A PROOF OF THE RIEMANN HYPOTHESIS

enabled by

a new integral and series representation of the meromorphic Zeta function occuring in the symmetrical form of the Riemann functional equation

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Dedicated to my wife Vibhuta
on the occasion of her 62th birthday, August 25, 2023

Abstract

For $s \neq \nu$, $(\nu \in \mathrm{Z})$ the meromorphic Zeta function

$$\xi^*(s) := \frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \int_1^\infty \psi(x^2)[x^s + x^{1-s}]\frac{dx}{x} - \frac{1}{2}\frac{1}{s(1-s)} = \xi^*(1-s)$$

is represented in the form

$$(*)\quad \xi^*(s) = \frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right) + \zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)} + \frac{1}{\pi}\sum_{n=0}^\infty (-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - 2\sum_{n=0}^\infty b_{2n}(s-\frac{1}{2})^{2n}$$

where

$$b_{2n} := \int_1^\infty \Phi(x)\left[\sum_{n=0}^\infty \frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}} \quad \text{and} \quad \Phi(x) := \sum_{n=1}^\infty (e^{-2\pi nx} - e^{-\pi n^2 x^2}) .$$

Correspondingly, the set of non-trivial zeros $\left\{z_n = \frac{1}{2} + it_n\right\}$ of the zeta function is characterized by two identical convergent (for $z_n^* \neq (2n - \frac{1}{2}) \pm it_n$) series representations

$$\sum_{n=0}^\infty b_{2n}(z_n - \frac{1}{2})^{2n} = \frac{1}{2\pi}\sum_{n=0}^\infty (-1)^n\left[\frac{\zeta(2n)}{2n-z_n} + \frac{\zeta(2n)}{(2n-1)+z_n}\right]$$

resp.

$$\sum_{n=0}^\infty (-1)^n b_{2n} t_n^{2n} = \frac{1}{2\pi}\sum_{n=0}^\infty (-1)^n\left[\frac{\zeta(2n)}{(2n-\frac{1}{2})-it_n} + \frac{\zeta(2n)}{(2n-\frac{1}{2})+it_n}\right]$$

from which it follows, that all t_n must be real. This proves the Riemann Hypothesis.

The poles of the representation (*) indicate an alternative entire Zeta function in the form

$$\xi^{**}(s) := \sin(\pi s)\,\xi^*(s) = \xi^{**}(1-s).$$

accompanied by the product representation

$$\xi^{**}(s) = \zeta(s)\pi^{1-\frac{s}{2}}\prod_{n=1}^\infty \frac{(1-\frac{s^2}{n^2})}{(1+\frac{s}{2n})} e^{\frac{s}{2}\left[\frac{1}{n}-\gamma\right]} .$$

Regarding Riemann's method for deriving the formula for the prime number density function $J(x)$ the representation of $\xi^{**}(s)$ in the form

$$\xi^{**}(s) = \sin(\pi s)\,\xi^*(s) := (1-s)\sin(\pi s)\left[\frac{1}{2}\pi^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}\right]\zeta(s)$$

ensures the link to the $li_1(x)$ − function, while the term $[\cdot\,\cdot]$ provides the link to the Mellin transform of the Kummer function ${}_1F_1\left(\frac{1}{2};\frac{3}{2},-x^2\right)$, which is restricted to the critical stripe domain. This enables the definition of a prime number density function, which is composed of $J(x)$, $(x > 1)$, and $J^*(x)$, $(0 < x < 1)$ (*).

(*) The concept is in line with the proposed Kummer function based Zeta function theory and a related alternative two-semicircle method to the Hardy-Littewood (major/minor arcs) circle method as proposed in (BrK).

1. Notations and Summary

For the notations we refer to (EdH). The baseline function for the Zeta function theory is given by, (EdH) 1.7,

$$\psi(x):=\sum_{n=1}^{\infty}e^{-\pi n^2 x}.$$

It is related to Jacobi's functional equation of the theta function ϑ enabling the symmetrical form of Riemann's functional equation in the form

$$\xi^*(s):=\frac{1}{2}\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s)=\int_1^\infty\psi(x^2)[x^s+x^{1-s}]\frac{dx}{x}-\frac{1}{2}\frac{1}{s(1-s)}=\xi^*(1-s)\ .$$

Our proposed alternative baseline function for the Zeta function theory is defined by

$$\Phi(x):=\sum_{n=1}^{\infty}(e^{-2\pi nx}-e^{-\pi n^2x^2})$$

accompanied by the series

$$b_{2n}:=\int_1^\infty\Phi(x)\left[\sum_{n=0}^{\infty}\frac{\log^{2n}(x)}{(2n)!}\right]\frac{dx}{\sqrt{x}}\ .$$

