OVERVIEW

The Riemann Hypothesis

The three Millennium problems: RH, NSE, YME

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August 24, 2018

Albert Einstein, "*we can't solve problems by using the same kind of thinking we used when we created them*",

Wolfgang E. Pauli, "*all things reach the one who knows how to wait*".

This homepage addresses the following three Millennium problems (resp. links to corresponding homepages):

A. The Riemann Hypothesis (RH)
B. The 3D-Navier-Stokes equations (NSE)
C. The Yang-Mills equations (YME)

A first helicopter view

From a helicopter point of view there is a common denominator of all solution concepts of this homepage: it is about a common mathematical frame to govern the "infinitesimal small" with respect to truly infinitesimal small "elements" and related "functions" and truly geometrical (i.e. equipped with an inner product) "function spaces", enabling a truly geometrical mathematical modelling framework with corresponding operators (including well defined domains and ranges).

Regarding the RH the "infinitesimal small" is about the challenge to represent the entire Zeta function as a (Mellin-) transform of a self-adjoint operator ([EdH] 10.3). The Hilbert transform is the proposed tool to build a self-adjoint operator enabling the Berry-Keating (Hilbert-Polya) conjecture, because of its property that any Hilbert transformed function has a vanishing constant Fourier term (see also Polya's Bessel function based alternative entire Zeta function representation, [EdH], 12.5).

Regarding the NSE and YME, beside the n-dimensional counterpart of the Hilbert transform, the Riesz transforms, the two applied central "objects" are the well-established "differentials" and the distributional "Hilbert scale" concept enabling Pseudo-Differential and Fourier multiplier (weak and strong singular integral) equations (Calderón) and the related (Stieltjes integral like) spectral representation of Hermitian operators.

In the context of the newly proposed "energy-space" $H_{1/2} = H_1 + H_1^{\perp}$ we also refer to the Bose-Einstein Condensation (BEC), where below the critical temperature T_C BEC "normal gas" particles coexist in equilibrium with "condensed" particles. Unlike a liquid droplet in a gas, here the "condensed" particles are not separated in space $(H_{-1/2} = H_0 + H_0^{\perp})$ from normal particles. Instead they are separated in standard momentum space H_1, which is a closed, compactly embedded subspace of the newly proposed "energy-space" $H_{1/2}$. The condensed standard particles all occupy a single quantum state of zero momentum, while normal standard particles all have finite momentum with respect to the $H_1 -$ norm.

A. The Riemann Hypothesis

All nontrivial zeros of the analytical continuation of the Riemann zeta function have a real part of $1/2$. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self adjoint operator.

We provide a solution for the RH building on a new Kummer function based Zeta function theory, alternatively to the current Gauss-Weierstrass function based Zeta function theory. This primarily enables a proof of the Hilbert-Polya conjecture (but also of other RH criteria like the Bagchi formulation of the Nyman-Beurling criterion or Polya criteria), whereby the imaginary parts of the zeros of the corresponding alternative Zeta function definition corresponds to eigenvalues of a bounded, self adjoint operator with (newly) distributional Hilbert space domain.

The proposed framework also provides an answer to Derbyshine's question, ("Prime Obsession")

... "*The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?*"

The answer, in a nutshell:

"identifying "fluids" or "sub-atomic particles" not with real numbers (scalar field, I. Newton), but with hyper-real numbers (G. W. Leibniz) enables a truly infinitesimal (geometric) distributional Hilbert space framework (H. Weyl) which corresponds to the Teichmüller theory, the Bounded Mean Oscillation (BMO) and the Harmonic Analysis theory. The distributional Hilbert scale framework enables the full power of spectral theory, while still keeping the standard L(2)=H(0)-Hilbert space as test space to "measure" particles' locations. At the same time, the Ritz-Galerkin (energy or operator norm minimization) method and its counterpart, the methods of Trefftz/Noble to solve PDE by complementary variation principles (A. M. Arthurs, K. Friedrichs, L. B. Rall, P. D. Robinson, W. Velte) w/o anticipating boundary values) enables an alternative "quantization" method of PDE models (P. Ehrenfest), e.g. being applied to the Wheeler-de-Witt operator.

Regarding the proposed alternative quantization approach we also refer to the Berry-Keating conjecture. This is about an unknown quantization ***H*** of the classical Hamiltonian H=xp, that the Riemann zeros coincide with the spectrum of the operator 1/2+i***H***. This is in contrast to canonical quantization, which leads to the Heisenberg uncertainty principle and the natural numbers as spectrum of the harmonic quantum oscillator. The Hamiltonian needs to be self-adjoint so that the quantization can be a realization of the Hilbert-Polya conjecture.

The challenge to represent to entire Zeta function as a (Mellin-) transform of a self-adjoint operator is explained in [EdH], 10.3:

Let $\gamma(s)$, $Z(s)$, $z(s)$ denote the Gamma function, the entire Zeta function ([EdH] 1.8 (1)) and the Zeta function ([EdH] 1.4 (3))

$$z(s) = \frac{\gamma(1-s)}{2\pi i}\int_{+\infty}^{+\infty}\frac{(-x)^s}{e^x-1}\frac{dx}{x},$$

which is analytic at all points of the complex s-plane except for a simple pole at $s=1$. It is equal to the Dirichlet function

$$z(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} \quad \text{for} \quad Re((s) > 1 \ .$$

Let further denote $f(x) := e^{-\pi x^2}$ denote the Gaussian function (with mean value $\hat{f}(0) = 1$) and

$$M[f](s) := \int_0^{\infty} f(x)x^s\frac{dx}{x} = \frac{1}{2}\gamma(\frac{s}{2})\pi^{-s/2} \quad , \quad G(x) := \sum_{n=1}^{\infty} f(nx).$$

Then it holds ([EdH] 1.6 (5), 12.5)

$$(*) \qquad \frac{1}{2}\gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}z(s) = \frac{1}{2}\gamma\left(\frac{1-s}{2}\right)\pi^{-\frac{1-s}{2}}z(1-s)$$

$$(**) \qquad Z(s) = \frac{1}{2}\int_0^{\infty} x^{1-s}\frac{d}{dx}\left(x^2\frac{d}{dx}G(x)\right)\frac{dx}{x} \ .$$

The function on the left hand side of (*) has poles at $s = 0,1$ ([EdH] 1.8), whereby the pole at $s=0$ is caused by the Gamma function. Therefore, Riemann multiplies it by $s(s-1)$ to define the entire function $Z(s)$; that is an analytical function, which is defined for all values of s , and the functional equation of the Zeta function is equivalent to $Z(s) = Z(1-s)$.

The functional equation is derived from the summation formula

$$\sum_{u\in Z^n} e^{-t|u|^2} = (\frac{\pi}{t})^{n/2}\sum_{u\in Z^n} e^{-|\pi u|^2/t} \quad , \ t > 0 \ .$$

It is an example applied to the Gaussian function of the general Poisson summation formula ([PeB], Corollary11.9)

$$\sum_{u\in Z^n}\hat{\varphi}(2\pi u) = \sum_{u\in Z^n}\varphi(u) \ \text{and} \ \sum_{u\in Z^n}\varphi(2\pi u) = (2\pi)^{-n}\sum_{u\in Z^n}\hat{\varphi}(u) \ .$$

The functional equation is applied to derive the Riemann (error) function for the density function ([EdH] 1.13)

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} log z(s)x^s\frac{ds}{s}.$$

The factor $(s-1)$ leads to the Li-function, while the factor $\frac{s}{2}$ is anticipated in the function $\gamma\left(1+\frac{s}{2}\right)$ leading to the famous Riemann error function ([EdH] 1.16). Its convergence property jeopardizes a proof of the RH criterion

$$\pi(x) = Li(x) + O\left(\sqrt{x}logx\right) = Li(x) + O(x^{\frac{1}{2}+\varepsilon}) \quad , \quad \varepsilon > 0 \ .$$

Our proposed Kummer function based Zeta function theory is based on the Hilbert transform of the Gaussian function, which is the Dawson function

$$F(z) := e^{-z^2}\int_0^z e^{-t^2}dt \quad \text{resp.} \quad \sqrt{\pi} f_H(x) := F(\sqrt{\pi}x)$$

and its corresponding Mellin transform

$$M[f_H](s) = \pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\gamma\frac{s}{2}.$$

We note that $F(z)$ satisfies the differential equation $F'(z) + 2F(z) = 1$, leading to the (polynomial) asymptotic $F(\sqrt{x}) = O(x^{-\frac{1}{2}})$, alternatively to the asymptotic $f(\sqrt{x}) = O(e^{-x})$.

With respect to Riemann's "trick" above to build an entire function, this corresponds to an alternative multiplication with the factor $(s-1)\tan(\frac{\pi}{2}s)$ of (*).
We note the identity $\tan\left(\frac{\pi}{2}s\right) = \cot\left(\frac{\pi}{2}(1-s)\right)$; on the critical line $(s = \frac{1}{2} + it, t \in R)$ it holds $(\tan\left(\frac{\pi}{2}s\right) = \cot\left(\frac{\pi}{2}\bar{s}\right)$ (see also ([EdH] 10.2, 10.3, 10.10).

Related to the $\cot(\pi z)$ –function L. Kronecker provided some representation of series by integrals ([HeK] p. 42 ff):

via integration of the function $\pi\cot(\pi z) f(z)$ along of two parallel abscissa axis (with corresponding domain and convergence assumptions regarding the function $f(z)$) it holds the partial fraction expansion

$$\pi\cot(\pi z) f(z) = \sum_{k=-\infty}^{\infty}\frac{f(k)}{z-k} \ .$$

Also via integration along two parallel axis of ordinates there is a corresponding representation in a different form given by

$$\frac{1}{2}\left\{\sum_{x_0<k<x_1} f(k) + \sum_{x_0\leq l\leq x_1} f(l)\right\} = -\frac{1}{2}\int_0^{\infty}(-1)^{\alpha}\pi\cot(x_\alpha + i\varepsilon y) f(x_\alpha + i\varepsilon y)dy$$

with $\alpha = 0,1, \varepsilon = -1,1$ whereby (x_0, x_1) , (y_0, y_1) are the two abscissa resp. ordinates values with $(x_0 < x_1$, $y_0 < y_1)$.

From [EdH] 1.11 we recall the Stieltjes integral representation with the integrant of the log-Zeta function given by

$$\int_0^{\infty} x^{-z}dJ(x) = \frac{1}{2}\int_0^{\infty} x^{-z}d\left\{\sum_{p^n<x}\frac{1}{n} + \sum_{p^l\leq x}\frac{1}{l}\right\} .$$

With respect to (**) above it formally also holds ([EdH] 10.3)

$$(***) \qquad \frac{Z(s)}{s(s-1)} = \frac{1}{2}\int_0^{\infty} x^{1-s}G(x)\frac{dx}{x}$$

that is, the function is formally the transform of the operator

$$g(s) \to \int_0^{\infty} g(xs)G(x)dx.$$

But the above operator has no transform at all, as the integral does not converge for any s.

Nevertheless, this statement can be given substance as follows ([EdH] 10.5), which motivated our proposed alternative transform representation below:

A continuous analog of Euler's ludicrous formula

$$\sum_{n=-\infty}^{\infty} x^n = (1 + x + x^2 + \dots) + (x^{-1} + x^{-2} + \cdots) = \frac{1}{1-x} + \frac{1}{x-1} = 0$$

is

$$\int_0^\infty x^{1-s} \frac{dx}{x} = \int_0^1 x^{1-s} \frac{dx}{x} + \int_1^\infty x^{1-s} \frac{dx}{x} = \frac{1}{1-s} + \frac{1}{s-1} = 0 \ .$$

This is, of course, nonsense because the values of s for which the above integrals converge are mutually exclusive – the first one being convergent for $Re(s) < 1$ and the second one being convergent for $Re(s) > 1$, but it does suggest that the formal transform of $g(x) \to \int_0^\infty g(ux)dx$ is zero.

In order to anticipate the non-vanishing constant Fourier term of the Gaussian function, Riemann modified the above Mellin transform representation in the form ([EdH] 10.3)

$$Z^{**}(s) = \frac{1}{2}\int_0^\infty x^{1-s} \frac{d}{dx}(x^2 \frac{d}{dx}(G(x) - 1))\frac{dx}{x}$$

leading to representations in the form ([EdH] 10.5)

$$\frac{Z(s)}{s(s-1)} = \frac{1}{2}\int_0^\infty x^{1-s}[G(x) - 1]\frac{dx}{x} \qquad \text{for } Re(s) < 0$$

$$\frac{Z(s)}{s(s-1)} = \frac{1}{2}\int_0^\infty x^{1-s}\left[G(x) - 1 - \frac{1}{x}\right]\frac{dx}{x} \qquad \text{for } 0 < Re(s) < 1$$

$$\frac{Z(s)}{s(s-1)} = \frac{1}{2}\int_0^\infty x^{1-s}\left[G(x) - \frac{1}{x}\right]\frac{dx}{x} \qquad \text{for } 1 < Re(s) \ ,$$

while destroying the (only formally valid) self-adjoint transform representation (***) above (see also ([EdH] 12.5 for Polya's alternative, entire Zeta function definition linked to the Bessel functions).

As any Hilbert transformed function has a vanishing constant Fourier term the Hilbert transform is the proposed additional tool to overcome the above challenge to build a self-adjoint operator enabling the Berry-Keating (Hilbert-Polya) conjecture, based on the Hilbert transformed Gaussian function, which is the Dawson function with the asymptotic

$$f_H(\sqrt{x}) = O(x^{-\frac{1}{2}}) \ , \ x \to \infty \ .$$

This (polynomial) convergence behavior is supposed to enable a verification of the RH according to the following convergence RH criterion:

RH is true if and only if $\pi(x) = Li(x) + O(x^{\frac{1}{2}} log x)$ if and only if $\pi(x) = Li(x) + O(x^{\frac{1}{2}+\varepsilon})$

In fact we claim that our alternatively proposed entire Zeta function $Z^*(s)$ below enables an alternative prime number counting function $\pi^*(x)$ fulfilling the following RH criterion

RH is true if and only if $\pi^*(x) = Li^*(x) + O(x^{\frac{1}{2}})$ if and only if $\pi^*(x) = Li^*(x) + O(x^{\frac{1}{2}})$.

