**Mathematical modelling frameworks for
SMEP (state of the art) and GUT (proposed)**

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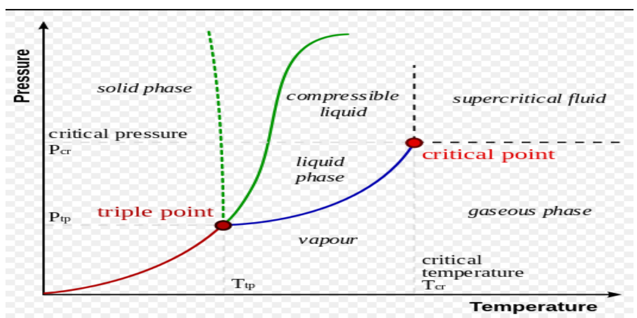
Some challenges of Standard Model of Elementary Particles (SMEP)

Quantum gravity is a field of theoretical physics that seeks to describe the force of gravity according to the principles of quantum mechanics. Ground state (or vacuum) energy modeling is concerned about necessarily "existing" "empty space" energy in the absence of matter.

Light consists of particles. The current of electrons increases with the increase of its frequency. In the same way it does for the current of light. But the current of light does not increase with the increase of the intensity (the "force" of the light). This phenomenon leads Einstein to the concept of photons with minimal quantum energy. But photons have no mass; nevertheless it holds the Einstein equation $E = m \cdot c^2$. In addition the light is an electro-magnetic wave in the sense of Maxwell equations.

The Higgs boson combines the existence of mass together with the action of the weak force. But why it provides especially to the quarks that much mass, is still a mystery.

The state of the art assumption is, that quantum fluctuation was the first mover in the early state of the universe; the 4 Nature forces arise at a later state after inflation. Energy breaking & phase transition are the current concepts to explain mass generation out of purely massless "particles" (photons).



[BlD]. Preface: "*There is a standard way to compute the field strength (or curvature) of a gauge potential (or connection). In case where $G = U(1)$ (or equivalent the group of rotations in the plane), the gauge potential is essentially the 4-vector potential of electromagnetism and the field strength is the electromagnetic field. H. Weyl introduced the concept of gauge transformation and gauge invariance. Yang and Mills introduced gauges prescribing a point-dependent choice of isotopic spin axis. In this case the group is SU(2) (or equivalent, the unit of quaternions).The Yang-Mills model was a precursor to the apparently successful model of Weinberg/Salam for weak interactions. The mechanism of spontaneous symmetry breaking allows gauge fields to acquire mass (consider, e.g. the massive "intermediate vector bosons" in the Weinberg-Salam model). In spite of these refinements, the basic fact remains that the existence of gauge fields is a consequence of the existence of gauge-invariant action densities for particle fields. …. The action density is a measure of the superfluous manifestations of the fields involved. Nature obeys the principle of least action.*"

[BlD] Bleecker D., Gauge Theory and Variational Principles, Dover Publications, Inc., Mineola, New York, 1981

Comparison table: current vs. proposed

	Standard Model of Elementary Particles (SMEP)	**Grand Unified Theory (GUT)** ***Proposed mathematical model***
Physical model interpretation	all "forces", which interact between different particles, are due to the interaction of related different quantum "types", resp. fields; Standard Model of Elementary Particles by the Yang-Mills field $SU(3)xSU(2)xU(1)=SU(3)xU(2)$,	no single force only "at" big bang and generation of 4 forces during inflation phase; 4 pseudo forces, only, which are measurable actions as a consequence of vacuum state energy as defined per considered variational PDE or PDO
Mathematical framework	additive Gauge fields concept per "force type" in combination with variational principles $G=U(1)$: group of rotations in the plane $G=SU(2)$: the unit of quaternions, Grassman-valued fermion field	single vacuum energy concept for all "pseudo-force" types in combination with variational principles per considered PDE/PDO
Hankel & Hilbert transforms	n/a	$H_\nu^2=-H^2=Id$ H_ν, H self-adjoint, L_2-isometric
Wave and wave package concept	"Dirac" function; its regularity depending from space dimension	distributional Hilbert space element of $H_{-1/2}$, the Hilbert space regularity is independent from the space dimension and better than "Dirac function" regularity for any space dimension
quantum state framework	$H_1 \subset H_0 \subset H_{-N/2-\varepsilon}, \varepsilon>0$	$H_{1/2} \subset H_0 \subset H_{-1/2}, N>0$
H_0 polynomial system	Hermite polynomials φ_n whereby φ_1 is the Gaussian function	Hilbert transformed Hermite polynomials φ_n^H : $(\varphi_n, \varphi_n^H)=0$, $\hat{\varphi}_n^H(0)=0$, i.e. the commutator $[x,P]=0$ for any convolution (singular integral) Pseudo-Differential Operator
$H_{-1/2}=H_0 \oplus H_0^\perp$	n/a	orthogonal projection operator $P_0: H_{-1/2} \to H_0$ $(P_0 u, v)_{-1/2}=(u,v)_{-1/2}\ \forall v \in H_0$ whereby for $u' \in H_{-1/2}$ $(u',v)_{-1/2}=(u,v)_0\ \forall v \in H_{-1/2}$; note: the least action principle is equivalent to operator norm minimization problem