The main result of our paper is

Main Theorem: For $s\neq\nu$, $(\nu\in Z)$ it holds

$$\xi^*(s)=\frac{\zeta(s)\sin\left(\frac{\pi}{2}(1-s)\right)+\zeta(1-s)\sin\left(\frac{\pi}{2}s\right)}{\sin(\pi s)}+\frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s}+\frac{\zeta(2n)}{(2n-1)+s}\right]-2\sum_{n=0}^{\infty}b_{2n}(s-\tfrac{1}{2})^{2n}\ .$$

Corollary: The set of non-trivial zeros $\left\{z_n=\frac{1}{2}+it_n\right\}$ of the zeta function is characterized by the two identical convergent series representations

$$\sum_{n=0}^{\infty}b_{2n}(z_n-\tfrac{1}{2})^{2n}=\frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-z_n}+\frac{\zeta(2n)}{(2n-1)+z_n}\right].$$

resp. (*)

$$\sum_{n=0}^{\infty}(-1)^n b_{2n}t_n^{2n}=\frac{1}{2\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{(2n-\frac{1}{2})-it_n}+\frac{\zeta(2n)}{(2n-\frac{1}{2})+it_n}\right].$$

Corollary: Regarding the two sets $\left\{2n-\frac{1}{2}\right\}$ and $\{t_n\}$ governed by the two types of symmetrical representations in (*) the set $\{t_n\}$ must be real, which proves the Riemann Hypothesis.

Remark: The poles of the meromorph $\xi^*(s)$-function representation indicate an alternative entire Zeta function in the form

$$\xi^{**}(s):=\sin(\pi s)\,\xi^*(s)$$

accompanied by the set of additional trivial zeros $\{\nu\}_{\nu\in Z}$.

Remark: The product representation of the entire Zeta function $\xi^{**}(s)$ is given by

$$\xi^{**}(s) = \sin(\pi s)\,\xi^{*}(s) = \pi^{1-\frac{s}{2}}\zeta(s)\prod_{n=1}^{\infty}\frac{(1-\frac{s^2}{n^2})}{(1+\frac{s}{2n})}e^{\frac{s}{2}\left[\frac{1}{n}-\gamma\right]}$$

which follows from the product representations, (LeB) p. 32:

i) $\quad \sin(\pi s) = 2\sin\left(\frac{\pi}{2}s\right)\cos\left(\frac{\pi}{2}s\right) = \pi s\prod_{n=1}^{\infty}(1-\frac{s^2}{n^2})$

ii) $\quad \Gamma\left(\frac{s}{2}\right) = \frac{2}{s}e^{-\gamma s/2}\prod_{n=1}^{\infty}\frac{e^{s/(2n)}}{(1+\frac{s}{2n})}$.

Remark: Technically speaking, Riemann built his entire Zeta function

$$\xi(s) := \pi^{-\frac{s}{2}}\frac{s}{2}\Gamma\left(\frac{s}{2}\right)(s-1)\zeta(s)$$

by multiplication of $\xi^{*}(s)$ with $s(s-1)$ to govern the two poles of the term $-\frac{1}{2}\frac{1}{s(1-s)}$. The proposed alternative entire Zeta function $\xi^{**}(s)$ is built by multiplication with $\sin(\pi s)$ accompanied by infinite poles at $s = \nu \in \mathbb{Z}$.

Remark: The principle term $\log(s-1)$ in Riemann's method for deriving the formula for the prime number density function $J(x)$ by substituting

$$\log\zeta(s) = \log\xi(s) = \log\pi^{-\frac{s}{2}} - \log\Gamma\left(1+\frac{s}{2}\right) - \log(s-1)$$

into

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^{s}\frac{ds}{s},\ (a>1)$$

results into the $li_1(x)-$ function, (EdH) 1.14,

$$li_1(x) := \lim_{\varepsilon\to 0}\int_0^{1-\varepsilon}\frac{dt}{\log t} + \int_{1+\varepsilon}^{x}\frac{dt}{\log t} = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log(s-1)}{s}\right]x^{s}ds\ \ (a>1).$$

The condition $a>1$ is a consequence of the Fourier inverse function

$$\frac{\log\zeta(s)}{s} = \int_0^{\infty}J(x)x^{-s-1}dx\ ,\ Re(s)>1.$$

Remark: Technically speaking, the formula for the prime number density function $J(x)$ is governed by the zeta function with domain outside of the critical stripe. Regarding the proposed alternative entire Zeta function $\xi^{**}(s)$ in order to keep the link to the $li_1(x)-$ function, while at the same time providing an additional link to the critical stripe domain of the zeta function, we propose the following „telescope" product representation

$$\xi^{**}(s) = \sin(\pi s)\,\xi^{*}(s) := (1-s)\sin(\pi s)\left[\frac{1}{2}\pi^{-\frac{s}{2}}\frac{\Gamma\left(\frac{s}{2}\right)}{1-s}\right]\zeta(s).$$