Putting $g(x) := e^{-x}$ and $I(x) := \int_{-\infty}^\infty \frac{g(t^2)}{x-t} dt$, this leads to the corresponding Mellin transforms replacements of $M[g](s)$ being replaced by $M[I(\sqrt{x})](s)$. i.e.

$$M[g](s) = \gamma(s) \qquad \to \qquad M[I(\sqrt{x})](s) = \pi \cdot \tan(\pi s) \cdot \gamma(s).$$

The definition of the entire Zeta function ([EdH] 1.8 (1))

$$Z(s) := (s-1)\pi^{-\frac{s}{2}}\gamma\left(1+\frac{s}{2}\right)z(s)$$

is based on the identities

$$M\left[-x\frac{d}{dx}f(x)\right](s) = sM[f](s) = \pi^{-\frac{s}{2}}\gamma(1+\frac{s}{2})$$

$$M\left[-\frac{d}{dx}(x^2\frac{d}{dx}(f(x))\right](s) = (s-1)sM[f](s) = (s-1)\pi^{-\frac{s}{2}}\gamma(1+\frac{s}{2}) \text{ .}$$

With

$$M\left[-\frac{d}{dx}(xf_H)\right](s) = (s-1)sM[f_H](s) = c_1\frac{tan\frac{\pi}{2}s}{\frac{\pi}{2}s}(s-1)\pi^{-\frac{s}{2}}\gamma(1+\frac{s}{2}) \text{ ,}$$

the corresponding alternative entire Zeta function definition is given by

$$Z^*(s) := c_1\frac{tan\frac{\pi}{2}s}{\frac{\pi}{2}s}Z(s) = c_1\frac{\cot(\frac{\pi}{2}(1-s)}{\frac{\pi}{2}s}Z(s) = c_2\left[-\frac{\pi}{2}(1-s)\cot(\frac{\pi}{2}(1-s))\right]\pi^{-\frac{s}{2}}\gamma(\frac{s}{2})z(s)$$

with a related Mellin transform representation in the form

$$Z^*(s) = \frac{1}{2}\int_0^\infty x^{1-s}\frac{d}{dx}(xG_H(x))\frac{dx}{x} \text{ .}$$

Considering the above replacement $-\frac{d}{dx}(x^2\frac{d}{dx}(f(x)) \rightarrow -\frac{d}{dx}(xf_H(x))$ we note the following rule for the Mellin transform

$$M\left[\frac{d}{dx}[xh(x]\right](s) = (1-s)M[h](s) \text{ .}$$

This results into

$$-M\left[\frac{d}{dx}\left(xf_H(x)\right)\right](s) = (s-1)M[f_H](s) \text{ .}$$

The fractional part function

$$\rho(x) := \{x\} := x - [x] = \frac{1}{2} - \sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu} \in L_2^{\#}(0,1)$$

is linked to the Zeta function by

$$\zeta(1-s) = (s-1)M[\rho](s-1) = M[-x\rho'(x)](s-1)$$

The Hilbert transform of the fractional part function is given by

$$\rho_H(x) = \sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu} = -\frac{1}{\pi}\log 2\sin(\pi x) \in L_2^{\#}(0,1) \text{ , } \hat{\rho}_H(0) = 0 \text{ , } \rho'_H \in H_{-1}^{\#}$$

Applying the idea of above leads to the replacement

$$M[-x\rho'(x)](s-1) = \zeta(1-s) \qquad \rightarrow \qquad M[-\rho_H(x)](s-1)$$

$$M[-\rho_H(x)](s-1) = M\left[\sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu})\right](s-1) = 2\pi\Gamma(\frac{1-s}{2})\Gamma^{-1}(\frac{s}{2})\cdot\chi(s)\cdot\frac{\zeta(s)}{s}$$

with same zeros as $\zeta(1-s)$.

The Gaussian function and its Hilbert transform are norm-equivalent with respect to the $L_2 = H_0$ −norm, i.e. both are equal in a weak L_2 −sense. The convergence of the transform representations in the critical stripe resp. on the critical line is ensured in a weak $H_{-\alpha}$ −sense ([EdH] 9.7, 9.8).

Both functions, $Z(s), Z^*(s)$ do have the same zeros. Therefore, in case the function $\theta^*(t) := Z^*\left(\frac{1}{2} + it\right)$ can be realized as a convolution $\theta(t) := (G * dF)(t)$ this would prove the RH ([CaD]).

The proposed Hilbert space framework of this paper provides corresponding PDO with appropriate symbols, which goes in line with corresponding convolution integral operators.

The Berry conjecture is that the nonimaginary solutions E_n of the zeros $z_n = 1/2 + E_n$ of the Riemann Zeta function are the eigenvalues of an appropriate Hermitian operator H , providing a model for the quantum chaos [BeM]. The definition of an appropriate operator requires always TWO elements: the definition of the mapping itself AND the definition of the underlying operator domain. The considerations above were all about the mapping definition only.

As the Gaussian function (resp. the fractional part function) and its Hilbert transform are equal in a weak L_2 −sense, i.e.

$$(f, v) = (f_H, v) \quad , \quad (\rho, v) = (\rho_H, v) \quad \forall v \in L_2 \ .$$

this indicates a domain definition of the to be built Hermitian operator in a weak H_α −sense with $\alpha \leq 0$.

The newly proposed Hilbert space Hilbert space for a quantum state Hilbert space framework is about a replacement of $L_2 \rightarrow H_{-1/2}$ with a corresponding replacement of the (energy space) domain of corresponding Schrödinger operator (in weak variation representation) $H_1 \rightarrow H_{1/2}$.

In a variational Hilbert space framework the proposed replacement above enables an in parallel reduction of the underlying Hilbert scale domains (in a weak variational representation) from $\alpha \rightarrow \alpha - 1/2$.

With respect to the entire Riemann Zeta function Z, the $cot -$ function and the Hermite polynomials h_n (with $\widehat{h_n}(\omega) = (-i)^n h_n$) we note that $Z \in H_{-1}$, $cot \in H_{-1}^{\#}$, $h_n \in L_2$, i.e. $(Z, logsin)_{-1/2}$, $(Z - cot, logsin)_{-1/2}, (Z, h_n)_{-1/2}$, $(Z - cot, h_n)_{-1/2}$ are defined. In this context we refer to

- the Bagchi reformulation of the Beurling RH criterion, as the $H_{-1/2}$ −Hilbert space is dense in H_{-1} with respect to the $\| \ \ \|_{-1}$ norm
- the wavelet related section below
- the theory of cardinal series ([WhJ] §11).

The approach above is basically about the definition of a “differential operator” in a weak Hilbert scale framework. Its counterpart in quantum mechanics is about a correspondingly defined Schrödinger momentum operator in the harmonic quantum oscillator model.

In the context of Pseudo-Differential operators this leads to the *Calderón-Zygmund integral-differential operator* with symbol

$$|\omega| = \sum_{k=1}^{n} \omega_k \frac{\omega_k}{|\omega_k|}$$

defined on corresponding domain given by ([EsG] example 3.4, [LiI] example 3.1.4, 3.1.6)

$$L[u] := \sum_{k=1}^{n} Y_k D_k u = \sum_{k=1}^{n} \frac{\gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \text{p.v.} \int_{-\infty}^{\infty} \frac{x_k - y_k}{|x-y|^{n+1}} \frac{\partial u(y)}{\partial y_k} dy$$

resp

$$\Lambda^{-1} u = \frac{\Gamma(\frac{n-1}{2})}{2\pi^{(n+1)/2}} \int_{-\infty}^{\infty} \frac{u(y)dy}{|x-y|^{n-1}}$$

where Y_k denote the Riesz operators, which are the n-dimensional generalizations of the Hilbert transform.

B. The Navier-Stokes Equations

The Navier-Stokes equations describe the motion of fluids. The *Navier–Stokes existence and smoothness* problem for the three-dimensional NSE, given some initial conditions, is to prove that smooth solutions always exist, or that if they do exist, they have bounded energy per unit mass.

We provide a global unique (weak, generalized Hopf) $H_{-1/2}$-solution of the generalized 3D Navier-Stokes initial value problem. The global boundedness of a generalized energy inequality with respect to the energy Hilbert space $H_{1/2}$ is a consequence of the Sobolevskii estimate of the non-linear term (1959).

The "standard" weak Hopf solution is not well posed due to not appropriately defined domains of the underlying velocity and pressure operators. Therefore, this is also the case for the corresponding classical solution(s).

The proposed solution also overcomes the "Serin gap" issue, as a consequence of the bounded non-linear term with respect to the appropriate energy norm.

The Navier-Stokes Equations (NSE) describes a flow of incompressible, viscous fluid. The three key foundational questions of every PDE is existence, and uniqueness of solutions, as well as whether solutions corresponding to smooth initial data can develop singularities in finite time, and what these might mean. For the NSE satisfactory answers to those questions are available in two dimensions, i.e. 2D-NSE with smooth initial data possesses unique solutions which stay smooth forever. In three dimensions, those questions are still open. Only local existence and uniqueness results are known. Global existence of strong solutions has been proven only, when initial and external forces data are sufficiently smooth. Uniqueness and regularity of non-local Leray-Hopf solutions are still open problems.

Basically the existence of 3D solutions is proven only for “large” Banach spaces. The uniqueness is proven only in “small” Banach spaces. The question of global existence of smooth solutions vs. finite time blow up is one of the Clay Institute millennium problems.

The existence of weak solutions can be provided essentially by the energy inequality. If solutions would be classical ones, it is possible to prove their uniqueness. On the other side for existing weak solutions it is not clear that the derivatives appearing in the inequalities have any meaning.

Basically all existence proofs of weak solutions of the Navier-Stokes equations are given as limit (in the corresponding weak topology) of existing approximation solutions built on finite dimensional approximation spaces. The approximations are basically built by the Galerkin-Ritz method, whereby the approximation spaces are e.g. built on eigen-functions of the Stokes operator or generalized Fourier series approximations.

It has been questioned whether the NSE really describes general flows: The difficulty with ideal fluids, and the source of the d'Alembert paradox, is that in such fluids there are no frictional forces. Two neighboring portions of an ideal fluid can move at different velocities without rubbing on each other, provided they are separated by a streamline. It is clear that such a phenomenon can never occur in a real fluid, and the question is how frictional forces can be introduced into a model of a fluid.

The question intimately related to the uniqueness problem is the regularity of the solution. Do the solutions to the NSE blow-up in finite time? The solution is initially regular and unique, but at the instant T when it ceases to be unique (if such an instant exists), the regularity could also be lost. Given a smooth datum at time zero, will the solution of the NSE continue to be smooth and unique for all time?

There is no uniqueness proof for weak solutions except for over small time intervals. The simplest possible model example how a singularity can appear, is the ODE

$$y'(t) = y^2(t) \quad y(0) = y_0$$

with its solution

$$y(t) = \frac{y_0}{1 - t \cdot y_0}$$

which becomes infinite in finite time. For n=3 every positive solution of

$$y'(t) = cy^3(t)$$

blows up, i.e. there is no global estimate by this method.

The global boundedness of our solution is a consequence of the Sobolevskii-estimate of the non-linear term enabling the generalized energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \le |(Bu,u)_{-1/2}| \le c \cdot \|u\|_{-1/2}\|u\|_1^2$$

Putting

$$y(t) := \|u\|_{-1/2}^2$$

one gets

$$y'(t) \le c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$$

This results into the a priori estimate

$$\|u(t)\|_{-1/2} \le \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \le c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\}$$

ensuring global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

C. The Yang-Mills Equations

The YME are concerned with quantum field theory. Its related Millennium problem is about an appropriate mathematical model to govern the current "mass gap" of the YME, which is the difference in energy between the vacuum and the next lowest energy field.

The classical Yang-Mills theory is a generalization of the Maxwell theory of electromagnetism where the *chromo*-electromagnetic field itself carries charges. For given distributions of electric charges and currents the Maxwell equations determine the corresponding electromagnetic field. The laws by which the currents and charges behave are unknown. The energy tensor for electromagnetic fields is unknown for elementary particles. Matter is built by electromagnetic particles, but the field laws by which they are constituted are unknown, as well. The original inertia law (before Einstein's gravity theory) forced to attribute physical-objective properties to the space-time continuum. Analog to the Maxwell equations (in the framework of a short distance theory) Einstein considered the inertia law as a field property of the space-time continuum.

As a classical field theory the Maxwell equations have solutions which travel at the speed of light so that its quantum version should describe massless particles (gluons). However, the postulated phenomenon of color confinement permits only bound states of gluons, forming massive particles. This is the mass gap. Another aspect of confinement is asymptotic freedom which makes it conceivable that the quantum Yang-Mills theory exists without restriction to low energy scales. The problem is to establish rigorously the existence of the quantum Yang-Mills theory and a mass gap. The being challenged concept is about the concept of a *displacement current*, which is “just” about a mathematical requirement to enabled consistent mathematical data model.

Based on an $H_{1/2}$ energy Hilbert space we propose (analog to the NSE solution) a corresponding (weak) variation Maxwell equation representation. Its corresponding generalization (as described above) leads to a modified QED model. In the same manner as the Serrin gap issue has been resolved (as a result of the reduced regularity requirements) the chromo-electromagnetic field /particles can now carry charges. The open "field law" question above and how "particles" are interacting with each other to exchange energy are modeled in same manner as the coherent/incoherent turbulent flows of its NSE counterpart. The corresponding "zero state energy" model is no longer built on the Hermite polynomials but on its related Hilbert transformed Hermite polynomials, which also span the $L_2 -$ Hilbert "test" space.

This provides a truly infinitesimal geometry (H. Weyl), enabling the concept of Riemann that force is a pseudo force only, which results from distortions of the geometrical structure. The baseline is a common Hilbert space framework (for all (nearby action) differential equations)

- providing the mathematical concept of a geometrical structure, while Riemann's manifold concept provides only a metric space and related affine connections

- building an integrated (no longer "force" dependent dynamical matter-field interaction laws) universal field model (including the gravity "force"), while replacing "force type" specific gauge fields and its combination model(s) for the electromagnetic, the strong and the weak nuclear power "forces"

As a consequence there are no "mass" and therefore no (YME-) "mass gap" anymore, but there is an appropriate vacuum (Hilbert) energy space, which is governed by the Heisenberg uncertainty principle.

In Plemelj's concept of an alternative simple and double layer potential ([PlJ]) with reduced regularity requirements to the underlying domains the corresponding differentials are named as "mass elements/ particles" in opposite to mass densities. The rotation invariance property of the Riesz operator ensures the "rotation invariance" of the considered differentials.

This concept leads to alternative Schrödinger momentum operator definition, which then also provides additional evidence to the famous constant $-\frac{1}{i}=i$ of the Schrödinger momentum operator in its classical form. For space dimension $\mathrm{m}=1$ it is given by $\mathrm{u(x)} \rightarrow P^{*}[u](\mathrm{x}) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{dx}}\mathrm{H[u]}(x) = -\mathrm{iH}[u_x](x)$ ([MeY], 7.1) with domain $H_{1/2}=H_0+H_0^{\neg}$ and a corresponding alternative ground state energy model with non-integer Hilbert space domain ([BrK1], [BrK2]). With respect to the $H_{-1/2}$ –inner product it holds $(P^{*}[u],v)_{-1/2} \cong (u,v)_0$.

In the context of section D below (following the notation in [ShF] p. 393) we mention the Vlasov formula for the plasma dielectric for longitudinal oscillations ε with

$$\varepsilon = 1-\frac{\omega_{pe}^2}{k^2}W\left(\frac{\omega}{k}\right) \quad \text{and} \quad W\left(\frac{\omega}{k}\right) := -H[\varphi_0'] \left(\frac{\omega}{k}\right) \text{ in same form as the operator above.}$$

Vlasov's assumption was that longitudinal oscillations set up initially in plasma with a non-pathological electron distribution function should be able to persist forever in the absence of dissipative collisions, i.e. it should be possible to consider real values for both ω and k.

Landau proved based on classical Laplace transform analysis that Vlasov's assumption was erroneously. The weak definition of the above alternative Schrödinger operator is proposed to be applied for a corresponding anaylsi in its related distribution Hilbert space framework.