Hamiltonian operator(s)	sum of Hamiltonian operators per considered gauge field combinations $H = H_{Dirac} + H_{Yang-Mills} + H_{Higgs} + H_{Einstein}$	single Hamiltonian operator per considered variational PDE/PDO
Harmonic oscillator, Hamiltonian operator $\underline{H} = -\frac{d^2}{dx^2} + x^2$ creation & annihilation operators with different domains H Hilbert transform operator	$\underline{A} = -\frac{d}{dx} + x : H_1 \to H_0$, $\underline{A}^* = \frac{d}{dx} + x : H_1 \to H_0$ $(\underline{A}u, v)_0$, $(u, \underline{A}^* v)_0 \quad \forall v \in H_0$ $\underline{H}$ self adjoint, eigenvalues real, $\underline{A}^* = -\underline{A}$ $\underline{H} = \underline{A}^* \underline{A} + 1 = \underline{A}\,\underline{A}^* - 1$	$\underline{a} = -\frac{d}{dx} + x : H_{1/2} \to H_{-1/2}$, $\underline{a}^* = \frac{d}{dx} + x : H_{1/2} \to H_{-1/2}$ $(\underline{a}u, v)_{-1/2}$, $(u, \underline{a}^* v)_{-1/2} \quad \forall v \in H_{-1/2}$ $\underline{a}, \underline{a}^*$ are self adjoint, because of $(u', v)_{-1/2} = (A[u'], v)_0 = (H[u], v)_0$ $= -(u, H[v])_0 = (u, A[-v'])_0 = (u, [-v'])_{-1/2}$
dependency of space dimension	yes	no
space-time dimensions	$N > 10$	4-dimensional scalar field $n = m + 1 = 4$
Poincare "conjecture"	n/a	applicable
Huygens' principle; spherical waves for temporal lines of space-time dimension	n/a	Cauchy & radiation problem for wave equation
particles vs. photons ratio	$6 \cdot 10^{-10}$	particles $\in H_0$, photons $\in H_{-1/2} - H_0$
Planck's action quantum	smallest action quantum, which can be measured in the test space H_0	smallest action quantum, which can be measured in the test space H_0
"time"	eigen-time of a photon is zero, i.e. a photon dies not realize that times goes by; how a photon can act? de Broglie time: $t = \frac{h}{E}$	R. Penrose, The Emperor's New Mind: "*the time of our perception does not "really" flow in quite the linear forward-moving way that we perceive it to do. The temporal ordering that we "appear" to perceive is, …, something that we impose upon our perceptions in order to make sense of them in relation to the uniform forward time-progression of an external physical reality*"
Schrödinger momentum operator $P := -i\hbar\nabla = \frac{\hbar}{i}\nabla$ $H := -\frac{\hbar^2}{2m}\Delta = \frac{1}{2m}\left(\frac{\hbar}{i}\nabla\right)^2$	$D(P) = H_0$: "eigen-functions" are plane-waves $f(x) = e^{ikx}$ which are not defined in H_0, i.e. $\notin H_0$	$D(P) = H_{-1/2}$: eigen-functions are plane-waves $f(x) = e^{ikx}$ which are defined in H_0, i.e. $\in H_0$

Harmonic quantum oscillator (generation & annihilation operators) $H_{osc} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 = \frac{1}{2}\hbar\omega(a^+ a + \frac{1}{2})$	$H_{osc} = \frac{1}{2}\hbar\omega(a^+ a + \frac{1}{2})$ $(H_{osc}\psi, \psi)_0 = \infty \quad \forall \psi \in H_0$ $E_n := \hbar\omega(n + \frac{1}{2}), \sum E_n = \infty$	$\underline{H} = c(\underline{a}^{*2} + \underline{a}^2)$ $(\underline{H}\psi, \psi)_{-1/2} < \infty \quad \forall \psi \in H_0$
“empty” vacuum	“a closed vacuum space cannot be diluted” (causing inflation/mass generation challenges in the early state of the universe)	no particles in a vacuum, “just” $H_{-1/2} - H_0$ – photons/waves
Heisenberg uncertainty principle	per affected gauge field/Hamiltonian operator	valid in $H_{-1/2}$ not valid in H_0
Schrödinger equation	classical mechanics relationship: continuity equation, Fourier waves $\frac{\partial}{\partial t}\|\psi\|_0^2 = \frac{\hbar}{2im}\left[(\psi^*, grad\psi)_0 - (\psi, grad\psi^*)_0\right]$	classical mechanics relationship continuity equation, Calderón wavelets $\frac{\partial}{\partial t}\|\psi\|_{-1/2}^2 = \frac{\hbar}{2im}\left[(\psi^*, grad\psi)_{-1/2} - (\psi, grad\psi^*)_{-1/2}\right]$ $\frac{\partial}{\partial t}\|\psi\|_{-1/2}^2 \cong \frac{\hbar}{2im}\left[(\psi^*, \psi)_0 - (\psi, \psi^*)_0\right] = 0, \quad \forall \psi \in H_0$
Generation of mass of sub-atomic particles and its mass value	energy breaking, phase transition; Lagrange (Klein-Gordon) density and potential of the Higgs field Φ, which is a kind of ether existing throughout the universe $L_{Higgs} = (D_\mu\Phi)(D^\mu\Phi)^+ - V(\Phi)$ $V(\Phi) := \mu^2\Phi\Phi^+ + \lambda(\Phi\Phi^+)^2$, D_μ covariante derivative	phase transition between $H_0 \subset H_{-1/2}$ and the closed sub-space $H_{-1/2} - H_0$
Casimir, Lamb effect	verification in test space H_0	verification in test space H_0
Elementary particles and the options $H := H_0$ or $H := H_{-1/2} = H_0 \oplus H_0^\perp$	Einstein-Podolsky-Rosen paradoxon $H = H_0$, $H_0^\perp = \{\ \}$ gauge/Higgs/(hypothetical) graviton bosons with only symmetric Schrödinger wave function solutions $H = H_0$, $H_0^\perp = \{\ \}$ “matter” particles fermions with only anti-symmetric Schrödinger wave function solutions $\psi = \psi_0 \in H_0$ with $\|\psi_0\|_0 = 1$ one therefore gets $\|\psi\|_0 = \|\psi_0\|_0 = 1$	$H = H_{-1/2}$ “wavelet” function/field space H_0 fermions (sub-) space = (matter) test space $\psi = \psi_0 + \psi_0^\perp \in H_{-1/2} = H_0 \otimes H_0^\perp$ with $\|\psi_0\|_0 = 1$, $\|\psi_0^\perp\|_{-1/2} =: \sigma < \infty$ one therefore gets $\|\psi\|_{-1/2}^2 \leq \sigma + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\sigma} x_i^2$

