A proof of the Riemann Hypothesis

enabled by

a new representation of Riemann`s meromorphic symmetric Zeta function

$$\boldsymbol{\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi^*(1-s)}$$

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origin paper
dedicated to my wife Vibhuta
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Abstract

For $s \neq \nu$, $\nu \in \mathbb{Z}$, for Riemann's meromorphic Zeta function

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s)$$

a new representation is derived in the form

$$\xi^*(s) + \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} = \sum_{n=0}^\infty b_{2n}^*(s-\frac{1}{2})^{2n} + \frac{1}{\pi}\sum_{n=0}^\infty(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right]$$

with

$$b_{2n}^* := -2b_{2n} := -2\int_1^\infty \Phi(x)\left[\sum_{n=0}^\infty \frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} \quad \text{and} \quad \Phi(x) := \sum_{n=1}^\infty(e^{-2\pi nx} - e^{-\pi n^2x^2}) .$$

Accordingly, the non-trivial zeros $\left\{s_n = \frac{1}{2} + it_n\right\}$ of the zeta function are characterized by the identity of the following two convergent alternating power series representations

$$\sum_{n=0}^\infty(-1)^n b_{2n}t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^\infty(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2 + t_n^2}\right] .$$

In case there are negative values $t_n^{2n} < 0$, the corresponding term on the left hand would change its sign, while the term on the right hand won`t change its sign. This proves the RH.

1. Notations and Main Theorem

For the notations we refer to (EdH). The baseline function for the Zeta function theory is given by $\psi(x^2) := \sum_{n=1}^{\infty} e^{-\pi n^2 x^2}$, (EdH) 1.7. It is related to Jacobi's functional equation resulting into the symmetrical form of Riemann's functional equation in the form, (EdH) 1.7,

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^{\infty} \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s) .$$

Riemann's related entire Zeta function is given by $\xi(s) = \frac{s}{2}(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)$, (EdH) 1.8.

Alternatively to $\psi(x^2)$ we consider the function

$$\Phi(x) := \varphi(x) - \psi(x^2) := \sum_{n=1}^{\infty} \Phi_n(x) := \sum_{n=1}^{\infty}(e^{-2\pi n x} - e^{-\pi n^2 x^2})$$

with

$$\varphi(x) := \sum_{n=1}^{\infty} e^{-2\pi n x} = \frac{1}{1-e^{-2\pi x}} = \frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)} .$$

The function $\Phi(x)$ resp. the related power series coefficients in the form

$$b_{2n} := \int_1^{\infty} \Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}}$$

enable an alternative $\xi^*(s)$ –function representation with three $s \leftrightarrow (1-s)$ symmetric summands.

Main Theorem: For $s \neq \nu$, $\nu \in Z$, it holds

$$\xi^*(s) = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n} .$$

In proving the Main Theorem (MT) the essential step is the

Lemma MT: For $s \neq \nu$, $\nu \in Z$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)} = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - \int_1^{\infty}[x^s + x^{1-s}]\varphi(x)\frac{dx}{x}.$$

Note:

$$\frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)}$$

Note (Grl) 3.552: The function $\varphi(x)$ is related to the Bernoulli polynomials in the form

$$\int_0^{\infty} x^{2m}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x} = \frac{|B_{2m}|}{2m}, \quad \int_0^{\infty} x^{2m}\frac{e^{-\pi x}}{\cosh(\pi x)}\frac{dx}{x} = (1-2^{1-2m})\frac{|B_{2m}|}{2m} .$$

Note (Grl) 3.552, 9.521: For $Re(s) > 0$ it holds

$$\int_0^{\infty} x^s\varphi(x)\frac{dx}{x} = \frac{\Gamma(s)}{(2\pi)^s}\zeta(s,1) = \frac{1}{\sin(\pi s)}\left[\sin\left(\frac{\pi}{2}s\right)\sum_{n=1}^{\infty}\frac{\cos(2\pi n)}{n^{1-s}} + \cos\left(\frac{\pi}{2}s\right)\sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{n^{1-s}}\right]$$

$$= \frac{1}{2}\left[\frac{1}{\cos\left(\frac{\pi}{2}s\right)}\sum_{n=1}^{\infty}\frac{\cos(2\pi n)}{n^{1-s}} + \frac{1}{\sin\left(\frac{\pi}{2}s\right)}\sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{n^{1-s}}\right].$$

