert scale: $H_{-\frac{1}{2}}, H_{0}, H_{\frac{1}{2}},$

$H_{1/2}=H_1 \otimes H_1^{\perp}$ is the proposed energy Hilbert space, govered by wavelets and the Heisenberg uncertainty relation; the discrete energy eigenfunctions are elements of H_1; the areas between the several discrete energy level lines reflect the „continuous" „transition energy" modelled as an (wave package) „element" of $H_1^{\perp}$.

In the context of the Berry-Keating conjecture and the proposed RH solution framework we recall, that for the imaginary parts of the zeros of the considered special Kummer function it holds the inequalities

$$n(n-\tfrac{1}{2}) < \sum_{k=1}^{n} 2\omega_k < n(n+\tfrac{1}{2}) \ .$$

(*) (ScE) p. 44: „*The different cases in the evaluation of „Z" arise thus: (a)* $n_s=0,1,2,3,4,\ldots.$ *(Bose-Einstein gas); (b)* $n_s=0,1$ *(Fermi-Dirac gas, Pauli´s exclusion principle). There may or may not be condition that the total number of particles is constant,* $n=\sum_s n_s$. …..*"*
(ScE) p. 50: „*Since in the Bose case we seem to be faced, mathematically, with simple oscillator of Planck type, of which the* n_s *is the quantum number, we may ask whether we ought not to adopt for* n_s *half-odd integers* $\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots\left(n+\frac{1}{2}\right),\ldots$ *rather than integers. Once must, I think, call that an open dilemma. From the point of view of analogy one would very much prefer to do so. For, the „zero-point energy"* $\frac{1}{2}h\mu$ *of a Planck oscillator is not only borne out by direct observation in the case of crystal lattices, it is also so intimately linked up with the Heisenberg uncertainty relation that one hates to dispense with it. …. Not until the idea of photons had gained considerable ground did Bose (about 1924) point out that we could, alternatively to the „holhraum" oscillator statistics, speak of photon statistics, but then we add to make it „bose statistics". Very soon after, Einstein applied the same to the particles of an ideal gas. And thereupon I pointed out that we could also in this case speak of ordinary statistics, applied to the wave-mechanical proper vibrations which correspond to the motion of the particles of the gas. … The wave point of view in both cases, or at least in all Bose cases, raises another interesting question. Since in the Bose case we seem to be faced, mathematically, with simple oscillator of the Planck type, of which the* n_s *is the quantum number, we may ask whether we ought not to adopt for* n_s *half-odd integers* $\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots n+\frac{1}{2},\ldots$ *rather then integers. One must, I think, call that an open dilemma. From the point of analogy one would very much prefer to do so. For, the "zero point energy" of a Planck oscillator is not only borne out by direct observation in the case of crystal lattices, it is also so intimitely linked up with the Heisenberg uncertainty relation that one hates to dispense with it. On the other hand, if we adopt it straightaway, we get into serious trouble, especially on contemplating changes of the volume (e.g. adiabatic compression of a given volume of black-body radiation), because in this process the (infinite) zero-point energy seems to change by infinite amounts! So we do not adopt it, and we continue to take for the* n_s *the integers, beginning with* 0."

4. The $H_{-1/2}$ quantum state space to model the non-linear (plasma) Landau damping phenomenon and gravity space-time

For this section we also refer to (BrK6). Quantum gravity is a field of theoretical physics that seeks to describe the force of gravity according to the principles of quantum mechanics.

Plasma is the fourth state of matter, where from general relativity and quantum theory it is known that all of them are fakes resp. interim specific mathematical model items.

An adequate model needs to take into account the *axiom of (quantum) state* (physical states are described by vectors of a separable Hilbert space ***H***) and the *axiom of observables* (each physical observable ***A*** is represented as a linear Hermitian operator A of the state Hilbert space). The corresponding mathematical model and its solutions are governed by the Heisenberg uncertainty inequality. As the observable space needs to support statistical analysis the L_2 −Hilbert space, this Hilbert space needs to be at least a subspace of ***H***.

At the same point in time, if plasma is considered as sufficiently collisional, then it can be well-described by fluid-mechanical equations. There is a hierarchy of such hydrodynamic models, where the magnetic field lines (or magneto-vortex lines) at the limit of infinite conductivity is "frozen-in" to the plasma. The "mother of all hydrodynamic models is the *continuity equation* treating observations with macroscopic character, where fluids and gases are considered as continua. The corresponding infinitesimal volume "element" is a volume, which is small compared to the considered overall (volume) space, and large compared to the distances of the molecules. The displacement of such a volume (a fluid particle) then is a not a displacement of a molecule, but the whole volume element containing multiple molecules, whereby in hydrodynamics this fluid is interpreted as a mathematical point.

The Hilbert space $H_{-1/2}$ is suggested as physical quantum (Hilbert) state space model accompanied by correspondingly defined variational (Differential, Pseudo Differential or singular integral operator) equations.

This section deals with the

(1) alternatively proposed quantum theory adequate Vlasov, resp. Landau model with a corresponding problem adequate norms (BrK6)

(2) consideration to apply same concepts (including the $H_{-1/2}$ based Maxwell equations) to the conceptual ideas of Wheeler and others to a space-time quantum geometrodynamics.

(1) alternatively proposed quantum theory adequate Vlasov, resp. Landau model with a corresponding problem adequate norms (BrK6)

Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. The continuity equation of ideal magneto-hydrodynamics is given by ((DeR) (4.1))

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

with $\rho = \rho(x,t)$ denoting the mass density of the fluid and $\boldsymbol{v}$ denoting the bulk velocity of the macroscopic motion of the fluid. The corresponding microscopic kinetic description of plasma fluids leads to a continuity equation of a system of (plasma) "particles" in a phase space $(\boldsymbol{x},\boldsymbol{v})$ (where $\rho(\boldsymbol{x},t)$ is replaced by a function $f(\boldsymbol{x},\boldsymbol{v},t)$) given by ([DeR] (5.1))

$$\frac{\partial}{\partial t}f + v \cdot \nabla_x f + \frac{dv}{dt} \cdot \nabla_v f + f\frac{\partial}{\partial v} \cdot \frac{dv}{dt} = 0 \ .$$

In case of a Lorentz force the last term is zero, leading to the so-called collisions-less (kinetic) Vlasov equation ((ShF) (28.1.2)).

The Vlasov equation is built under the assumptions, that the plasma is sufficiently hot i.e. „plasma particle" collisions can be neglected: the mathematical tool to distinguish between unperturbed cold and hot plasma is about the Debye length and Debye sphere ([DeR]). The corresponding interaction (Coulomb) potential of the non-linear Landau damping model is based on the (Poisson) potential equation with corresponding boundary conditions. A combined electro-magnetic plasma field model needs to enable "interaction" of cold and hot plasma "particles", which indicates Neumann problem boundary conditions. The corresponding double layer (hyper-singular integral) potential operator of the Neumann problem is the Prandtl operator $\overline{\mathrm{P}}$, fulfilling the following properties ([LiI] Theorems 4.2.1, 4.2.2, 4.3.2):

i) the Prandtl operator $\overline{P}: H_r \to \hat{H}_{r-1}$ is bounded for $0 \leq r \leq 1$

ii) the Prandtl operator $\overline{\mathrm{P}}: \mathrm{H}_\mathrm{r} \to \hat{\mathrm{H}}_{\mathrm{r}-1}$ is Noetherian for $0 < r < 1$

iii) for $1/2 \leq r < 1$, the exterior Neumann problem admits one and only one generalized solution.

Therefore, the Prandtl operator enables a combined (conservation of mass & (linear & angular) momentum balances) integral equations system, where the two momentum balances systems are modelled by corresponding momentum operator equations with corresponding domains according to $H_{1/2} = H_1 \otimes H_1^{\perp} = H_{-1/2}^*$.

The Landau equation is a model describing time evolution of the distribution function of plasma consisting of charged particles with long-range interaction. It is about the Boltzmann equation with a corresponding Boltzmann collision operator where almost all collisions are grazing.

The Landau damping phenomenon ("*wave damping w/o energy dissipation by collision in plasma*") is an observed plasma/quantum physical phenomenon. In (MoC) this phenomenon has been „proven" for the non-linear Vlasov equation based on analytical norm estimates, which is about differentiability requirements beyond C^∞; even the mathematical model of the GRT (which is not consistent to the quantum mechanics mathematical model of „discrete" „quantum leaps") works out with *differentiable* manifolds, only, whereby the differentiability requirement is already w/o any physical meaning (!); we claim, that the proof in (MoC) is not a proof of the physical phenomenon, but provides evidence, that the Vlasov equation is not the adequate mathematical model of the Landau phenomenon. This statement is in alignment with the critisism of Landau regarding Vlasov´s equation.

Vlasov's mathematical argument against the Landau equation (leading to the Vlasov equation) was, that "*this model of pair collisions is formally (!) not applicable to Coulomb interaction due to the divergence of the kinetic terms*". This argument is being overcome by the proposed distributions framework.

Vlasov's formula for the plasma dielectric for the longitudinal oscillators is based on the integral ((ShF) p. 392)

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} \quad .$$

As Landau pointed out, this model overlooks the important physical phenomenon of electrons travelling with exactly the same material speed $v_\varphi = \frac{\omega}{k}$ and the wave speed v.

In ((ShF) p. 395) the correct definition (as provided by Landau) for the Vlasov formula is given, which is basically a threefold integral definition depending from the value ω_I the imaginary part of $\omega = \omega_R + i\omega_I$:

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} \qquad \text{for } \omega_I < 0$$

$$W\left(\frac{\omega}{k}\right) = -p.v.\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} - \pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I = 0$$

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{}'(v)dv}{\frac{\omega}{k}-v} - 2\pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I > 0$$

If ω_I were to continue and become positive (damped disturbance), then analytical continuation yields, in addition to the integral along the real line (which also presents no difficulty of interpretation), a full residue contribution.

$Im(\omega)$ arises from the pole at $v = v_\varphi$, which is about the pole of the above integral, when the path of integration lies on the x-axis ((ChF) 7). Consequently, the effect is connected with those particles in the distribution that have a velocity nearly equal to the phase velocity – the "resonant particles". These particles travel along with the wave and do not see a rapidly fluctuating electric field: They can, therefore, exchange energy with the wave effectively. The easiest way to understand this exchange of energy is to picture a surfer trying to catch an ocean wave. If the surfboard is not moving, it merely bobs up and down as the wave goes by and does not gain any energy on the average. Similarly, a boat propeller much faster than the wave cannot exchange much energy with the wave. However, if the surfboard has almost the same velocity as the wave, it can be caught and pushed along by the wave; this is, after all, the main purpose of the exercise. In that case, the surfboard gains energy, and therefore the wave must lose energy and is damped. On the other hand, if the surfboard should be moving slightly faster that the wave, it would push on the wave as it moves as it moves uphill; then the wave could gain energy. In plasma, there are electrons both faster and slower than the wave. A Maxwellian distribution, however, has more slow electrons than fast ones. Consequently, there are more particles taking energy from the wave than vice versa, and the wave is damped. As particles with $v \approx v_\varphi$ are trapped in the wave, $f(v)$ is flattened near the phase velocity.

In the nonlinear case when the amplitude of an electron or ion wave exited, say, by a grid is followed in space, it is often found that the decay is not exponential, as predicted by linear theory, if the amplitude is large ((ChF) 8.7). Instead, one typically finds that the amplitude decays, grows again, and then oscillates before settling down to a steady state value. Although other effects may also be operative, these oscillations in amplitude are exactly what would be expected from the nonlinear effect of particle trapping. Trapping of a particle of velocity v occurs when its energy in the wave frame is smaller than the wave potential. … Small waves will trap only these particles moving at high speeds near v_φ. .... When the wave is large, its linear behavior can be expected to be greatly modified…. The quantity ω_B is called the bounce frequency of oscillation of a particle trapped at the bottom of sinusoidal potential well. The frequency ω of the equation of motion is not constant unless x is small. The condition $\omega_B \geq \omega$ turns out to define the breakdown of linear theory even when other processes besides particle trapping are responsible. Another type of nonlinear Landau damping involves the beating of two waves….

To the author´s opinion the non-linear Landau damping phenomenon is still an open modelling and corresponding model solution problem. The $H_{-1/2} = H_0 \otimes H_0^{\perp}$ quantum state (Hilbert) space is proposed as appropriate framework; its solution follows the same conceptual approach as for the 3-D nonlinear, nonstationary Navier Stokes initial-boundary equations.

The proof of the quantum-physical non-linear Landau phenomenon in (MoC) is built on analytical norm estimates based on the Vlasov equation, which

- is built on the continuity equation of ideal magneto-hydrodynamics, a mass density of the fluid, a velocity of the *macroscopic* motion of the fluid with assumed Lorentz force, only, leading to a collisions-less model
- neglects the important physical phenomenon of "*electrons travelling with exactly the material speed and the wave speed*" ((ShF) p. 392). The applied analytical norms in (MoC) are even far beyond the classical L_2 -based of statistics norms where the linear Landau phenomenon has been proven following standard techniques and which are governed by the Heisenberg uncertainty relation.

The proof of the linear Landau phenomenon is based standard classical L_2 -estimates. The proof in (Moc) for the corresponding non-linear Vlasov equation (which is only about statistical distribution function, as in Kolmogorov's turbulence theory) requires solutions, which are analytical („hybrid, gliding", i.e. beyond C^{∞} regularity requirements). In other words, the proof providees evidence, that the Vlasov equation is a not appropriate model for the non-linear Landau damping phenomenon.

In (BrK6) we provide a distributional Hilbert space framework to enable a proof of the non-linear Landau damping phenomenon based on the non-linear Landau collision operator. The eigen-pair solutions of the related Oseen operator is proposed to be applied to build the problem adequate Hilbert scale. The appropriate physical model of the nonlinear Landau damping is built by the *weak variational representation* of a (Pseudo) Differential operator equation with a correspondingly defined domain, including appropriate initial and/or boundary conditions. The current classical related PDE system representation is interpreted as the approximation solution to it and not the other way around.

Let S^1 and H denote the integral operator of order -1 resp. the Hilbert transform operator of order zero(BrK3). The simplified model problem of the Landau damping phenomenon is then about the (Vlasov type, collision-free) model operator in a $H_{-1/2}$ variational framework, anticipating the Penrose stability condition (the Hilbert transform of first derivative of the homogeneous equilibrium f^0 of $f(t,x,v) = f^0(v) + h(t,x,v)$ neglecting the quadratic term in the nonlinear Vlasov equation) by the following norm equivalence

$$(H[f^{0\prime}], g)_{-1/2} = (S^{-1}\left[H[f^{0\prime}]\right], g)_0 = (H[H[f^0]], g)_0 \cong (f^0, g)_0$$

The Hilbert scale framework with its underlying „polynomial decay" norms is extended with the "exponential decay" Hilbert scale inner product resp. norm

$$(x,y)^2_{(t)} := \sum_{i=1} e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i) \text{ resp. } \|x\|^2_{(t)} := (x,x)^2_{(t)}.$$

It governs the Hilbert scale norms by the inequality ($\alpha > 0$)

$$\|x\|^2_{-\alpha} \leq \delta^{2\alpha}\|x\|^2_0 + e^{t/\delta}\|x\|^2_{(t)}.$$

Then, the corresponding Hilbert scale estimates overcomes the „*analytical velocity profile*" f^0 estimate assumptions in the form (MoC)

$$sup\{|\widetilde{f^0} \cdot e^{2\pi\mu|y|}| \leq C_0 \mid y \in R^n\} \quad , \quad \sum_{n \in R^d} \frac{\mu^n}{n!} \|\nabla^n_v\|_{L^1(dv)}.$$

In other words, the apropriate Hilbert scale framework avoids the purely mathematical requirements (analytical norms and the Penrose stability criterion). This criterion becomes „natural" part of the weak variational model, anticipating also the physical phenomenon of "*electrons travelling with exactly the material speed and the wave speed*" ((ShF) p. 392). The extension of the above exponential decay" inner norm to a combined position-velocity generalized Fourier series representation is straightforward.