The extension to space dimension $\mathrm{m}>1$ leads to the Pseudo differential operator

$$\mathrm{L[u]} := \sum_{\mathrm{k}=1}^{\mathrm{n}} \mathrm{Y_k D_k u} = \sum_{\mathrm{k}=1}^{\mathrm{n}} \frac{\gamma\left(\frac{\mathrm{n}+1}{2}\right)}{\pi^{\frac{\mathrm{n}+1}{2}}} \mathrm{p.v.} \int_{-\infty}^{\infty} \frac{\mathrm{x_k}-\mathrm{y_k}}{|\mathrm{x}-\mathrm{y}|^{\mathrm{n}+1}} \frac{\partial \mathrm{u(y)}}{\partial \mathrm{y_k}} \mathrm{dy}$$

and

$$(P^{*}[u],v)_{-1/2} := (-i\mathrm{L}[u],\mathrm{v})_{-1/2} \quad \forall v \in H_{-1/2} \,.$$

In the context of the newly proposed "energy-space" $H_{1/2}=H_1+H_1^{\neg}$ we also refer to the Bose-Einstein condensation, where below the critical temperature T_C BEC "normal gas" particles coexist in equilibrium with "condensed" particles. Unlike a liquid droplet in a gas, here the "condensed" particles are not separated in space ($H_{-1/2}=H_0+H_0^{\neg}$) from normal particles. Instead they are separated in momentum space. The condensed particles all occupy a single quantum state of zero momentum, while normal particles all have finite momentum.

In case the domain of such a compact operator is the L_2 Hilbert space the corresponding eigenfunctions build the basis of this Hilbert space. The concept of "wave package" enables also continuous spectra. Therefore, such "wave packages" require a domain extension (e.g. $\mathrm{L_2} \rightarrow \mathrm{H_{-1/2}}$) in order to ensure convergent inner products and related norms. "Wave packages" are also called "eigen-differentials" (H. Weyl), playing a key role in quantum mechanics in the context of the spectral representation of Hermitian operators (D. Hilbert, J. von Neumann, P. A. M. Dirac).

C.1 Some mathematical aspects regarding "differentials" and "distributional Hilbert scale"

The current quantum state Hilbert space is identical with the "measurement /observation /statistical" function space L_2. The newly proposed distributional Hilbert spaces are $\mathrm{H}_{-1/2}$ and H_{-1}. The corresponding inner product $((x,y)_{-1}$ can be put into relation to an inner product in the form $((dx,dy))$. The role of the L_2 as the quantum state Hilbert space is being replaced by $\mathrm{H}_{-1/2}$.

As the L_2 is a closed, compactly embedded subspace of $\mathrm{H}_{-1/2}$ with respect to the $\mathrm{H}_{-1/2}$-norm, resp. as the $\mathrm{H}_{-1/2}$ is a closed, compactly embedded subspace of H_{-1} with respect to the H_{-1}-norm, compactness arguments can be applied in combination with corresponding Garding type inequalities of related operator representation in the form $B=A+K$ ([AzA], [BrK]).

The differential "objects" can be interpreted as "ideal point" (non-standard numbers, "monads"), which then are going to replace the real numbers. Both fields do have the same cardinality, but the non-standard numbers allow infinitesimal small number "objects" in the neighborhood of each "real" number. In a certain sense the field of real numbers is compactly embedded into the field of non-standard numbers. The Riesz theorem below provides the corresponding relationship to a closed subspace X_0 of a Hilbert space X.

In [GeR] ideal point for the space-time are considered to model "singular points" and "points at infinity". It leads to the notions of "indecomposable past-set (IP)", with the two categories of proper IP (PIP) and terminals IP (TIP), defining the ideal points of the future.

The advantages with respect to the three considered problems are the following

1. the "measurement of real numbers is already an approximation by rational numbers, i.e. truly "observations" of irrational number "objects" are not possible; each irrational number is already a full universe, i.e. an approximation of an infinite numbers of rational numbers; extending those number field to ideal numbers is just the same mystery with same cardinality; the key differentiator is related to a measurement of length axiom by given "unit of measure" length

2. the $\mathrm{H}_{-1/2}$ Hilbert space provides an alternative model to the current $\mathrm{H}_0=L_2$ "quantum state" Hilbert space, which avoids the Dirac "function" concept with its handicap of space dimension depending regularity requirements ($\delta\in\mathrm{H}_{-\frac{n}{2}-\varepsilon},\varepsilon>0$)

3. The physical "observation" L_2 Hilbert space (supporting also statistical analysis) is still valid and applicable; it is a subspace of $\mathrm{H}_{-1/2}$, whereby its corresponding complementary closed subspace $H_0^{\neg}$ of $H_{-1/2}=H_0+H_0^{\neg}=L_2+L_2^{\neg}$ enables an alternative modelling of "wave packages" ("eigen-differentials")

4. the Zeta function Z on the critical line is an element of the H_{-1} Hilbert space. Therefore, the proposed framework enables the Bagchi-Beurling (density) criterion, as both, the fractional part function $\rho(x)$ and the $\log(\sin(x))$ function, are elements of the Hilbert space L_2, i.e. the inner products $(Z,\rho)_{-1/2}$, $(Z,\log(sin))_{-1/2}$ are defined, and as the Hilbert space $H_{-1/2}$ is dense in H_{-1}

5. the $\mathrm{H}_{-1/2}$ Hilbert ("fluid") space enables the Sobolevskii-estimate of the non-linear term of the 3-dimensional, non-stationary, non-linear variation NSE representation, leading to the bounded, generalized energy inequality

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^{2}+\|u\|_{1/2}^{2}\leq\left|(Bu,u)_{-1/2}\right|\leq c\cdot\|u\|_{-1/2}\|u\|_{1}^{2}$$

for $u \in \mathrm{H}_1$.

The theorem of Riesz ensures "quasi-optimal" approximation properties of each "object" of a closed subspace X_0 of a Hilbert space X:

Theorem (Riesz): *For each ε with $0 < \varepsilon < 1$ there exists a $y = x_\varepsilon \in X$ with $\|y\| = 1$ and*

$$inf\{\|x-y\| \quad |x \in X_o\} \geq \varepsilon$$

Remark 1*:* A separable Hilbert scale can be built from the solutions of an eigenvalue equation

$$\mathrm{Kx} = \mathrm{\sigma x}$$

where K *denotes a symmetric and compact operator:*

Lemma*: for no more than countable values σ_i the equation $Kx = \sigma x$ possesses non-trivial solutions x_i and $lim_{i\to\infty}\sigma_i = 0$.*

Remark 2*: as 0 is an element of X_0 this means that the inf-term above is at* most *equal 1; therefore the theorem states that this value can be arbitrarily close approximated. This can be interpreted as counterpart of the approximation of an irrational number by rational numbers.*

Remark 3: Regarding "ideal points in space-time" we refer to [GeR], where …*"a prescription is given for attaching to a space-time M, subject only to a causality condition, a collection of additional "ideal points". In particular, for any asymptotically simple space-times, the ideal points can be interpreted as the boundary at conformal infinity. The concept makes possible an extension of the domain-of-dependence concept to causal spaces. … Some of these ideal points can be interpreted as singular points of M, others as points of infinity*." In [PeR] the concept of a "conformal cyclic cosmology (CCC)" in combination with a revisited second principle of thermodynamics is proposed.

Theorem: *Let H_1 and H_0 be Hilbert spaces (H_1 being a subspace of H_0) with*

i) $\|x\|_0 \leq \|x\|_1$ *for* $x \in H_1$

ii) H_1 *is dense in* H_0

iii) the unit ball of H_1 *is relatively compact in* H_0 *.*

Then there exists an operator A with $D(A) = H_1$, $R(A) = H_0$ and $\|x\|_1 = \|Ax\|_0$, whereby the operator A is

i) positive definite, self-adjoint

ii) A^{-1} *is compact.*

The corresponding eigenvalue problem

$$A\varphi_i = \sigma_i \varphi_i$$

has infinite solutions $\{\sigma_{i,}\varphi_i\}$ *with* $\sigma_i \to \infty$ *and* $(\varphi_i, \varphi_k) = \delta_{i,k}$ *, and for each element* $x \in H_1 = A^{-1}H_0$ *it holds the representation*

$$x = \sum_{i=1}^{\infty} (x, \varphi_i)\, \varphi_i$$

Inner products with corresponding norms of a distributional Hilbert scale can be defined based on the eigen-pairs of an appropriately defined operator in the form

$$(x, y)_\alpha := \sum_i^\infty \lambda_i^\alpha (x, \varphi_i)(y, \varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

Additionally, for $t > 0$ there can be an inner product resp. norm defined for an additional governing Hilbert space with an "exponential decay" behavior in the form

$$e^{-\sqrt{\lambda_i}t}$$

given by

$$(x, y)_{(t)}^2 := \sum_{i=1} e^{-\sqrt{\lambda_i}t} (x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 := (x, x)_{(t)}^2$$

The distributional $\mathrm{H}_{-1/2}$ – Hilbert space is proposed to model quantum states, alternatively to the Hilbert space H_0. A mathematical (wavelet microscopic) analysis of those states is then about an analysis of the "objects"

$$x = x_0 + x_0^\perp \in \boldsymbol{H}_0 \otimes \boldsymbol{H}_0^\perp$$

with

$$\|x_0\|_0 = 1 \qquad \sigma := \|x_0^\perp\|_{-1/2}^2$$

As it holds for any $\mathrm{t}, \delta, \alpha > 0$ and $\lambda \geq 1$ the inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$$

the following inequality is valid for any $\in H_0$, governing the approximation "quality" of a quantum state with respect to the norm of H_0:

$$\|x\|_{-1/2}^2 \leq \delta \|x\|_0^2 + e^{t/\delta} \|x\|_{(t)}^2 = \sigma \|x\|_0^2 + e \|x\|_{(\sigma)}^2 = \sigma \|x\|_0^2 + \sum_{i=1}^{\infty} e^{1 - \sqrt{\lambda_i}\sigma} x_i^2$$

It balances the "continuous" view of the overall state with its "discrete" components.

C. 2 The Maxwell equations: baseline for the Yang-Mills and the gravity field equations

[PeR] p. 141: "*Mathematically, Yang-Mills theory is basically just Maxwell theory with some "extra internal indices", so that the single photon is replaced by a multiplet of particles. In the case of strong interactions, things called quarks and gluons are respective analogues of the electrons and photons of electromagnetic theory. The quarks, but not the gluons, are massive, with masses considered to be directly linked to the Higgs*. *In the standard theory of weak interactions, the photon is considered to be part of a multiplet containing three other particles, all of which are massive, referred to as W^+, W^-, and Z. Again, these masses are considered to be coupled to that of the Higgs. Thus, according to current theory, when that mass – providing ingredient is removed, at the extremely high temperatures back near the Big Bang …. Then full conformal invariance should be restored."*

The Maxwell equations (and its related (complementary) variational principles with its underlying $U(1)$ symmetry group) build the baseline for all related SMEP Lagrange density formulations and the Einstein field equations.

[EiA] p. 52 ff: „*die Maxwellschen Gleichungen bestimmen das elektrische Feld, wenn die Verteilung der elektrische Ladungen und der Ströme bekannt sind. Die Gesetze aber, nach denen sich Ströme und Ladungen verhalten, sind uns nicht bekannt. Wir wissen wohl, dass die Elektrizitäten in Elementarkörperchen (Elektronen, positiven Kernen) bestehen, aber wir begreifen es nicht vom theoretischen Standpunkt aus. … Wir kennen daher, <u>falls wir überhaupt die Maxwellschen Gleichungen zugrunde legen dürfen</u>, den Energietensor für die elektromagnetischen Felder nur außerhalb der Elementarteilchen*. …. *Wir wissen heute, daß die Materie aus elektrischen Elementarteilchen aufgebaut ist, sind aber nicht im Besitz der Feldgesetze, auf welchen die Konstitution jener Elementarteilchen beruht. Wir sind daher genötigt, uns bei deren Behandlung der mechanischen Probleme einer ungenauen Beschreibung der Materie zu bedienen, welche der von der klassischen Mechanik verwendeten entspricht. Die Dichte σ der ponderablen Substanz und die hydrodynamischen Druckkräfte (Flächenkräfte) sind die Grundbegriffe, auf die eine derartige Beschreibung sich stützt*".

[HaS] p. 72: "*Force-carrying particles can be grouped into four categories* (gravitational force, electromagnetic force, weak nuclear force, strong nuclear force) *according to the strength of the force that they carry and the particles with which they interact<u>. It should be emphasized that this division into four classes is man-made</u>; it is convenient for the construction of partial theories, but it may not correspond to anything deeper. Ultimately, most physicists hope to find a unified theory that will explain all four forces as different aspects of a single force."*

Schrödinger E.: "*Indeed there is no observation concerned with the geometrical shape of a particle or even with an atom."*

Gravitational force: it is universal, that every particle feels the force of gravity, according to its mass or energy

Electromagnetic force: it interacts with electrically charged particles like electrons and quarks, but not with uncharged particles such as gravitons

Weak nuclear force: it is responsible for radioactivity and which acts on all matter particles of spin ½, but not on particles of spin 0, 1, or 2

Strong nuclear force: it holds the quarks together in the proton and neutron, and it holds the protons and neutrons together in the nucleus of an atom.

In classical electromagnetic field theory one deals with the following quantities: **E** = electric field, **H** = magnetic field, **B** = magnetic induction**,** **J** = electric current density, **D** = dielectric displacement ρ = charge density. In the language of exterior forms this leads to ([FlH], 4.6)

$$d\alpha = 0 \quad , \quad d\beta = -4\pi\gamma \quad (\alpha = \alpha_{E,B},\ \beta = \beta_{H,D})$$

while $d\gamma = 0$ corresponds to $\dot{\rho} + divJ = 0(\ \gamma = \gamma_{J,\rho})$. Maxwell equations in free spaces are simply

$$d\alpha = 0 \quad d * \alpha = 0\ .$$

With respect to the "rotation" invariance topic of this paper we note the integrated Maxwell field equations by Lorentz ([WeH] §20) based on the assumption that the distribution of charges and currents are known:

The equation $div\boldsymbol{B} = 0$ is satisfied by setting $-\boldsymbol{B} = curl\boldsymbol{f}$ in which $\boldsymbol{f}$ is the *vector potential*. By substituting this in the first Maxwell equation $curl\boldsymbol{E} + \frac{\boldsymbol{1}}{\boldsymbol{c}}\frac{\partial\boldsymbol{B}}{\partial t} = 0$ one gets that $\boldsymbol{E} - \frac{\boldsymbol{1}}{\boldsymbol{c}}\frac{d\boldsymbol{f}}{dt} = 0$ is irrotational so that one can set $\boldsymbol{E} - \frac{\boldsymbol{1}}{\boldsymbol{c}}\frac{d\boldsymbol{f}}{dt} = grad\varphi$ in which φ is the *scalar potential*. Finally the vector and scalar potential are defined by correspond wave equations in the following form (based on the differential form representation of the Maxwell equations)

$$\frac{1}{c^2}\ddot{\varphi} - \Delta\varphi = -\rho\ ,\ \frac{1}{c^2}\ddot{\boldsymbol{f}} - \Delta\boldsymbol{f} = -\boldsymbol{s}\ \ .$$

We note that the Klein-Gordon equation is the relativistic counterpart of the wave equation. The Maxwell equations reflect Faraday's near action theory approach. They are based on Gauss's law for electric fields, which are formulated in integral and differential form requiring different regularity assumptions for the different domains of the corresponding operators. Therefore, the first step to enable a "rotation invariance" principle based on common domains for all SMEP affected operators, is to consider only the integral form of the Gauss's electric fields, i.e.

$$\oint_S \vec{E}\circ\vec{n}da = \frac{q_{encl}}{\varepsilon_0}$$

with the electric field $\vec{E}$, the permittivity of free space (or "vacuum permittivity") ε_0, the amount of the enclosed contributing charge q_{encl} and Maxwell's displacement current

$$I_d \coloneqq \varepsilon_0\frac{d}{dt}\oint_S \vec{E}\circ\vec{n}da.$$

The second (main) step is to apply Plemelj's alternative definition of simple and double layer potentials (to further reduce regularity requirements of the affected operator domains), which goes along with Plemelj's revisited definitions of "*mass & flux*" vs. "*mass densities*" (see also the problem B related homepage: www.navier-stokes-equations.com). Plemelj's key concept (for space dimension $n = 2$) replaces the normal derivative concept by the integral (σ, σ_0 elements of the boundary!)

$$\bar{U}(\sigma) = -\int_{\sigma_0}^{\sigma}\frac{dU}{dn}ds,$$

whereby $d\bar{U}(\sigma)$ may be defined, but not $\frac{dU}{dn} = -\frac{d\bar{U}(\sigma)}{d\sigma}$. The new term is named "current strength" flowing through the corresponding boundary piece, also called "current or flux".

