Lommel Polynomials

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taken from [GWa] source

The Lommel polynomials $g_n(x)$, defined by ([GWa] 9-6)

$$g_n(x) = \sum_{0}^{\leq[n/2]} (-1)^m \frac{(\text{n - m})!}{\text{m}!(\text{n - 2m})!} \frac{\Gamma(\text{n} + 1 - \text{m})}{\text{m}!} x^m \quad ,$$

fulfill

$$g_{n+1}(x) = (n+1) g_n(x) - x g_{n-1}(x) \quad , \; g_0(x) := g_1(x) := 1 \quad .$$

Putting

$$h_n(\frac{1}{2x}) := x^{-n} g_n(x^2)$$

a relation between the modified Lommel polynomials and the Bessel function is given by Hurwitz's asymptotic formula ([GWa] 9-65):

$$J_0(\frac{1}{x}) = \lim_{n\to\infty} \frac{(2x)^{-n}}{n!} h_n(x) \quad .$$

Lemma: Being $\{\alpha_k\}_{k\in N}$ resp. $\{j_k\}_{k\in N}$ the zeros of $J_0(2\sqrt{x})$ resp. $J_0(x)$ i.e. $\{\alpha_k = j_k{}^2/4\}_{k\in N}$, putting $\sigma_{m+1} := \sum_{1}^{\infty} \frac{1}{\alpha_n{}^{m+1}}$ then it holds ([TCh] 7, II, theor. 6.4, [DDi])

i) $$-\frac{d}{dx} J_0(2\sqrt{x}) = \frac{J_1(2\sqrt{x})}{\sqrt{x}} = J_0(2\sqrt{x}) \sum_{m=0}^{\infty} \sigma_{m+1} x^m$$

whereby $\sigma_1 = \sum_{1}^{\infty} \frac{1}{\alpha_n} = 1$ is the only integer value for the σ_{m+1}.

ii) $$\sum_{1}^{\infty} \frac{1}{2\sqrt{\alpha_k}} L_n(+/-\alpha_k) L_m(+/-\alpha_k) = \delta_{n,m}$$

iii) $$J_0(2\sqrt{x}) = \lim_{n\to\infty} \frac{g_n(x)}{n!} = \lim_{n\to\infty} \frac{x^{n/2}}{n!} h_n(\frac{1}{2\sqrt{x}}) = \lim_{n\to\infty} \frac{x^{n/2}}{n!} \frac{1}{\sqrt{n+1}} L_n(x)$$

iv) $$J_0(\frac{2}{\sqrt{x}}) = \lim_{n\to\infty} \frac{g_n(\frac{1}{x})}{n!} = \lim_{n\to\infty} \frac{x^{-n/2}}{n!} h_n(\frac{\sqrt{x}}{2}) = \lim_{n\to\infty} \frac{x^{-n/2}}{n!} \frac{1}{\sqrt{n+1}} L_n(\frac{1}{x})$$

v) *for large values of* m *it holds* $\frac{j_m}{j_n} - \frac{n}{m} = O(\frac{1}{n})$

vi) (*) $$\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{1}{j_k{}^2} h_m(\frac{1}{j_k}) h_n(\frac{1}{j_k}) = \frac{\delta_{n,m}}{2(n+1)}$$ with $h_n(\frac{1}{2x}) = x^{-n} g_n(x^2)$.

From [GWa] 6-5, 13-6, 13-24, we recall

$$\frac{\Gamma(s)}{\Gamma(1-s)}=\int_0^\infty x^s J_0(2\sqrt{x})\frac{dx}{x}=(1-s)\int_0^\infty x^{s-\frac{1}{2}}J_1(2\sqrt{x})\frac{dx}{x} \quad \text{for } 0<\mathrm{Re}(s)<1.$$

We summaries the above in

Proposition 1: For the Lommel polynomials the following relations hold true:

i) $$J_0(2\sqrt{x})=\lim_{n\to\infty}\frac{g_n(x)}{n!} \quad , \quad \hat{J}_0(2\sqrt{x})=\lim_{n\to\infty}\frac{\hat{g}_n(x)}{n!}$$

ii) $$\frac{\Gamma(s)}{\Gamma(1-s)}=\int_0^\infty x^s J_0(2\sqrt{x})\frac{dx}{x}=\lim_{n\to\infty}\frac{1}{n!}\int_0^\infty x^s g_n(x)\frac{dx}{x} \quad \text{for } 0<\mathrm{Re}(s)<1$$

iii) $$\frac{1}{2}\sum_{\nu=1}^\infty \frac{1}{2\alpha_\nu}\frac{1}{\alpha_\nu{}^n}g_n(\alpha_\nu)\frac{1}{\alpha_\nu{}^m}g_m(\alpha_\nu)=\frac{\delta_{n,m}}{n+1}$$ (note $\alpha_\nu \approx \nu^2$).