Single and double layer potential	Lebesgue integrals, mass density $\mu'(s) = \frac{d\mu}{ds}$ and existing normal derivative with corresponding (mathematically required) regularity assumptions $\frac{dU}{dn}$	Stieltjes integrals, Plemelj's concepts of an mass element $d\mu_s$ and related "current of a force" **on** the boundary through the boundary element ds $\overline{U}(\sigma) = -\int_{\sigma_0}^{\sigma} \frac{dU}{dn_s} ds$
Quantum theory of radiation, intensity = surface force density $[\text{int}\,ensity]$ $= [surface\_force\_density]$ $= [energy\_density]$ $= \frac{N}{cm^2} == \frac{Ncm}{cm^3} = Pascal$	regularity requirement/assumptions of an existing normal derivative with exterior/interior domain w/o physical meaning	very poor regularity assumptions in the form $\int_{\sigma_0}^{\sigma} \lvert d\mu \rvert < \infty$ to ensure continuous single layer potential; concept of an apparent size of the element ds as "seen" from the interior point p and double layer potential
Maxwell-Lorentz equations, (retarded) potentials and corresponding field equations	Lorentz gauge $A \to A - \nabla\chi$ $\phi \to \phi + \frac{\dot{\chi}}{c}$ because of the assumption $H = curlA$, $curl(E + \frac{\dot{A}}{c}) = 0$ and therefore $E + \frac{\dot{A}}{c} = -\nabla\phi$ Vector field A is not completely determined by the magnetic field H	Vector field A is completely determined by the magnetic field H; no differentiability assumption required; no calibration required due to this purely mathematical assumptions (w/o physical meaning; same situation as for the "differential manifolds" assumptions for the Einstein field equations)

[PlJ] Plemelj J., Potentialtheoretische Untersuchungen, B. G. Teubner Verlag, Leipzig, 1911

[ShF] Shu F. H., The Physics of Astrophysics, Volume I, Radiation, University Science Books, Mill Valley, California,1991

Appendix

The Eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|...\|$. We will consider mappings $K: H \to H$. Unless otherwise noticed the standard assumptions on K are:

i) K is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$
ii) K is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent,
iii) K is injective, i.e. $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} \quad .$$

Lemma: Let K be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|} \quad .$$

Theorem: There exists a countable sequence $\{\lambda_i, \varphi_i\}$ of eigenelements and eigenvalues $K\varphi_i = \lambda_i \varphi_i$ with the properties

i) the eigenelements are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_k) = \delta_{i,k}$
ii) the eigenvalues tend to zero, i.e. $\lim_{i \to \infty} \lambda_i$
iii) the generalized Fourier sums

$$S_n := \sum_{i=1}^{n} (x, \varphi_i)\varphi_i \to x \quad \text{with } n \to \infty \text{ for all } x \in H$$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \varphi_i)^2$$

holds for all $x \in H$.

Hilbert Scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|...\|$ and A be a linear operator with the properties

i) A is self-adjoint, positive definite
ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{\lambda_i, \varphi_i\}$ with

$$A\varphi_i = \lambda_i \varphi_i \quad , \quad (\varphi_i, \varphi_k) = \delta_{i,k} \text{ and } \lim_{i\to\infty} \lambda_i$$

ii) any $x \in H$ is represented by

$$(*) \qquad x = \sum_{i=1}^{\infty} (x, \varphi_i)\varphi_i \quad \text{and} \quad \|x\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2 \; .$$

Lemma: Let $x \in D(A)$, then

$$(**) \qquad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i)\varphi_i \quad , \quad \|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2 \; ,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i) \; .$$

Because of (*) there is a one-to-one mapping I of H to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} \mid \hat{x} = (x_1, x_2, ...)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i) \; .$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_{1}^{\infty} (x, \varphi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x},\hat{y})_{\alpha}=\sum_{i}^{\infty}\lambda_i^{\alpha}(x,\varphi_i)(y,\varphi_i)=\sum_{i}^{\infty}\lambda_i^{\alpha}x_i y_i$$

and the norms

$$\|\hat{x}\|_{\alpha}^{2}=(\hat{x},\hat{x})_{\alpha}$$

Let $\hat{H}_{\alpha}$ denote the set of all sequences with finite $\alpha-$norm. then $\hat{H}_{\alpha}$ is a Hilbert space. The proof is the same as the standard one for the space l_2.

Similarly one can define the spaces H_{α}: they consist of those elements $x\in H$ such that $Ix\in\hat{H}_{\alpha}$ with scalar product

$$(x,y)_{\alpha}=\sum_{i}^{\infty}\lambda_i^{\alpha}(x,\varphi_i)(y,\varphi_i)=\sum_{i}^{\infty}\lambda_i^{\alpha}x_i y_i$$

and norm

$$\|x\|_{\alpha}^{2}=(x,x)_{\alpha}\cdot$$

Because of the Parseval identity we have especially

$$(x,y)_0=(x,y)$$

and because of (**) it holds

$$\|x\|_{2}^{2}=(Ax,Ax)_0\ ,\ H_2=D(A)\cdot$$

The set $\{H_{\alpha}|\alpha\ge 0\}$ is called a Hilbert scale. The condition $\alpha\ge 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha<0$: $\lim\lambda_i^{\alpha}\to 0$. Then there exist sequences $\hat{x}=(x_1,x_2,...)$ with

$$\|\hat{x}\|_{2}^{2}<\infty\ ,\ \|\hat{x}\|_{0}^{2}=\infty\ \cdot$$

Because of Bessel's inequality there exists no $x\in H$ with $Ix=\hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_{\alpha}|\alpha\ge 0\}$ for different indices:

Lemma: Let $\alpha<\beta$. Then

$$\|x\|_{\alpha}\le\|x\|_{\beta}$$

and the embedding $H_{\beta}\to H_{\alpha}$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \le \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with

$$\mu = \frac{\gamma - \beta}{\gamma - \alpha} \text{ and } \nu = \frac{\beta - \alpha}{\gamma - \alpha}.$$

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \le t^{\beta-\alpha}\|x\|_\beta$

ii) $\|x - y\|_\beta \le \|x\|_\beta$, $\|y\|_\beta \le \|x\|_\beta$

iii) $\|y\|_\gamma \le t^{-(\gamma-\beta)}\|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \le t^{\beta-\rho}\|x\|_\beta$ for $\alpha \le \rho \le \beta$

ii) $\|y\|_\sigma \le t^{-(\sigma-\beta)}\|x\|_\beta$ for $\beta \le \sigma \le \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \text{ and } \|x\|_2 = \|Ax\|.$$

Extensions and generalizations

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i} t} (x, \varphi_i)(y, \varphi_i)$$
$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 \ .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^{α}.