The term $(1-\mathrm{s})$ ensures the link to the $li_1(x)-$ function. The term $[\cdot\,\cdot]$ relates to the Mellin transform of the Kummer function ${}_1\mathrm{F}_1\left(\frac{1}{2};\frac{3}{2},-\mathrm{x}^2\right)$ restricted to the critical stripe domain. The corresponding Mellin transform is given by the

Lemma (Grl) 7.612: For $0<\mathrm{Re}(\mathrm{s})<1$ it holds

$$\frac{1}{2}\int_0^\infty \mathrm{x}^{\frac{s}{2}}\,{}_1\mathrm{F}_1\left(\frac{1}{2};\frac{3}{2},-\mathrm{x}\right)\frac{\mathrm{dx}}{\mathrm{x}}=\int_0^\infty \mathrm{x}^{\mathrm{s}}\,{}_1\mathrm{F}_1\left(\frac{1}{2};\frac{3}{2},-\mathrm{x}^2\right)\frac{\mathrm{dx}}{\mathrm{x}}=\frac{1}{2}\frac{\Gamma(\frac{s}{2})}{1-\mathrm{s}}.$$

Technically speaking, the term $[\cdot\,\cdot]$ enables the definition of an additional density function governed by the zeta function in the critical stripe.

2. Proof of the Main Theorem

Main Theorem: For $\mathrm{s}\neq\nu$, $(\nu\in\mathrm{Z})$ it holds

$$\xi^*(\mathrm{s})=\frac{\zeta(\mathrm{s})\sin\left(\frac{\pi}{2}(1-\mathrm{s})\right)+\zeta(1-\mathrm{s})\sin\left(\frac{\pi}{2}\mathrm{s}\right)}{\sin(\pi\mathrm{s})}+\frac{1}{\pi}\sum_{\mathrm{n}=0}^\infty(-1)^{\mathrm{n}}\left[\frac{\zeta(2\mathrm{n})}{2\mathrm{n}-\mathrm{s}}+\frac{\zeta(2\mathrm{n})}{(2\mathrm{n}-1)+\mathrm{s}}\right]-2\sum_{\mathrm{n}=0}^\infty \mathrm{b}_{2\mathrm{n}}(\mathrm{s}-\tfrac{1}{2})^{2\mathrm{n}}.$$

Remark: In proving this theorem the essential step is the following Lemma MT. Its proof in the following section is based on a novel integral and series representation of the Riemann zeta function $\zeta(s)$ as provided in (MiM).

Lemma MT: For $\mathrm{s}\neq\nu$, $\nu\in\mathrm{Z}$, it holds

$$-\frac{1}{2}\frac{1}{\mathrm{s}(1-\mathrm{s})}=\frac{1}{2}\left[\frac{\zeta(\mathrm{s})}{\sin\left(\frac{\pi}{2}\mathrm{s}\right)}+\frac{\zeta(1-\mathrm{s})}{\cos\left(\frac{\pi}{2}\mathrm{s}\right)}\right]+\frac{1}{\pi}\sum_{\mathrm{n}=0}^\infty(-1)^{\mathrm{n}}\left[\frac{\zeta(2\mathrm{n})}{2\mathrm{n}-\mathrm{s}}+\frac{\zeta(2\mathrm{n})}{(2\mathrm{n}-1)+\mathrm{s}}\right]-\int_1^\infty[\mathrm{x}^{\mathrm{s}}+\mathrm{x}^{1-\mathrm{s}}]\frac{1}{2}\frac{\mathrm{e}^{-\pi\mathrm{x}}}{\sinh(\pi\mathrm{x})}\frac{\mathrm{dx}}{\mathrm{x}}.$$

Before proving Lemma MT we show that Main Theorem is a consequence. With

$$\varphi(\mathrm{x}):=\frac{1}{2}\frac{\mathrm{e}^{-\pi\mathrm{x}}}{\sinh(\pi\mathrm{x})}=\frac{1}{\mathrm{e}^{2\pi\mathrm{x}}-1}=\sum_{\mathrm{n}=1}^\infty \mathrm{e}^{-2\pi\mathrm{n}\mathrm{x}},\ \mathrm{x}>1\ ,^{(*)}$$

the third summand is a consequence of the identity

$$-\int_1^\infty \Phi(\mathrm{x})[\mathrm{x}^{\mathrm{s}}+\mathrm{x}^{1-\mathrm{s}}]\frac{\mathrm{dx}}{\mathrm{x}}=\int_1^\infty \psi(\mathrm{x}^2)[\mathrm{x}^{\mathrm{s}}+\mathrm{x}^{1-\mathrm{s}}]\frac{\mathrm{dx}}{\mathrm{x}}-\int_1^\infty[\mathrm{x}^{\mathrm{s}}+\mathrm{x}^{1-\mathrm{s}}]\varphi(\mathrm{x})\frac{\mathrm{dx}}{\mathrm{x}}.$$