In case of the logarithmic potential (for space dimension $n = 2$) the current is determined by a simple measure: it is the change of the conjugate potential (which is related to the Hilbert transform) between the points σ_0, σ. The generalization to the space dimensions $n > 2$ is provided in [StE] based on the Riesz operator concept. With respect of the relationship of the YME and the Hilbert transform we refer to [DuR].

Plemelj's quotes [PlJ]: §5: "*es handelt sich hier um eine Verallgemeinerung, wie es die Erweiterung differentiierbarer Funktionen auf die stetigen ist*". §8: *„Bisher war es üblich für das Potential $V(p)$ die form*

$$\int log \frac{1}{r_{ps}} \acute{\mu}(s) ds$$

vorauszusetzen, wobei dann $\acute{\mu}(s)$ die Massendichtigkeit der Belegung genannt wurde. Eine solche Annahme erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potentiale der größte Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der (Stieltjes-schen) Form

$$\int log \frac{1}{r_{ps}} d\mu_s$$

verwendbar.

It enables a replacement of the

- "*Dichte (density) $\sigma = \mu(s)ds$ der ponderablen Substanz*" ([EiA] p. *52)* by the Plemelj *"mass element $d\mu_s$"*
- "*hydrodynamischen Druckkräfte (Flächenkräft*e *(area forces))*" ([EiA] p. 52) by the Plemelj „current"
- co-variant (Maxwell electromagnetic *potential*) vector by the Plemelj "*potential"*.

In other words, Plemelj's (Stieltjes integral based) mass element and potential definitions transform Newton's (Lebesgue integral based) mass density $\mu(s)ds$ and potential definition moving from a far distance action theory to a near action theory. In combination with a correspondingly defined (rotation invariant) inner product for those "mass elements" an adapted variational representation of the Maxwell equations is proposed as model to describe the behavior of currents and charges of an electromagnetic field. We note that the co-variant derivative of a scalar field corresponds to the partial derivative.

In the appendix we briefly sketch the relationship of Plemelj's alternative potential definition, the Hilbert transform and the Laplacian equation, which may be solved using simple layer potential. The 3-D analogy of the Cauchy-Riemann differential equations is provide in [RuC]. The Riesz transforms on spheres are provide in [ArN].

The achieved regularity reduction, while the Green formula keep still valid, enables a redefined Dirichlet (energy) integral $D(u,v) \coloneqq (\nabla u, \nabla v)$ in the form ($\alpha = 0$ being replaced by $\alpha = -1/2$)

$$(u, \Delta v)_\alpha + (\nabla u, \nabla v)_\alpha = (u, \Delta v)_\alpha + (u, v)_{1*\alpha} = \langle u, v_n \rangle_\alpha \ .$$

With respect to the Lorentz integration of the Maxwell equations above this goes along with a variational representation in case of the first wave equation (which denotes a wave disturbance travelling with velocity c) in the form

$$\varphi \in H_{1/2} : \quad \frac{1}{c^2}(\ddot{\varphi}, \dot{\vartheta})_{-1/2} + (\varphi, \vartheta)_{1/2} + \langle \varphi_n, \vartheta \rangle_{-1/2} = -(\rho, \vartheta)_{-1/2} \quad \forall \vartheta \in H_{1/2}$$

The wave equation plays also a key role in the harmonic quantum oscillator analysis. The variational representation is basically about hyper-singular integral equations. In [WeP], [WeP1] the Laplace operator with respect to electric and magnetic boundary conditions is analyzed. In [ChF] a method is introduced and analyzed allowing appropriate approximations w/o the necessity of a Lagrange multiplier.

The corresponding Klein-Gordon equation $\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta+\frac{m^2c^2}{h^2}\right]\varphi(x)=0$ is derived from the energy-momentum equality

$$(*)\quad \frac{1}{c^2}E^2-\boldsymbol{P}^2=m^2c^2I$$

whereby $E:=ih\frac{\partial}{\partial t}$ and $\boldsymbol{P}:=-ih\nabla$ denotes the standard Schrödinger operator. Dealing with the quadratic terms of (*) leads to also negative energy values E ([HeW] (134)).

Replacing the standard Schrödinger operator in (*) by the proposed one of this paper, leads to $\boldsymbol{P}^{*\boldsymbol{2}}=-hH^2\left[\frac{d^2}{dx^2}\right]=h\frac{d^2}{dx^2}$ (because of $H^2=-I$) and therefore, to

$$(**)\quad \frac{1}{c^2}E^2+\boldsymbol{P}^{*2}=m^2c^2I$$

which indicates a Helmholtz type equation given by

$$-\nabla^2\mathrm{u}-\mu^2u=\begin{cases}\delta_i\\0\end{cases} \text{ in the considered domain}$$

(δ_i denotes the Dirac function at source i corresponding to the fundamental solutions vs. zero for the general solution) with its underlying distance variable (Hankel function based) solutions governed by the divergence (conservation) theorem and the Sommerfeld radiation condition at infinity ([BrK4], [KyP]). It enables the so-called distance function wavelet analysis based on Helmholtz-Fourier transforms and corresponding Helmholtz-Fourier series (e.g. [AmS], [ChW]). For its related Riesz transforms on spheres we refer to [ArN].

We mention that the Helmholtz (decomposition) theorem states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field.

The Helmholtz equation is e.g. used to approximate model wave propagation in inhomogeneous media or for the determination of a radiation field surrounding a source of radiation. The corresponding Cauchy problem is ill posed in the sense of [HaJ]), i.e. in general the Cauchy problem suffers nonexistence and instability of the solution. A similar situation is given in the context of linear and non-linear parabolic equations, in case certain compatibility relations of the initial value function are not given ([BrK5], [BrK6]). In case of the Cauchy problem for the Helmholtz equation at a fixed frequency

$$\Delta\mathrm{u}(\mathrm{t},\mathrm{x})+\mu^2u(t,x)=0 \qquad , x\in R^n \ \ , t\in(0,1)$$

$$\mathrm{u}(0,\mathrm{x})=g(x)\ ,\ u_t(0,\mathrm{x})=0\ ,\ x\in R^n$$

($\Delta:=\frac{\partial^2}{\partial t^2}+\sum_{k=1}^n\frac{\partial^2}{\partial x_k^2}$) and its corresponding Fourier transformed equation

$$\hat{\mathrm{u}}_{tt}(\mathrm{t},\omega)+(\mu^2-|\omega|^2)\hat{u}(t,\omega)=0$$

$$\hat{\mathrm{u}}(0,\omega)=\hat{g}(\omega)\ ,\ \hat{u}_t(0,\omega)=0\ ,\ \omega\in R^n$$

the solution is given by

$$\mathrm{u}(\mathrm{t},\mathrm{x})=\frac{1}{(2\pi)^{n/2}}\iint e^{ix\omega}\cosh\left(t\sqrt{|\omega|^2-\mu^2}\right)\hat{g}(\omega)d\omega.$$

Its kernel $\cosh\left(t\sqrt{|\omega|^2-\mu^2}\right)$ increase rapidly with exponential order as $|\omega|\to\infty$, by which $\hat{g}(\omega)$ must decay rapidly. In order to enable a corresponding analysis to [BrK7] we propose a governing Hilbert space framework with an "exponential decay" inner product as provided in the previous section, given by

$$((\mathrm{u},\mathrm{v}))_{k.(t)}=\sum_{i=1}^{\infty}\sigma_i^k e^{-\sqrt{\sigma_i}t}(u,\varphi_i)(v,\varphi_i)$$

$$\||u\||_{k.(t)}^2=((\mathrm{u},\mathrm{u}))_{k.(t)}^2 \quad .$$

The considered eigenvalues σ_i are still to be defined properly. The underlying Hilbert space construction of [BrK1] is based on the eigen-pair solutions of the Laplacian operator, resp. the corresponding boundary integral operators (single layer potential, double layer potential and the normal derivative of the double layer potential) which are strongly elliptic pseudodifferential operators of integer orders. The corresponding Galerkin approximation theory is provided in [BrK8]. In the considered case above the corresponding eigen-pair solution of the Helmholtz operator should be the preferred one. In [AmS] a corresponding overview is given, including the corresponding eigen-pair solutions for the Helmholtz operator, as provided in [KrR]: while for the Laplacian model problem on the unit circle the eigenvalues for the common eigen-functions $e^{\pm imt}$ are given by

$$(1)\ \omega_m=\begin{cases}0 & m=0\\ \frac{1}{2|m|} & m\neq 0\end{cases}\quad,\ (2)\ \ \omega_m=\begin{cases}-1/2 & m=0\\ 0 & m\neq 0\end{cases}\quad,\ ,\ (3)\ \ \omega_m=\begin{cases}0 & m=0\\ -|m|/2 & m\neq 0\end{cases}$$

the corresponding eigenvalues for the Helmholtz operator, depending also on the wave number μ, are given by

$$(1)\ \omega_m=\frac{i\pi}{2}J_m(\mu)H_m(\mu)\ \ ,\ (2)\ \ \omega_m=-\frac{1}{2}+\frac{i\pi}{2}k\acute{J}_m(\mu)H_m(\mu)\ \ ,\ ,\ (3)\ \ \omega_m=-\frac{i\pi}{2}\mu^2\acute{J}_m(\mu)\acute{H}_m(\mu)$$

In [ChS] the behavior, in the important high frequency limit $\mu\to\infty$, with respect to the standard L_2 – norm is considered.

[KiA] p. 6/9: A solution of the Maxwell equation system in a vacuum can be described by a divergence free solution of one of the two vector valued wave equations and defining the other field by Amperes Law or by Faraday's Law of Induction, respectively In case the fields allow a Fourier transformation in time in the form

$$\widehat{\boldsymbol{E}}_t(x,\omega):=\int_R E(x,t)e^{i\omega t}dt$$

one speaks of time-harmonic fields. Those fields are then complex valued. In a vacuum without external current density this results into the vector Helmholtz equations

$$\Delta\widehat{\boldsymbol{E}}_t(x,\omega)+\mu^2\widehat{\boldsymbol{E}}_t(x,\omega)=0\ \text{ and }\ \ \Delta\widehat{\boldsymbol{H}}_t(x,\omega)+\mu^2\widehat{\boldsymbol{H}}_t(x,\omega)=0\ .$$

A vector field $\widehat{\boldsymbol{E}}_t(x,\omega)$ combined with $\widehat{\boldsymbol{H}}_t(x,\omega)\coloneqq\frac{1}{i\omega\varepsilon_0}curl\widehat{\boldsymbol{E}}_t(x,\omega)$ provides a solution of the time-harmonic Maxwell equations (w/o external current density) if and only if $\widehat{\boldsymbol{E}}_t(x,\omega)$ is a divergence free solution of the vector Helmholtz equations, that is

$$\Delta\widehat{\boldsymbol{E}}_t(x,\omega)+\mu^2\widehat{\boldsymbol{E}}_t(x,\omega)=0\ \text{ and }\ div\widehat{\boldsymbol{E}}_t(x,\omega)=0\quad .$$

Analogously, a divergence free solution of the vector Helmholtz equation $\widehat{\boldsymbol{H}}_t(x,\omega)$ combined with $\widehat{\boldsymbol{E}}_t(x,\omega)\coloneqq-\frac{1}{i\omega\varepsilon_0}curl\widehat{\boldsymbol{H}}_t(x,\omega)$ leads to a solution of Maxwell's equations in vacuum.

C. 3 A "helicopter view" regarding Schrödinger's "purely quantum wave" vision

The Schrödinger (differentiation) operator is not bounded with respect to the norm of L_2, i.e. only on a dense subspace of L_2 a corresponding spectral representation of this operator can be defined. The not vanishing constant Fourier term of the baseline Hermite polynomial (which is the Gaussian function) leads to mathematical challenges with respect to the creation and annihilation operators of the related Hamiltonian operator of the quantum oscillator model. The Hilbert transform of a function f has always vanishing constant Fourier terms. As a consequence, the Hilbert-transformed Schrödinger operator form with extended domain $H_{-1/2}$ is bounded (with respect to the norm of L_2) leading to a bounded Hermitian operator with corresponding spectral form representation.

Based on the newly defined common Hilbert space domain spectral theory can be applied, while

- the (physical) test space keeps the same, i.e. $L_2 = H_0$
- the current domains of the considered operators are extended to enable a (convergent) energy norm $\|x\|_{1/2}$ and a corresponding weak variation representation of the considered operator equations with respect to the inner product $(\boldsymbol{x}, \boldsymbol{y})_{-1/2}$.

The corresponding notions from variation theory are "energy norm" and "operator norm" with correspondingly defined minimization problems ("energy" resp. "action" minimization problems). The corresponding eigenvalue problem of an operator T is then related to the inner product $(Tx, x)_{-1/2}$.

With respect to the newly proposed Pseudo-differential and Fourier multiplier operators with extended fractional Hilbert scale domain we note the following:

- The Maxwell equations are represented by differential equations or integral equations. Both representations are considered as equivalent.
- The Lagrange ("force") and the Hamiltonian ("energy") formalisms are considered as equivalent. The mathematical proof is based on the Legendre transform, i.e. the equivalence is only valid if the assumptions of the Legendre transform are fulfilled.

In both cases, corresponding (mathematical) regularity assumptions are required to enable those propositions. A restriction of the domain regularity of the considered operators leads to no longer well-defined classical differential equations resp. to no longer valid Lagrange formalism. In other words, the provided consistent model in the distributional framework represents the mathematical/transcendental view of the considered physical world, while the corresponding classical solutions of the several differential equations are mathematical approximations to those physical models. This concept also overcomes the "physical interpretation" challenge of the "Neumann PDE" representation of the pressure p in the NSE model.

As a consequence there is only a "one-energy" (field) concept and corresponding (PDE specific) manifestations/ forms of considered "Nature forces".

The proposed variation Hilbert space frame is built on the space-time frame with dimension $n = m + 1 = 4$. Therefore the Huygens' Principle (which is also valid for the initial value problem of the wave equation) is valid for all considered "wave" PDE, overcoming e.g. the $n > 10$ requirement of the string theory. At the same time, the characteristics roles of a space-time dimension = 4 is also underlined by the specific role of undistorted spherical travelling waves (Courant-Hilbert, "*methods of mathematical physics*", II, VI, §10.3).