Obviously we have

$$\|x\|_{(t)} \le c(\alpha, t)\|x\|_{\alpha} \text{ for } x \in H_{\alpha}$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the $(t)-norm$ is weaker than any $\alpha - norm$. On the other hand any negative norm, i.e. $\|x\|_{\alpha}$ with $\alpha < 0$, is bounded by the $0 - norm$ and the newly introduced $(t) - norm$. As it holds for any $t, \delta, \alpha > 0$ and $\lambda \ge 1$ the inequality

$$\lambda^{-\alpha} \le \delta^{2\alpha} + e^{t(\delta^{-1} - \sqrt{\lambda})}$$

one gets

Lemma: Let $\alpha > 0$ be fixed. The $\alpha - norm$ of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha - norm$, which can be reformulated in the form ($\mu, \nu > 0, \mu + \nu > 1$)

$$\|x\|_{\beta}^2 \le \nu\varepsilon\|x\|_{\gamma}^2 + \mu e^{-\nu/\mu}\|x\|_{\alpha}^2$$

applying Young's inequality to

$$\|x\|_{\beta}^2 \le (\|x\|_{\alpha}^2)^{\mu} (\|x\|_{\gamma}^2)^{\nu} \ .$$

The counterpart of the fourth lemma above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\|x - y\| \le \|x\|$

ii) $\|y\|_1 \le \delta^{-1}\|x\|$

iii) $\|x - y\|_{(t)} \le e^{-t/\delta}\|x\|$.

In this paper we are especially concerned with the $H_{-1/2}$ – Hilbert space, as the proposed alternative framework to model quantum states. With respect to the above lemma this means, that for any bounded $x \in H_0$ it holds with $\sigma := t := \delta$

$$\|x\|_{-1/2}^2 \le \delta\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2 = \sigma\|x\|_0^2 + e\|x\|_{(\sigma)}^2 = \sigma\|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\sigma} x_i^2 \, .$$

For

$$\psi = \psi_0 + \psi_0^{\perp} \in H_0 \otimes H_0^{\perp}$$

with

$$\|\psi_0\|_0 = 1 \, , \; \|\psi_0^{\perp}\|_{-1/2}^2 =: \sigma$$

one therefore gets

$$\|x\|_{-1/2}^2 \le \sigma\|x\|_0^2 + \sum_{i=1}^{\infty} e^{1-\sqrt{\lambda_i}\sigma} x_i^2 \, .$$

Eigen-functions and Eigen-differentials

Let H *be* a (infinite dimensional) Hilbert space with inner product $(.,.)$, the norm $\|...\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1} compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum. We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$
$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \varphi_\mu (\varphi_\mu,*) d\mu \quad , \quad \mu \in [\lambda_0, \infty) \ ,$$

i.e.

$$dE_\lambda = \varphi_\lambda (\varphi_\lambda,*) d\lambda \ .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ. The set E_λ fulfills the following properties:

i) E_λ is a projection operator for all $\lambda \in R$
ii) for $\lambda \le \mu$ it follows $E_\lambda \le E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$
iii) $\lim_{\lambda \to -\infty} E_\lambda = 0$ and $\lim_{\lambda \to \infty} E_\lambda = Id$
iv) $\lim_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f : [a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) \ .$$

Symbolically one writes

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda \ .$$

Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax, y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \qquad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$
$$(A^2 x, y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ , \quad \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function.

It hold the following eigen-pair relations

$$A\varphi_i = \lambda_i \varphi_i \qquad A\varphi_\lambda = \lambda \varphi_\lambda \qquad \|\varphi_\lambda\|^2 = \infty \ , \ (\varphi_\lambda, \varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu) \ .$$

The φ_λ are not elements of the Hilbert space. The so-called eigen-differentials, which play a key role in quantum mechanics, are built as superposition of such eigen-functions.

Let I be the interval covering the continuous spectrum of A. We note the following representations:

$$x = \sum_1^\infty (x,\varphi_i)\varphi_i + \int_I \varphi_\mu (\varphi_\mu, x) d\mu \ , \quad Ax = \sum_1^\infty \lambda_i (x,\varphi_i)\varphi_i + \int_I \lambda \varphi_\mu (\varphi_\mu, x) d\mu$$

$$\|x\|^2 = \sum_1^\infty |(x,\varphi_i)|^2 + \int_I |(\varphi_\mu, x)|^2 d\mu \ ,$$

$$\|x\|_1^2 = \sum_1^\infty \lambda_i |(x,\varphi_i)|^2 + \int_I \lambda |(\varphi_\mu, x)|^2 d\mu$$

$$\|x\|_2^2 = \|Ax\|^2 = \sum^\infty \lambda_i^2 |(x,\varphi_i)|^2 + \int_I \lambda^2 |(\varphi_\mu, x)|^2 d\mu \ .$$

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\varphi_\lambda(x) = \lambda \varphi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\varphi_\lambda(x)$ it results into elements of H. The practical way to find Eigen-differentials is looking for solutions of a distribution equation.