Analogue to Riemann's approach deriving his famous power series representation for $\xi(\mathrm{s})$, (EdH) 1.8 (**), one gets for $\mathrm{b}_{2\mathrm{n}}:=\int_1^\infty \Phi(\mathrm{x})\left[\sum_{\mathrm{n}=0}^\infty\frac{\log^{2\mathrm{n}}(\mathrm{x})}{(2\mathrm{n})!}\right]\frac{\mathrm{dx}}{\sqrt{\mathrm{x}}}$ the power series representation

$$-\int_1^\infty \Phi(\mathrm{x})[\mathrm{x}^{\mathrm{s}}+\mathrm{x}^{1-\mathrm{s}}]\frac{\mathrm{dx}}{\mathrm{x}}=-2\sum_{\mathrm{n}=0}^\infty \mathrm{b}_{2\mathrm{n}}(\mathrm{s}-\tfrac{1}{2})^{2\mathrm{n}}.$$

(*) $\varphi\left(\frac{1}{x}\right)$, $0<x<1$; (Grl) 3.552: $\int_1^\infty x^{2m}\varphi(x)\frac{dx}{x}=\frac{|B_{2m}|}{2m}$; (PoG) p. 65: $f(t)>0, f'(t)<0, f''(t)<0$ for $0\leq t\leq 1$, then the even function $F(z)=\int_0^1 f(t)\cos(zt)dt$ has infinite many, only real zeros

(**) $[\mathrm{x}^{\mathrm{s}}+\mathrm{x}^{1-\mathrm{s}}]=2\sqrt{\mathrm{x}}\left[\cosh\left(\mathrm{s}-\frac{1}{2}\right)\log\mathrm{x}\right]$ and $\cosh(\mathrm{y})=\sum_{\mathrm{n}=0}^\infty\frac{\mathrm{y}^{2\mathrm{n}}}{(2\mathrm{n})!}$ with $\mathrm{y}:=\left(\mathrm{s}-\frac{1}{2}\right)\log\mathrm{x}$.

3. Proof of Lemma MT

Lemma MT: For $s \neq \nu$, $\nu \in \mathbb{Z}$, it holds

$$-\frac{1}{2}\frac{1}{s(1-s)} = \frac{1}{2}\left[\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} + \frac{\zeta(1-s)}{\cos\left(\frac{\pi}{2}s\right)}\right] + \frac{1}{\pi}\sum_{n=0}^{\infty}(-1)^n\left[\frac{\zeta(2n)}{2n-s} + \frac{\zeta(2n)}{(2n-1)+s}\right] - \int_1^{\infty}[x^s + x^{1-s}]\frac{1}{2}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

The Lemma is a consequence of the integral and series representations

$$\frac{\zeta(s)}{\sin\left(\frac{\pi}{2}s\right)} = \frac{1}{s-1} - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s} + \int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}$$

$$\frac{\zeta(1-s)}{\sin\left(\frac{\pi}{2}(1-s)\right)} = \frac{1}{-s} - \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{(2n-1)+s} + \int_1^{\infty}x^{s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x}.$$

as provided in (MiM) in section 4 *„Special cases, 4.1 The case $c = 0$“*

For the special case $c = 0$ the integral

$$\zeta(s) = -\pi^{s-1}\frac{\sin\left(\frac{\pi}{2}s\right)}{s-1}\int_0^{\infty}\frac{x^{1-s}}{\sinh^2(x)}dx\,, \qquad Re(s) < 0 \qquad \textit{(MiM) (4.1)}$$

can be broken into two parts $\zeta(s) = \zeta_0(s) + \zeta_1(s)$ where

$$\zeta_1(s) = \frac{\sin\left(\frac{\pi}{2}s\right)}{s-1} + \sin\left(\frac{\pi}{2}s\right)\int_1^{\infty}x^{1-s}\frac{e^{-\pi x}}{\sinh(\pi x)}\frac{dx}{x} \qquad \textit{(MiM) (4.6)}$$

$$\zeta_0(s) = -\frac{2}{\pi}\sin\left(\frac{\pi}{2}s\right)\sum_{n=0}^{\infty}(-1)^n\frac{\zeta(2n)}{2n-s} \qquad \textit{(MiM) (4.8)}$$

which are both valid for all s.

4. References

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