Schrödinger's "purely quantum wave" vision is about half-odd integers, rather than integers to be applied to wave-mechanical vibrations which correspond to the motion of particles of a gas resp. the eigenvalues and eigen-functions of the harmonic quantum oscillator still governed by the Heisenberg uncertainty inequality. The alternatively proposed $H_{1/2}$ energy space enables Schrödinger's vision ([ScE] (7.23) ff):

let ω denotes the angular frequency, h the (h-bar) Planck constant and

$$e := \frac{\omega h}{2}$$

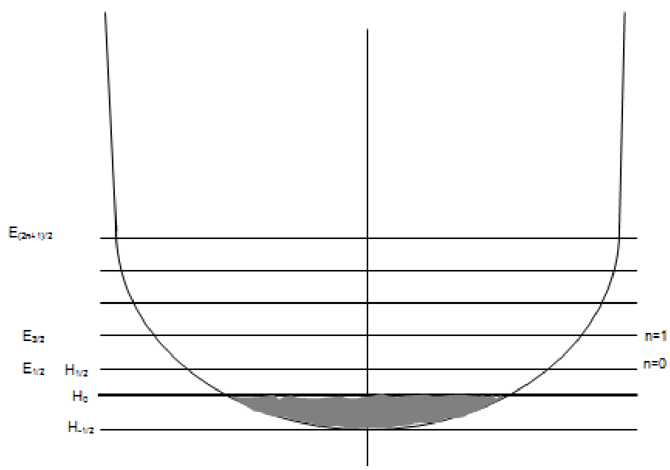
Then Schrödinger's "half-odd integer vision" is about the following replacement:

$n = 0$	$E_0 = e$	$\rightarrow$	$E_{1/2} = 1 * e$
$n > 0$	$E_1 = 1 * \omega h$	$\rightarrow$	$E_{3/2} = 2 * e$
	$E_2 = 2 * \omega h$	$\rightarrow$	$E_{5/2} = 3 * e$
...			
	$E_n = n * \omega h$	$\rightarrow$	$E_{(2n+1)/2} = (n+1) * e,\ n = 0,1,2,...$

...

As a consequence the corresponding eigenvalue and eigenfunction solutions of the number operator (i.e. the product of generation and annihilation operators) start with index $n = 1$, not already with $n = 0$.

With respect to the ladder operators of the harmonic quantum 0scillator the proposed alternative quantum state and related energy Hilbert scales can be visualized by



(Hilbert scale: $H_{-\frac{1}{2}}, \boldsymbol{H_0}, H_{\frac{1}{2}}$)

Corresponding Hermite polynomials being linked to Weber's (Whittaker's) parabolic cylindrical polynomials by ([AbM] (13.1.32) [BuH] p. 215) are provided in section G.

We note that both, the Hermite polynomial system and the Hilbert transformed Hermite polynomial system build orthogonal systems of the Hilbert space $H_{0,} = L_2$. Therefore, generation and annihilation operators can be defined accordingly, whereby corresponding Hermite polynomials (original and Hilbert transform) are orthogonal with respect to the $H_{0,} = L_2$ inner product, as a consequence of the corresponding Hilbert transform property.

C. 4 The “Higgs field” and “Kant’s ether field”

The gauge invariance is the main principle in current SMEP theory.

[BlD] 10.3: “*It is fine that the gauge field of electromagnetism has zero mass because there the force is mediated by photons, which are massless. However, Yang-Mills type forces must arise from the exchange of massive particles because of the observed short range of these forces. The Higgs mechanism helps in two ways. First, gauge fields can acquire mass by the symmetry breaking. Second, the undesirable Goldstone bosons (which arise in the symmetry-breaking process) can be usually gauged away.”*

The Higgs effect (or mechanism) builds on an extended from global to local $U(1)$ transformations symmetry group of the underlying Lagrangian. It explains the mass of the gauge W- and Z-(weak interaction) bosons of the weak “nuclear-force”.

[HiP]: “*Within the framework of quantum field theory a “spontaneous” breakdown of symmetry occurs if a Lagrangian, fully invariant under the internal symmetry group, has such a structure that physical vacuum is a member of a set of (physically equivalent) states which transform according to a nontrivial representation of the group. This degeneracy of the vacuum permits non-trivial multiplets of scalar fields to have nonzero vacuum expectation values (or “vacuons”), whose appearance leads to symmetry-breaking terms in propagators and vertices. … When the symmetry group of the Lagrangian is extended from global to local transformations by introduction of coupling with a vector gauge field the original scalar massless boson as a result of spontaneous breakdown of symmetry then becomes the longitudinal state of a massive vector (Higgs) boson whose transverse state sare the quanta of the transverse gauge field. A perturbative treatment of the model is developed in which the major features of these phenomena are present in zero order*.”

The Higgs boson is supposed to be a heavy elementary particle (with non-zero rest mass of about 125 GeV with spin 0). The Higgs field is supposed to fill the whole universe interacting with each particle, which “moves” through it by a kind of frictional resistance, i.e. which has kinetic energy. Therefore, the Higgs effect (i.e. generating mass particles) requires a Higgs field with not vanishing amplitudes in the ground state. The corresponding Lagrange density is described by the (Lorentz-invariant) Klein-Gordon equation (i.e. it is about a relativistic theory) equipped with the Higgs potential, which is a “special” “Mexican hat” potential in the form

$$V(\varphi) \coloneqq \mu^2 \varphi\varphi^* + \beta(\varphi\varphi^*)^2.$$

The corresponding local gauge invariant quantum field theory is about the related Lagrange density, which is invariant under a local $U(1)$ gauge transformation, i.e. there is an infinitesimal symmetry transformation of matter & gauge fields (fermions & bosons).
The Higgs effect overcomes the issue of the weak nuclear power SM model that there can be existed no particles with mass w/o violating the gauge invariance of the describing Lagrange density.

The current understanding of all known "particles" in the universe is, that there is a split into two groups of those "particles" to overcome the contact body (body-force interaction) problem

1. spin(1/2)-"matter"-"particles", which are "objects" with a spin(1/2), i.e. those "objects" *look the same* only after the second rotation

2. spin(0,1,2)-"force"-"particles", which are "objects" with spins 0,1,2, interacting with spin(1/2)-"matter" "objects".

The first group goes back to Dirac, who introduced this *purely mathematical* concept to "explain" why spin(1/2)-"matter"-"particles", especially the electrons, can exist as "separate" "objects", while not merging to one big "soup" ("object"?). Dirac's theory enables consistency of the quantum mechanics and the special relativity theory.

In order to avoid the same ("soup" disaster) effect Pauli postulated his exclusion principle in order to ensure that spin(1/2)-"matter"-"particles" under the influence of spin(0,1,2)-"force"-"particles" do not collapse to a state of extremely high density.

We mention the following open challenges:

- *the Higgs effect still leads to oppositions for energies not in the range of* $7GeV < m_H < 1TeV$
- *the gauge invariance principle based on the Lagrange formalism leads to bosons with mass to generate fermions, while the original concept of fermions and bosons was to distinguish between particle with mass (fermions) and mass-less "force" interaction particles between them (bosons)*
- *a "gravitation" "boson" is not possible the be adapted to the SMEP, while with the Higgs boson there is now a "particle" included, which "acts" in the whole universe, especially in the vacuum (which fills most of the space, in the cosmology world, but also in the quantum world), but only affecting weak nuclear-forces and its related W- and Z-bosons*
- *generally, the Higgs boson combines the existence of mass together with the action of the weak force. But why it provides it especially to the quarks that much mass, is still a mystery.*

The alternative approach of this paper is about a replacement of the gauge invariance principle per defined (problem specific) Langrage density by a "rotation invariance" principle per defined Hilbert space framework (which is common for all physical problem specific Langrage density representations). The beauty of the gauge methodology to gauge away undesirable (e.g. Goldstone) bosons has its counterpart in the Einstein field equation by the covariant derivative concept going in line with *differentiable* manifolds (whereby the differentiability requirement is w/o any physical meaning). The corresponding concept, which replaces the inertial system concept of the Newton gravity model, is an infinitesimal displacement (tensor) fields A_j^k with $dA^k - \delta A^k = A_j^k dx^j$. Our alternative approach avoids the above purely mathematical model driven concepts, which are w/o any physical meaning.

As a consequence, the proposed perturbative treatment of the Higgs model ([HiP]) is replaced by a corresponding approximation analysis of compactly embedded closed spaces in the considered Hilbert space framework (see e.g. ([AzA], [CoD]).

The mass of a proton consists nearly exclusively of the energy of the gluons, which are the interconnection particles, which hold together the quark. In this sense, mass is essentially the manifestation of the vacuum energy (which could be interpreted as a projection operator from a "vacuum" field into a "matter field). The "Higgs" field, interpreted as a kind of ether which exists in the whole universe, goes along with "Kant's conception of Ether as a field" ([WoW]), and his ontology of space, which is anti-Newtonian in the sense that space is not an object, but the form of representing a physical object. As there is only about 10% of matter of the universe "covered" by (obviously inconsistent) mathematical physics models (e.g. quantum and gravitation field theory) while about 90% of the remaining matter is unknown "dark" matter with not-known form and substance "embedded (?)" in a "universe space", which mainly "contains" "vacuum" besides the above 100% matter it is might be worth to review Kant's dynamic view of space ([FöE]).

With respect to the alternative proposed "quantum state" Hilbert space model of this paper (where the "Higgs" particles might be just another name of those quantum particles) we note that it enables

- a projection operator $P: H_{-\alpha} \to H_0$ to model "matter creation" ("*mass being considered as condensed energy*") from purely mass-less generalized "wave/energy" quantum states out of the complementary closed space $H_{-\alpha} - H_0$
- the Hamiltonian formalism based on the variational "energy" representation of appropriate PDO equations with reduced regularity requirements compared to the Lagrange formalism; in those cases the Legendre ("contact" (!)) transformation cannot be applied to prove the equivalence of both formalisms
- an infinitesimal *rotation invariant* symmetry (invariance), based on a variational (weak) Hamiltonian formalism, i.e. any "renormalization" requirements based on local, (only) affine infinitesimal transformations are not required anymore.

D. A combined L_2- based Fourier wave and $(H_{-1/2}-H_0)-$ based Calderón wavelet (Non-standard MEP) analysis tool and the Landau damping

The purpose of the section is threefold:

1. leverage on variation space-time integrator concept for Maxwell equations
2. suggest a combined (wave (L_2) & wavelet ($L_2^{\neg}$)) analysis technique for the "fluid" Hilbert space $H_{-1/2}=L_2 x L_2^{\neg}$ for a Landau equation based proof of the (non-linear) Landau damping effect, anticipating the physical explanation of the by experience verified phenomenon
3. enable a proof of the Landau damping phenomenon, which is about verifying of a quantum physical (plasma fluid) effect, as described by the Landau equation, alternatively to the proof of [MoC], [ViC] applying analytical norms based on the classical PD Vlasov equation with its incorporated analytical distribution functions(*). The Landau damping phenomenon is about "*wave damping w/o energy dissipation by collisions in plasma*", because electrons are faster or slower than the wave and a Maxwellian distribution has a higher number of slower than faster electrons as the wave. As a consequence, there are more particles taking energy from the wave than vice versa, while the wave is damped ([BiJ]).

*) *The proof of the proposition that* "the Euler constant is an irrational number with a probability of 100%" *is straightforward in an analytical (distribution function) Banach space framework, as the set of rational numbers is a zero (sub) set of the field of real numbers with respect to the L_2 (Lebesgue) inner product.*

A L_2- based Fourier wave analysis is the baseline for statistical analysis, as well as for PDE and PDO theory. There are at least two approaches to wavelet analysis, both are addressing the somehow contradiction by itself, that a function over the one-dimensional space R can be unfolded into a function over the two-dimensional half-plane (see appendix). The Fourier transform of a wavelet transformed function f is given by ([LoA], [MeY]):

$$\widehat{W_\vartheta[f]}(a,\omega):=(2\pi|a|)^{\frac{1}{2}}c_\vartheta^{-\frac{1}{2}}\hat{\vartheta}(-a\omega)\hat{f}(\omega) \quad .$$

For $\varphi,\vartheta\in L_2(R)$, $f_1,f_2\in L_2(R)$,

$$0<\left|c_{\vartheta\varphi}\right|:=2\pi\left|\int_R \frac{\hat{\vartheta}(\omega)\overline{\hat{\varphi}}(\omega)}{|\omega|}d\omega\right|<\infty$$

and $\left|c_{\vartheta\varphi}\right|\le c_\vartheta c_\varphi$ one gets the duality relationship ([LoA])

$$(W_\vartheta\ f_1,W_\varphi^* f_2)_{L_2(R^2,\frac{dadb}{a^2})}=c_{\vartheta\varphi}(f_1,f_2)_{L_2}$$

i.e.

$$W_\varphi^* W_\vartheta\ [f]=c_{\vartheta\varphi}f \quad \text{in a } L_2-\text{sense.}$$

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$$(W_\vartheta\ f_1,W_\varphi^* f_2)_{L_2(R^2,\frac{dadb}{a^2})}=c_{\vartheta\varphi}(f_1,f_2)_{L_2}$$

i.e.

$$W_\varphi^* W_\vartheta\ [f]=c_{\vartheta\varphi}f \quad \text{in a } L_2-\text{sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "*microscope observations*" of *two* wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula. The prize to be paid is about additional efforts, when re-building the reconstruction wavelet. We further note that for a convenient choice of the two wavelet functions the Gibbs phenomenon disappears ([HoM] 2.7).

We note the Gaussian function related "Mexican hat" (wavelet) function

$$g(x) := -\frac{d^2}{dx^2}(e^{-\frac{x^2}{2}}) = (1-x^2)e^{-\frac{x^2}{2}}.$$

being successfully applied e.g. in wavelet theory (see also below section D), as well as the Poisson wavelet ([HoM], example 7.0.2). We further mention that the Hilbert transform of a wavelet is again a wavelet.

The proposed alternative quantum state Hilbert space $H_{-1/2}$ provides an alternative concept to the "Dirac function" calculus. This overcomes current handicaps concerning the regularity of the Dirac function, which depends from the space dimension, i.e. $\delta \in H_{-s}(R^n)$ for $s > n/2$.

The alternatively proposed Hilbert space $H_{-1/2}$ provides a truly "microscopic" mathematical frame (independently from the space dimension), while still supporting the existing physical observation (statistical analysis) subspace. It is also proposed to replace the (continuous & differentiable) manifold concept (and exterior products of differential forms) in Einstein's field theory.

The extended admissibility condition above indicates that wavelet "pairs" in the form $(\varphi, \vartheta) \in L_2 x H_{-1} \cong H_{-1/2} x H_{-1/2}$ would be an appropriate good baseline to start from, when analyzing in the Hilbert space frame $H_{-1/2} = L_2 x L_2^{\neg}$, , resp. $L_2^{\neg}$, where $L_2^{\neg}$ denote the complementary space of L_2 with respect to the $H_{-1/2}$ −norm, while still analyzing the "observation measurement" Hilbert space L_2 by Fourier waves.