Definition: Let $H = L_2^*(\Gamma)$ with $\Gamma = S^1(R^2)$, i.e. Γ is the boundary of the unit disk. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β Fourier coefficients and norms are defined by

$$u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx \ , \quad \|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \ .$$

The Fourier coefficients of the convolution operator

$$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2}u(y)dy =: \oint k(x-y)u(y)dy$$

are given by

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|}u_\nu \ .$$

The operator A enables characterizations of the Hilbert spaces $H_{-1/2}$ and H_{-1} in the form

$$H_{-1/2} = \left\{\psi \,\middle|\, \|\psi\|_{-1/2}^2 = (A\psi,\psi)_0 < \infty\right\} , \ H_{-1} = \left\{\psi \,\middle|\, \|\psi\|_{-1}^2 = (A\psi,A\psi)_0 < \infty\right\} \ .$$

This requires “differentiating / momentum building” of “less regular” “functions than $H_0 := L_2^*(\Gamma)$.
It holds

$$(u,v)_{-1} = (Au,Av)_0 \ , \ (u,v)_{-1/2} = (Au,v)_0 \ , \ (u',v')_{-1} = (Au',Av')_0 = (Hu,Hv)_0 \ ,$$

whereby

$$(Au')(x) = -\oint \log 2\sin\frac{x-y}{2}du(y) = \oint \cot\frac{x-y}{2}u(y)dy = (Hu)(x) \ .$$

Lemma: It holds

i) if $\{\psi_n\}$ is an orthogonal system of $L_2^{\#}(0,1)$, then $\{\overline{\psi}_n\} := \{H\psi_n\}$ is also an orthogonal system of $L_2^{\#}(0,1)$

ii) $(\overline{\psi}_n,\psi_n)_0 = 0$

iii) $\overline{\psi}_n' \approx n\overline{\psi}_n$ is an orthogonal system of $H_{-1}^{\#}(0,1)$

iv) for $g \in H_0^{\#}$ it holds

$$\|g'\|_{-1} = \|Ag'\|_0 = \|Hg\|_0 = \|g\|_0 \ .$$

Some key properties of the Hilbert transform

$$(Hu)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x-y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x-y} dy$$

are given in

Lemma: i) The constant Fourier term vanishes, i.e. $(Hu)_0 = 0$

ii) $$H(xu(x)) = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y)dy$$

iii) For odd functions it hold $H(xu(x)) = x(Hu)(x)$

iv) If $u, Hu \in L_2$ then u and Hu are orthogonal, i.e.

$$\int_{-\infty}^{\infty} u(y)(Hu)(y)dy = 0$$

v) $\|H\| = 1$, $H^* = -H$, $H^2 = -I$, $H^{-1} = H^3$

vi) $H(f * g) = f * Hg = Hf * g$ $\quad f * g = -Hf * Hg$

vii) If $(\varphi_n)_{n \in N}$ is an orthogonal system, so it is for the system $(H(\varphi_n))_{n \in N}$, i.e.

$$(H\varphi_n, H\varphi_n) = -(\varphi_n, H^2 \varphi_n) = (\varphi_n, \varphi_n)$$

viii) $\|Hu\|^2 = \|u\|^2$, i.e. if $u \in L_2$, then $Hu \in L_2$.

Proof:

ii) Consider the Hilbert transform of $xu(x)$

$$H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{x-y} dy \quad .$$

The insertion of a new variable $z = x - y$ yields

$$H(xu(x)) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(x-z)u(x-z)}{z} dz = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(xu(x-z)}{z} dz - \frac{1}{\pi} \int_{-\infty}^{\infty} u(x-z)dz = xH(u(x)) - \frac{1}{\pi} \int_{-\infty}^{\infty} u(y)dy$$

iv) $$\int_{-\infty}^{\infty} u(y)(Hu)(y)dy = \frac{i}{2\pi} \int_{-\infty}^{\infty} sign((\omega)|\hat{u}(\omega)|^2 d\omega$$ whereby $|\hat{u}(\omega)|^2$ is even .

An ‘optimal order‘ shift theorem for the heat equation

For the parabolic model problem

$$\dot{u} - \Delta u = f \qquad \text{in } \Omega \times (0,T]$$
$$u = 0 \qquad \text{on } \partial\Omega \times (0,T]$$
$$u(0) = u_0 , \qquad \text{in } \Omega \; .$$

an ‘optimal order’ shift theorem in the Sobolev Hilbert space framework analogue to the Laplace equation is given in the form

Lemma: i) $$\int_0^T \|w\|_{k+2}^2 dt \le c \int_0^T \|f\|_k^2 dt$$

ii) $$\int_0^T t^{-1/2} \|w\|_{k+2}^2 dt \le c \int_0^T t^{-1/2} \|f\|_k^2 dt$$

Proof: Let $w_i := (w, \varphi_i)$ resp. $f_i := (f, \varphi_i)$ being the generalized Fourier coefficient related to the eigen pairs $-v_i'' = \lambda_i v_i$. Then it holds

$$\dot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \quad \text{and} \quad w_i(0) = 0 \; .$$

with the solution

$$w_i(t) = \int_0^t e^{-\lambda_i (t-\tau)} f_i(\tau) d\tau \; .$$

By changing the order of integration one gets

$$\int_0^T \|w\|_{k+2}^2 dt = \sum \lambda_i^{k+2} \int_0^T w_i^2(t) dt \le \sum \lambda_i^{k+2} \int_0^T \left[\int_0^t e^{-\lambda_i (t-\tau)} d\tau \right] \left[\int_0^t e^{-\lambda_i (t-\tau)} f_i^2(\tau) d\tau \right] dt$$
$$\le \sum \lambda_i^{k+2} \int_0^T \lambda_i^{-1} \left[\int_0^t e^{-\lambda_i (t-\tau)} f_i^2(\tau) d\tau \right] dt = \sum \lambda_i^{k+1} \int_0^T f_i^2(\tau) \left[\int_\tau^T e^{-\lambda_i (t-\tau)} dt \right] d\tau \; .$$

For the initial value problem it holds

$$\dot{z} - z'' = 0 \qquad \text{in } (0,1) \times [0,T]$$
$$z(0,t) = z(1,t) = 0 \qquad \text{for } t \in (0,T]$$
$$z(x,o) = g(x) \qquad \text{for } x \in (0,1) \; .$$