For a complete Plasma phenomenon model (combining hot and cold plasma) the additional Landau collision operator can be split into a linear operator of order $m = 2\alpha$ with corresponding domain and a compact disturbance governed by the Garding inequality $(Bu, u) \quad \geq c_1\|u\|_\alpha^2 - c_2\|u\|_\beta^2$, where H_β is compactly embedded into H_α.
The conceptual approach as above is the same as for the proposed solution of the still open 3-D nonlinear, nonstationary Navier Stokes initial-boundary equations, the still open mass problem of the Yang-Mills equations by re-visited Maxwell equations in the sense of above, being followed by a re-visited (Lorentz transformation based) special relativity theory for a correspondingly extention to the general relativity theory regarding „mass-gravity pheonmenon" and related geometric properties (i.e. the inner product) of an appropriately defined Hilbert space framework.

(2) consideration to apply same concepts (including the $H_{-1/2}$ based Maxwell equations) to the conceptual ideas of Wheeler and others to a space-time quantum geometrodynamics.

In ((BrK), (BrK1), (BrK7), (BrK8) we propose a $H_{-1/2}$(weak) quantum gravity model, which overcomes the current „quantum state stage background dependency" problem (as a consequence of key principles of GRT and all related „philosophical" aspects of „space-time and the transcendental external world" ((AnE), (BoD), (RoC0-3), (ScE1-2), (SmL), (WeH3), (WhJ0-2)). The proposed quantum space-time gravity model ensures „quantum state stage background indepencency", going along with non-linear elasticity PDO equations (note: the origin of the notion „tensor" came for elasticity theory), in our case, embedded in a weak $H_{-1/2}$ variational PDO framework, governed by the least action principle.

A proper Cauchy problem formulation of the Einstein-Vacuum equations $\boldsymbol{R}_{\alpha\beta}(\boldsymbol{g})=0$, where $\boldsymbol{g}$ is an unknown four dimensional Lorentz metric and $\boldsymbol{R}_{\alpha\beta}$ is its Ricci curvature tensor, is about finding a metric g on Σ_0 coinciding with the Riemannian metric g_{ij} and that the tensor k_{ij} is the second fundamental form of the hypersurface $\Sigma_0=t=0$. The latter property can be expressed as follows. Let T denote the unit vector field normal to the level hypersurfaces of the time foliation Σ_t . Then $k_{ij}=-\frac{1}{2}L_T g_{ij}\rceil\Sigma_0$, where L_T denotes the Lie derivative in the direction of the vector field T (KlS). The Einstein field equations are overdetermined, i.e. from a mathematical point of view they are not well defined. As a „physical problem" consequence, there is the so-called *„gauge freedom"* of the Einstein field equations, allowing a special choice of gauge to resolve given ambiguities, e.g. special wave coordinates $x^\alpha, \alpha=0,\dots$ with $\nabla g_{\alpha\beta}(0)\in H^{s-1}(\Sigma_0)$ and $\dot{g}_{\alpha\beta}(0)\in H^{s-1}(\Sigma_0)$ for $s\geq 2+\varepsilon$ (KlS).

The GRT is built on Riemann´s mathematical concept of „manifolds"; we emphasis, that the mathematical model of the GRT requires *„differentiable"* manifolds, whereby only *continuous* manifolds might be required by physical GRT modelling aspects, w/o taking into account any appropriate quantum theoretical modelling requirements. Therefore, challenging the „continuity" concept, taking into account also its relationship to the quantum theory Hilbert space framework H_a and the related Sobolev embedding theorem, leads to the proposed replacement of the Dirac function concept by an alternative $H_{-1/2}$ −quantum state Hilbert space, which is independently defined and applicable from any space dimension n.

An mathematical framework with reduced regularity assumptions puts the spot on the Lagrangian Hilbert-Einstein functional, where ist critical points of this action satisfies the Einstein field equations ((LeP). Alternatively to the considered „Buchdahl field equations" in (LeP) based on an alternative defining function $f(R)$ of the scalar curvature function R, we propose to consider a weak variational representation of the Einstein field equations as a Hilbert-Einstein action functional minimization problem formulated in a Hamiltonian formalism. A similar missunderstanding as the „equivalence" of integral vs. differential form representation of the Maxwell equations is given in case of the „equivalence" of the Lagragian and the Hamiltonian formalism. The proof is enabled by the Legendre (contact (!)) transformation $L(x,y)\rightarrow H(x,\frac{dL}{dy})$

$$g(x,y):=y\cdot\vartheta(x,y)-f(x,y,)$$

which requires certain regularity requirements according to $dg=y\cdot d\vartheta+(d\vartheta\cdot dy)-\frac{\partial f}{\partial x}dx=(y+dy)d\vartheta-\frac{\partial f}{\partial x}dx$. In standard theory the product $d\vartheta dy$ is usually neglected to be infinitesimal small of second order compared to dx.

A weak variational representation of the Einstein field equations as action minimization functional in a Hamiltonian formalism in a $H_{-1/2}$ −quantum state Hilbert space framework would provide same opportunities like for the NSE and Maxwell equations to overcome e.g. current blow-up (Ricci flow) and singularity (Big Bang) challenging.

The appropriate plasma collisions (dynamics) model is another central building block for the related geometrodynamics problem/solution area. The proposed framework is also suggested to be applied to build a unified quantum field and gravity field theory based on the conceptual thoughts of Wheeler/deWitt (CiI), and and the related Loop Quantum Theory (LQT), which is a modern version of the theory of Wheeler and deWitt, where "*the variables of the theory describe the fields that form matter, photons, electrons, other components of atoms and the gravitational field - all on the same level*" ((RoC1) section 8, "dynamics as relation").

(CiI) 2.8: *Einstein's "general relativity" or ""geometric geometry of gravitation" or "geometrodynamics", has two central ideas: (1) Space-time geometry "tells" mass-energy how to move, (2) mass-energy "tells" space-time geometry how to curve. The concept (1) is automatically obtained by the Einstein field equations, (CiI) (2.3.14), basically as the covariant divergence of the Einstein tensor is zero. At the same point in time there are multiple tests of the geometrical structure and of the geodesic equation of motion, e.g. gravitational deflection and delay of electromagnetic waves, de Sitter and Lense-Thirring effect, perihelion advance of Mercury, Lunar Laser Ranging with its relativistic parameters: time dilation or gravitational redshift, periastron advance, time delay in propagation of pulse, and rate of change of orbital period*, (CiI) 3.4.
(CiI) 3.5: "*Hilbert used a variational principle and Einstein the requirement that the conservation laws for momentum and energy for both, gravitational field and mass-energy, be satisfied as a direct consequence of the field equations. ... Einstein geometrodynamics, ..., has the important and beautiful property the the equations of motion are a direct mathematical consequence of the Bianchi identities*."

5. The $H_{-1/2}$ quantum state space replacing the Dirac distributions space and a corresponding alternative Coulomb potential

With respect to the below we note that the Dirac theory with its underlying concept of a Dirac "function" is proposed to be replaced by (fluid/quantum/... state) "elements" of the distributional Hilbert space $\mathrm{H}_{-1/2}$. We note that the regularity of the Dirac distribution "function" depends from the space dimension, i.e. it is an element of $\mathrm{H}_{-1/2-\varepsilon}$ ($\varepsilon > 0$, n = space dimension). Therefore, the alternative $\mathrm{H}_{-1/2}$ quantum state concept avoids space dimension depending regularity assumptions for quantum mechanics "wave packages" / "eigen-functions" / "momentum functions" with corresponding continuous spectrum. We note that for signals on R the spectrum of the Hilbert transform is (up to a constant) given by the distribution $\mathrm{v.p.}(\frac{1}{x})$, whereby the symbol "v.p." denotes the Cauchy principal value of the integral over R. Its corresponding Fourier series is given by $-\mathrm{i}\cdot\mathrm{sgn}(\mathrm{k})$ with its relationship to "positive" and "negative" Dirac "functions" and the unit step function $Y(x)$. The $\mathrm{H}_{-1/2}$ framework, replacing the Dirac "function" concept, enables a generalization to dimensions n>1 without any corresponding additional regularity requirements.

The probability density function and the reliability function of the standard normal law are defined by

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2} \quad , \quad \bar{\Phi}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty \varphi(t)dt \ .$$

The Mill's ration function of the standard normal law is then given by

$$M(x) := \frac{\bar{\Phi}(x)}{\varphi(x)} = e^{x^2} \int_x^\infty e^{-t^2} dt \ .$$

Its reciprocal is the so-called failure (hazard) rate. Its Continued Fractions (CF) representation is provided in (AbM) 7.1.13, as well as the inequalities

$$\frac{1}{x+\sqrt{x^2+2}} < M(x) \leq \frac{1}{x+\sqrt{x^2+\frac{4}{\pi}}} \ .$$

The Mill's ratio function enjoys a series of beautiful properties ((BaÁ) Theorem 2.5, Corollary 2.6 and Theorem 3.2, (RuM)). For example, the Mill´s ratio is strictly log-convex (for corresponding properties of log-convex function, see section 1.i), which is also relevant for the alternative Kummer function Zeta function theory in the context of the usage of the Gamma (auxiliary) function with ist functional equations and ist relationship to the Euler integrals and the Gauss product formula. The functions

$$V_m(x) = \frac{2}{\Gamma(m+1)} e^{x^2} \int_x^\infty e^{-t^2} (t^2 - x^2)^m dt \ , \ m > -1$$

can be considered as one-dimensional regularizations of the Coloumb potential $1/|x|$, which are finite at the original for $\mathrm{m} > -1$, and which has applications in the study of atomes in magnetic fields and which is in fact a particular case of the Tricomi confluent hypergeometric function ((BaÁ1) and references cited there). It holds

$$V_{-1}(x) = \frac{1}{|x|} \quad , \quad \frac{1}{2} V_0(x) = M(x).$$

An alternative $\mathrm{H}_{-1/2}$ quantum state concept replacing the $\mathrm{H}_{-1/2-\varepsilon}$ ($\varepsilon > 0$, n being the space dimension) „Dirac" Hilbert space enables an alternative Coloumb potential by the above regularizations, which can be extended to all space dimensions n, i.e. in the one-dimensional case the standard Coloumb potentail became its corresponding approximation in a „Dirac world". This goes in line with the proposed concept of this paper, that the „distributional variational mathematical model" is the more „realistic/appropriate" physical model, and the classical corresponding PDE or PDO equation are the corresponding approximations to those.

6. The new ground state energy model and its (re-interpreted) related background radiation (energy) term in the Einstein gravitation tensor

The mathematical model concept of the newly proposed ground state energy model is about an inner product definition for differentials in the context of Plemelj's alternatively proposed definition of a „potential" in the form $\langle\langle du, dv\rangle\rangle \cong (u,v)_{-1}$, $\langle\langle du, v\rangle\rangle \cong (u,v)_{-1/2}$, (PlJ)$^{(*)}$. Plemelj's correspondingly alternatively proposed „flux" is defined by

$$\bar{U}(\sigma) := -\oint_{\sigma_0}^{\sigma} \frac{\partial U}{\partial n} d\sigma \quad (\sigma_0, \sigma_0 \in surface!),$$

whereby in case $\bar{U}(\sigma)$ is differentiable, it holds $\frac{d\bar{U}(\sigma)}{d\sigma} = -\frac{dU}{dn}$. In case $\frac{dU}{dn}$ is not defined (i.e. $\bar{U}(\sigma)$ is not differentiable), the „flux" $\bar{U}(\sigma)$ is a still well defined term. The concept is developped for logarithmic potential ($n = 2$), which is related to the Cauchy-Riemann differential equations and the $\bar{U}$ being the conjugate of $U(\sigma)$, resp. its Hilbert transform. The generalization to dimensions $n > 2$ ($divA = 0$, $rotA = 0$ (RuC)) leads to the concept of Riesz transforms (StE1).

From (LaC) p. 4 (see also (WoW) p. 411) we recall: „*The analytical approach (of variational methods) to the problem of motion is quite different. The particle is no longer an isolated unit but part of a „system*". $^{(**)}$

Plemelj´s alternative concept of a „current"/"flux" through a surface is based on his alternative definition of a potential, which is mathematically speaking a replacement of the Lebesgue integral by the Stietjes integral (PlJ). Its definition is purely built on infinitely small boundary/surface „elements", i.e. it does not need any regularity requirements of the potential (solution) function within the enclosed interior or exterior domains of the considered PDE system.

The Maxwell equations govern the electromagenic field, when the distribution of the electric changes and currents are known $^{(**)}$. The Lions conjecture is about vector fields with $divE(x) = curlB(x) = 0$ that the scalar product $E(x) \cdot B(x) \in H^1$ belongs to the Hardy space H^1 (MeY), with its obvious relationship to the Hilbert space $H_{1/2}$ with the related decomposition $H_{1/2} = H_1 + H_1^{\perp}$.

The alternative „current" concept is proposed being applied to the Maxwell equations in a weak variational $\in H_{-1/2}$ framework (going along with a corresponding generalized Fourier expansion concept) to transport energy between quantum state „elements" $\in H_{-1/2}$, providing an alternative concept to the sophisticated „displacement current" concept. With respect to the below „*electromotive motional force*" concept of the Maxwell-Lorentz equations the concept above indicates a modelling split between „electro"- and „magnetic"-waves, whereby the current „combination" model is still valid restricted by to „observation" (statistical) Hilbert space framework $H_0 = L_2$ and still governed by the light velocity.

(*) Bisher war es üblich, für das Potential die Form (*) zu nehmen. Eine solche Einschränkung erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potentiale der grösste Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form (**) verwendbar."

(**) (EiA) p. 52: „Die Gesetze aber, nach denen sich Ströme und Ladungen verhalten, sind uns nicht bekannt. Wir wissen wohl, dass die Elektrizitäten in Elementarkörperchen (Elektronen, positiven Kernen) bestehen, aber wir begreifen es nicht vom theoretischen Standpunkt aus. Wir kennen die energetischen Faktoren nicht, welche die Anordnung der Elektrizität in Körperchen von bestimmter Grösse und Ladung bewirken, und alle Versuche, die Theorie nach dieser Seite hin zu vervollständigen, sind bisher gescheitert. Wir kennen daher, falls wir überhaupt die Maxwellschen Gleichungen zugrunde legen dürfen, den Energietensor für die elektroagnetischen Felder nur ausserhalb der Elementarteilchen. An diesen Stellen, den einzigen, wo wir einen vollständigen Ausdruck für den Energietensor aufgestellt zu haben glauben, gilt $\frac{\partial T_{ij}}{\partial x_j} = 0$. „… p.54: „wir wissen heute, dass die Materie aus elektrischen Elementarteilchen aufgebaut ist, sind aber nicht im Besitz der Feldgesetze, auf welchen die Konstitution jener Elementarteilchen beruht." … p. 81. „Für ein Feldgesetz der Gravitation muss die Poissongleichung der Newtontheorie zum Muster dienen. … Die Untersuchungen der speziellen Relativittätstheorie haben uns gezeigt, dass an die Stelle des Skalars der Massendichte der Tensor der Energiedichte zu treten hat. In diesem ist nicht nur der Tensor der Energie der ponderabeln Materie, sondern auch der der elektromagnetischen Energie enthalten. Wir haben sogar gesehen, dass unter dem Gesichtspunkte einer tieferen Analyse der Energietensor der Materie nur ein vorläufiges, wenig tiefgreifendes Darstellungsmittel für die Materie anzusehen ist. In Wahrheit besteht ja die Materie aus elektrischen Elementarteilchen und ist selbst Teil, ja als der Hauptteil des elektromagnetischen Feldes anzusehen. Nur der Umstand, dass die wahren Gesetze des elektromangentischen Feldes für sehr intensive Felder noch nicht hinreichend bekannt sind, zwingt uns vorläufig dazu, die wahre Struktur dieses Tensors bei der Darstellung der Theorie unbestimmt zu lassen."