In line with the proposed distributional $H_{-1/2}$ −Hilbert space concept of this paper, we suggest to define "continuous entropy" in a weak $H_{-1/2}$ − frame in the form

$$h(X) := (f, \log\frac{1}{f})_{-1/2} \ ,$$

where X denotes a continuous random variable with density $f(x)$. In this case it can be derived from a Shannon (discrete) entropy in the limit of n, the number of symbols in distribution $P(x)$ of a discrete random variable X ([MaC]):

$$H(X) := \sum_i P(x_i) \log(\frac{1}{P(x_i)}) \ .$$

This distribution $P(x)$ can be derived from a set of axioms. This is not the case, in case of the standard entropy (which cannot be derived from dynamic laws (!), anyway, [PeR]) in the form

$$h(X) := (f, \log\frac{1}{f})_0 .$$

Plasma is the fourth state of matter, where from general relativity and quantum theory it is known that all of them are fakes resp. interim specific mathematical model items. Plasma is an ionized gas consisting of approximately equal numbers of positively charged ions and negatively charged electrons. One of the key differentiator to neutral gas is the fact that its electrically charged particles are strongly influenced by electric and magnetic fields, while neutral gas is not. As a consequence the quantitative fluid/gas behavior as it is described by the Euler or the Navier-Stokes equations cannot be applied as adequate mathematical model. Even it would be possible there is no linkage to the quantitative fluid/gas/plasma behavior and its corresponding turbulence behavior as it is described by the Euler or the Navier-Stokes equations. The approach in statistical turbulence is about low- and high-pass filtering Fourier coefficients analysis which is about a "local Fourier spectrum" analysis. As pointed out in [FaM] this is a contradiction in itself, as, either it is non-Fourier, or it is nonlocal. The proposal in [BrK7] is about a combination of the wavelet based solution concept of [FaM], [FaM1], with a revisited CLM equation model in a physical $\mathrm{H}_{-1/2}$ Hilbert space framework.

In fluid description of plasmas (MHD) one does not consider velocity distributions. It is about number density, flow velocity and pressure. This is about moment or fluid equations (as NSE and Boltzmann/landau equations), which got new opportunities with respect to still open problems regarding the proposed concepts of this paper ($H_{-1/2}$ −based variational theory & Maxwell/Helmholtz/Sommerfeld/Plemelj PDO equations):

The 2nd topic above enables a turbulent $\mathrm{H}_{-1/2}$ signal which can be split into two components: coherent bursts and incoherent noise. Additionally the model enables a localized Heisenberg uncertainty inequality in the closed ("noise"/"wave packages") subspace $\mathrm{H}_{-1/2}-\mathrm{H}_0$, while the momentum-location commutator vanishes in the (coherent bursts) test space H_0. As a first trial we propose the Morlet wavelet, which is a sin wave that is windowed (i.e. multiplied point by point) by a Gaussian, having a mean value of zero.

The proposed 3D-Navier-Stokes equations (NSE) solution of this paper is based on a distributional Hilbert space concept to derive appropriate energy norm estimates of the non-linear, non-stationary 3-D-Navier-Stokes equations. In [BrK3] we propose problem adequate norms to derive corresponding adequate a priori estimates for the transport equation, which are in line with the results of [LiP], [LiP1].

With respect to the previous sections we note that the Hilbert transform operator (which is valid for every Hilbert scale) is the "natural" partner to the wavelet-transform operator, as it is skew-symmetric, rotation invariant and each Hilbert transformed "function" has vanishing constant Fourier term. The example in the context above is the Hilbert transform of the Gaussian distribution function, the (odd) Dawson function, with the "polynomial degree" point of zero at +/- infinite.

With respect to the variational space-time integrators for Maxwell's equations we refer to [StA]: The concept combines spatial and time discretization in developing *geometric numerical integrators*. The approach preserve, by construction, various geometric properties and invariants of the continuous physical systems that they approximate. With respect to "*the nonlinear Landau damping for general interaction*" we note that the *variational space-time integrators* concept treats electromagnetic Lagrange density as a discrete differential 4-form in space-time, while in [MoC], [ViC] the analysis is concerned

1. with Fourier analysis with respect to the space dimension in combination with C^0/L_∞ −estimates with respect to the time dimension. It ends up in "analytical norm" estimates requiring mathematical regularity assumptions, which are not appropriate for the underlying physical (plasma) models (e.g. KAM theory).

2. with different regularity requirements concerning the most critical underlying interaction potential W ($W \in H_{-1/2-\varepsilon}$, $W \in H_{-1}$) of the Vlasov-equation.

The Boltzmann equation is a (non-linear) integro-differential equation which forms the basis for the kinetic theory of gases. This not only covers classical gases, but also electron /neutron /photon transport in solids & plasmas / in nuclear reactors / in super-fluids and radiative transfer in planetary and stellar atmospheres. The Boltzmann equation is derived from the Liouville equation for a gas of rigid spheres, without the assumption of "molecular chaos"; the basic properties of the Boltzmann equation are then expounded and the idea of model equations introduced. Related equations are e.g. the Boltzmann equations for polyatomic gases, mixtures, neutrons, radiative transfer as well as the Fokker-Planck and Vlasov equations. The treatment of corresponding boundary conditions leads to the discussion of the phenomena of gas-surface interactions and the related role played by proof of the Boltzmann H-theorem.

The Landau equation (a model describing time evolution of the distribution function of plasma consisting of charged particles with long-range interaction) is about the Boltzmann equation with a corresponding Boltzmann collision operator where almost all collisions are grazing (based on a Fourier multiplier representation ([LiP1]) in line with Oseen kernels, Laplace and Fourier analysis techniques [LeN] and scattering problem analysis techniques based on Garding type (energy norm) inequalities (like the Korn inequality). *Its solutions enjoy a rather striking compactness property.*

The collision operator is given by

$$Q(f,f)=\frac{\partial}{\partial v_i}\left\{\int_{R^N}a_{ij}(v-w)\left[f(w)\frac{\partial f(v)}{\partial v_j}-f(v)\frac{\partial f(w)}{\partial w_j}\right]dw\right\}$$

with

$$a_{ij}(z)=\frac{a(z)}{|z|}\left\{\delta_{ij}-\frac{z_i z_j}{|z|^2}\right\}=\frac{a(z)}{|z|}P(z):=\frac{1-[1-a(z)]}{|z|}[Id-Q](z)\quad Q(z):=(R_iR_j)_{1\le i,j\le N}$$

and $a(z)$ symmetric, non-negative and even in z and R_i denote the Riesz operators.

A variation theory approach in appropriately defined Hilbert spaces is proposed to prove the Landau damping based on the Landau equation. A weak variational representation (which is anyway the proper framework for quantum theory) would overcome Vlasov's mathematical argument against the Landau equation that "this model of pair collisions is formally (!) not applicable to Coulomb interaction due to the divergence of the kinetic terms". At the same point in time the "analytical norm" based proof of the Landau damping ([MoC]) is based on the Vlasov equation, which does not reflect the underlying physical explanation of this effect. This is about "wave damping w/o energy dissipation by collisions in plasma", because electrons are faster or slower than the wave and a Maxwellian distribution has a higher number of slower than faster electrons as the wave. As a consequence, there are more particles taking energy from the wave than vice versa, while the wave is damped ([BiJ]). Technically, Villani's proof is based on the Vlasov's formula for the plasma dielectric for the longitudinal oscillators based on the integral ([ShF] p. 392)

$$W\left(\frac{\omega}{k}\right)=-\int_{-\infty}^{\infty}\frac{F_0{}'(v)dv}{\frac{\omega}{k}-v}\quad .$$

This model overlooks the important physical phenomenon of electrons travelling with exactly the same material speed $\frac{\omega}{k}$ and the wave speed v. This is about the pole of the above integral, when the path of integration lies on the x-axis. Mathematically, this issue can be addressed by the "principle-value integral", which still would neglect the underlying physical interpretation issue. In ([ShF] p. 395&) the correct definition for the correct classical definition is given (as provided by Landau), which is basically a threefold integral definition depending from the value ω_I (<0,=0,>0), the imaginary part of $\omega = \omega_R + i\omega_I$:

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{\prime}(v)dv}{\frac{\omega}{k}-v} \qquad \text{for } \omega_I < 0$$

$$W\left(\frac{\omega}{k}\right) = -p.v.\int_{-\infty}^{\infty} \frac{F_0{\prime}(v)dv}{\frac{\omega}{k}-v} - \pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I = 0$$

$$W\left(\frac{\omega}{k}\right) = -\int_{-\infty}^{\infty} \frac{F_0{\prime}(v)dv}{\frac{\omega}{k}-v} - 2\pi i\, F_0{}'\left(\frac{\omega}{k}\right) sgn(k) \qquad \text{for } \omega_I > 0$$

If ω_I were to continue and become positive (damped disturbance), then analytical continuation yields, in addition to the integral along the real line (which also presents no difficulty of interpretation), a full residue contribution.

Needless to state that in the proposed (distributional) Hilbert space framework the Hilbert transform based interpretation becomes now valid also from a physical interpretation perspective, in line with our proposed NMEP and the underlying alternative Schrödinger momentum operator given by

$$\mathrm{u(x)} \rightarrow P^*[u](\mathrm{x}) \coloneqq -\mathrm{i}\frac{\mathrm{d}}{\mathrm{dx}}\mathrm{H[u]}(x) = -\mathrm{iH}[u_x](x)$$

with domain $H_{1/2} = H_0 + H_0^{\neg}$. The corresponding formulation with respect to Plemelj's mass element concept results into a (weak variational) representation with respect to the $H_{-1/2}-$ inner product of the Stietjes integral operator

$$W_{F_0}\left(\frac{\omega}{k}\right) = -p.v.\int_{-\infty}^{\infty} \frac{dF_0(v)}{\frac{\omega}{k}-v} \quad .$$

For a variational formulation of the plasma physics model we propose to consider the Landau equation starting with a "model problem" collision operator given by

$$\tilde{Q}(f,f) = \frac{\partial}{\partial v_i}\left\{\int_{R^N} a_{ij}(v-w)\left[f(w)\frac{\partial f(v)}{\partial v_j} - f(v)\frac{\partial f(w)}{\partial w_j}\right]dw\right\}$$

with

$$(*) \quad a_{ij}(z) = \frac{1}{|z|}\left\{\delta_{ij} - \frac{z_i z_j}{|z|^2}\right\} =\coloneqq \frac{1}{|z|}[Id - Q](z) \quad Q(z) \coloneqq (R_i R_j)_{1\le i,j\le N}$$

(where the $\mathrm{R_i}$ denote the Riesz operators) in combination of the Riesz operators based alternative Schrödinger operator, building on the Calderón-Zygmund operator.

Obviously the operator $\tilde{Q}$ enables an inner product definition related to the $H_{-1/2}-$ inner product in the form

$$(\tilde{Q}v, w)_0 \cong (v, w)_{-1/2} \quad .$$

The remaining part to the collision operator Q given by

$$\frac{[1-a(z)]}{|z|}[Id - Q](z) \quad .$$

It can be interpreted as a compact disturbance, which is governed by the Garding type inequality (as e.g. also Korn's second inequality) following standard functional analysis techniques (e.g. [AzA], [BrK8], [LeN]). For corresponding (complementary) variational methods (e.g. the method of Noble) we refer to [ArA], [VeW], but lso to [BrA]. Consequently, the analysis of the related Cauchy problem follows the same approach as for the NSE (section B) and the Helmholtz equations (section C).

E. A (truly infinitesimal geometry) Hilbert space based quantum gravity theory

The relativistic cosmology is based on the three assumptions, the cosmological principle, the Weyl postulate and the GRT. The GRT is based on the Riemann geometry with its underlying mathematical axiom, that the *Pythagoras theorem* is valid only for the case *when two points are infinitely near*.

We note that the Legendre (contact body) transform, which is applied to prove the equivalent of the Lagrange and Hamiltonian formalism, applied to differentials, is neglecting the $dxdy$ term in

$$d(x+y)=dx+dy \quad , \quad d(xy)=xdy+ydx+\boldsymbol{dxdy}=xdy+ydx$$

which is kind of contradiction to the Pythagoras axiom above.

The framework of the GRT is the affine connected manifolds ([WeH] p. 123). The metric character of a manifold is characterized relatively to a system of reference (= (1) *co-ordinate system* + (2) *calibration*) by two fundamental forms, namely a (1) quadratic differential form and a (2) linear one. They remain invariant during transformations to new co-ordinate systems.

The Weyl curvature tensor is a measure of a pseudo-Riemann manifold. It expresses the "tidal force" that a body feels when moving along a geodesic. It conveys the information how the shape of the body is distorted by this "tidal force". The Ricci curvature tensor, which expresses the trace components of the Riemann curvature tensor, contains the information about how the volumes change in the presence of "tidal forces", so the Weyl tensor is traceless component of the Riemann tensor. It is a tensor that has the same symmetries as the Riemann tensor with the extra condition that it be trace-free. The Weyl curvature is the only component of curvature for Ricci-flat manifolds and always governs the characteristics of the field equations of an Einstein manifold.

[WeH] p. 91: *The transition from Euclidean geometry to that of Riemann is founded in principle on the same idea as that which led from physics based on action at a distance to physics based on infinitely near action The Ohm Law find by the observation, that the current flowing along a conducting wire is proportional to the difference of potential between the ends of the wire. Also the Coulomb Law deals with "actions at a distance". In order to model the physical model in its most general form, one accordingly deduces this law by reducing the measurements obtained to an infinitely small portion of wire. This results in the expression*

$$curl\vec{E}=0 \quad , \quad div\vec{E}=\rho$$

on which Maxwell's theory is founded. Proceeding in the reverse direction, one derives from this differential law by mathematical processes the integral law, which we observed directly, on the supposition that conditions are everywhere similar *(homogeneity). … The fundamental fact of Euclidean geometry is that the square of the distance between two points is a quadratic form of the relative co-ordinates of the two points (Pythagoras Theorem).*

We propose a replacement of the "Pythagoras theorem" in the infinitesimal small by the concept of "rotating differentials", leveraging on the Riesz operators property to be rotation invariant. This goes along with reduced regularity requirements to the corresponding operator domain. It also anticipate Leibniz' living force concept (see below).

At the same time it enables a replacement of the affine connected manifolds concept and its underlying invariant *exterior* (covariant) derivative concept (of p-forms) (e.g. enabling Hodge's potential theory of closed Riemann manifolds based on differential forms w/o vector fields) by a distributional Hilbert space concept with a corresponding *inner* (differential) derivative product (defining a corresponding (norm-) metric). The p-forms representations of the Riemann curvature tensor are e.g. given in ([FlH]). The Riemann geometry requires *differentiable* manifold w/o any physical meaning. The alternative (distributional) Hilbert space framework avoids this purely mathematical requirement, enabling also an alternative "orthogonality concept" as being applied in the Weyl postulate, where the world lines of the fluid particles, which act as the source of the gravitational field and which are often taken to model galaxies, should be *hypersurface orthogonal.*

In the context of the newly proposed "energy-space" $H_{1/2} = H_1 + H_1^{\neg}$ (where $H_1^{\neg}$ represents the vacuum energy space) we note that already for the vacuum field equations ($R_{\alpha,\beta} = 0$) there are two solutions, the Minkowski and the Schwarzschild metrics. Therefore those metrics are not compatible with the uniquely defined Hilbert space metric/norm. Gödel's example of a new type of cosmological solutions with non-vanishing density of matter (and with a cosmological term $\neq 0$) of Einstein's field equations provides a system with a rotation of matter relatively to the compass of inertia. This solution, or rather the properties of the four-dimensional space it defines are also provided ([GöK]).

The complementary variational analysis is proposed to characterize the solutions of the quantum gravitational field equations ([ArA]). The method of Noble is based on a system of two operator equations, which is analogue to the Euler differential equations, covering not only (non-linear) partial differential equations, but also integral equations. It is basically about a characterization of the solution as a saddle point of a minimization functional based on a "Hamiltonian" function $W(w', u)$, which is convex with respect to w', and concave with respect to u ([VeW] 6.2.4).