The compatibility relations

$$g(1) = 0 \;,\; g'(0) = 0 \;,\; g''(1) = g'^2(1) \;, \text{etc.}$$

ensure corresponding regularity of the solution z

In case the initial value function has reduced regularity the following estimates are valid

$$\|z(t)\|_k^2 \le c t^{-(k-l)} \|g\|_l^2 \quad , \quad \int_0^T t^{-1/2} \|z'\|_{-1/2}^2 dt \le c \|g\| \; .$$

An unusual 'optimal order' shift theorem for the wave equation

We consider the hyperbolic model problem

$$\begin{aligned} \ddot{u} - \Delta u &= f && \text{in } \Omega \times (0,T] \\ u &= 0 && \text{on } \partial\Omega \times (0,T] \\ u(0) = u_0,\ \dot{u}(0) &= u_1 && \text{in } \Omega . \end{aligned}$$

An 'optimal order' shift theorem in the Sobolev Hilbert space framework analogue to the Laplace and the heat equation is not necessarily ensured for hyperbolic problems. This can be proven by a simple counter example:

Let

$$\Psi(x,t) := e^{-(\frac{1}{2}-(x-t))^2} \quad \text{and} \quad f(x,t) := 2\Psi(x,t) - 4t\Psi'(x,t)$$

then, because of $\dot{\Psi} + \Psi' = 0$ and $\ddot{\Psi} = \Psi''$, it holds

Lemma: The function $u(x,t) := t^2 \cdot \Psi(x,t)$ is a solution of the wave equation with $u_0 = u_1 = 0$ fulfilling

$$\|u''\|_{L_2(L_2)} \approx \|\Psi''\|_{L_2(L_2)}$$

while, at the same time, it holds

$$\|f\|_{L_2(L_2)} = \|Au\|_{L_2(L_2)} \approx \|\Psi'\|_{L_2(L_2)} .$$

The situation changed in case of the Hilbert space framework $H_{(-t)}$ with the exponential decay $e^{-\sqrt{\lambda_i}t}$ (and also for the related Hilbert scale $H_{\beta,(-t)}$):

Theorem: The hyperbolic self-adjoint, positive definite wave equation operator fulfills an 'optimal order' shift theorem in the form

$$\|u\|_{L_2(H_{k+2,(-t)})} \le c \cdot \|Au\|_{L_2(H_{k,(-t)})} .$$

Proof: The Fourier coefficients of the solution of the wave equation are given by

$$u_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\sqrt{\lambda_i}(t-\tau) f_i(\tau) d\tau .$$

With this one gets

$$\int_0^T e^{-\sqrt{\lambda_i}t} u_i^2(t) dt = \frac{1}{\lambda_i} \int_0^T e^{-\sqrt{\lambda_i}t} (\int_0^t \sin\sqrt{\lambda_i}(t-\tau) f_i(\tau) d\tau)^2 dt \le \frac{1}{\lambda_i} \int_0^T e^{-\sqrt{\lambda_i}t} (\int_0^t e^{\sqrt{\lambda_i}(t-\tau)} \sin^2\sqrt{\lambda_i}(t-\tau) d\tau) \cdot (\int_0^t e^{-\sqrt{\lambda_i}(t-\tau)} f_i^2(\tau) d\tau)) dt .$$

Applying the formula ([GrI] Gradshteyn/Ryzhik, table of integrals, 2.663)

$$\int e^{ax} \sin^2(bx) dx = \frac{e^{ax}}{2a} - \frac{e^{ax}}{a^2+4b^2} \left\{ \frac{a}{2}\cos(2bx) + b\sin(2bx) \right\}$$

this leads to (by changing the order of integration)

$$\le \frac{1}{\lambda_i^{3/2}} \int_0^T e^{-\sqrt{\lambda_i}t} \int_0^t f_i^2(\tau) d\tau dt = \frac{1}{\lambda_i^{3/2}} \int_0^T f_i^2(\tau) \int_\tau^T e^{-\sqrt{\lambda_i}t} dt d\tau \le \frac{1}{\lambda_i^2} \int_0^T e^{-\sqrt{\lambda_i}\tau} f_i^2(\tau) d\tau .$$

Harmonic music "noise" signals
"exactly" "between" red (Brownian) and white noise

Brownian noise or red noise is the kind of signal noise produced by Brownian motion. A Brownian motion (i.e. a Wiener process) is a continuous stationary stochastic process having independent increments, i.e. $B(t)-B(0)$ is a normal random variable with mean μt and variance $\sigma^2 t$, μ,σ^2 constant real numbers. The density function of a Brownian motion is given by

$$f_{B(t)}(x)=\frac{1}{\sqrt{2\pi\sigma^2 t}}e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$$

The sample paths of Brownian motion are not differentiable, a mathematical fact explaining the highly irregular motions of small particles. The total variation of Brownian motion over a finite interval $[0,T]$ is infinite. It holds

$$Var\left[\frac{B(t)}{t}\right]=\frac{\sigma^2}{t}.$$

If $W(t)$ is a Wiener process on the interval $[0,\infty)$, then, as well the process

$$W^*(t):=\begin{cases} tW(1/t), & t>0 \\ 0 & ,t=0 \end{cases}.$$

White noise can be defined as the derivative of a Brownian motion (i.e. a Wiener process) in the framework of infinite dimensional distribution theory, as the derivative $B'(t)$ of $B(t)$, does not exist in the ordinary sense. Not only $B'(t)$, but also all derivatives of Brownian motion are generalized functions on the same space. For each t, the white noise $B'(t)$ is defined as a generalized function (distribution) on an infinite dimensional space. A Brownian motion is obtained as the integral of a white noise signal $dB(t)$, i.e.

$$B(t)=\int_0^t dB(\tau)$$

meaning that Brownian motion is the integral of the white noise $dB(t)$ whose power spectral density is flat

$$S_0=\left|Fourier[B'](\omega)\right|^2=const.$$

This means, that the spectral density S_0 for white noise is flat, i.e. $S_0/\omega^0=c$ i.e. it is inversely proportional to ω^0. It holds $Fourier[B'](\omega)=i\omega Fourier[B](\omega)$. Therefore the power spectrum of Brownian noise is given by

$$S(\omega)=\left|Fourier[B](\omega)\right|^2=\frac{S_0}{\omega^2}.$$

This means, that the spectral density of Brownian (red) noise is S_0/ω^2, i.e. it is inversely proportional to ω^2, meaning it has more energy at lower frequencies, even more so than pink noise.
The above indicates a spectral density for harmonic music "noise" signals given by

$$S^*(\omega)=\left|Fourier[B^*](\omega)\right|^2=\frac{S_0}{\omega}.$$

Distribution solution of the 1D wave equation

The „vibration string" equation

$$u_{tt} - k^2 u_{xx} = 0$$

has a solution $u(x,t) = f(x-kt)$ for any function of one variable f, which has the physical interpretation of a „traveling wave" with „shape" $f(x)$ moving at velocity k.

