# A toolbox
# to solve the Riemann Hypothesis
# &
# to build a non-harmonic Fourier series based two-semicircle method

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August 25, 2021
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Dedicated to my wife Vibhuta
on the occasion of her 60th birthday, August 25, 2021

Contents

## Scope of application

This paper provides a toolbox for a newly proposed Kummer-function based Zeta function theory to prove the RH and to enable a non-harmonic Fourier series based two-half-circles method (replacing the major/minor arcs division of the Hardy-Littlewood circle method) to solve open binary number theoretical problems.

Technically speaking a missing appropriate approximating Paley-Wiener function to the entire Zeta function $\xi(s)$ is the root of evil of the only conditional convergent term $\sum_{Im(\rho)>0} Li(x^{\rho}) + Li(x^{1-\rho})$ in the formula for the Riemann density $J(x)$. The zeros $y_n$ of the even Zeta function $\Xi(s) := \xi(1/2 + is)$ enable the product representations $\Xi(s) = \Xi(0) \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{y_n^2}\right)$. Its related (absolute value) counterpart $H(s) := \Xi(0) \prod_{n=1}^{\infty} (1 - \frac{s^2}{\lambda_n^2})$, (with $\lambda_n := |y_n|$ and $\sum_1^{\infty} \lambda_n^{-2} < \infty$), is symmetrical to the x-axis, but w/o information about a critical line symmetry. In a Kummer-function based Zeta function theory $H(s)$ is replaced by Paley-Wiener functions in the form

$$\frac{1}{2} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\kappa_n}\right)\left(1 - \frac{s}{1-\kappa_n}\right), \frac{1}{2} \prod_{n=1}^{\infty} \left(1 - \frac{s}{\kappa_n}\right)\left(1 - \frac{s}{\sigma_n}\right), \quad \kappa_n := \frac{1}{4} + 2\pi i \omega_n, \ \sigma_n := \frac{3}{4} + 2\pi i w_n{}^{(*)},$$

enabling an appropriate analysis of the critical, only conditional convergent term $\sum_{Im(\rho)>0} Li(x^{\rho}) + Li(x^{1-\rho})$.

Putting $f(x) := e^{-\pi x^2}$ the entire Zeta function $\xi(s)$ is given by, (EdH) 1.8,

$$(*) \quad \xi(s) := \frac{s}{2} \Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = (1-s)\zeta(s) M[-xf'(x)](s) = \xi(1-s).$$

The alternatively proposed entire Zeta function is based on the Hilbert transform of $f(x)$, defined by

$$\xi^*(s) := (1-s)\zeta(s) M[f_H(x)](s) = \frac{1}{2}(1-s)\zeta(s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \tan(\frac{\pi}{2}s)$$

with the related duality equation

$$\xi^*(s) \cot(\frac{\pi}{2}s) = \xi^*(1-s) \cot(\frac{\pi}{2}(1-s)).$$

It enables a product representation in the form

$$\zeta(s)\zeta(1-s) = \sqrt{\pi}\xi^*(s)\xi^*(1-s) \frac{\sin\left(\frac{\pi}{2}s\right)}{\frac{\pi}{2}s} \Gamma\left(1 - \frac{s}{2}\right) \frac{\sin\left(\frac{\pi}{2}(1-s)\right)}{\frac{\pi}{2}(1-s)} \Gamma\left(1 - \frac{1-s}{2}\right),$$

where the left-hand side represents the Euler product in the form

$$\zeta(s)\zeta(1-s) = \prod_p \frac{1}{1-p^{-s}} \prod_q \frac{1}{1-q^{s-1}}, \quad Re(s) > 1$$

resp.

$$\zeta\left(\frac{1}{2}+s\right)\zeta\left(\frac{1}{2}-s\right) = \prod_p \frac{\sqrt{p}}{\sqrt{p}-p^{-s}} \frac{\sqrt{p}}{\sqrt{p}-p^{s}} \quad Re(s) > \frac{1}{2}.$$

The right-hand side is linked to several Paley-Wiener type product representations, where the most prominent example is given by, (LeB) p. 32,

$$\sin(\frac{\pi}{2}s) = (\frac{\pi}{2}s) \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{4n^2}\right).$$

The zeros $\alpha_n$ of the Digamma function enable the product representation, (MeI),

$$\frac{\Psi(\frac{s}{2})}{\Gamma(\frac{s}{2})} \cdot \frac{\Psi\left(-\frac{s}{2}\right)}{\Gamma\left(-\frac{s}{2}\right)} = \prod_{n=0}^{\infty} \left(1 - \frac{s}{2\alpha_n}\right)\left(1 + \frac{s}{2\alpha_n}\right).$$

The zeros $z_n$ of the Kummer functions ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right)$ are the zeros of ${}_1F_1\left(1; \frac{3}{2}, -z\right)$, enabling a product representation in the form

$${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right) \ {}_1F_1\left(1, \frac{3}{2}; z\right) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right)\left(1 + \frac{z}{z_n}\right).$$