In continuum mechanics, the infinitesimal strain theory is about the deformation of a solid body. The displacement gradient is a 2nd order tensor, where is it possible to perform a geometric linearization of any one of the (infinitely many possible) strain tensors, e.g. the (Lagrange) strain tensor. Considering the linearized strain tensor as the "primary" unknown, instead of the displacement in the pure traction problem of three-dimensional linearized elasticity leads to a well-posed minimization problem, constrained by a weak form of the St Venant compatibility conditions. This approach also provides a new proof of Korn's inequality ([CiP]).

Regarding the NSE and YME the "infinitesimal small" of "fluids" and "quanta" is and will be all the time out of scope for any human observations. Mathematics is a purely descriptive science with well-established concepts to deal with any kind and "size" of "infinity" (e.g. Cauchy, Dedekind, Bolzano, Weierstrass, Kronecker, Cantor, Gödel, Brouwer). The mathematical tool managing physical "observations" are Partial Differential Equations (PDE), mathematical statistics and approximation theory. Those concepts are also applied in quantum mechanics and quantum field theory. The essential mathematical "objects" are the real numbers (while "nearly all" of those objects are far away from being "real") and the Lebesgue integral building the Lebesgue (Hilbert) space L_2, where all rational numbers build a null set measured by its corresponding norm.

F. A "helicopter view" on the unknown mathematical model of a "big bang" conform initial or radiation PDO equation model

Regarding the "big bang" singularity for $t \to 0$ and the newly proposed Hilbert space framework $\mathrm{H}_{-1/2}$ we note that already the solution of the initial value problem of the (simplest, purely parabolic, linear) heat equation diverges for $t \to 0$ in case the initial value function is measurable, i.e. $u_0 \in \mathrm{H}_0 = L_2$ ($\|u\|_k \leq ct^{-k/2}\|u_0\|$), while with reduced regularity assumption ($u_0 \in \mathrm{H}_{-1/2}$) it holds $\|u\|_k \leq c\|u_0\|_{-1/2}$. The corresponding situation is valid for the non-stationary Navier-Stokes equations and the (free boundary) Stefan problem.

Beside the (well established) theory of initial value (parabolic) PDE problems there is the still open "Courant-Hilbert" conjecture concerning the most simple (hyperbolic, linear) initial value wave equation. The Huygens principle is valid under the same prerequisites for the initial value problem, as well as for the corresponding radiation problem ([CoR] VI, §9.1, §10.3). The "Courant-Hilbert conjecture" is about the reverse, i.e. distortion-free, progressive, spherical waves do only exist, if the Huygens principle is valid. In combination with the Hadamard conjecture ([CoR] §9.1) this would characterize the 4-dimensional space-time continuum and its related Maxwell theory.

Having in mind that already each irrational number is its own mathematical universe (i.e. it is defined by a sequence of an infinite numbers of rational numbers) the solution of the problem above seems to be a necessarily question to be answered, before a mathematical model of the "big bang" "situation" can be formulated.

In [PeR] a revisited entropy concept is proposed leading to a proposed conformal cycle cosmology, which is basically a periodical solution of a corresponding mathematical model, *that the universe as a whole is to be seen as an extended conformal manifold consisting of a (possible infinite) succession of aeons, and each appearing to be an entire expanding universe history* ([PeR]p. 147):

"*According to current particle-physics about how to masses of basic particles actually come about, a particle's rest-mass ought to arise through the agency of a special particle (or perhaps a family of such special particles) referred to as a Higgs boson(s). … In the very early universe, when the temperature (=energy form !!) was so high as to have provided energies greatly in excess of this Higgs value, all particles would then, according to standard ideas, indeed become effectively massless, like a photon*."

The physical field, of which photons provide the quantum constituents, is the Maxwell electromagnetic field. It is completely conformal invariant.

In [FrH] the "hyperboloidal initial value" problem for Einstein's conformal vacuum field equations (reduced to Cauchy problems for first order quasilinear symmetric hyperbolic systems) is considered. For initial data of the class $\mathrm{H}_s, s \geq 4$, there is a unique development which is a solution of the conformal vacuum field equations of class $\mathrm{H}_s, s \geq 4$. This provides a solution of Einstein's vacuum field equations which has a smooth structure at past null infinity (note that according to the Sobolev embedding theorem it holds that the Hilbert space H_s *is a subset of* C^0 *for* $s > n/2$. The regularity of this solution is still a strong PDE solution, i.e. the correspondingly defined weak solution would even allow even in the standard case (w/o the proposed alternative energy space framework) regularity assumption of the class $\mathrm{H}_s, s \geq 2$, which is still more regular than in the above considered parabolic initial value problems (heat equation and Stefan problem, as well as the NSE).

In [TaN] the theory of Cauchy problems for solutions of elliptic equations is provided covering PDO in the space of distributions on closed sets, generalized form of capacity associated with a semi normed space applied to systems of differential equations with injective (surjective) symbols.

The analogue concept of the Riemann curvature tensor in elasticity theory is the fourth-rank (elasticity) tensor, which links the stress tensor to the strain tensor by Hook's law $\sigma_{jk} = c_{jkmn}\varepsilon_{mn}$.

The first boundary value problem of elasticity is given by ([FiG])

$$\text{given } \underline{f} \in \underline{L_2} \text{ find } \underline{u} \in \underline{H_2} \text{ such that}$$

$$-\nabla\sigma(\underline{u}) = \underline{f} \;;\;\; -(\sigma_{ik,k}(\underline{u}) = f_i$$

It holds the shift theorem: for $\underline{f} \in \underline{L_2}$ the solution $\underline{u} \in \underline{H_2}$ exists uniquely and $\|\underline{u}\|_{H_2} \leq c \left\|\underline{f}\right\|_{L_2}$, i.e. the operator $-\nabla(\sigma(\underline{u})$ behaves like the Laplacian operator. The corresponding weak variational representation of the boundary value problem is given by (Nitsche's method for contact problems)

$$a(\underline{u}, \underline{w}) = (\underline{f}, \underline{w}) \;,\;\; \underline{w} \in H_1$$

with

$$a(\underline{v}, \underline{w}) := (\sigma_{ik}(\underline{v}), \varepsilon_{ik}(\underline{w}) + \oint n_i\{\sigma_{ik}(\underline{v})w_k + \sigma_{ik}(\underline{w})v_k\}ds \;.$$

In [ShI] necessary and sufficient solvability conditions of ill-posed non-homogeneous Cauchy problems for PDE (as the Cauchy problem for the Lamé operator) with injective symbol of order ≥ 1 are provided.

In [CiP] an approach to the purely traction problem of the 3-dimensional linearized elasticity problem is provided, whose novelty consists in considering the linearized strain tensor as the "primary" unknown, instead of the displacement itself as is customary. The approach leads to a well-posed minimization problem, constrained by a weak form of the St Venant compatibility conditions. The corresponding complementary extremal problem (as alternative to the minimization problem of the potential energy) is based on the principle of Castigliano leading to the quadratic form $W(\varepsilon) = \frac{1}{2}c_{jkmn}\varepsilon_{jk}\varepsilon_{mn}$ ([VeW] 4.2.6).

With regards to the Maxwell equations we note that the components of the electric and magnetic field forces $\underline{E}$, $\underline{H}$ build the 4-dimensional electromagnetic field force tensor $F_{ik} = (\underline{E}, \underline{H})$. The Maxwell stress tensor is given by

$$\sigma_{ik} = \frac{1}{4\pi}\left\{-E_iE_k - H_iH_k + \frac{1}{2}\delta_{ik}(E^2 + H^2)\right\}.$$

With regards to "initial data for the Cauchy problem in general relativity" we refer to the corresponding lecture notes [PoD]:

Lecture 1: Introduction to Lorentzian geometry and causal theory
Lecture 2: The Einstein equations from a PDE perspective. The constraint equations and the local existence theorem of Choquet-Bruhat
Lecture 3: Solving the constraint equations via conformal method
Lecture 4: Topological censorship from the initial data point of view.

We further note the Weyl tensor representation

$$C_{iklm} = R_{iklm} - \frac{1}{2}R_{il}g_{km} + \frac{1}{2}R_{im}g_{kl} + \frac{1}{2}R_{kl}g_{im} - \frac{1}{2}R_{km}g_{il} + \frac{1}{6}R(g_{il}g_{km} - g_{im}g_{kl}) \;.$$

The reduced Einstein equations representation are given by ([PoD] lecture 2)

$$R^H_{\alpha,\alpha\beta} := -\frac{1}{2}g^{\gamma,\gamma}g_{\alpha\beta,\gamma\delta} + Q(g.\partial g) = 0 \;.$$

G. Some comments regarding related philosophical concepts

We can think (hear and watch) the Yoda quote "may the FORCE be with us" and mathematics can model this FORCE/POWER/ENERGY in a way that all corresponding physical (law) models are consistent; ... the bad (or good?) news is, that's it and that's all! From a philosophical perspective we are back to

- Leibniz's ontology of force (e.g. "Fünf Schriften zur Logik und Metaphysik" and "Monadologie")

  "the primitive active and passive forces, the form and matter are in the monadological view understood as features of the perceptions of the monads … in this way the notion of force, … loses its foundational status: primitive force gets folded into the perceptual life of non-extended perceiving things", Garber's monograph: Leibniz: Body, Substance, Monad, 2009)

- Kant's conception of physical matter and the existence of ether, which fills the whole space and time with its moving forces ([WoW])

  "it is the moving forces of the ether that affect us"; …"There exists a matter, distributed in the whole universe as a continuum, uniformly penetrating all bodies, and filling (all spaces) (thus not subject to displacement). Be it called ether, or caloric, or whatever…", … "space is hypostatically", …" Space which can be sensed (the object of the empirical intuition of space) is the complex of moving forces of matter – without which, space would be no object of possible experience", …"Matter does not consist of simple paets, but each part is, in turn, composite…", Each part of matter is a quantum; i.e. matter does not consist of metaphysically simple parts, and Laplace's talk of material points (which were to be regarded as parts of matter) would, understood literally, contain a contradiction." … "Atomism is a false doctrine of nature", …forces "fill a space (both) extensively and intensively", …

- Schrödinger's "*(my) view of the world*" with respect to "*reasons for abandoning the dualism of thought and existence, or mind and matter*"

  "The objective world has only been constructed at the prize of taking the self, that is, mind, out of it remaking it; mind is not part of it; obviously, therefore, it can neither act on it nor be acted on by any of its parts. If this problem of the action of mind on matter cannot be solved within the framework of our scientific representation of the objective world, where and how can it be solved?" ... "No single man can make a distinction between the realm of his perceptions and the realm of things that cause it, since however detailed the knowledge he may have acquired about the whole world, the story is occurring only once and not twice. The duplication is an allegory suggested mainly by communication with other beings."

  A TENTATIVE ANSWER:'"A single experience that is never to repeat itself is biologically irrelevant. Biologic value lies only in learning the suitable reaction to a situation that offers itself again and again, in many cases periodically, and always requires the same response if the organism is to hold its ground." ... But whenever the situation exhibits a relevant differential - let us say the road is up at the place where we used to cross it, so that we have to make a detour - this differential and our response to it intrude into consciousness, from which, however, they soon fade below the threshold, if the differential becomes a constantly repeating feature. .... Now in those fashion differentials, variants of response, bifurcations, etc., are piled up one upon the other in unsurvey able abundance, but only the most recent ones remain in the domain of consciousness, only those with regard to which the living substance is still in the stage of learning or practicing.

  ... I would summarize my general hypothesis thus: consciousness is associated with the learning of living substance; it's knowing how (Können) is unconscious"

- Heidegger's notion of „mathematical" physics ("Holzwege, die Zeit des Weltbildes" (72) ff: die neuzeitliche Physik heisst mathematische, weil sie … eine ganz bestimmte Mathematik anwendet. … Keineswegs wird das Wesen des Mathematischen durch das Zahlenhafte bestimmt. … Durch sie (math. Physik) und für sie wird in einer betonten Weise etwas als das Schon-Bekannte im vorhinein ausgemacht. Der sich geschlossene Bewegungszusammenhang raum-zeitlich bezogener Massenpunkte. … Kein Zeitpunkt hat vor einem anderen einen Vorzug. Jede Kraft bestimmt sich nach dem, … was sie an Bewegung ..in der Zeiteinheit zur Folge hat.

Appendix

Some formulas

Let H and M denote the Hilbert and the Mellin transform operators. For the Gaussian function f(x) it holds

$$M[f](s)=\frac{1}{2}\pi^{-s/2}\Gamma(\frac{s}{2}) \quad , \quad M[-xf'(x)](s)=\frac{s}{2}\pi^{-s/2}\Gamma(\frac{s}{2})=\frac{1}{2}\pi^{-s/2}\Pi(\frac{s}{2})$$

The corresponding entire Zeta function is given by

$$\xi(s):=\frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s)=(1-s)\cdot\zeta(s)M[-xf'(x)](s)=\xi(1-s).$$

The central idea is to replace

$$M[xf'(x)](s)\to M[f_H(x)](s)$$

by

$$M[f_H(x)](s)=2\pi\cdot M\left[x\cdot{}_1F_1(1,\frac{3}{2},-\pi x^2)\right](s)=\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s)$$

This enables the definition of an alternative entire Zeta function in the form

$$\xi^*(s):=(1-s)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s)\cdot\zeta(s)$$

with same zeros as $\xi(s)$.

We briefly sketch the relationship of Plemelj's alternative potential definition, the Hilbert transform and the Laplacian equation, which may be solved using simple layer potential:

For the fundamental solution of the Laplacian

$$\gamma(s-t):=-\frac{1}{\pi}log|\theta(s)-\theta(t)|$$

the potential

$$v(s)=\int\gamma(s-t)u(t)dt$$

is replaced by

$$v(s)=\int\gamma(s-t)du(t).$$

By partial integration one gets

$$v(s)=-\int\partial_t\gamma(s-t)u(t)\,dt$$

For smooth boundaries one gets

$$\partial_t\gamma(s-t)=\frac{1}{2\pi}cot\frac{s-t}{2}+g(s,t)$$

where $g(s,t)$ is correspondingly regular.

The half-odd integers are related to the Fourier coefficients of the convolution (integral) equation

$$\mathrm{Gu} = \mathrm{f}$$

where

$$\mathrm{G[u](y)} := \int_{-\infty}^{\infty} \mathrm{g(y-x)u(x)dx} \ , \quad \mathrm{g(x)} := \frac{1}{\cosh(\mathrm{x})}$$

with the secans hyperbolicus function ([GrI], 1.232, 1.411)

$$\mathrm{g(x)} := \mathrm{sech(x)} := \frac{1}{\cosh(\mathrm{x})} = \mathrm{cn(x;1)} = \mathrm{dn(x;1)} = 2\sum_{n=0}^{\infty}(-1)^n \, e^{-(2n+1)x}$$

$$= -2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(n+\frac{1}{2})\pi}{(n+\frac{1}{2})^2\pi^2+x^2} = 1+2\sum_{n=1}^{\infty}\frac{E_{2n}}{(2n)!}x^{2n}$$

defining a distribution function similar to the normal distribution function.