Plemalj´s mass element concept is related to ideal numbers (non-standard/hyperreal number). *„Their relationship to the principle of relativity is simple"* (*).

The reduced regularity requirements of a $H_{-1/2}$ based variational representation goes along with a usage of the integral representation of the Maxwell-Lorentz equations (FlD). It is also related to Plemelj's alternative *„mass element"* and *„current/flux"* concept (with corresponding reduced regularity requirements to the standard Green formulas) and the corresponding Gauss' (electromagnetic) and Faraday's (connecting) field laws.

The integral representation of the Maxwell equations are given by (FlD):

1. the Gauss laws (relating the electric flux through a closed surface to the charge enclosed by that surface):

$$\oint_S \vec{E}\circ\hat{n}da = \frac{q_{encl}}{\varepsilon_0} \quad \text{(electric field)} \quad ; \quad \oint_S \vec{B}\circ\hat{n}da = 0 \quad \text{(magnetic field)}$$

2. the Faraday law (two effects connecting electric field circulation with a changing magnetic field)

a) the magnetic induction (= magnetic flux *density*) (Faraday)

$$\oint_S \vec{E}\circ d\vec{l} = -\oint_S \frac{\partial\vec{B}}{\partial t}\circ\hat{n}da \quad \text{(electric field circulation = change of magnetic field)}$$

b) the acting *„electromotive motional force"* (emf), involving the movement of a charged particle through the magnetic field:

$$emf = -\frac{d}{dt}\oint_S \vec{B}\circ\hat{n}da \qquad \text{(flux rule).}$$

By formally operating with the „rot" operator for the „vacuum" case (i.e. $divE = divB = 0$, in combination with the formula $-grad(divA) = \Delta A + rot(rotA)$) to the differential form of the „combined" (2a/b) Faraday law

$$\nabla\times\vec{E} = rot\vec{E} = -\frac{\partial\vec{B}}{\partial t} \quad \text{(integral form: } \oint_S \vec{E}\circ d\vec{l} = -\frac{d}{dt}\oint_S \vec{B}\circ\hat{n}da)$$

leads to the wave equations the form ($c^2 = \varepsilon\cdot\mu$, ε permittivity, μ permeability)

$$\frac{\partial^2}{\partial t^2}E - c^2\Delta E = \frac{\partial^2}{\partial t^2}B - c^2\Delta B = 0.$$

It describes the propagation of electromagnetic waves with light velocity through a vacuum.

(*) (PoP): In Einstein´s theory the rule of speed addition is used, when adding units does not lead to endless increase of the sum, it is limited by the maximum velocity-of-light limit. But in this case the matter is not in the breaking up of the Eudocks-Archimedean axiom, but in the special features of Lorentz transformations, actual for pseudo-Euclidean continuum of space-time. Obviously, it can be admitted, that the analogical rule of addition will work when dealing with simple quantities, such as the length or the time space. But still, it is not clear why we must limit the endless space with some set of radius, to which the sum of the added quantities would aspire. The prospect law exists, but we do understand that lessening of length within the distance is the optic illusion, but not the characteristic of the spacial metrics.

Now let u stake the quantum mechanics. It is known, that the so-called „ultra-violet-catastrophe" was the direct consequence from the formulae of the classical mathematical analysis – for the balance of radiation in the field of high frequencies the result was endless quantity of energy. But the way out was found not in the modification of mathematical principles, but in realizing experimential data: Max Planck´s hypothesis put the limit to the endless energy subdivision $E = h\nu$ appered to be non-divided. And at the moment the clinical formulae of analysis being used, and what concerns all „disturbing" modern physic-theoretic learnt as Richard Feynman said, to „sweep them under the rug".

There is no absolute motion, two points can be move only with regard to each other. If we take one of them for standard point, we believe it is stable, and the second one moves with regard to the first one. And vice versa: we can take the second moving point for the stable starting point and consider the first one to be moving. The notion of motion quite naturally and necessarily requires the principle of relativity as the distance change between these two points BETWEEN THEM with some time. Sketchily the principle of relativity is explained with the example of two points A and C. We take one of them fort he starting point, the other moves with regards to the starting point, and vice versa. Let us imagine, in space there are two points (mathematically size less), separated by some distance. Now let us try to imagine that the distance changes… But how can we check this „change"? Anri Poincare, illustrating these cases, made the imaginary experience- he asked: what would happen if the distance between the two points becomes twice bigger? And he answered: the world would not notice it. I think it is clear. To be able to speak of the change of the distance between two points, there must be one more point which would be stable with regard to one of the two given points.

The Dirichlet integral $D(u,v) \coloneqq (\nabla u, \nabla v)$, as part of the Green formula, which is derived from the Gauss „law" in combination with the formula $div(A \cdot gradB) = A \cdot \Delta B + gradA \cdot gradB$, defines the inner product of the „energy" Hilbert space H_1. We note that this means that even for the (weak) standard variational representation of the wave equations the minimum regularity requirements is $E, B \in H_1$. Due to the Sobolev embedding theorem this means that the solutions $E, B \in H_1$ are only continuous in case space dimension $n = 1$.

A similar „formal operating with an „*appropriate operator*"" „trick" (whereby we note that an „operator" definition necessarily requires the definition of a corresponding domain*)* is also applied for the Navier-Stokes equations resulting into *„Neumann problem for the pressure field $p(\vec{x},t)$ ($\vec{n}$ denotes the outward unit normal to the domain G)*

$$\Delta p = \rho(\vec{v} \cdot \nabla \vec{v} - \vec{f}) \text{ in } G$$
$$\frac{\partial p}{\partial n} = -\left[\mu \Delta \vec{v} - \rho \vec{v}_1 \cdot \nabla \vec{v} - \vec{f}\right] \cdot \vec{n} \quad \text{ at } \partial G.$$

It follows that the prescription of the pressure at the bounding walls or at the initial time independently of $\vec{v}$, could be incompatible with the initial and boundary conditions of the NSE PDE system, and therefore, could render the problem ill-posed (GaG)", (HeJ).

The above *Ricci* ODE type estimate is valid for all non-linear evolution PDE, as well as for the *Ricci flow* theory further below.

Therefore, the Maxwell equations and Navier-Stokes equations solutions in combination with e.g. non-regular initial or boundary value „functions" require coresponding modified a priori „energy norm" estimates (see also section 3.a). Concerning *the singularities in the general relativity theory* we quote the abstract from (TrH):

„Regular solutions of EINSTEIN's equations mean very different things. In the case oft he empty-sapce equations, $R_{ik} = 0$, such solutions must be metrics $g_{ik}(x^l)$ without additional singular „field sources" (EINSTEIN's „Particle problem"). However the „phenomenological matter" is defined by the EINSTEIN equations $R_{ik} - \frac{1}{2} g_{ik} R = -\mu T_{ik}$ itselves. Therefore if 10 regular functions $g_{ik}(x^l)$ are given (which the inequalities of LORENTZ signatur fulfil) then these g_{ik} define 10 functions $T_{ik}(x^l)$ without singularities. But, the matter-tensor T_{ik} must fulfill the two inequalities $T \geq 0$, $T_0^0 \geq \frac{1}{2} T$ only and therefore the EINSTEIN-equations with „phenomenological matter" mean the two inequalities $R \geq 0$, $R_0^0 \leq 0$ which are incompatible with permanently regular metric with LORENTZ-signature, generally. But exactly those inequalities for the space-time curvature generate the collaps; resp. generate the anti-collaps of stars, star systems and the whoel cosmos („Big bang")."

The ground state energy level is principlely not measurable. On the other side, "„*real*" *is only that, which is measurable*" (M. Planck), and *mass is essentially the manifestation of the „vaccum" energy*.

Claim: the alternatively proposed ground state energy concept in (BrK), (BrK1), (BrK8) is about rotating differentials (monads, ideal point) governing the energy „transport" from one „particle" to the other; the inner product of those „monads" is given by the H_{-1} Hilbert space. When such a „monad" state is being „tested/measured" against H_0 „distributions", it results into an approximation „element" $\in H_{-1/2}$ (alternatively to the standard quantum status $\in H_0$). We note that the regularity of this Hilbert space is „infinitely small" ($\varepsilon > 0$) better that the best possible Dirac function (in case of space dimension $n = 1$). The corresponding „energy" Hilbert space replacing H_1 is the Hilbert space $H_{1/2}$ with the related decomposition $H_{1/2} = H_1 \otimes H_1^{\perp} = H_{-1/2}^{*}$. The orthogonal projection $H_1^{\perp} \rightarrow H_1$ projection onto the compactly embedded Hilbert space of H_1 into $H_{1/2}$ with respect of the inner product of $H_{1/2}$ provides an alternative model for so-called „symmetry breaking" effects (*), i.e. H_1 –mass is the manifestation of the $H_1^{\perp}$ – ground state energy.

The classical Yang-Mills theory is the generalization of the Maxwell theory of electromagnetism where chromo-electromagnetic field itself carries charges. As a classical field theory it has solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. *This is the mass gap*. Another aspect of confinement is asymptotic freedom which makes it conceivable that quantum Yang-Mills theory exists without restriction to low energy scales. The above concept enables mass „elements" manifestated a mass densities as intrinsic modelling element of the variational framework of the considered PDE system. In other words, postulating the weak variational representation as the „truly" physical model / „law" (while the corresponding classical represention becomes the approximation solution, going along with moving from Differential Operators to Pseudo Differential Operators with correspondingly reduced domain regularity assumptions) the concept of „force" (**) acting on an physical mass „density-element" becomes an „measurable" model specific observation modelled by the corresponding projection operator.

(*) **(1)** *„there is no continuous infinitesimal transformation for charge conjugation. No states exist that carry charge values in a continuum from the -e electric charge of an electron to the +e of the positron, or between the $I_z = \pm 1/2$ isospin eigenvalues. How do we define invariance for discrete symmetries?*" ((NeD) 9.1); **(2)** *„In the early universe, pressures and temperature prevented the permanent establishment of elementary particles. Even quarks and leptons were unable to form stabe objects until the Universe had cooled beyond the supergravity phase. If the fundamental bulding blocks of Nature (elementary particles) or space-time itself were not permanent then what remained the same? The answer is symmetry.*" Source: abyss.uoregon.edu); **(3)** *„Inflation theory connotes problems such as fine-tuning and a low-entropy initial state. In the framework of Einstein's general relativity (GR), the question on the initial state of the universe is hard to answer because GR actually predicts that a singularity appears at the beginning of the universe. … An interesting proposal is to assume that the universe begins with no origin through eternal inflation. However, inflation requires extremely special initial conditions. One may consider the universe to have a quantum cosmological origin. In fact, this suggestion naturally give rises to the necessity of quantum gravity and quantum corrections to GR. One may expect that, with proper consideration of quantum gravitational effects, a gravitational theory must be free from singularities. In this work, we show that the Eddington-inspired Born-Infeld (EiBI) theory of gravity provides an opportunity to analyze the initial state of the universe.*" (KiH); **(4)** *the Wigner-Eckart-theorem to govern the „observable" variables in H_1;* **(5)** *the Higgs mechanism with the Higgs (quadratic term) potential of the Higgs field;* **(6)** *the quadratic Zeeman effect: in solid state physics the quadratic Zeeman effect (and the related Landau levels) has been recognized to be of importance for even moderate magnetic field, where the frequently small effective mass of an electron in a crystal magnifies the influence of the field (HeE); ….* **(7)** *anti-symmetry in the Landau levels in models for atoms in strong magnetic fields.*

(**) in the Newton classical sense the notion „force" is about the temporal change of the momentum; in quantum physics this is about the interaction of elementary particles, more specifically, the generation and destruction of an elementary particle.

„The set of prime numbers are related to the set of nontrival zeros of the zeta function of Riemann. Alain Connes‛ spectral interpretation of the critical zeros of the Riemann zeta function is about eigenvalues of an absorption spectrum of an unbounded operator in a suitable Hilbert space.

In (LaG) a spectral interpretation is provided of the zeros of the Dedekind zeta function of an algebraic number field K of degree n in an automorphic setting. If K is a complex quadratic field, the torical forms are functions defined on modular surfaces X, such that the sum of this function over the „gauss set" of K is zero, and Eisenstein seires provide such torical forms. … Alternatively, the torical forms are othogonal to orbital series in X".

In (EhP) contemporary infinitesimalist alternatives to the Cantor-Dedekind theory of continua are provided.

The Hilbert scale framework and related compactness arguments (based on Garding type inequalities like Korn´s 2nd inequality for the strain and stress tensor (AzA)), enables the full power of functional analysis (e.g. *Gateaux* differential defined on convex linear manifolds in combination with complementary extremal problem solutions (VeW), replacing the Co-variant derivative concept; Lipschitz continuous Gateaux F_u , Hölder norms and fixpoint arguments to show unique solution of non-linear problems) spectral theory ($B := A + K$ with a positive definite, self-adjoint H_α operator A of order $m = 2\alpha$, A^{-1} compact and a compact disturbance operators K; Garding‘s inequality: $(Bv, v) \geq c_1\|v\|_\alpha^2 - c_2\|v\|_\beta^2$, H_α compactly embedded in H_β) and approximation theory in Hilbert scales (NiJ1) going along with reduced regularity of variational problem solutions based on Newtonian potentials (NiJ2) for a combined quantum and gravitation theory.

In (KlS) regularity and geometric properites of solutions of the Einstein-vacuum equations are considered concerning the study of rough solutions to the initial value problem expressed relatively to the wave coordinates. Based on Strichartz type inequalities a gain of half a derivative relatively to the classical result is achieved.

We emphasis the correspondingly defined Hilbert space of the Hilbert space $L^2_{2\delta}(R^{\times}_{+}) =: L^2_{2\delta}$ ($\delta \in R$) with the norm $\|f\|^2 := \int_0^{\infty} |f(x)^2| \cdot (1 + log^2 x)^{\delta} d^{\times}x < \infty$ fulfilling the following embedding properties

$$H^{\delta}(R) \subset L^2(R) \subset H^{-\delta}(R).$$

In (KiA1) relatively undistorted wave solutions of the form $u = gf(\theta)$, of the wave equation in three space variables are considered, where $\theta = \theta(x, y, z, t)$ and $g = g(x, y, z, t)$ are the phase and the amplitude, and f is an arbitrary wave form function of a single variable. The plane and spherical waves are explicite solutions of this problem. In those cases the phase does not determine the amplitude uniquely. In the plane wave, g can be multiplied by an arbitrary harmonic function, and in the spherical wave, g can be multiplied by any arbitrary function harmonic on the sphere. Choosing the delta function as f one obtain a diffusionsless solution of the two-dimensional wave equation, which still not contradict to the Huygens principle.

From (CoR), p. 763 we recall: „*relatively undistorted spherical waves relate to the problem of transmitting with perfect fidelity signals in all directions. All we can do here is to formulate a conjecture which will be given some support in article 3: Families of spherical waves for arbitrary time-like lines exist only in the case of two or four variables, and then only if the differential equation is equivalent to the wave equation. A proof of this conjecture would show that the four-dimensional physical space-time world of classical physics enjoys an essential distinction*.“

The Bateman solution is a model for the *electrical* wave motion on the basis of the Maxwell equations.

In (WeP), (WeP1) self-adjoint extensions and related spectral properties of the Laplace operator with respect to electric and magnetic boundary conditions are considered. Two integral representations of the electric and the magnetic fields E resp. H (produced by a given current distribution in the presence of perfectly conducting bodies with given boundaries governed by the Maxwell equations) allow a complete discussion about the asymptotic behavior of the solutions E resp. H as $t \to \infty$.