Lemma (Fourier-Bessel expansion [GWa] 18): Let $f(x)$ be defined arbitrarily in the interval (0,1) and let $\int_0^1 \sqrt{x}f(x)dx$ exist and (if it is an improper integral) let it absolutely convergent and j_m being the zeros of $J_0(x)$. Let

$$a_m := \frac{2}{J^2{}_{\nu+1}(j_m)}\int_0^1 xf(x)J_\nu(j_m x)dx \quad \text{where } \nu\geq -1/2.$$

Let x be any interval point of an interval (a,b) such that $0<a<b<1$ and such that $f(x)$ has limited total fluctuation in (a,b). Then the series

$$\sum_1^\infty a_m J_\nu(j_m x)$$

is convergent and its sum is $\frac{f(x+0)+f(x-0)}{2}$, whereby $a_m=O(\frac{1}{m})$.

We recall Sheppard's result from [GWa] 18-27, i.e.

$$a_m J_\nu(j_m x)=\frac{2J_\nu(j_m x)}{J^2{}_{\nu+1}(j_m)}\int_0^1 xf(x)J_\nu(j_m x)dx=O(\frac{1}{j_m}) \quad \text{for } 0<x\leq 1 .$$

In [DDi] proposition 1, iii) is proven building a proper Riemann-Stieltjes integral:

The term

$$\lambda'(x) = \frac{J_1(\frac{1}{x})}{J_0(\frac{1}{x})}$$

is analytic outside any circle that contains the finite zeros of $J_0(\frac{1}{x})$. Hence it possesses a Laurent expansion about the origin that converges uniformly on and in any annulus whose inside boundary has the finite zeros of $J_0(\frac{1}{x})$ in its interior. Let C be the contour that encircles the origin in a positive direction and that lies within the annulus. Then it holds [DDi]

$$\frac{1}{2\pi i}\int_C x^k h_n(x)\lambda'(x)dx = \begin{cases} 0 & k < n \\ \frac{1}{2^{n+1}(n+1)} & k = n \end{cases}$$

Let $\alpha(x)$ the non-decreasing step function having increase of

$$\frac{1}{j_n^2} \quad \text{at the point} \quad x = \frac{1}{j_n} \quad \text{for} \quad n = 0, \pm 1, \pm 2, ...$$

then it holds [DDi]

$$\int_{1/j_0}^{1/j_1} h_n(x)h_m(x)d\alpha(x) = \frac{\delta_{n,m}}{2^{n+1}(n+1)} \quad .$$

Remark:

The Stieltjes inverse formula gives the relation to hyper-functions.

The Riemann-Stieltjes integral representation then leads to proposition 1 iii) and

The Lommel polynomials build an orthogonal polynomial system of a Hilbert space $H_{-\beta}$ with $\beta > 0$.

The relation to the Bagchi Formulation of the Nyman RH criterion is obvious [KBr].

We recall Euler's analysis of the zeros of the Bessel functions. With the notations we follow [GWa] 15-41, 15-5. We use the abbreviation $j_{-k} := -j_k$ to write the zeros of $J_0(x)$ in the form $\{j_k\}_{k\in Z-\{0\}}$. The zeros of $J_0(2\sqrt{x})$ are taking to be $\alpha_1, \alpha_2, \alpha_3, ...$ i.e. $\{\alpha_k\}_{k\in N}$ and it holds for $\{\alpha_k = j_k{}^2/4\}_{k\in N}$

$$J_0(2\sqrt{x}) = J_0(2\sqrt{0})\prod_{n=1}^{\infty}(1-\frac{x}{\alpha_n})$$

In order to determinate the smallest zeros of $J_0(2\sqrt{x})$ Euler differentiated logarithmically to conclude

$$-\frac{d}{dx}\log J_0(2\sqrt{x}) = \sum_{n=1}^{\infty}\frac{1}{\alpha_n - x} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{x^m}{\alpha_n{}^{m+1}}$$

provided that $|x| < \alpha_1$, and the last series is absolute convergent.

Putting

$$\sigma_{m+1} := \sum_{1}^{\infty}\frac{1}{\alpha_n{}^{m+1}}$$

and change the order of summations results into

$$-\frac{d}{dx}J_0(2\sqrt{x}) = \frac{J_1(2\sqrt{x})}{\sqrt{x}} = J_0(2\sqrt{x})\sum_{m=0}^{\infty}\sigma_{m+1}x^m \quad .$$

Based on this formula Euler obtained a system of equations, which allow to calculate the σ_k and from that to deduce the smallest values of α_k, i.e. Euler calculated

$$\sigma_1 = 1, \sigma_2 = 1/2, \sigma_3 = 1/3, \sigma_4 = 11/48, \sigma_5 = 19/120, \sigma_6 = 473/4320, ...$$

to deduce e.g.

$$\alpha_1 = 1.445795.., \alpha_2 = 7.6658...., \alpha_3 = 18.72....$$

We summaries Euler's results above in the

Lemma Being $\{j_k\}_{k\in Z-\{0\}}$ the zeros of $J_0(y)$ and $\{\alpha_k = j_k{}^2/4\}_{k\in N}$ the zeros of $J_0(2\sqrt{x})$ it holds

i) $-\frac{d}{dx}\left[\log J_0(2\sqrt{x})\right] = \frac{J_1(2\sqrt{x})}{\sqrt{x}J_0(2\sqrt{x})} = 1 + \sum_{m=1}^{\infty}\sigma_{m+1}x^m = 1 + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j_n}\left[\frac{x}{j_n}\right]^k$

ii) $-\frac{d}{dx}\left[\log J_0(2\sqrt{x})\right] = \sum_{n=1}^{\infty}\frac{1}{\alpha_n - x}$ for $|x| < \alpha_1 = 1.445795..$

iii) $\sigma_1 = \sum_{1}^{\infty}\frac{1}{\alpha_n} = 1$ is the only integer value for the σ_m.