The Mellin, the Hilbert and the Fourier transforms of $\mathrm{g(x)}$ are given by ([GrI], 3.523, 3.981)

$$\mathrm{M}[g](\mathrm{s}) = \int_0^{\infty} g(t)t^s\frac{dt}{t} = 2^{1-s}\gamma(s)\ {}_1F_1(-1;s;\tfrac{1}{2}),$$ (where γ denotes the Gamma function)

$$\mathrm{H}[g](\mathrm{x}) = 2\pi\int_0^{\infty}\frac{\sin(2\pi xy)}{\coth(\frac{\pi}{2}\pi y)}dy = \frac{1}{\pi}\frac{\sinh(4x)}{1+\cosh(4x)} = \frac{1}{\pi}\frac{\cosh(4x)-1}{\sinh(4x)}$$

$$= -2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x}{(n+\frac{1}{2})^2\pi^2+x^2}$$

$$\mathrm{F}[g](\mathrm{x}) := \hat{\mathrm{g}}(\omega) := \int_{-\infty}^{\infty} g(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty}\frac{e^{(1-i\omega)t}}{e^{2t}+1}dt = 2\,\frac{\frac{\pi}{2}}{\cosh(\frac{\pi}{2}\omega)}$$

$$= \pi g(\frac{\pi}{2}\omega) = -2\sum_{n=0}^{\infty}(-1)^{n-1}\frac{(n+\frac{1}{2})}{(n+\frac{1}{2})^2+(\frac{1}{2}\omega)^2}$$

where

$$z_n := -\mathrm{i}\pi(\tfrac{1}{2}+n) \ , \ \mathrm{n}\epsilon\mathrm{Z}$$

are the simple poles of the integrand above with the residuals

$$Res_{z=z_n}\frac{e^{(1-i\omega)z}}{e^{2z}+1} = \frac{e^{(1-i\omega)z_n}}{e^{2z_n}+1} = \frac{i}{2}(-1)^n e^{-\pi\omega(n+\frac{1}{2})}$$

The solution of the above integral equation is then given by

$$\mathrm{u(x)} = \frac{1}{4\pi}\left[\frac{2}{\pi}\cosh(\frac{\pi}{2}\omega)\hat{f}(\omega)\right](-x) \ .$$

The generalized Hermite polynomials satisfy different differential equations for even and odd polynomials enabling corresponding spectral analysis ([KrA]). For the special case of the Schrödinger differential equation the spectrum of its related Schrödinger equation operator L is discrete, consisting of the odd integers. The corresponding eigen-functions form a complete orthogonal set in the weighted-L_2 space ([DaD]).

The Galois group of the equations $K_n^{(i)}(x) = 0$, $i = 0,1$, is the symmetric group S_n, where

$$K_n^{(0)}(x^2) = H_{2n}(x) \quad , \quad xK_n^{(1)}(x^2) = H_{2n+1}(x)$$

and $H_n(x)$ denote the Hermite polynomials ([ScW]), i.e. the polynomials $K_n^{(i)}(x)$ are non-affine.

Our alternatively proposed Schrödinger (Calderón) equation operator differs from the standard operator L by its combination with the Hilbert-transform operator H based on extended domain, which is an unitary operator with corresponding spectral theorem and a representation

$$H = \cos(A) + i * \sin(A) = e^{iA}$$

whereby A denotes a Hermitian operator with a corresponding spectrum on the unit circle. This enables a spectral representation of the alternatively proposed Schrödinger equation operator, whereby the L_2 space is governed by a discrete spectrum, while the corresponding complementary space of L_2 is governed by a continuous spectrum, modelling the "ground state zero" "eigen-differential" / "wave packages".

From [GrI] 1.317, 1.411, we recall the following identities

$$\frac{\pi}{2}\tan\left(\frac{\pi}{2}x\right) = \frac{\pi}{2}\cdot\frac{1-\cos(\pi x)}{\sin(\pi x)} = \frac{\pi}{2}\cdot\frac{sin(\pi x)}{1+\cos(\pi x)}$$

$$\frac{\pi}{2}\tan\left(\frac{\pi}{2}x\right) = \frac{1}{x}\cdot\sum_{k=1}^{\infty}\frac{2^{2k-1}}{(2k)!}|B_{2k}|\cdot(\pi x)^{2k} \quad , \quad x^2 < 1 \ .$$

The expansion of the tan – resp. the $xcot(x)$ –term in series of simple fractions are given by ([GrI] 1.421)

$$\frac{tan\frac{\pi}{2}x}{\frac{\pi}{2}x} = \frac{8}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2-x^2} \quad , \quad -\frac{\pi}{2}(1-x)\cot\left(\frac{\pi}{2}(1-x)\right) = -1 + 2\sum_{k=1}^{\infty}\frac{(1-x)^2}{4k^2-(1-x)^2} \quad .$$

With respect to the relationship of the Zeta function and the $\pi xcot(\pi x)$ function we further note the following formulas ([BeB] 5, [TiE] 4.14, [AnG] entry 3, entry 44)

i. $\sum_{n>x}\frac{1}{n^s} = \frac{1}{2\pi x}\int_{x-i\infty}^{x+i\infty} x^{-s}(-\pi zcot(\pi z)dz$

ii. $z(s) = \sum_{n\leq x}\frac{1}{n^s} - \frac{x^{1-s}}{1-s} + O(x^{-\sigma}) \quad , \ \sigma \geq \sigma_0 > 0 \ , \ |t| \leq \frac{2\pi x}{c} \ , \ c > 1$

iii. $xcoth(x) = 1 + \frac{x^2}{3+}\frac{x^2}{5} + \frac{x^2}{7} \ldots\ldots = 1 + \frac{x^2}{3} - \frac{x^2}{9}\left[\frac{x^5}{5} + \frac{\frac{4\cdot5}{2\cdot3}x^2}{7} + \frac{\frac{2\cdot3}{4\cdot5}x^2}{9} + \frac{\frac{6\cdot7}{4\cdot5}x^2}{11} + \frac{\frac{4\cdot5}{6\cdot7}x^2}{13} + \cdots\right]$

iv. $(\frac{\pi}{2}s)\cdot coth\left(\frac{\pi}{2}s\right) = 1 + \frac{s^2}{1+}\frac{1^2(s^2+1^2)}{3} + \frac{}{+}\frac{2^2(s^2+2^2)}{5}\frac{}{+}\frac{3^2(s^2+3^2)}{7} \ldots\ldots \ s \in C.$

We note that the periodical continuation of the log-Gamma-function $log\gamma(s)$ with domain $(0,1)$ is $\in H^{\#}_{-1/2}(0,1)$, also anticipating the odd sin-Fourier terms in the form $\frac{\log(2\pi n)}{2\pi n}+\frac{\gamma}{2\pi n}$ ([GrI] 6.443). The log-Gamma-function is linked to the log-sin-function by the formula

$$log\gamma(x)+log\gamma(1-x)=\log(2\pi)+log\frac{1}{2\sin(\pi x)}=\log(2\pi)+\sum_{k=1}^{\infty}\frac{\cos(2\pi kx)}{k}$$

$$\frac{\pi}{2}\cot\left(\frac{\pi}{2}x\right)-\frac{\pi}{2}\cot\left(\frac{\pi}{2}(x-1)\right)=\frac{2\pi}{2\sin(\pi x)}\ .$$

The Stieltjes continued fraction theory provides an integral representation of the continued fraction

$$\frac{1|}{|a_1-z}-\frac{b_1^2|}{|a_2-z}-\frac{b_2^2|}{|a_3-z}-\cdots\qquad,\quad a_n\in R, b_n\in R-\{0\}, z\in Z$$

in the form ([HeE], [BrK1])

$$\mathrm{u(z)}=\mathrm{S}[\sigma](z):=\int_{-\infty}^{\infty}\frac{d\sigma(\mu)}{z-\mu}=\frac{1|}{|a_1-z}-\frac{b_1^2|}{|a_2-z}-\frac{b_2^2|}{|a_3-z}-\cdots$$

provided that the series $\sum_{n=1}^{\infty}|p_n(z)|^2$ diverge for at least one non-real $z\in Z-R$ (and therefore for all $z\in Z-R$) whereby the polynomials $p_n(z)$ are defined by the linear homogenous equations

$$(a_1-z)p_1(z)-b_1p_2(z)=0$$

$$-b_{n-1}p_{n-1}(z)+(a_n-z)p_n(z)-b_np_{n+1}(z)=0\quad .$$

The above is about a real bounded J-fraction ([WaH], theorem 27.4). The equivalent function of a positive definite J-fraction can be represented as a Stieltjes transform ([WaH], theorems 65.1 & 66.1).

Just as a side remark we note the result from P. Bundschuh: the number

$$\sum_{n=0}^{\infty}\frac{1}{n^2+1}=\frac{1}{2}+\frac{\pi}{2}\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}=\frac{1}{2}+\frac{\pi}{2}\coth(\pi)$$

is transcendental.

The above formula is related to the series ([ChK] VI, §2)

$$\frac{1}{2}-\frac{\pi}{2}z\cot(\pi z)=\frac{1}{2}+\frac{\pi}{2}\mathrm{z}\left(\frac{d}{dz}\rho_H(\mathrm{z})\right)=\sum_{n=1}^{\infty}zeta(2n)z^{2n}\qquad,\ |z|<1\quad .$$

Putting $a_n := (2\pi)^{-1/2}(-2)^n n!$, $(n = 0,1,2 \ldots.)$ the Hermite polynomials are linked to Weber's (Whittaker's) parabolic cylindrical polynomials by ([AbM] (13.1.32) [BuH] p. 215)

$$M_{\left(n+\frac{1}{2}\right)-\frac{1}{4};-\frac{1}{4}}(z) := \sqrt{2}\frac{a_n}{(2n)!} z^{\frac{1}{4}} e^{-\frac{z}{2}} He_{2n}\left(\sqrt{2z}\right) = \frac{n!}{\gamma\left(n+\frac{1}{2}\right)} z^{\frac{1}{2}} z^{-\frac{1}{4}} e^{-\frac{z}{2}} L_n^{-\frac{1}{2}}(z) = z^{\frac{1}{4}} e^{-\frac{z}{2}} M(-n, \frac{1}{2}; z)$$

$$M_{\left(n+\frac{1}{2}\right)+\frac{1}{4};+\frac{1}{4}}(z) := 2\frac{a_n}{(2n+1)!} z^{\frac{1}{4}} e^{-\frac{z}{2}} He_{2n+1}\left(\sqrt{2z}\right) = \frac{n!}{\gamma\left(n+\frac{3}{2}\right)} z^{\frac{1}{2}} z^{\frac{1}{4}} e^{-\frac{z}{2}} L_n^{\frac{1}{2}}(z) = z^{-\frac{1}{4}} e^{-\frac{z}{2}} \left[xM(-n, \frac{3}{2}; z)\right]$$

resp. ([AbM] (13.6) ([GrI] (9.231)

$$M\left(-n, \frac{1}{2}; \frac{z^2}{2}\right) = \frac{n!}{(2n)!} (-\frac{1}{2})^{-n} He_{2n}(z)$$

$$xM\left(-n, \frac{3}{2}; \frac{z^2}{2}\right) = \frac{n!}{(2n+1)!} (-\frac{1}{2})^{-n} He_{2n+1}(z)$$

where

$$M_{\left(n+\frac{1}{2}\right)+\mu;\mu}(z) = \frac{z^{\frac{1}{2}-\mu} e^{\frac{z}{2}}}{(2\mu+1)(2\mu+2)\ldots(2\mu+n)} \frac{d^n}{dz^n} (z^{n+2\mu} e^{-z}) \ .$$

The Mellin transform of $h_{\rho;\sigma}(z) := e^{-\frac{z}{2}} M_{\sigma;\rho}(z)$ is given by ([GrI] (7.621)

$$\int_0^\infty x^s \, h_{\rho;\sigma}(s) \frac{dx}{x} = \frac{\gamma(1+2\sigma)\gamma(\rho-s)\gamma(\frac{1}{2}+\sigma+s)}{\gamma(\frac{1}{2}+\sigma+\rho)\gamma(\frac{1}{2}+\sigma-s)} \quad , \quad Re\left(-\frac{1}{2}-\sigma\right) < Re(s) < Re(\rho) \ .$$

There are at least two approaches to wavelet analysis, both are addressing the somehow contradiction by itself, that a function over the one-dimensional space R can be unfolded into a function over the two-dimensional half-plane.

A wavelet transform W_ϑ

$$W_\vartheta[f](a,b) := |a|^{-1/2} \int_R f(t)\vartheta\left(\frac{t-b}{a}\right) dt$$

of a function f with respect to a wavelet function ϑ is an isometric mapping. The admissibility condition is given by

$$0 < c_\vartheta := \int_R \frac{|\hat{\vartheta}(\omega)|^2}{|\omega|} d\omega < \infty$$

The corresponding adjoint operator W_ϑ^* is given by the inverse wavelet transform on its range:

$$W_\vartheta^*[g](a,b) := c_\vartheta^{-1/2} \int_R \int_R |a|^{-\frac{1}{2}} g(a,b) \frac{1}{a} \vartheta\left(\frac{t-b}{a}\right) \frac{da}{a} db$$

The Fourier transform of a wavelet transformed function f is given by

$$\widehat{W_\vartheta[f]}(a,\omega) := (2\pi|a|)^{\frac{1}{2}} c_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-a\omega)\hat{f}(\omega) \quad .$$

For $\varphi, \vartheta \in L_2(R)$, $f_1, f_2 \in L_2(R)$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\overline{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \le c_\vartheta c_\varphi$ one gets the duality relationship ([LoA])

$$(W_\vartheta \ f_1, W_\varphi^* f_2)_{L_2(R^2, \frac{dadb}{a^2})} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$$

i.e.

$$W_\varphi^* W_\vartheta \ [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{-sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula. The prize to be paid is about additional efforts, when re-building the reconstruction wavelet. The extended admissibility condition above indicates that wavelet "pairs" in the form $(\varphi, \vartheta) \in L_2 x H_{-1}$ would be an appropriate good baseline to start from, when analyzing in the Hilbert space frame $H_{-1/2} = L_2 x L_2^{\neg}$, where $L_2^{\neg}$ denote the complementary space of L_2 with respect to the $H_{-1/2}$ −norm. The Hilbert transform operator (which is valid for every Hilbert scale) is the "natural" partner to the wavelet-transform operator, as it is skew-symmetric, rotation invariant and each Hilbert transformed "function" has vanishing constant Fourier term. The example in the context above is the Hilbert transform of the Gaussian distribution function, the (odd) Dawson function, with the "polynomial degree" point of zero at +/- infinite.

With respect to the proposed alternative Schrödinger momentum operator and the above we recall the Heisenberg uncertainty inequality:

Let $g \in H_0 = L_2(R)$, $\|g\|_0 = 1$, then

$$\mu(g) := t_0\omega_0 := (\int_R (t-t_0)^2|g(t)|^2)(\int_R (\omega-\omega_0)^2|\hat{g}(\omega)|^2) \geq \frac{1}{4}$$

whereby t_0 denotes the mean value of the location of the particle and

$$\omega_0 := \int_R \omega|\hat{g}(\omega)|^2\, d\omega = (-i\frac{d}{dx}\varphi, \varphi)_0$$

the mean value of the momentum of the particle. The localization "uncertainty" $\mu(g)$ of the function g at the phase point (t_0, ω_0) is defined, provided that $-\infty < t_0, \omega_0 < \infty$, i.e. $g \in H_{1/2}$.

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