The standard approach to solve the time-harmonic Maxwell equations

$$curl^2 E - k^2 E = 0 \ , \ \ k := \omega\sqrt{\mu_0 \varepsilon_0}.$$

by a single layer potential, weigthed with an element $a \in (\partial D, C^3)$ of a tangential field, leads to the following results:

Theorem 3.33 (KiA): the curl of the potential $v = \int_{\partial D} a(y)\theta(x,y)ds(y)$ with Hölder-continuous tangential field $a \in (\partial D, C^3)$ can be continuously extended from D to $\overline{D}$ and from $R^3 - \overline{D}$ to $R^3 - \overline{D}$ with limiting values of the tangential components given by (3.44).

Theorems 3.35/3.36 (KiA): For every $f \in C^{0,\alpha}_{Div}(\partial D)$ the uniqueness and existence of the exterior boundary value problem

$$curlE^s - i\omega\mu_0 H^s = 0 \ , \ curlH^s + i\omega\mu_0 E^s = 0 \text{ in } R^3 - \overline{D}$$

fulfilling the Silver-Müller radiation condition

$$\sqrt{\varepsilon_0}E(x) - \sqrt{\mu_0}H(x) \times \frac{x}{|x|} = O(\frac{1}{|x|^2}) \quad ,$$
$$\sqrt{\mu_0}H(x) - \sqrt{\varepsilon_0}E(x) \times \frac{x}{|x|} = O(\frac{1}{|x|^2})$$

(which is equivalent to a corresponding Sommerfeld radiation condition ((KiA) theorem 3.30)) is given by the solutions

$$E^s(x) = curl \int_{\partial D} a(y)\theta(x,y)ds(y) \ , \ H^s(x) = \frac{1}{i\omega\mu_0} curlE^s(x).$$

For the related cavity problem (variational formulation, existence and generalized Fourier series representation) in the context of the Helmholtz decomposition we refer to the theorems 4.32, 4.34, lemma 4.35, corollary 4.36.

The scattering theory is based on eigenfunction expansions associated with the Schrödinger operator and its underlying properties of the spectrum of the Schrödinger operator, where eigenfunction solutions of the Schrödinger equation are unique solutions of an integral equation, representing the distored plane wave (i.e. the plane wave plus the outgoing scattered wave) (IkT).

The results are being extended in (LuL) for the Klein-Gordon equation specifying the class of potentials by transforming the K-G equation into a representation in the form $i\dot{\theta}=A[\theta]$ with $\theta(x,t):=\{u(x,t),i\dot{u}(x,t)\}$ in an appropriately defined Hilbert space. Its norm is basically the norm of the $H_0=L_2$ −space, i.e. the reduced regularity assumption to the initial data $\in H_{-1/2}$ in combination with the non-regular inital value properties of parabolic inital value problem ($H_{a;t}\sim t^{-a/2}H_0 \sim H_{-\frac{1}{2};t=0}$) can be applied in combination with the genralized Fourier transform ((IkT) theorem 5)

$$\hat{f}(k)=\frac{1}{(2\pi)^{\frac{3}{2}}}l.i.m\int_E \overline{\varphi(x,k)}f(x)dx$$

where $l.i.m\int_E \dots dx$ mean the limit in the mean of the function $l.i.m\int_{K(N)}\dots dx$, $N\rightarrow\infty$, $K(N)=K(0,N)$, and $K(x,N)$ denotes the sphere of radius N with ist centre at x.

In (AhJ) the „scattering trinity" is considered for the Helmholtz equation with radiation condition as a model for exterior scattering problem. The trinity is about the null-field method, modified Green functions (for the Dirichlet and Neumann problem) and the corresponding reproducing kernel (the difference between the modifed Dirichlet/ Neumann Green functions). In the light of the alternative Plemelj potential concept the exterior integral relation ((AhJ) (5.2))

$$\int_{\partial B}\frac{\partial K(x,y)}{\partial n(y)}u(y)dS_y=\int K(x,y)\frac{\partial u(y)}{\partial n(y)}dS_y$$

(and its related „trinity" partners) will enable a corrsponding Hilbert space in line with the proposed „quantum state Hilbert space $H_{-1/2}=H_0\otimes H_0^{\perp}=H_{1/2}^{*}$, resp. the corresponding energy Hilbert space Hilbert space $H_{1/2}=H_1\otimes H_1^{\perp}=H_{-1/2}^{*}$, whereby the spectrum of H_1 are governed by Fourier waves, while the spectrum of $H_1^{\perp}$ are governed by Calderón wavelets.

With respect to the Einstein field equation (which are even formulated as classical PDE system) Plemelj's „curent/flux" concept would avoid the concept of infinite numbers of tangent spaces and the related manifold concept with its underlying concepts of „diffeomorphism" and „covariant derivative". Also the requirement of *differentiable* manifolds for the Einstein field equations, which are w/o any physical meaning, would be avoided.

We note that the most significant difficulty with the Ricci equation $Ricci(g) = 0$ (which is vacuum Einstein equation) from the PDE point of view (and the related Cauchy problem with appropriate inital data) is the highly degree of non-uniqueness, which is due to the coordinate or diffeomorphism invariance, leading to the $\varphi^*(Ricci(g)) = Ricci(\varphi^* g)$. Thus, if g is a solution, then also $\varphi^* g$.

The „observation" limitations go in line with the light velocity boundary, while the model itself provides concepts of an extended definition of „dark energy", which is called „dark", because it does not appear to interact with observable „electro"-waves. This dark energy covers then $> 4.6\% + 23\% + 72\% = 99.6\%$ of the total universe space. From a mathematical model point of view this is reflected by the compactly embeddedness of H_1 into $H_{1/2}$.

In case there are any mathematical obstacles when applying the Ricci flow to the several research topics (CaH1), why then not (as an interim „solution") defining a proper „displacement Ricci flow", to align the fragmented current results?

The building principle of the Einstein gravitation tensor $G \coloneqq Ricci - \frac{1}{2} Sg$ puts the Ricci tensor in relationship to the energy-momentum tensor T by $G = 8\pi T$ (with normed Newton gravitation constant $C \coloneqq 1$), i.e. the matter density, as described by the energy-momentum operator, generates the curvature of the space-time and particles moves along corresponding surveyors (geodesics). The term $-\frac{1}{2} Sg$ was added by Einstein in order to ensure the „conservation of energy" „law/principle" (divergence-free opertor) for the to-be-built-curvature tensor, which is valid for the energy-momentum operator. The same argument also allows the famous „cosmological constant" term, which seems to get a new role in the „dark energy" modelling, whereby the constant is assumed to be extremely small.

The curvature scalar $S = Ricci_i^i$ is the trace of Ricci. It is related to the Gauss curvature K (the product of the two main curvatures of a surface) and the Weingarten mapping W by $S = 2K = 2 \cdot \det(W)$. The Ricci and the Gauss curvature scalar are intrinsic parts of the surface, i.e. they are parts of the „interior" surface geometry. The scalar value is independent from the coordinate system. In other words, the Ricci curvature scalar is constant throughout space-time, as the trace of the energy-momentum, but the Ricci tensor is not. The mathematically required „correction" term is called Weyl tensor; its physical interpretation is about the so-called „tidal effect", which keeps the matter density, while deforming the volume of a small ball of matter is being attracted by a large „ball" of matter.

On semi-Riemannian manifolds it holds $div(Ricci) = \frac{1}{2}dS = dK$ ((OlR) 11.5). The Ricci potential definition indicates the „extended" Plemelj potential definition based on dK:

„The Riemannian metric is a Euclidean scalar product on each tangent space, $T_m(M)$, $m \in M$. Although the situation is infinitesimally Euclidean (i.e. on each tangent space) it is not locally. The defect to being locally Euclidian is given by the curvature. … The most important object is the Ricci curvature which is an average of the sectional curvature. … It is a bilinear form of same nature as the Riemannian metric, which governs the volume element at a distance r from m in the direction of a unit vector u by the following asymptotic expansion

$$D_t \coloneqq dvol = (1 - \frac{r^2}{6}Ricci_m(u,u) + o(r^2))dvol_{eucl}.$$

… The simplest curvature although the weakest is the scalar curvature which is, at each point, the trace of the bilinear from Ricci with respect to the Euclidian structure g. It is a smooth real valued function on M"(BeG).

Claim: In the context of the newly proposed ground state energy model we propose the Ricci tensor becoming the single curvature tensor, $\tilde{G} := Ricci \coloneqq T + dK$, while the Weyl tensor is being re-interpreted as background radiation energy. The trace of the combined energy terms is then constant throughout space-time and positive. Then the Ricci curvature is being governed by the electromagnetic energy-momentum operator T (the Newton gravitation constant and the factor $c\pi$ have beed omitted) and the background energy-radiation operator $dK = div(Ricci)$ based on no longer differentiable (or continuous) manifolds domains, but on definite or indefinite product spaces ((BoJ)).

From the stand-point of the ether hypothesis the new concept overcomes the *„remarkable, conceptual difference between the gravitation field and the electromagnetic field"* (*). It enables the concepts of convex topology, hypersingular integral operators, Sobolev-Slobodetskii spaces, Neumann problem and double layer potential, as well as, spaces with of fractional quotients and related integral projectors ((EsG), (LiI)).

The newly proposed concept also fits to Gödel´s new type of examples (GoK), allowing cosmological solutions of Einstein´s field equations of gravitation in a four dimensional space with rotational symmetry, being stationary and spacially homogeneous, where any two world lines of matter are equidistant and a positive direction of time can consistently be introduced in the whole solution. The latter topic in the context of „space-time is a quantum", „quanta of space", „a single quantum of a Faraday´s line of a gravitation fields", spin networks", „quantum mechanics can not deal with the curvature of space-time", „general relativity cannot account for quanta" and „the flux of time" are considered in ((RoC0-3)). For the conceptual baseline to those topics we refer to (BoD).

(*) (EiA1): „…; There can be no space nor any part of space without gravitational potentials; for these confer upon space its metrical qualities, without which it cannot be imagined at all. The existence of the gravitational field is inseparably bound up with the existence of space. On the other hand a part of space may very well be imagined without an electromagnetic field; thus in contrast with the gravitational field, the electromagnetic field seems to be only secondarily linked to the ether, the formal nature of electromagnetic field being as yet in no way determined by that of gravitational ether. From the present state of theory it looks as if the electromagnetic field, as opposed to the gravitational field, rests upon an entirely new formal motif, as though nature might just as well have endowed the gravitational ether with fields of quite another tpye, for example, with fields of a scalar potential, instead of fields of the electromagnetic type. …; wesentlich ist ja nur, dass neben den beobachtbaren Objekten noch ein anderes, nicht wahrnehmbares Ding als real angesehen werden muss, um die Beschleunigung bzw. die Rotation als etwas Reales ansehen zu können."

The newly proposed model puts the spot on the „Ricci flow" concept, as being successfully applied in the context of the geometrization of 3-manifolds (e.g. (AnM), (CaH), (CaJ), (HaJ), (ThW), (YeR)). For an overview to the several related topic areas (e.g. parabolic re-scaling, evlution of curvature and geometric quantities under Ricci flow, existence theory, Perelman's W entropy functional, gradient formulation, total scalar curvature $E(g) \coloneqq \int_M SdV$ with gradient $\nabla E(g) = -Ricci + \frac{S}{2}g$ and its relation to the Einstein tensor) we refer to (ToP). We note that the scalar curvature S satisfies $S_t = \frac{\partial}{\partial t}S = \Delta S + 2|Ricci|^2$, so by the maximum principle its minimum is non-decreasing along the flow (HaR1).

In (CaJ) the Ricci potential function is considered, if g is a rotationally symmetric metric R^n. Then every two-dimensional linear subspace W of R^n which passes through the origin is a totally geodesic submanifold of (R^n, g). Let $K(x)$ be the Gauss curvature of $W$with respect to the metric induced by g and dA the induced area element, so that KdA is the curvature form of W.

Then the Ricci potential function $w_g(t)$ is defined by

$$w_g(t) \coloneqq \frac{1}{2\pi}\int_{D_t} KdA_g$$

where $D_t \coloneqq \{(s,\theta) \in W | 0 \leq s \leq t\}$ in a disk centered at the origin of W.

The rotationally symmetric metric links back to Gödel´s new type of examples (GoK).

The main results related to Riemannian and Einstein metrics and Ricci flow are (YeR):

(1) Ricci pinched stable Riemannian metrics can be deformed to Einstein metrics through the Ricci flow of R. Hamilton;

(2) (suitably) negatively pinched Riemannian manifolds can be deformed to hyperbolic space forms through Ricci flow;

(3) (3) L2-pinched Riemannian manifolds can be deformed to space forms through Ricci flow.

„The use of direct variational methods becomes difficult, since there exist compact manifolds which do not carry any Einstein metric; ... a proof of the existence of an Einstein metric by variational method should require some geometric assumption on the manifold" ((BeA), chapter 4). The theorem of deTurck ((BeA) theorem 5.14) is about regularity assumptions in Hölder spaces $C^{m.\alpha}$ to ensure the existence of a Riemannian metric $g \in C^{m.\alpha}$ $(m > 2;\ 0 < \alpha < 1)$ such that $Ricci(g) = r \in C^{m.\alpha}$. At the same point in time there is no Riemannian metric with Ricci curvature in any neighborhood of the origin, however $r = \sum_{i,j} x^i x^j\, dx^i \times dx^j$ is the Ricci curvature of a Riemannian metric in $R^n - \{0\}$ ((BeA) Example 5.8). The Bianchi identity is a genuine obstruction to solve the Ricci equations, even locally in the special case, where the Ricci candidate r is a linear polynomial ((BeA)example 5.12).

We note that even elastodynamic Stokes´ fundamental solution is singular at origin zero ((BoM) 7.7). In (NiJ2) an unusual shift for the equation is provided concerning the regularity of the solution of the right hand side based on standard estimates of the Newtonian potential. In (NiJ3) a free boundary problem for the Stokes flow is considered, where its transformed partial differential equations with fixed domain are highly non-linear. They consist in an elliptic linear system coupled with an ordinary differential equation representing the free boundary. On the free boundary the conditions are (i) the free boundary is a streamline, (ii) the tangential force vanishes, (iii) the normal force is proportional to the mean curvature of the boundary. By straightening the boundary the problem is reduced to one in a fixed domain. The analytical properties of the solutions are analyzed in a Hölder space framework. We note that the elliptic system coupled with an ODE show some analogy to the gradient flow formulation of the Ricci flow coupled with the solution of the heat operator ((ToP) 6.4).

An appropriately defined Hilbert space framework would enable the full power of analysis techniques of non-linear parabolic problems with non-regular initial value data ((BrK2), 3-D-non-linear and non-stationary NSE-problem)), related $t{\sim}0$ singularity behavior ((BrK2), §6) in the form

$$H_{a;t} \sim t^{-a/2} H_0 \quad , \quad H_{a;t} \sim t^{1/4-a/2} H_{1/2}$$

and corresponding Hilbert scale resp. Hölder space approximation theory ((NiJ), (NiJ1)). The corresponding analysis techniques is about uniqueness and existence of Cauchy type PDO equations with appropriately choosen Hilbert space anticipating singularity behavior $t{\sim}0$ and blowup effects for $t \to T$ due to the fact, that there is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE $y'(t) = y^2(t)$, $y(0) = y_0$ with the solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time. The Ricci flow analysis is also governed by this simple ODE constraint. The Constatin-Lax-Majda (CLM) model for the three-dimensional vorticity equation might provide an additonal useful modelling tool for further elaborations on this topic (CoP).