With respect to [DDi] we define the bounded variation function

$$\mu(x) := -\log J_0(2\sqrt{x}),$$

which fulfills

$$\mu'(x) = \frac{d\mu}{dx} = -\frac{d}{dx}\left[\log J_0(2\sqrt{x})\right] = \frac{J_1(2\sqrt{x})}{\sqrt{x}J_0(2\sqrt{x})} = \frac{1+\sum_1^\infty a_{k+1}b_k(x)}{1+\sum_1^\infty b_k(x)} \qquad \text{for } x \neq \alpha_k ,$$

whereby $\{\alpha_k\}_{k\in N}$ are the zeros of $J_0(2\sqrt{x})$ and $a_k := \frac{1}{k}$ and $b_k(x) := \frac{(-x)^k}{(k!)^2}$.

We note the relation

$$1 = \int_0^\infty \frac{1}{4\sqrt{x}} J_0^{\,2}(2\sqrt{x}) \frac{J_1(2\sqrt{x})dx}{\sqrt{x}J_0(2\sqrt{x})} = \int_0^\infty \frac{1}{4\sqrt{x}} J_0^{\,2}(2\sqrt{x})d\mu = \int_0^\infty \sqrt{x}\left[\frac{J_0(2\sqrt{x})}{2\sqrt{x}}\right]^2 d\mu \ .$$

Hermite Polynomials

The Hermite polynomials $H_n(x)$ fulfill the recursion formula

$$H_n(\sqrt{2\pi}x) = 2xH_{n-1}(\sqrt{2\pi}x) - (n-1)b_n\varphi_{n-2}(x) - 2(n-1)H_{n-2}(\sqrt{2\pi}x) \ .$$

Putting $a_n := \sqrt{\frac{2(n-1)!}{n!}}$ $\quad b_n := \sqrt{\frac{(n-2)!}{n!}}$, $\varphi_0(x) := \pi^{-1/4}e^{-\frac{x^2}{2}}$, $\varphi_1(x) := 2^{-1/2}\pi^{-1/4}xe^{-\frac{x^2}{2}}$

this gives the recursion formula

$$\varphi_n(x) := a_n x\varphi_{n-1}(x) - (n-1)b_n\varphi_{n-2}(x) \ ,$$

with $\hat{\varphi}_0(t) := \sqrt{2}\pi^{1/4}e^{-\omega^2/2}$ and $[H(\varphi_0)]^{\wedge}(\omega) = -i\,\mathrm{sgn}(\omega)\hat{\varphi}_0(\omega)$. Applying the inverse Fourier transform then gives

$$[H(\varphi_0)](t) = \sqrt{2}\pi^{1/4}\int_{-\infty}^{\infty}(-i\,\mathrm{sgn}(\omega))e^{-\omega^2/2}e^{-i\omega t}d\omega \ .$$

Since $\mathrm{sgn}(\omega)e^{-\omega^2/2}$ is odd we have

$$[H(\varphi_0)](t) = 2\sqrt{2}\pi^{1/4}\int_0^{\infty}e^{-\omega^2/2}\sin(\omega t)d\omega \ .$$

Putting $f(x) = \pi^{1/4}\varphi_0(\sqrt{2\pi}x)$ it follows

$$\pi^{1/4}[H(\varphi_0)](\sqrt{2\pi}x) = 2\sqrt{2\pi}\int_0^{\infty}e^{-\omega^2/2}\sin(\sqrt{2\pi}\omega x)d\omega \ .$$

Substituting the variables $\omega = \sqrt{2\pi}\xi$ then leads to

$$[H(f)](x) = 4\pi\int_0^{\infty}e^{-\pi\xi^2}\sin(2\pi\xi x)d\xi \ .$$

From the Hermite polynomials recursion formula the Hilbert transforms can be calculated by

$$\hat{\varphi}_n(x) := a_n\left[x\hat{\varphi}_{n-1}(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_{n-1}(y)dy\right] - (n-1)b_n\hat{\varphi}_{n-2}(x)$$

$$\hat{\varphi}_0(x) = \pi^{1/4}\int_{-\infty}^{\infty}e^{-\frac{\omega^2}{2}}\sin(\omega x)d\omega \ .$$

The Hermite polynomials and the Hilbert transformed Hermite polynomials build a orthogonal system of $L_2(-\infty,\infty)$.

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