Note: Riemann's density function $J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^s\frac{ds}{s}$, $a > 1$, is zero for $0 \leq x < 2$, (EdH) 1.11. The functions $\Phi_n(x)$ enable a modified non-zero density function $J^*(x)$ for $1 \leq x < 2$:

$\Phi_n(x) \geq 0$ for $n \geq 2, x \geq 1$; $\Phi_1(x) < 0$ for $1 \leq x < 2$; $\Phi_1(2) = 0$; $\Phi_1(x) > 0$ for $x > 2$.
Putting $\Phi^*_{1,2}(x) := -\Phi_1(x)$, $\Phi^*_{0,1}(x) := -\Phi_1\left(\frac{1}{x}\right)$, for $1 \leq x < 2$, $\Phi^*_{1,2}(x) = \Phi^*_{0,1}(x) = 0$ for $x \geq 2$, $\Phi^*_{2,\infty}(x) := \Phi(x)$ for $x \geq 2$, $\Phi^*_{2,\infty}(x) = 0$ for $x < 2$, the three terms of the sum $\Phi^*_{0,1}(x) + \Phi^*_{1,2}(x) + \Phi^*_{2,\infty}(x) > 0$ have the disjunct domains $0 < x < 1$, $1 \leq x < 2$, $2 \leq x < \infty$.

Corollary: The set of non-trivial zeros $\left\{s_n = \frac{1}{2} + it_n\right\}$ of the zeta function are characterized by the following identity of two convergent series representations

$$\sum_{n=0}^{\infty} b_{2n}(s_n - \tfrac{1}{2})^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-z_n} + \frac{1}{(2n-1)+s_n}\right]$$

resp.

$$\sum_{n=0}^{\infty}(-1)^n b_{2n}t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{4n-1}{(2n-\frac{1}{2})^2+t_n^2}\right].$$

In case there would exist a negative value $t_n^{2n} < 0$, the affected term on the left side changes its sign, while the corresponding term on the right side would not. This proves the RH.

2. The proofs of the MT and the Lemma MT

Lemma MT: For $s \neq \nu$, $\nu \in Z$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)} = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\cos(\frac{\pi}{2}s)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - \int_1^{\infty}[x^s + x^{1-s}]\varphi(x)\frac{dx}{x}.$$

Proof: From (MiM) we recall the combination of the two formulae (MiM) 4.6 and 4.8 (valid for all $s \in C$)

$$\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} = \frac{1}{s-1} - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s} + \int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x} \quad , \quad \frac{\zeta(1-s)}{\sin(\frac{\pi}{2}(1-s))} = \frac{1}{-s} - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{(2n-1)+s} + \int_1^{\infty}x^s\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

With $\frac{1}{s-1} + \frac{1}{-s} = \frac{1}{s(s-1)}$ one gets

$$-\frac{1}{2}\frac{1}{s(s-1)} = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\sin(\frac{\pi}{2}(1-s))}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - \int_1^{\infty}[x^{1-s} + x^s]\varphi(x)\frac{dx}{x}.$$

Main Theorem: For $s \neq \nu$, $\nu \in Z$, it holds

$$\xi^*(s) = -\frac{1}{2}\left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\cos(\frac{\pi}{2}s)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s - \tfrac{1}{2})^{2n}.$$

Proof: From the lemma MT one gets

$$\xi^*(s) = \int_1^{\infty}\psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$$

$$= \int_1^{\infty}[\psi(x^2) - \varphi(x)][x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\sin(\frac{\pi}{2}(1-s))}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right]$$

$$= -\int_1^{\infty}\Phi(x)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\sin(\frac{\pi}{2}(1-s))}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right].$$