The proposition 6.3.1 in (ToP) is about an inner product to enable a weak variational representation of the Ricci (gradient) flow analogue to the standard heat equation equation. In (BrK10) pp. 19/22, appropriate Hilbert and Hölder space inner products are provided enabling optimal shift theorems of the corresponding operator equations for both cases, homogeneous heat and non-linear parabolic equations with non-regular initial value functions, as well as for the inhomogenous heat equation with zero initial value function. The corresponding generalized variational representation ((ToP) proposition 6.3.1) in combination with a non-regular (distributional) initial value „function" provides some further evidence of too high, classical) regulary assumptions for the Einstein field equations concerning continuous (even differentiable) manifolds.

In (KiH) the „initial state" of an anisotropic universe in Eddington-inspired Born-Infeld (EiBI) gravity filled with a scalar field, is considered, whose potential has various forms.

„The EiBI theory of gravity is described by the action

$$S_{EiBI} = \frac{1}{\kappa} \int \sqrt{-\left|g_{\mu\nu} + R_{\mu\nu}(\Gamma)\right|} - \lambda \sqrt{-\left|g_{\mu\nu}\right|} + S_M(g, \Phi)$$

where $\left|g_{\mu\nu}\right|$ denotes the determinat of $g_{\mu\nu}$, λ is a dimensionless parameter that is related to the cosmological constant, and κ is an additional parameter. In this theory, the metric $g_{\mu\nu}$ and the connection $\Gamma^{\rho}_{\mu\nu}$ are treated as independent fields. The Ricci tensor $R_{\mu\nu}(\Gamma)$ is evaluated solely from the connection, and the matter field Φ is coupled only to the gravitational field $g_{\mu\nu}$.

There is a maximal pressure solution (MPS) of the equation of motion of the EiBI gravity, where the early-time behavior of the MPS describes the initial state of the universe including the early-time behavior of the universe for various nonsingular potentials (the perfect-fluid analogy is inappropriate to describe the behavior of the early universe). A nonsingular initial state exists as an exact MPS, when the asymtotic form of the scalar potential does not increase faster than the quadratic power for large-field values. The physical relevant form of a MPS is stable as the pertubations grow exponentially at early times. This implies that the MPS is a fixed point in the past."

In (TsB) an integral form of the Einstein equations is provided in combination with a covariant formulation of the Mach principle.

7. NSE, YME and plasma/geometrodynamics problem/solution areas

The common Hilbert scale is about the Hilbert spaces H_α with $\alpha=-1,-\frac{1}{2},0,\frac{1}{2},1$ with its corresponding inner products $(*,*)_\alpha$. The proposed mathematical concepts and tools are especially correlated to the names of Plemelj, Stieltjes and Calderón.

The newly proposed "fluid/quantum state" Hilbert space $\mathrm{H}_{-1/2}$ with its closed orthogonal subspace of H_0 goes also along with a combined usage of L_2 waves governing the H_0 Hilbert space and "orthogonal" wavelets governing the $\mathrm{H}_{-1/2}-\mathrm{H}_0$ space. The wavelet "reproducing" ("duality") formula provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets, where the "*microscope observations*" of *two* wavelet (optics) functions can be compared with each other (LoA). The prize to be paid is about additional efforts, when re-building the reconstruction wavelet.

In fluid description of plasmas (MHD) one does not consider velocity distributions (e.g. (GuR)). It is about number density, flow velocity and pressure. This is about moment or fluid equations (as NSE and Boltzmann/Landau equations). In (EyG) it is proven that smooth solutions of non-ideal (viscous and resistive) incompressible magneto-hydrodynamic (plasma fluid) equations satisfy a stochastic (conservation) law of flux. It is shown that the magnetic flux through the fixed Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons.

One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. The continuity equation of ideal magneto-hydrodynamics is given by ((DeR) (4.1))

$$\frac{\partial}{\partial t}\rho+\nabla\cdot(\rho\boldsymbol{v})=0$$

with $\rho=\rho(x,t)$ denoting the mass density of the fluid and $\boldsymbol{v}$ denoting the bulk velocity of the macroscopic motion of the fluid. The corresponding microscopic kinetic description of plasma fluids leads to a continuity equation of a system of (plasma) “particles” in a phase space $(\boldsymbol{x},\boldsymbol{v})$ (where $\rho(\boldsymbol{x},t)$ is replaced by a function $f(\boldsymbol{x},\boldsymbol{v},t)$) given by ((DeR) (5.1))

$$\frac{\partial}{\partial t}f+v\cdot\nabla_x f+\frac{dv}{dt}\cdot\nabla_v f+f\frac{\partial}{\partial v}\cdot\frac{dv}{dt}=0\ .$$

In case of a Lorentz force the last term is zero, leading to the so-called collisions-less (kinetic) Vlasov equation ((ShF) (28.1.2)).

In (GuR) it is shown that the magnetic flux through the fixed surface is equal to the average of the magnetic fluxes through the ensemble of surfaces at earlier times for any (unit or general) value of the magnetic Prandtl number. For divergence-free $\vec{z}=(\vec{\mathrm{u}},\vec{B})\in C([t_0,t_f],C^{k.\alpha}),(\vec{\mathrm{u}}(0),\vec{B}(0))\in C^{k.\alpha}$ the key inequalities are given by

- unit magnetic Prandtl number:

$$e^{-2\gamma(t_f-t_0)}\left\|\vec{z}(t_f)\right\|_2^2+2\int_{t_0}^{t_f}e^{-2\gamma(t-t_0)}[\varepsilon\|\vec{z}(t)\|_2^2+\mu\|\nabla\vec{z}(t)\|_2^2]dt\le\|\vec{z}(0)\|_2^2$$

- general magnetic Prandtl number (→ stochastic Lundquist formula):

$$e^{-2\gamma(t_f-t_0)}\left\|\vec{B}(t_f)\right\|_2^2+2\int_{t_0}^{t_f}e^{-2\gamma(t-t_0)}\left[\varepsilon\left\|\vec{B}(t)\right\|_2^2+\mu\left\|\nabla\vec{\mathrm{B}}(t)\right\|_2^2\right]dt\le\left\|\vec{B}(0)\right\|_2^2\ .$$

The mathematical tool to distinguish between unperturbed cold and hot plasma is about the Debye length and Debye sphere (DeR). The corresponding interaction (Coulomb) potential of the non-linear Landau damping model is based on the (Poisson) potential equation with corresponding boundary conditions.

A combined electro-magnetic plasma field model needs to enable "interaction" of cold and hot plasma "particles", which indicates Neumann problem boundary conditions. The corresponding double layer (hyper-singular integral) potential operator of the Neumann problem is the Prandtl operator $\overline{\mathrm{P}}$, fulfilling the following properties ((LiI) Theorems 4.2.1, 4.2.2, 4.3.2):

iv) the Prandtl operator $\overline{\mathrm{P}}: \mathrm{H}_{\mathrm{r}} \to \widehat{\mathrm{H}}_{\mathrm{r}-1}$ is bounded for $0 \leq \mathrm{r} \leq 1$

v) the Prandtl operator $\overline{\mathrm{P}}: \mathrm{H}_{\mathrm{r}} \to \widehat{\mathrm{H}}_{\mathrm{r}-1}$ is Noetherian for $0 < \mathrm{r} < 1$

vi) for $1/2 \leq \mathrm{r} < 1$, the exterior Neumann problem admits one and only one generalized solution.

Therefore, the Prandtl operator enables a combined (conservation of mass & (linear & angular) momentum balances) integral equations system, where the two momentum balances systems are modelled by corresponding momentum operator equations with corresponding domains according to $H_{1/2} = H_1 \otimes H_1^{\perp} = H_{-1/2}^{*}$. For a correspondingly considered variational representation (e.g. for the (Neumann) potential equation or the corresponding Stokes equation) it requires a less regular Hilbert space framework than in standard theory. Basically, the domain H_1 of the standard (Dirichlet integral based) "energy" (semi) inner product $\mathrm{a}(\mathrm{u},\mathrm{v}) = (\nabla u, \nabla v)$ is extended to $H_{1/2}$ with a corresponding alternative (semi) inner product in the form $\mathrm{a}(\mathrm{u},\mathrm{v}) = (\nabla u, \nabla v)_{-1/2} = (u,v)_{1/2}$.

The corresponding situation to the plasma model above of the fluid flux of an incompressible viscous fluid leads to the Navier-Stokes equations. They are derived from continuum theory of non-polar fluids with three kinds of balance laws: (1) conservation of mass, (2) balance of linear momentum, (3) balance of angular momentum ((GaG)). Usually the momentum balance conditions are expressed on problem adequate "force" formula derived from the Newton formula $= m \cdot \frac{dv}{dt}$. For getting any well-posed (evolution equation) system is it necessary to define its corresponding initial-boundary value conditions.

The NSE are derived from the (Cauchy) stress tensor (resp. the shear viscosity tensor) leading to liquid pressure force $\frac{\partial p}{\partial x_i} = -\frac{\partial T_{ji}}{\partial x_j} + \mu \Delta v_i$. In electrodynamics & kinetic plasma physics the linear resp. the angular momentum laws are linked to the electrostatic (mass "particles", collision, static, quantum mechanics, displacement related; "fermions") Coulomb potential resp. to the magnetic (mass-less "particles", collision-less, dynamic, quantum dynamics, rotation related; "bosons") Lorentz potential.

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space H_1 (KiA). On can, however, make it coercive by adding a certain bilinear form on the boundary of the domain (vanishing on a subspace of H_1), which causes a change in the natural boundary conditions (CoM).

We propose modified Maxwell equations with correspondingly extended domains according to the above. This model is proposed as alternative to SMEP, i.e. the modified Maxwell equation are proposed to be a "Non-standard Model of Elementary Particles (NMEP)", i.e. an alternative to the Yang-Mills (field) equations. The conceptual approach is also applicable for the Einstein field equations. Mathematical speaking this is about potential functions built on correponding "density" functions. The source density is the most prominent one. Physical speaking the source is the root cause of the corresponding source field. Another example is the invertebrate density (=rotation) with its corresponding rotation field. The Poincare lemma in a 3-D framework states that source fields are rotation-free and rotation fields are source-free.

The physical interpretation of the rotation field in the modified Maxwell equations is about rotating "mass elements w/o mass" (in the sense of Plemelj) with corresponding potential function. In a certain sense this concept can be seen as a generalization of the Helmholtz decomposition (which is about a representation of a vector field as a sum of an irrotational (curl-free) and a solenoidal (divergence-free) vector field): it is derived applying the delta "function" concept. In the context of the proposed distributional Hilbert space framework, the Dirac function concept (where the regularity of those "function" depends from the space dimension) is replaced by the quantum state Hilbert space $H_{-1/2}$. The solution $u \in H_{-1/2}$ of the Helmholtz equation in terms of the double layer potential is provided in ((LiI), 7.3.4). From the Sobolev embedding theorem it follows, that for any space dimension n>0 the modified Helmholtz equation is valid for not continuous vector fields.

7a. The related NSE problem/solution area

This section is about a straightforward solution of the NSE Millenium problem by closing the Serrin gap provided that the $H_{-1/2}$ Hilbert space is the accepted underlying fluid state model.

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for “large” Banach spaces. The uniqueness is proven only in “small” Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems. The existence of weak solutions can be provided, essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning. Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigenfunctions of the Stokes operator or generalized Fourier series approximations. It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

The NSE are derived from the (Cauchy) stress tensor (resp. the shear viscosity tensor) leading to liquid pressure force. In electrodynamics & kinetic plasma physics the linear resp. the angular momentum laws are linked to the electrostatic (mass “particles”, collision, static, quantum mechanics, displacement related; “fermions”) Coulomb potential resp. to the magnetic (mass-less “particles”, collision-less, dynamic, quantum dynamics, rotation related; “bosons”) Lorentz potential.

We note that the solution of the Navier-Stokes equation are related to the considered degenerated hypergeometric functions by its corresponding integral function representation (PeR1).

A "3D challenge" like the 3-D nonlinear, non-stationary NSE is also valid, when solving the monochromatic scattering problem on surfaces of arbitrary shape applying electric field integral equations. From (IvV) we recall that the (integral) operators A and A(t): $\mathrm{H}_{-1/2} \rightarrow \mathrm{H}_{1/2}$ are bounded Fredholm operators with index zero. The underlying framework is still the standard one, as the domains are surfaces, only. An analog approach as above with correspondingly defined surface domain regularity is proposed.
The initial boundary value problem determines the initial pressure $p_0(x)$ by the Neumann problem

$$\Delta p_0 = (f_0 - u_0 \cdot \nabla u_0 \qquad \text{in} \quad \Omega$$

$$\frac{\partial p_0}{\partial n} = [\Delta u_0 - u_0 \cdot \nabla) u_0 + f_0] \cdot n \qquad \text{at} \quad \partial\Omega$$

with$f_0 := \lim_{t \to 0} f(\cdot, t)$. Applying formally the div-operator to the classical NSE the pressure field must satisfy the following Neumann problem ((GaG))

$$\Delta p = (u \cdot \nabla) u - f \qquad \text{in} \quad \Omega$$

$$\frac{\partial p}{\partial n} = [\Delta u - (u \cdot \nabla) u + f] \cdot n \qquad \text{at} \quad \partial\Omega$$

where n denotes the outward unit normal to $\partial\Omega$. As it holds that

$$[\Delta u - (u \cdot \nabla) u + f] \cdot n|_{\partial\Omega} \quad \rightarrow \quad \left[\Delta u_0 - (u_{\cdot 0} \nabla) u_0 + f_0\right] \cdot n\Big|_{\partial\Omega} \quad \text{in} \quad H_{-1/2}(\partial\Omega)$$

and

$$\nabla \cdot [f - u \cdot \nabla) u]|_{\partial\Omega} \quad \rightarrow \nabla \cdot [f_0 - u_0 \cdot \nabla) u_0]|_{\partial\Omega} \qquad \text{in} \quad H_{-1/2}(\partial\Omega)$$

the pressure p tends to p_0 in the sense that $\|\nabla(p(\cdot,t) - p_0\| \rightarrow 0$ as $t \rightarrow 0$.

As a consequence the prescription of the pressure at the boundary walls or at the initial time independently of u, could be incompatible with and, therefore, could retender the NSE problem ill-posed.

With respect to the relationship to the considered Hilbert space $H_{-1/2}$ we emphasis that the Prandtl operator with domain $H_{1/2}$ and range $H_{-1/2}$ is bounded and coercive and the corresponding exterior Neumann problem admit one and only on generalized solution (BrK2).

A $H_{-1/2}$ (fluid state) Hilbert space framework is also applied to derive optimal finite element approximation estimates for non-linear parabolic problems with not regular initial value data (BrK2).

Kolmogorov's turbulence theory is a purely statistical model (based on the H_0 (observation/test) Hilbert space), which describes (only!) the qualitative behavior of turbulent flows. There is no linkage to the quantitative fluid behavior as it is described by the Euler or the Navier-Stokes equations. The physical counterpart of his low- and high-pass filtering Fourier coefficients analysis is a "local Fourier spectrum", which is a contradiction in itself, as, either it is non-Fourier, or it is nonlocal ((FaM)). WE propose to combine the wavelet based solution concept of (FaM) with a revisited CLM equation model in a physical $H_{-1/2}$ Hilbert space framework to enable a turbulent $H_{-1/2}$ signal which can be split into two components: coherent bursts and incoherent noise. The model enables a localized Heisenberg uncertainty inequality in the closed (noise) subspace $H_{1/2} = H_1 \otimes H_1^{\perp} = H_{-1/2}^{*}\ L_2^{\perp} = H_0^{\perp} = H_{-1/2} - H_0$, while the momentum-location commutator vanishes in the (coherent bursts) test space H_0.

7b. The related YME problem/solution area

This section is about a straightforward solution of the YME mass gap problem provided that the $H_{-1/2}$ Hilbert space is the accepted underlying quantum state model.