There is no physical reason for the "shape" to be differentiable, but if it is not, the differential equation is not satisfied at some points. In order to not through away physically meaningful solutions because of technicalities, the concept of distributions can be applied.

If the equation above is also meaningful, if u is a distribution, then u is called a weak solution of it. If u is twice continuously differentiable and the equation holds, one calls u a strong or classical solution. Each classical solution is a weak solution. In case of the equation above it's also the other way around. The same is NOT TRUE for the elliptic Laplace equation (counter example is the classical solution $u(x.y) := \log(x^2+y^2)$, but not a weak solution as it holds there $\Delta \log(x^2+y^2) = 4\pi\delta$). In order to see this we show that for $u(x.y) := f(x-kt) \in L^1_{loc}(R^2)$ it holds

$$(*) \qquad (u_{tt} - k^2 u_{xx}, \varphi) = 0 \;.$$

From the following identities

i) $(u_{tt}, \varphi) = (u, \varphi_{tt}) = \int\int f(x-tk)\varphi_{tt}\,dxdt$

ii) $(u_{xx}, \varphi) = (u, \varphi_{xx}) = \int\int f(x-tk)\varphi_{xx}\,dxdt$

it follows

$$(u_{tt} - k^2 u_{xx}, \varphi) = \int\int f(x-tk)\left[\varphi_{tt} - k^2\varphi_{xx}\right]dxdt \;.$$

Substituting the variable in the form $y = x - kt$ and $z = x + kt$ means

$$\frac{\partial(y,z)}{\partial(x,t)} = \begin{pmatrix} 1 & -k \\ 1 & k \end{pmatrix} \text{ and } 2k\,dxdt = dydz \;.$$

From this it follows

$$(u_{tt} - k^2 u_{xx}, \varphi) = -2k\int\int f(y)\varphi_{yz}\,dzdy = -2k\int_{-\infty}^{\infty} f(y)\Big(\int_{-\infty}^{\infty}\varphi_{yz}\,dz\Big)dy \;.$$

As

$$\int_{-\infty}^{\infty}\varphi_{yz}\,dz = \varphi_y\Big|_{z=-\infty}^{z=\infty} = 0$$

this proves (*) above.

The time-symmetry of the classical wave equation

The time-symmetry of the classical wave equation is one of the main challenges of the big bang theory. It is linked to the miss-understanding that the ground state energy is fixed and uniquely defined:

The definition of the Hamiltonian

$$\underset{-}{H} := \frac{p^2}{2m} + \frac{1}{2}\omega^2 x^2 =: T + V$$

defines the non-measurable ground state energy as the state of the lowest energy, i.e. at the point $(x = 0, p = 0)$ in the phase space, and defines this energy as zero.

The kinetic energy of strings with mass ρ are given by

$$T = \rho \int_0^l \frac{1}{2} u_x^2(x,t) dx \ .$$

The internal forces of strings (being looked at as mechanical systems) are built on strains, depending proportionally from its lengths

$$L = \int_0^l \sqrt{1 + u_x^2(x,t)} dx \ .$$

For small displacements this is replaced by

$$L = l + \Delta l = \int_0^l \left[1 + \frac{1}{2} u_x^2(x,t) + ... \right] dx \quad \text{with} \quad \Delta l \approx \int_0^l \frac{1}{2} u_x^2(x,t) dx \ .$$

Correspondingly the potential energy $V(x)$ is approximately defined by

$$V(L) = V(l + \Delta l) \approx V(l) + \Delta l \frac{dV}{dL}\Big|_{L=l} \ .$$

Putting as the “strain constant”=”tension”

$$\sigma_s := \frac{dV}{dL}\Big|_{L=l} \quad \text{and} \quad V(l) := 0$$

(the latter one in order to just simplify the algebraic term) the potential energy is then given in the form

$$V \approx \sigma_s \int_0^l \frac{1}{2} u_x^2(x,t) dx \ .$$

Defining the corresponding “string velocity” by

$$c_s := \sqrt{\frac{\sigma_s}{\rho}}$$

then gives the wave equation of strings in the form

$$u_{tt} - c_s^2 u_{xx} = 0 \ .$$

Complementary variational principles

The topic is about „saddle point problems and non-linear minimization problems on convex manifolds". The scope is about saddle point problems and non-linear minimization problems on sub Hilbert space $U = H_0 \subset H_{-1/2}$.

Let $a(\cdot,\cdot): V \times V \to R$ a symmetric bilinear form with energy norm

$$\|u\|^2 := a(u,u) \; .$$

Let further $u_0 \in V$ and $F(\cdot): V \to R$ a functional with the following properties:

i) $F(\cdot): V \to R$ is convex on the linear manifold $u_0 + U$,
i.e. for every $u, v \in u_0 + U$ it holds $F((1-t)u + tv) \le (1-t)F(u) + tF(v)$ for every $t \in [0,1]$
ii) $F(u) \ge \alpha$ for every $u \in u_0 + U$
iii) $F(\cdot): V \to R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \to R$ with

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = F_u(v) \, .$$

Then the minimum problem

$$J(u): a(u,u) - F(u) \to \min \quad , \qquad u - u_0 \in U \, .$$

is equivalent to the variational equation

$$a(u,\varphi) + F_u(\varphi) = 0 \quad \text{for every} \quad \varphi \in U$$

and admits only an unique solution.