If the term $[x^s + x^{1-s}] = 2\sqrt{x}\left[\cosh\left(s - \frac{1}{2}\right)\log x\right]$ is expanded in the usual power series $\cosh(y) = \sum_{n=0}^{\infty}\frac{y^{2n}}{(2n)!}$ with $y := \left(s - \frac{1}{2}\right)\log x$ this gives $\int_1^{\infty}\Phi(x)[x^s + x^{1-s}]\frac{dx}{x} = 2\int_1^{\infty}\Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}(s - \frac{1}{2})^{2n}\right]\frac{dx}{\sqrt{x}}$. From this it follows

$$\xi^*(s) = -2\int_1^{\infty}\Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}(s - \tfrac{1}{2})^{2n}\right]\frac{dx}{\sqrt{x}} - \frac{1}{2}\left[\frac{\zeta(s)}{\sin(\frac{\pi}{2}s)} + \frac{\zeta(1-s)}{\sin(\frac{\pi}{2}(1-s))}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right]$$

$$= -2\sum_{n=0}^{\infty} b_{2n}(s - \tfrac{1}{2})^{2n} + \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\zeta(2n)\left[\frac{1}{2n-s} + \frac{1}{(2n-1)+s}\right].$$

3. Some relations to Kummer functions

The representation provided by the Main Theorem

$$\xi^*(s) = -\frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right)+\zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^{\infty} b_{2n}(s-\tfrac{1}{2})^{2n}$$

indicates an alternative entire Zeta function in the form

$$\xi^{**}(s) = \xi^*(s)\sin(\pi s) = (1-s)\frac{\sin(\pi s)}{\pi s}\left[\pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s}\right]\zeta(s)\ .$$

In the critical stripe the Mellin transform of the Kummer function ${}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)$ is given by, (GrI) 7.612,

$$M\left[\ {}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi x^2\right)\right](s) = \pi^{-\frac{s}{2}}\frac{\Gamma\left(1+\frac{s}{2}\right)}{1-s} \quad ,\ \ 0 < \mathrm{Re}(s) < 1\ .$$

Therefore, formally only, for $\omega(x) \coloneqq \sum_{n=1}^{\infty}\ {}_1F_1\left(\frac{1}{2};\frac{3}{2},-\pi n^2 x^2\right)$ the zeta function $\zeta(s)$ may be represented in the form

$$\frac{\zeta(s)}{1-s}\Gamma\left(1+\frac{s}{2}\right) = \int_0^{\infty} x^s\omega(x)\frac{dx}{x}\ .$$

Note: Riemann built his famous power series representation of his entire Zeta function $\xi(s) := \pi^{-\frac{s}{2}}\frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\zeta(s)$ by multiplication of $\xi^*(s) = \int_1^{\infty}\psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)}$ with $s(s-1)$ to govern the two poles of the term $-\frac{1}{2}\frac{1}{s(1-s)} = -\frac{1}{2}\frac{1}{s(1-s)} = -\frac{1}{2}\left[\frac{1}{s} + \frac{1}{1-s}\right]$; multiplication of this term with $\sin(\pi s)$ gives $\frac{1}{2}\frac{\sin(\pi s)}{s(1-s)} = \sin\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{2}s\right)\left[\frac{1}{s} + \frac{1}{1-s}\right] = \cos\left(\frac{\pi}{2}s\right)\left[\frac{\sin\left(\frac{\pi}{2}s\right)}{s} + \frac{\sin\left(\frac{\pi}{2}(1-s)\right)}{1-s}\right]$.

Note: The term $(1-s)$ resp. the term $\log(s-1)$ provides the principle term of the Riemann density function $J(x)$, i.e. $li_1(x) = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^s ds\ \ (a > 1)$, (EdH) 1.14.

Note: the term $\frac{\sin(\pi s)}{\pi s}$ is an entire function of order one and order type $\sigma = \pi/2$ with $\frac{\sin(\pi s)}{\pi s} = \prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right) = \prod_{n=-\infty}^{\prime\, n=\infty}\left(1 - \frac{s}{n}\right)e^{s/n}$, (LeB) p. 32.

4. References

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