We propose an alternative mathematical framework for the Standard Model of Elementary Particles (SMEP), which replaces gauge theory and variational principles: The underlying concepts of exterior derivatives and tensor algebra are replaced by (distributional) Hilbert scales and (purely Hamiltonean) variational principles. As a consequence, the vacuum energy becomes an intrinsic part of the variational principles, i.e. it is identical for all considered Lagrange resp. Hamiltonian mechanisms of all related differential equations, while the corresponding "force" becomes an observable of the considered (Hamiltonean) minimization problem.

In some problem statements of the YME there are basically two assumptions made:

1. the energy of the vacuum energy is zero

2. all energy states can be thought of as particles in plane-waves.

As a consequence the mass gap is the mass of the lightest particle.

Our challenge of proposition 1 is about the measure of the vacuum energy, which gives the value "zero". While the energy norm in the standard H_1 Hilbert space might be zero, the value of the quantum state with respect to the energy norm of the sub-space $\mathrm{H}_{1/2}$ still can be >0.

Our challenge of proposition 2 is going the same way: an elementary particle can be tested against the test space H_0, (condensed energy). It still can be analyzed by "wavelets" in the closed complementary space $\mathrm{H}_{-1/2} - \mathrm{H}_0$, where the test space is "just" compactly embedded. Those "wavelets" might be interpreted as all kinds of today's massless "particles" (neutrinos and photons) with related "dark energy". As a consequence there is no YME mass gap anymore, but there is a new concept of vacuum energy (wave packages, eigen-differentials, rotation differential) governed by the Heisenberg uncertainty principle. This is about an alternative harmonic quantum energy model enabling a finite "quantum fluctuation = total energy", while replacing Dirac's Delta function by $\mathrm{H}_{-1/2}$ distributions enabling and an alternative Schrödinger's momentum operator (BrK7).

A physical interpretation could be about "rotating differentials" ("quantum fluctuations"), which corresponds mathematically to Leibniz's concept of monads. The mathematical counterparts are the ideal points (or hyper-real numbers). This leads to non-standard analysis, whereby the number field has same cardinality than the real numbers. It is "just" the Archimedean principle which is no longer valid.

The electromagnetic interaction has gauge invariance for the probability density and for the Dirac equation. The wave equation for the gauge bosons, i.e. the generalization of the Maxwell equations, can be derived by forming a gauge-invariant field tensor using generalized derivative. There is a parallel to the definition of the covariant derivative in general relativity. With respect to the above there is an alternative approach indicated, where the fermions are modelled as elements of the Hilbert space H_0, while the complementary closed subspace $H_0^\perp$ is a model for the "interaction particles, bosons". For gauge symmetries the fundamental equations are symmetric, but e.g. the ground state wave function breaks the symmetry. When a gauge symmetry is broken the gauge bosons are able to acquire an effective mass, even though gauge symmetry does not allow a boson mass in the fundamental equations. Following the above alternative concept the "symmetry state space" is modelled by H_0, while the the ground state wave function is an element of the closed subspace $H_0^\perp$ of $\mathrm{H}_{-1/2}$ (BrK).

Reformulated Maxwell or gravitation field equations in a weak $\mathrm{H}_{-1/2}$-sense leads to the same effect, as dealing with an isometric mapping $g \rightarrow H[g]$ in a weak H_0 sense (H denotes the Hilbert transform) alternatively to a second order operator in the form $x \cdot P(g(x))$ in a weak $\mathrm{H}_{-1/2}$ sense. This results into some opportunities as

- the solutions of the Maxwell equations in a vaccum do not need any callibration transforms to ensure wave equation character; therefore, the arbitrarily chosen Lorentz condition for the electromagnetic potential (to ensure Lorentz invariance in wave equations) and its corresponding scalar function ((FeR), 7th lecture) can be avoided

- enabling alternative concepts in GRT to e.g. current (flexible") metrical affinity, affine connexions and local isometric 3D unit spheres dealing with rigid infinitesimal pieces, being replaced by geometrical manifolds, enabling isometrical stitching of rigid infinitesimal pieces ((CiI), (ScP)).

When one wants to treat the time-harmonic Maxwell equations with variational methods, one has to face the problem that the natural bilinear form is not coercive on the whole Sobolev space. On can, however, make it coercive by adding a certain bilinear form on the boundary of the domain (vanishing on a subspace of H_{-1}), which causes a change in the natural boundary conditions.

In SMEP (Standard Model of Elementary Particles) symmetry plays a key role. Conceptually, the SMEP starts with a set of fermions (e.g. the electron in quantum electrodynamics). If a theory is invariant under transformations by a symmetry group one obtains a conservation law and quantum numbers. Gauge symmetries are local symmetries that act differently at each space-time point. They automatically determine the interaction between particles by introducing bosons that mediate the interaction. $U(1)$ (where probability of the wave function (i.e. the complex unit circle numbers) is conserved) describes the elctromagnetic interaction with 1 boson (photon) and 1 quantum number (charge Q). The group $SU(2)$ of complex, unitary (2x2) matrices with determinant I describes the weak force interaction with 3 bosons W^+, W^-, Z, while the group $SU(3)$ of complex, unitary $3x3$ matrices describes the strong force interaction with 8 gluon bosons.

The gauge invariance is the main principle in current SMEP theory.

(BlD) 10.3: "*It is fine that the gauge field of electromagnetism has zero mass because there the force is mediated by photons, which are massless. However, Yang-Mills type forces must arise from the exchange of massive particles because of the observed short range of these forces. The Higgs mechanism helps in two ways. First, gauge fields can acquire mass by the symmetry breaking. Second, the undesirable Goldstone bosons (which arise in the symmetry-breaking process) can be usually gauged away."*

The Higgs effect (or mechanism) builds on an extended from global to local $U(1)$ transformations symmetry group of the underlying Lagrangian. It explains the mass of the gauge W- and Z- (weak interaction) bosons of the weak "nuclear-force".

(HiP): "*Within the framework of quantum field theory a "spontaneous" breakdown of symmetry occurs if a Lagrangian, fully invariant under the internal symmetry group, has such a structure that physical vacuum is a member of a set of (physically equivalent) states which transform according to a nontrivial representation of the group. This degeneracy of the vacuum permits non-trivial multiplets of scalar fields to have nonzero vacuum expectation values (or "vacuons"), whose appearance leads to symmetry-breaking terms in propagators and vertices. … When the symmetry group of the Lagrangian is extended from global to local transformations by introduction of coupling with a vector gauge field the original scalar massless boson as a result of spontaneous breakdown of symmetry then becomes the longitudinal state of a massive vector (Higgs) boson whose transverse state sare the quanta of the transverse gauge field. A perturbative treatment of the model is developed in which the major features of these phenomena are present in zero order*."

The Higgs boson is supposed to be a heavy elementary particle (with non-zero rest mass of about 125 GeV with spin 0). The Higgs field is supposed to fill the whole universe interacting with each particle, which "moves" through it by a kind of frictional resistance, i.e. which has kinetic energy. Therefore, the Higgs effect (i.e. generating mass particles) requires a Higgs field with not vanishing amplitudes in the ground state.

7c. The related plasma/ geometrodynamics problem/solution area

This section is about a space-time quantum model, enabling a space-time stage background independency, provided that the $H_{-1/2}$ Hilbert space is the accepted underlying quantum state model.

Replacing the affine connexion and the underlying covariant derivative concept by a geometric structure with corresponding inner product puts the spot on the

Thurston conjecture*: The interior of every compact 3-manifold has a canonical decomposition into pieces which have geometric structure (ThW).*

This conjecture asserts that any compact 3-manifold can be cut in a reasonably canonical way into a union of geometric pieces. In fact, the decomposition does exist. The point of the conjecture is that the pieces should all be geometric. There are precisely eight homogeneous spaces (X, G) which are needed for geometric structures on 3-manifolds. The symmetry group $SU(2)$ of quaternions of absolute value one (the model for the weak nuclear force interaction between an electron and a neutrino) is diffeomorph to S^3, the unit sphere in R^4. The latter one is one of the eight geometric manifolds above (ScP). We mention the two other relevant geometries, the Euclidean space E_3 and the hyperbolic space H_3. It might be that our universe is not an either... or ..., but a combined one, where then the "connection" dots would become some physical interpretation. Looking from an Einstein field equation perspective the Ricci tensor is a second order tensor, which is very much linked to the Poincare conjecture, its solution by Perelman and to S^3 (AnM). The geometrodynamics provides alternative (pseudo) tensor operators to the Weyl tensor related to H_3(CiI). In (CaJ) the concept of a Ricci potential is provided in the context of the Ricci curvature equation with rotational symmetry. The single scalar equation for the Ricci potential is equivalent to the original Ricci system in the rotationally symmetric case when the Ricci candidate is nonsingular. For an overview of the Ricci flow regarding e.g. entropy formula, finite extinction time for solutions on certain 3-manifolds in the context of Prelman's proof of the Poincare conjecture we refer to (KlB), (MoJ).

The single scalar equation for the Ricci potential (CaJ) might be interpreted as the counterpart of the CLM vorticity equation as a simple one-dimensional turbulent flow model in the context of the NSE.

In an universe model with appropriately connected geometric manifolds the corresponding symmetries breakdowns at those "connection dots" would govern corresponding different conservation laws in both of the two connected manifolds. The Noether theorem provides the corresponding mathematical concept (symmetry --> conservation laws; energy conservation in GRT, symmetries in particle physics, global and gauge symmetries, exact and broken). Those symmetries are associated with "non-observables". Currently applied symmetries are described by finite- (rotation group, Lorentz group, ...) and by infinite-dimensional (gauged $U(1)$, gauged $SU(3)$, diffeomorphisms of GR, general coordinate invariance...) Lie groups.

A manifold geometry is defined as a pair (X, G), where X is a manifold and G acts transitively on X with compact point stabilisers (ScP). Related to the key tool "Hilbert transform" resp. "conjugate functions" of this page we recall from (ScP), that Kulkarni (unpublished) has carried out a finer classification in which one considers pairs (G, H), where G is a Lie group, H is a compact subgroup and G/H is a simple connected 3-manifold and pairs (G_1, H_1) and (G_2, H_2) are equivalent if there is an isomorphism $G_1 \rightarrow G_2$ sending H_1 to a conjugate of H_2. Thus for example, the geometry S^3 arises from three distinct such pairs, (S^3, e), $(U(2), SO(2))$, $(SO(4), SO(3))$. Another example is given by the Bianchi classification consisting of all simply connected 3-dimensional Lie groups up to an isomorphism.

References

(AbM) Abramowitz M., Stegun A., Handbook of mathematical functions, Dover Publications Inc., New York, 1970

(AhJ) Ahner J. F., A scattering trinity: the reproducing kernel, null-field equations and modified Green’s functions, Q. J. Mech. Appl. Math. Vol. 39, No. 1, 153-162, 1986

(AlF) Almgren F. J., Plateau’s Problem, An Invitation to Varifold Geometry, American Mathematical Society, New York, 2001

(AnE) Anderson E., The Problem of Time, Springer, Cambridge, UK, 2017

(AnJ) Annett J. F., Superconductivity, Superfluids and condensates, Oxford University Press, Oxford, 2004

(AnM) Anderson M. T., Geometrization of 3-manifolds via the Ricci flow, Notices Amer. Math. Sco. 51, (2004) 184-193

(ApT) Apostol T. M., Introduction to Analytic Number Theory, Springer Verlag, 2000

(ArA) Arthurs A. M., Complementary Variational Principles, Clarendon Press, Oxford, 1970

(ArE) Artin E., The Gamma Function, Holt, Rinehart and Winston, New York, Chicago, San Francisoc, Toronto, London, 1964

(ArN) Arcozzi N., Li X., Riesz transforms on spheres, Mathematical Research Letters, 4, 401-412, 1997

(AzA) Aziz A. K., Kellog R. B., Finite Element Analysis of Scattering Problem, Math. Comp., Vol. 37, No 156 (1981) 261-272

(AzT) Azizov T. Y., Ginsburg Y. P., Langer H., On Krein's papers in the theory of spaces with an indefinite metric, Ukrainian Mathematical Journal, Vol. 46, No 1-2, 1994, 3-14

(AzT1) Azizov, T. Y., Iokhvidov, I. S., Dawson, E. R., Linear Operators in Spaces With an Indefinite Metric, Wiley, Chichester, New York, 1989

(BaB) Bagchi B., On Nyman, Beurling and Baez-Duarte’s Hilbert space reformulation of the Riemann Hypothesis, Indian Statistical Institute, Bangalore Centre, (2005), www.isibang.ac.in

(BaÁ) Baricz À., Mills‘ ratio: Monotonicity patterns and functional inequalities, J. Math. Anal. Appl. 340, 1362-1370, 2008

(BaÀ) Baricz Á., Pogány T. K., Inequalities for the one-dimensional analogous oft he Coloumb potential, Acta Polytechnica Hungarica, Vol. 10, No. 7, 53-67, 2013

(BaR) Barnard R. W., Gordy M., Richards K. C., A note on Turán type and mean inequalities fort he Kummer function, J. Math. Anal. Appl. 349 (1), 259-263, 2009

(BeA) Besse A., L., Einstein Manifolds, Springer-Verlag, Berlin, Heidelberg, 1987

(BeB) Berndt B. C., Ramanujan's Notebooks, Part I, Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

(BeB1) Berndt B. C., Number Theory in the Spirit of Ramanajan, AMS, Providence, Rhode Island, 2006

(BeG) Besson G., The geometrization conjecture after R. Hamilton and G. Perelman, Rend. Sem. Mat. Pol. Torino, Vol. 65, 4, 2007, pp. 397-411

(BeI) Belogrivov, I. I., On Transcendence and algebraic independence of the values of Kummer’s functions, Translated from Sibirskii Matematicheskii Zhurnal, Vol. 12, No 5, 1971, 961-982

(BeL) Bel L., Introduction d'un tenseur du quatrième ordre, C. R. Acad. Sci. Paris, 247, 1094-1096, 1959

(BlD) Bleecker D., Gauge Theory and Variational Principles, Dover Publications, Inc., Mineola, New York, 1981

(BiN) Bingham N. H., Goldie C. M., Teugels J. L., Regular variation, University Press, Cambridge, 1989

(BiN1) Bingham N. H., Szegö's theorem and its probabilistic descendants, Probability Surveys, Vol. 9, 287-324 2012

(BiP) Biane P., Pitman J., Yor M., Probability laws related to the Jacobi theta and Riemann Zeta functions, and Brownian excursion, Amer. Math. soc., Vol 38, No 4, 435-465, 2001

(BoD) Bohm D., Wholeness and the Implicate Order, Routledge & Kegan Paul, London, 1980

(BoJ) Bognar J., Indefinite Inner Product Spaces, Springer-Verlag, Berlin, Heidelberg, New York, 1974

(BoM) Bonnet M., Boundary Integral Equations Methods for Solids and Fluids, John Wiley & Sons Ltd., Chichester, 1995

(BrH) Bezis H., Asymptotic Behavior of Some Evolution Systems, In: Nonlinear Evolution Equations (M. C. Crandall ed.). Academic Press, New York, 141-154, 1978

(BrK) Braun K., A new ground state energy model, www.quantum-gravitation.de

(BrK1) Braun K., An alternative Schroedinger (Calderon) momentum operator enabling a quantum gravity model

(BrK2) Braun K., Global existence and uniqueness of 3D Navier-Stokes equations

(BrK3) Braun K., Some remarkable Pseudo-Differential Operators of order -1, 0, 1

(BrK4) Braun K., A Kummer function based Zeta function theory to prove the Riemann Hypothesis and the Goldbach conjecture

(BrK5) An alternative trigonometric integral representation of the Zeta function on the critical line

(BrK6) Braun K., A distributional Hilbert space framework to prove the Landau damping phenomenon