In case the sub space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \to R$ is continuous with respect to convergence in the energy norm, then there exists a solution.

We note that the energy functional is even strongly convex in whole V .

The method of Noble provides the link of the Hamiltonian function to the complementary variational principles:

Let $(E,(\cdot,\cdot))$ and $(E',\langle\cdot,\cdot\rangle)$ be Hilbert spaces and $T: E \to E'$, $T^*: E' \to E$ linear operators fulfilling

$$(u',Tu) = \langle T^* u',u\rangle .$$

Let further $W: E'xE \to R$ be a functional fulfilling

$$T = \frac{\partial W(u',\cdot)}{\partial u'} \quad \text{and} \quad T^* = \frac{\partial W(\cdot,\cdot u)}{\partial u}$$

i.e. the operators and derivatives from in the sense of Gateaux. Putting

$$W(u',u) = \frac{1}{2}(u',u') - F(u)$$

the minimization problem

$$(*) \qquad J(u) := (Tu,Tu) + 2F(u) \to \min \ , \ u \in U \subset E$$

leads to

$$Tu = u' \ \text{and} \ (T^* u',\cdot) = -F_u(\cdot) .$$

Lemma: If $F(\cdot)$ is a convex functional it follows that $W(u',u)$ is convex concerning u' and concave concerning u . Then the minimization problem (*) is equivalent to the variational equation

$$(v',\varphi) + F_u(\varphi) = 0 \ \text{ for every } \ \varphi \in U$$

resp.

$$(T^* v',\varphi) = -F_u(\varphi) \ \text{ for every } \ \varphi \in U \ .$$

In other words, there is a characterization of the solution of (*) as a saddle point.

References

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Velte W., Direkte Methoden der Variationsrechnung, Teubner Studienbücher, B. G. Teubner Stuttgart,1976

Non Linear Problems and optimal FEM L_∞ – error estimates

Let the problem be given by

$$F(x,u)=0$$

with the (roughly) regularity assumptions:

i) there is a unique solution

ii) F, F_u are Lipschitz continuous.

The approximation problem is given by:

find $\varphi \in S_h \quad (F(\cdot,\varphi),\chi)=0$ for $\chi \in S_h$.

Error analysis

Put

$$f(x):=F_u(x,u(x)) \quad \text{and} \quad \varphi = u-e$$

Then

$$(fe,\chi)=(R,\chi)$$

with a remainder term

$$R:=R(e):=F(\cdot,u-e)+fe$$

resp.

$$(fe,\chi)=(fu-R(e),\chi).$$

Let P_h denote the L_2 – projection related to $(f\cdot,\cdot)=(R,\chi)$, then

$$\varphi = P_h(u-\frac{1}{f}R(e))$$

resp.

$$e=(I-P_h)u+P_h\frac{1}{f}R(e)) =: T(e) \; .$$

Therefore the difference $e=u-e$ is a fix point of T .

For the following we use the abbreviations

$$B_{\kappa\bar{\varepsilon}} := \left\{ e \middle| \|e\|_{L_\infty} \le \kappa\bar{\varepsilon} \right\} \text{ and } \bar{\varepsilon} := \inf_{\chi\in S_h} \|u-\chi\|_{L_\infty} \; .$$

Key properties of the operator T

Lemma:

i) There is a $\kappa > 0$ such that for $\bar{\varepsilon}$ sufficiently small, then T maps the ball $B_{\kappa\bar{\varepsilon}}$ into itself.

ii) for $\bar{\varepsilon}$ sufficiently small, T is a contradiction in $B_{\kappa\bar{\varepsilon}}$.

Proof: i) Because of P_h and f^{-1} are being bounded it holds

$$\|I - P_h\|_{L_\infty} \le c_1 \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty} = \bar{\varepsilon}$$

and

$$\left\|P_h(\frac{1}{f} R(e))\right\|_{L_\infty} \le c_2 \|R(e)\|_{L_\infty} \quad .$$

It is

$$\|F(\cdot, u - e) + fe\|_{L_\infty} \le c_3 \|e\|_{L_\infty}^2 = c_3 \kappa^2 \bar{\varepsilon}^2$$

with c_3 being the Lipschitz constant of F_u . Therefore

$$\|T(e)\|_{L_\infty} \le c_1 \bar{\varepsilon} + c_3 c_2 \kappa^2 \bar{\varepsilon}^2 \quad .$$

Now fixing $\kappa > c_1$ and choosing $\bar{\varepsilon}_0$ according to $\kappa = c_1 + c_3 c_2 \kappa^2 \bar{\varepsilon}_0$ gives i)

ii) it holds

$$\|T(e_1) - T(e_2)\|_{L_\infty} = \left\|P_h(\frac{1}{f}(R(e_1) - R(e_2))\right\|_{L_\infty} \le c_2 \|R(e_1) - R(e_2)\|_{L_\infty}$$

and

$$R(e_1) - R(e_2) = F(\cdot, u - e_1) - F(\cdot, u - e_2) = (F_u(\cdot, \vartheta) - F_u(u)(e_1 - e_2) \quad .$$

With

$$F_u(\cdot, \vartheta) = F_u(\cdot, u - \vartheta e_1 - (1 - \vartheta) e_2)$$

one gets

$$\|F_u(\cdot, \vartheta) - F_u(\cdot, u)\| \le \kappa \bar{\varepsilon} c_3 \quad .$$

Choosing

$$\bar{\varepsilon} < Min(\varepsilon_0, \frac{1}{c_2 c_3 \kappa})$$

then proves ii).

Consequence: The operator T has a unique fix-point in the ball $B_{\kappa\bar{\varepsilon}}$
From this it follows the

Theorem: The FEM admits the error estimate

$$\|u - \varphi\|_{L_\infty} \le c \inf_{\chi \in S_h} \|u - \chi\|_{L_\infty} \quad .$$

With respect to "*The regularity of piecewise defined functions with respect to scales of Sobolev spaces*" we refer to the corresponding paper of J. A. Nitsche.