(BrK7=BrK1) Braun K., An alternative Schroedinger (Calderón) momentum operator enabling a quantum gravity model

(BrK8) Braun K., Comparison table, math. modelling frameworks for SMEP and GUT

(BrK9) Braun K., Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations, Jap. Journal of Applied Mathematics, 3, 1, 59-72, 1986

(BrK10) Braun K., J. A. Nitsche´s footprints to NSE problems, www.navier-stokes-equations.com

(BrK related papers) www.navier-stokes-equations.com/author-s-papers

(BuH) Buchholtz H., The Confluent Hypergeometric Function, Springer-Verlag, Berlin, Heidelberg, New York, 1969

(BrR) Brent R. P., An asymptotic expansion inspired by Ramanujan, rpb@cslab.anu.edu.au

(CaD) Cardon D., Convolution operators and zeros of entire functions, Proc. Amer. Math. Soc., 130, 6 (2002) 1725-1734

(CaH) Cao H.-D., Zhu X.-P., Hamilton-Perelman's Proof of the Poincare Conjecture and the Geometrization Conjecture, arXiv:math/0612069v1, 2006

(CaH1) Cao H. D., Chow B., Chu S. C., Yau S. T., Collected Papers on Ricci Flow, International Press2003

(CaJ) Cao J., DeTurck D., The Ricci Curvature with Rotational Symmetry, American Journal of Mathematics 116, (1994), 219-241

(CeC) Cercignani C., Theory and application of the Boltzmann equation, Scottish Academic Press, Edinburgh, Lonson, 1975

(ChD) Christodoulou D., Klainerman, Asymtotic properties of linear field equations in Minkownski space, Comm. Pure Appl. XLIII, 137-199, 1990

(ChD1) Christodoulou D., Klainerman, The Global Nonlinear Stability of the Minkowski Sapce, Princeton University Press, New Jersey, 1993

(ChF) Chen F., F., Introdcution to Plasma Physics and Controlled Fusion, Volume I: Plasma Physics, Plenum Press, New York, 1984

(ChH) Chen H., Evaluations of Some Variant Euler Sums, Journal of Integer Sequences, Vol. 9, (2006) 1-9

(ChK) Chandrasekharan K., Elliptic Functions, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985

(ChK1) Chaudhury K. N., Unser M., On the Hilbert transform of wavelets, arXiv:1107.4619v1

(ChJ) Choi J., Srivastava H. M., The Clauen function and ist related Integrals, Thai J. Math., Vol 12, No 2, 251-264, 2014

(CiI) Ciufolini I., Wheeler J. A., Gravitation and Inertia, Princeton University Press, Princeton, New Jersey, 1995

(CoF) Coffey M. W., Polygamme theory, Li/Kneiper constantts, and validity of the Riemann Hypothesis, http://arxiv.org

(CoP) Constatin P., Lax P. D., Majda A., A simple one-dimensional model fort he three-dimensional vorticity model, Communications on Pure and Applied Mathematics, Vol. XXXVIII, 715-724, 1985

(CoR) Courant R., Hilbert D., Methods of Mathematical Physics Volume II, J. Wiley & Sons, New York, 1989

(DeJ) Derezinski J., Richard S., On Radial Schrödinger Operators with a Coulomb Potential, Ann. Henri Poincaré 19 (2018), 2869-2917

(DeJ) Derbyshire J., Prime Obsession, Joseph Henry Press, Washington, D. C., 2003

(DeR) Dendy R. O., Plasma Dynamics, Oxford Science Publications, Oxford, 1990

(DrM) Dritschel M. A., Rovnyak, J., Operators on indefinite inner product spaces

(EbP) Ebenfelt P., Khavinson D., Shapiro H. S., An inverse problem for the double layer potential, Computational Methods and Function Theory, Vol. 1, No. 2, 387-401, 2001

(EdH) Edwards Riemann's Zeta Function, Dover Publications, Inc., Mineola, New York, 1974

(EhP) Ehrlich P., Contemporary infinitesimalist theories of continua and their late 19th- and early 20th-century forerunners, arXiv.org 180.03345, Dec 2018

(EiA) Einstein A., Grundzüge der Relativitätstheorie, Vieweg & Sohn, Braunschweig, Wiesbaden, 1992

(EiA1) Einstein A., Äther und Relativitätstheorie, Julius Springer, Berlin, 1920

(EiA2) Einstein A., Podolsky B., Rosen N., Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? Physical Review, Vol. 47, 1935

(ElL) Elaissaoui L., Guennoun Z. El-Abidine, Relating log-tangent integrals with the Riemann zeta function, arXiv, May 2018

(ElL1) Elaissaoui L., Guennoun Z. El-Abidine, Evaluation of log-tangent integrals by series involving zeta(2n+1), arXiv, May 2017

(EsG) Eskin G. I., Boundary Value Problems for Elliptic Pseudodifferential Equations, Amer. Math. Soc., Providence, Rhode Island, 1981

(EsO) Esinosa O., Moll V., On some definite integrals involving the Hurwitz zeta function, Part 2, The Ramanujan Journal, 6, p. 449-468, 2002

(EsR) Estrada R., Kanwal R. P., Asymptotic Analysis: A Distributional Approach, Birkhäuser, Boston, Basel, Berlin, 1994

(EyG) Eyink G. L., Stochastic Line-Motion and Stochastic Conservation Laws for Non-Ideal Hydrodynamic Models. I. Incompressible Fluids and Isotropic Transport Coefficients, arXiv:0812.0153v1, 30 Nov 2008

(FaK) Fan K., Invariant subspaces of certain linear operators, Bull. Amer. Math. Soc. 69 (1963), no. 6, 773-777

(FaM) Farge M., Schneider K., Wavelets: application to turbulence, University Warnick, lectures, 2005

(FaM1) Farge M., Schneider K., Wavelet transforms and their applications to MHD and plasma turbulence: a review, arXiv:1508.05650v1, 2015

(FeR) Feynman R. P., Quantum Electrodynamics, Benjamin/Cummings Publishing Company, Menlo Park, California, 1961

(FlD) Fleisch D., A Student's Guide to Maxwell's Equations, Cambridge University Press, 2008

(GaA) Ganchev A. H., Greenberg W., van der Mee C. V. M., A class of linear kinetic equations in Krein space setting, Integral Equations and Operator Theory, Vol. 11, 518-535, 1988

(GaG) Galdi G. P., The Navier-Stokes Equations: A Mathematical analysis, Birkhäuser Verlag, Monographs in Mathematics, ISBN 978-3-0348-0484-4

(GaL) Garding L., Some points of analysis and their history, Amer. Math. Soc., Vol. 11, Providence Rhode Island, 1919

(GaW) Gautschi W., Waldvogel J., Computing the Hilbert Transform of the Generalized Laguerre and Hermite Weight Functions, BIT Numerical Mathematics, Vol 41, Issue 3, pp. 490-503, 2001

(GiY) Giga Y., Weak and strong solutions of the Navier-Stokes initial value problem, Publ. RIMS, Kyoto Univ. 19 (1983) 887-910

(GöK) Gödel, K., An Example of a New Type of Cosmological Solutions of Einstein´s Field Equations of Gravitation, Review of Modern Physics, Vol. 21, No. 3, 1949

(GrI) Gradshteyn I. S., Ryzhik I. M., Table of integrals series and products, Academic Press, New York, San Franscisco, London, 1965

(GrJ) Graves J. C., The conceptual foundations of contemporary relativity theory, MIT Press, Cambridge, Massachusetts, 1971

(GuR) Gundersen R. M., Linearized Analysis of One-Dimensional Magnetohydrodynamic Flows, Springer Tracts in Natural Philosophy, Vol 1, Berlin, Göttingen, Heidelberg, New York, 1964

(HaE) Haidemenakis E. D., Physics of Solids in Intense Magnetic Fileds, Plenum Press, New York, 1969

(HaG) Hardy G. H., Riesz M., The general theory of Dirichlet's series, Cambridge University Press, Cambridge, 1915

(HaJ) Havil J., Gamma, exploring euler’s constant, Princeton University Press, Princeton and Oxford, 2003

(HaJ1) Hartle J. B., Hawking S. W., Wave function of the Universe, Physical Review, D, Vol. 28, No. 12, 1983

(HaR) Hamilton R. S., Non-singular solutions of the Ricci flow on three-manifolds, Com. Anal. and Geometry, Vol. 7, No. 4, pp. 695-729, 1999

(HaR1) Hamilton R. S., Three manifolds with postivie Ricci curvature, Lour. Diff. Geom. 17, pp. 255-306, 1982

(HaS) Hawking S. W., Penrose R., The Singularities of Gravitational Collapse and Cosmology, The Royal Society, Vol. 314, Issue 1519, 1970

(HaS1) Hawking S. W., Particle Creation by Black Holes, Commun. Math. Phys. 43, 199-220, 1975

(HeJ) Heywood J. G., Walsh O. D., A counter-example concerning the pressure in the Navier-Stokes equations, as $t \to 0^+$, Pacific Journal of Mathatics, Vol. 164, No. 2, 351-359, 1994

(HeM) Heidegger M., Holzwege, Vittorio Klostermann, Frankfurt a. M., 2003

(HeW) Heisenberg W., Physikalische Prinzipien der Quantentheorie, Wissenschaftsverlag, Mannhein, Wien, Zürich, 1991

(HiP) Higgs P. W., Spontaneous Symmetry Breakdown without Massless Bosons, Physical Review, Vol. 145, No 4, p. 1156-1162, 1966

(HoM) Hohlschneider M., Wavelets, An Analysis Tool, Clarendon Press, Oxford, 1995

(HoA) Horvath A. G., Semi-indefinite-inner-product and generalized Minkowski spaces, arXiv

(HoM) Holschneider M., Wavelets, An Analysis Tool, Clarendon Press, Oxford, 1995

(HuA) Hurwitz A., Über einen Satz des Herrn Kakeya, Zürich, 1913

(IvV) Ivakhnenko, V. I., Smirnow Yu. G., Tyrtyshnikov E. E., The electric field integral equation: theory and algorithms, Inst. Numer. Math. Russian of Academy Sciences, Moscow, Russia

(JoF) John F., Formation of singularities in elestic waves, Lecture Notes in Phys., Springer-Verlag, 190-214, 1984

(KaD) Karp D., Sitnik S. M., Log-convexity and log-concavity of hypergeometric-like functions, J. Math. Anal. Appl. 364, 384-394, 2010

(KaD1) Kazanas D., Cosmological Inflation: A Personal Perspective, Astrophys. Space Sci. Proc., (2009) 485-496

(KaM) Kaku M., Introduction to Superstrings and M-Theory, Springer-Verlag, New York, Inc., 1988

(KaM1) Kac M., Probability methods in some problems of analysis and number theory, Bull. Am. Math. Soc., 55, 641-655, (1949)

(KeL) Keiper J. B., Power series expansions of Riemann's Zeta function, Math. Comp. Vol 58, No 198, (1992) 765-773

(KiA) Kirsch A., Hettlich F., The mathematical theory of time-harmonic Maxwell´s equations, expansion-, integral-, and variational methods, Springer-Verlag, Heidelberg, New York, Dordrecht, London, 2015

(KiA1) Kiselev A. P., Relatively Undistorted Cylindrical Waves Depending on Three Spacial Variables, Mathematical Notes, Vol. 79, No. 4, 587-588, 2006

(KlB) Kleiner B., Lott J., Notes on Perelman s papers, Mathematics ArXiv

(KiH) Kim H., Origin of the Universe: A Hint from Eddington-inspired Born-Infeld gravity, Journal of the Korean Physical Society, Vol. 65, No. 6, pp. 840-845, 2014

(KlS) Klainermann S., Rodnianski, Regularity and geometric properties of solutions of the Einstein-Vacuum equations, Journées équations aux dérivées partielles, No. 14, p. 14, 2002

(KlS1) Klainerman S., Nicolò, The Evolution Problem in General Relativity, Birkhäuser, Boston, Basel, Berlin, 1950

(KlS2) Klainerman S., Remarks on the global Sobolev inequalities in Minkowski space, Comm. Pure. Appl. Math., 40, 111-117, 1987

(KlS3) Klainerman S., Uniform decay estimates and the Lorentz invariance oft he classical wave equation, Comm. Pure Appl. Math., 38, 321-332, 1985

(KnA) Kneser A., Das Prinzip der kleinsten Wirkung von Leibniz bis zur Gegenwart, B. G. Teubner, Leipzig, Berlin, 1928

(KoA) Kolmogoroff A., Une contribution à l'étude de la convergence des séries de Fourier, Fund. Math. Vol. 5, 484-489

(KoH) Koch H., Tataru D., Well-poseness for the Navier-Stokes equations, Adv. Math., Vol 157, No 1, 22-35, 2001

(KoJ) Korevaar J., Distributional Wiener-Ikehara theorem and twin primes, Indag. Mathem. N. S., 16, 37-49, 2005

(KrA) Krall A. M., Spectral Analysis for the Generalized Hermite Polynomials, Trans. American Mathematical Society, Vol. 344, No. 1 (1994) pp. 155-172

(KrR) Kress R. Linear Integral Equations, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, Hong Kong, 1941

(KrR1) Kraußhar R. S., Malonek H. R., A charachterization of conformal mappings in R^4 by a formal differentiability condition, Bulletin de la Societè Royale de Liège, Vol. 70, Vol. 1, 35-49, 2001

(LaC) Lanczos C., The variational principles of mechanics, Dover Publications Inc., New York, 1970

(LaC1) Langenhof C. E., Bounds on the norm of a solution of a general differential equation, Proc. Amer. Math. Soc., 8, 615-616, 1960

(LaE) Landau E., Die Lehre von der Verteilung der Primzahlen Vol 1, Teubner Verlag, Leipzig Berlin, 1909

(LaE1) Landau E., Ueber die zahlentheoretische Function phi(n) und ihre Beziehung zum Goldbachschen Satz, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, Vol 1900, p. 177-186, 1900

(LaE2) Landau E., Die Lehre von der Verteilung der Primzahlen Vol 2, Teubner Verlag, Leipzig Berlin, 1909

(LaE3) Landau E., Über eine trigonometrische Summe, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, Vol 1928, p. 21-24, 1928

(LaE4) Landau E. Vorlesungen über Zahlentheorie, Erster Band, zweiter Teil, Chelsea Publishing Company, New York, 1955

(LaE5) Landau E., Die Goldbachsche Vermutung und der Schnirelmannsche Satz, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Klasse, 255-276, 1930

(LaE6) Landau E., Über die Fareyreihe und die Riemannsche Vermutung, Göttinger Nachrichten (1932), 347-352

(LaG) Lachaud G., Spectral analysis and the Riemann hypothesis, J. Comp. Appl. Math. 160, pp. 175190, 2003

(LaJ) An Elementary Problem Equivalent to the Riemann Hypothesis, https://arxiv.org

(LeN) Lebedev N. N., Special Functions and their Applications, translated by R. A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, New York, 1965

(LeN1) Lerner N., A note on the Oseen kernels, in Advances in phase space analysis of partial differential equations, Siena, pp. 161-170, 2007

(LeP) LeFloch P. G., Ma Y., The global nonlinear stability of Minkowski space, arXiv: 1712.10045v1, 28 DEc 2017

(LiI) Lifanov I. K., Poltavskii L. N., Vainikko G. M., Hypersingular Integral Equations and their Applications, Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D. C., 2004

(LiI1) Lifanov I. K., Nenashaev A. S., Generalized Functions on Hilbert Spaces, Singular Integral Equations, and Problems of Aerodynamics and Electrodynamics, Differential Equations, Vol. 43, No. 6, pp. 862-872, 2007

(LiJ) Linnik J. V., The dispersion method in binary additive problems, American Mathematical Society, Providence, Rhode Island, 1963

(LiP) Lions P. L., On Boltzmann and Landau equations, Phil. Trans. R. Soc. Lond. A, 346, 191-204, 1994

(LiP1) Lions P. L., Compactness in Boltzmann's equation via Fourier integral operators and applications. III, J. Math. Kyoto Univ., 34-3, 539-584, 1994

(LiX) Li Xian-Jin, The Positivity of a Sequence of Numbers and the Riemann Hypothesis, Journal of Number Theory, 65, 325-333 (1997)

(LoA) Lifanov I. K., Poltavskii L. N., Vainikko G. M., Hypersingular Integral Equations and Their Applications, Chapman & Hall/CRC, Boca Raton, London, New York, Washington, D. C. 2004

(LoJ) Long J. W., The Grad Student's Guide to Kant's Critique of Pure Reason, iUniverse, Bloomington, 2016

(LuL) Lusternik L. A., Sobolev V. J., Elements of Functional Analysis, A Halsted Press book, Hindustan Publishing Corp. Delhi, 1961

(MaJ) Mashreghi, J., Hilbert transform of log(abs(f)), Proc. Amer. Math. Soc., Vol 130, No 3, p. 683-688, 2001

(MaJ1) Marsden J. E., Hughes T. J. R., Mathematical foundations of elasticity, Dover Publications Inc., New York, 1983

(MeJ) Meiklejohn J. M. D., The Critique of Pure Reason, By Immanuel Kant, Translated by J. M. D. Meiklejohn, ISBN-13 978-1977857477, ISBN-10: 1977857477

(MeY) Meyer Y., Coifman R., Wavelets, Calderón-Zygmund and multilinear operators, Cambridge University Press, Cambridge, 1997

(MiJ) Milnor J., Morse Theory, Annals of Mathematical Studies, Number 51, Princeton University Press, Princeton, 963

(MiT) Mikosch T., Regular Variation, Subexponentiality and Their Application in Probability Theory, University of Groningen

(MoC) Mouhot C., Villani C., On Landau damping, Acta Mathematica, Vol. 207, Issue 1, p. 29-201, 2011

(MoJ) Morgan J. W., Tian G., Ricci Flow and the Poincare Conjecture, Mathematics ArXiv

(MoM) Morse M., Functional Topology and Abstract Variational Theory, Proc. N. A. S., 326-330, 1938

(NaC) Nasim C. On the summation formula of Voronoi, Trans. American Math. Soc. 163, 35-45, 1972

(NaP) Naselsky P. D., Novikov D. I., Noyikov I. D., The Physics of the Cosmic Microwave Background, Cambridge University Press, Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, Sao Paulo, 2006

(NaS) Nag S., Sullivan D., Teichmüller theory and the universal period mapping via quantum calculus and the H space on the circle, Osaka J. Math., 32, 1-34, 1995

(NeD) Neuenschwander D. E., Emmy Noether's Wonderful Theorem, The John Hopkins University Press, Baltimore, 2011

(NiJ) Nitsche J. A., lecture notes, approximation theory in Hilbert scales

(NiJ1) Nitsche J. A., lecture notes, approximation theory in Hilbert scales, extensions and generalizations

(NiJ2) Nitsche J. A., Direct Proofs of Some Unusual Shift-Theorems, Anal. Math. Appl., Gauthier–Villars, Montrouge, 1988, pp. 383–400

(NiJ3) Nitsche J. A., Free boundary problems for Stokes´s flows and finite element methods, Equadiff 6, 2006, pp. 327-332

(NiN) Nielsen N., Die Gammafunktion, Chelsea Publishing Company, Bronx, New York, 1965

(ObF) Oberhettinger, Tables of Mellin Transforms, Springer-Verlag, Berlin, Heidelberg, New York, 1974

(OlF) Olver F. W. J., Asymptotics and special functions, Academic Press, Inc., Boston, San Diego, New York, London, Sydney, Tokyo, Toronto, 1974

(OlF1) Olver F. W. J., Lozier D. W., Boisvert R. F., Clark C. W., NIST Handbook of Mathematical Functions

(OlR) Oloff R., Geometrie der Raumzeit, Vieweg & Sohn, Braunschweig/Wiesbaden, 1999

(OsK) Oskolkov K. I., Chakhkiev M. A., On Riemann „Nondifferentiable‟ Function and the Schrödinger Equation, Proc. Steklov Institude of Mathemtics, Vol. 269, 2010, pp. 186-196

(OsH) Ostmann H.-H., Additive Zahlentheorie, erster Teil, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1956

(PeB) Petersen B. E., Introduction the the Fourier transform and Pseudo-Differential operators, Pitman Advanced Publishing Program, Boston, London, Melbourne, 1983

(PeM) Perel M., Gorodnitskiy E., Representations of solutions of the wave equation based on relativistic wavelet, arXiv:1205.3461v1, 2012

(PeO) Perron O., Die Lehre von den Kettenbrüchen, Volumes 1-2, Nabu Public Domain Reprint, copyright 1918 by B. G. Teubner in Leipzig

(PeR) Penrose R., Cycles of Time, Vintage, London, 2011

(PeR1) Peralta-Fabi, R., An integral representation of the Navier-Stokes Equation-I, Revista Mexicana de Fisica, Vol 31, No 1, 57-67, 1984

(PeR2) Penrose R., Structure of space-time, Batelle Rencontre, C. M. DeWitt and J. M. Wheeler, 1967

(PeR3) Penrose R., Zero rest mass fields including gravitation: asymptotic behaviours, Proc. Toy. Soc. Lond., A284, 159-203, 1962

(PeR4) Penrose R., The Emperor's New Mind: Concerning Computers, Minds, and the Laws of Physics, Oxford Univ. Press, 1989

(PeR5) Penrose R., Rindler W., Spinors and Space-Time, Cambridge University Press, Cambridge, 1984

(PhR) Phillips R., Dissipative operators and hyperbolic systems of partial differential equations, Trans. Amer. Math. Soc. 90 (1959), 193-254

(PiS) Pilipovic S., Stankovic B., Tauberian Theorems for Integral Transforms of Distributions, Acta Math. Hungar. 74, (1-2) (1997), 135-153

(PiS1) Pilipovic S., Stankovic B., Wiener Tauberian theorems for distributions, J. London Math. Soc. 47 (1993), 507-515

(PlJ) J. Plemelj, Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911

(PoE) Postnikov E. B., Singh V. K. Continuous wavelet transform with the Shannon wavelet from the point of view of hyperbolic partial differential equations, arXiv:1551.03082

(PoG) Polya G., Über Nullstellen gewisser ganzer Funktionen, Math. Z. 2 (1918) 352-383

(PoG1) Polya G., Über eine neue Weise bestimmte Integrale in der analytischen Zahlentheorie zu gebrauchen, Göttinger Nachr. (1917) 149-159

(PoG2) Polya G. Über die algebraisch-funktionentheoretischen Untersuchungn von J. L. W. V. Jensen, Det Kgl. Danske Videnskabernes Selskab., Mathematisk-fysiske Meddeleler. VII, 17, 1927

(PoG3) Polya G., Über Potenzreihen mit ganzzahligen Koeffizienten, Math. Ann. 77, 1916, 497-513

(PoG4) Polya G., Arithmetische Eigenschaften der Reihenentwicklung rationaler Funktionen, J. Reine und Angewandte Mathematik, 151, 1921, 1-31

(PoD) Pollack D., Initial Data for the Cauchy Problem in General Relativity, General Relativity Spring School 2015, Junior Scientist Andrejewski Days, March 22nd to April 4th, 2015, Brandenburg an der Havel, Germany

(PoP) Poluyan P., Non-standard analysis of non-classical motion; do the hyperreal numbers exist in the quantum-relative universe?

(PrK) Prachar K., Primzahlverteilung, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1957

(RiB) Riemann B., Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe, Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen, transcribed by D. R. Wilkins, 2000

(RoC) Rovelli C., Quantum Gravity, Cambridge University Press, Cambridge, 2004

(RoC1) Rovelli C., The Order of Time, Penguin Random House, 2018

(RoC2) Rovelli C., Reality is not what it seems, Penguin books, 2017

(RoC3) Rovelli C., Seven brief lessons on physics, Penguin Books, 2016

(RoJ) Roberts J. T., Leibniz on Force and Absolute Motion, Philosophy of Science, Vol 70, No 3, pp. 553-573, 2003

(RuB) Russel B., The Philosophy of Leibniz, Routledge, London, New York, paperback edition, 1992

(RuC) Runge C., Über eine Analogie der Cauchy-Riemannschen Differentialgleichungen in drei Dimensionen, Nach. v. d. Gesellschaft d. Wissenschaften zu Göttingen, Math-Phys. Klasse Vol 1992, 129-136, 1992

(RuM) Ruskai M. B., Werner E., Study of a Class of Regularizations of $1/|x|$ using Gaussian Integrals, arXiv:math/990212v2

(RyG) Rybicki, G. B., Dawson's integral and the sampling theorem, Computers in Physics, 3, 85-87, 1989

(ScE) Schrödinger E., Statistical Thermodynamics, Dover Publications, Inc., New York, 1989

(ScE1) Schrödinger E., My View of the World, Ox Bow Press, Woodbridge, Connecticut, 1961

(ScE2) Schrödinger E., *What is Life?* and *Mind and Matter*, Cambridge University Press, Cambridge, 1967

(ScP) Scott P., The Geometries of 3-Manifolds, Bull. London Math. Soc., 15 (1983), 401-487

(SeA) Sedletskii A. M., Asymptotics of the Zeros of Degenerated Hypergeometric Functions, Mathematical Notes, Vol. 82, No. 2, 229-237, 2007

(SeE) Seneta E., Regularly Varying Functions, Lecture Notes in Math., 508, Springer Verlag, Berlin, 1976

(SeH) Seifert H., Threlfall W., Variationsrechnung im Grossen, Chelsea Publishing Company, New York, 1951

(SeJ) Serrin J., Mathematical Principles of Classical Fluid Mechanics

(ShF) Shu F. H., Gas Dynamics, Vol II, University Science Books, Sausalito, California, 1992

(ShM) Scheel M. A., Thorne K. S., Geodynamics, The Nonlinear Dynamics of Curved Spacetime

(ShM1) Shimoji M., Complementary variational formulation of Maxwell s equations in power series form

(SiT) Sideris T., Formation of singularities in 3-D compressible fluids, Comm. Math. Phys., 101, 47-485, 1985

(SmL) Smolin L., Time reborn, Houghton Miflin Harcourt, New York, 2013

(SmL1) Smith L. P., Quantum Effects in the Interaction of Electrons With High Frequency Fields and the Transition to Classical Theory, Phys. Rev. 69 (1946) 195

(SoH) Sohr H., The Navier-Stokes Equations, An Elementary Functional Analytical Approach, Birkhäuser Verlag, Basel, Boston, Berlin, 2001

(SoP) Sobolevskii P. E., On non-stationary equations of hydrodynamics for viscous fluid. Dokl. Akad. Nauk SSSR 128 (1959) 45-48 (in Russian)

(StE) Stein E. M., Conjugate harmonic functions in several variables

(StE1) Stein E. M., Harmonic Analysis, Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, New Jersey, 1993

(SzG) Szegö, G., Orthogonal Polynomials, American Mathematical Society, Providence, Rhode Island, 2003

(ThW) Thurston W. P., Three Dimensional Manifolds, Kleinian Groups and Hyperbolic Geometry, Bulletin American Mathmematical society, Vol 6, No 3, 1982

(TiE) Titchmarsh E. C., The theory of the Riemann Zeta-function, Clarendon Press, London, Oxford, 1986

(TrH) Treder H.-J., Singularitäten in der Allgemeinen Relativitätstheorie, Astron. Nachr. Bd. 301, H. 1, 9-12, 1980

(TsB) Al´Tsuler B. L., Integral form of the Einstein equations and a covariant formulation oft he Mach´s principle, Soviet Physics Jetp, Vol. 24, No. 4, 1967

(VaM) Vainberg M. M., Variational Methods for the Study of Nonlinear Operators, Holden-Day, Inc., San Francisco, London, Amsterdam, 1964

(VeG) Veneziano G., A simple/short introduction to pre-big-bang physics/cosmology, in „Erice 1997, Highlights of subnuclear physics‟ 364-380, talk given at conference: C97-08-26.2 p. 364-380

(VeW) Velte W., Direkte Methoden der Variationsrechnung, B. G. Teubner, Stuttgart, 1976

(ViI) Vinogradov I. M., The Method of Trigonemotrical Sums in the Theory of Numbers, Dover Publications Inc., Minelola, New York 2004

(ViI1) Vinogradov, I. M., Representation of an odd number as the sum of three primes, Dokl. Akad. Nauk SSSR 15, 291-294 (1937)

(ViJ) Vindas J., Estrada R., A quick distributional way to the prime number theorem, Indag. Mathem., N.S. 20 (1) (2009) 159-165

(ViJ1) Vindas J., Local behavior of distributions and applications, Dissertation, Department of Mathematics, Louisiana State University, 2009

(ViJ2) Vindas J., Introduction to Tauberian theory, a distributional approach, https://cage.ugent.be

(ViM) Villarino M. B., Ramanujan's Harmonic Number Expansion Into Negative Powers of a Trangular Number, Journal of Inequalities in pure nd applied mathematics, Vol. 9, No. 3 (2008), Art. 89, 12 pp.

(VlV) Vladimirow V. S., Drozzinov Yu. N., Zavialov B. I., Tauberian Theorems for Generalized Functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 1988

(WeD) Westra D. B., The Haar measure on SU(2), March 14, 2008

(WeH) Weyl H., Space, Time, Matter, Cosimo Classics, New York, 2010

(WeH1) Weyl H., Matter, structure of the world, principle of action, ..., in (WeH) §34 ff.

(WeH2) Weyl H., Was ist Materie? Verlag Julius Springer, Berlin, 1924

(WeH3) Weyl H., Philosophy of Mathematics and Natural Science, Princeton University Press, Princeton and Oxford, 2009

(WeH) Weyl H., Über die Gleichverteilung von Zahlen mod. Eins, Math. Ann., 77, 1914, 313-352

(WeP) Werner P., Self-Adjoint Extension of the Laplace Operator with Respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. Appl., 70, 1979, pp. 131-160

(WeP1) Werner P., Spectral Properties of the Laplace Operator with respect to Electric and Magnetic Boundary Conditions, J. Math. Anal. Appl., 92, 1983, pp. 1-65

(WhJ1) Whittaker J. M., Interpolatory Function Theory, Cambridge University Press, Cambridge, 1935

(WhJ2) Whittaker J. M., The „Fourier‟ Theory of Cardinal Functions, Proceedings of the Edinburgh Mathematical Society, Vol. 1, Issue 3, pp. 169-176, 1928

(WhJ) Wheeler J. A., On the Nature of Quantum Geometrodynamics

(WhJ1) Wheeler J. A., Awakening to the Natural State, Non-Duality Press, Salisbury, 2004

(WhJ2) Wheeler J. A., At home in the universe, American Institute of Physics, Woodbury, 1996

(WoJ) Wohlfart J., Werte hypergeometrischer Funktionen, Inventiones mathematicae, Vol. 92, Issue 1, 1988, 187-216

(WoW) Wong W. Kant's Conception of Ether as a field in the Opus posthumum, Proc Eighth Intern. Kant Congress, Marquestte University Press, Vol. II, Memphis 1995

(YeR) Ye R., Ricci flow, Einstein metrics and space forms, Trans. Americ. Math. Soc., Vol. 338, No. 2, 1993

(ZeA) Zemanian A. H., Generalized Integral Transformations, Dover Publications, Inc. New York, 1968

(ZhB) Zhechev B., Hilbert Transform Relations

(ZyA) Zygmund A., Trigonometric series, Volume I & II, Cambridge University Press, 1959