For the related Whittaker function $M(z) := z^{-\frac{1}{4}} e^{-\frac{z}{2}} M_{\frac{1}{4},\frac{1}{4}}(z) = z^{\frac{1}{2}} \ {}_1F_1\left(1, \frac{3}{2}, -z\right)$ the one-side Laplace transform is given by

$$\int_0^{\infty} t^{\frac{s}{2}} M(nt) \frac{dt}{t} = n^{-\frac{s}{2}} \frac{\sqrt{\pi}}{2} \Gamma\left(\frac{s}{2}\right) \tan(\frac{\pi}{2}s), \qquad -1 < Re(s) < 1.$$

It enables an alternative definition of Riemann's contour integral representation

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}.$$

(*) With respect to the RH-Polya criterion, (CaD),(PoG), we note that if $G(iz)$ takes real values for real $z$, and all zeros of $G(iz)$ are real, then also the function $G(i(z+a)) + G(i(z-a))$ has only real zeros for any positive constant $a$.

Riemann's contour integral representation is derived from the integral representation of the Dirichlet series

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}=\frac{1}{\Gamma(1-s)}\int_0^{\infty}\frac{x^s}{e^x-1}\frac{dx}{x}\ ,\ Re(s)>1.$$

For the 1st derivative of the Hilbert transform of the fractional part function it holds

$$\rho_H'(x)=-2\sum_1^{\infty}\sin 2\pi\nu x=-\cot(\pi x)\in H_{-1}^{\#}(0,1).$$

Correspondingly, in the critical stripe the „function"

$$\zeta^*(s):=\frac{1}{2}M[\rho_H'(x)](1-s)=(2\pi)^{s-1}\Gamma(1-s)\cos(\frac{\pi}{2}s)\zeta(1-s)$$

is defined as a complex-valued $H_{-1}^{\#}(0,1)$ based distributional Mellin integral representation.
For $h_n:=\sum_{k=1}^{n}\frac{1}{2k-1}$ one gets

$$\zeta_h^*(s):=\sum_{n=1}^{\infty}\frac{h_n}{n}\frac{1}{n^s}=\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}(\sum_{n=1}^{\infty}\frac{h_n}{n}e^{-nx})\,dx \text{ for } Re(s)>0.$$

The „function" $\zeta_h^*(s)$ can be interpreted as a complex-valued $L_2^{\#}(0,1)$ based approximation distributional integral representation of $\zeta^*(s)$. The related density function of $\zeta_h^*(s)$ is given in the appreciated form

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\zeta_h^*(s)x^s\frac{ds}{s}=\sum_{n=1}^{x}\frac{h_n}{n}\log(\frac{x}{n}).$$

We note that in the product representation of $\zeta(s)\zeta(1-s)$ the term $(1-s)$, generating the $li(x)$ term, „is consumed" by one of the product representations. Putting

$$f_1(x):=\log(\pi\sin(\frac{\pi}{2}x))dx\ ,\quad f_2(x):=\log(\frac{\pi}{2}\tan\left(\frac{\pi}{2}x\right))$$

then for any divergent sequence of positive numbers $0<a_1^{(k)}\le a_2^{(k)}\le a_3^{(k)}\le\cdots.,\ (k=1,2)$ it holds, (PoG1)

$$(+)\ \lim_{n\to\infty}\frac{1}{\log\pi-\log 2}\frac{\log n}{n}\sum_{i\le n}f_k(\frac{a_i^{(k)}}{n})=1=\int_0^1 f_k(t)dt.$$

In other words, for appropriately defined sequences $\{a_i^{(k)}\}_{i\in N}$, $k=1,2,$ there is a pair of two number theoretical distribution functions each with asymptotics $\sim n/\log n$.

The distribution functions pair $(f_1(x),f_2(x))$ is supposed as the appropriate tool

i) to define an alternative $li(x)$ function approximation term $\pi^*(x)$ of $\pi(x)$, simplifying the proof of the RH criterion based on the term $[li(x)-\pi^*(x)]+[\pi^*(x)-\pi(x)]=O(\sqrt{x}\log x)$

ii) to prove the binary Goldbach conjecture based on the term $\pi^*(2x-p)$.

The building of the sequences $\{a_i^{(k)}\}_{i\in N}$, $k=1,2,$ is based on the zeros of the considered Kummer functions and the Digamma function. They enable gaps and density theorems dealing with the determination of the rate of growth of analytic functions from their growth on sequences of points $a_n$:

All zeros $\{z_n\}_{n\in N}$ of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ are complex-valued. For the absolute values of their imaginary parts $|Im(z_n)|=2\pi\omega_n$ and the negative zeros $w_n$ of the Digamma function it holds

$$(++)\ \ n-\frac{1}{2}<\omega_n,-w_n<n \quad \text{with} \quad 2\pi\omega_n:=|Im(z_n)|.$$

**Remark**: the Kadec $1/4$ –theorem provides the link to the theory of Paley-Wiener functions; those are entire functions of exponential type at most $\pi$ that are square integrable on the real axis, which are isomorphic to the functions of the Hilbert space $L_2^{\#}(-\pi,\pi)$.

**Remark**: Appropriately condensed sequences of $\omega_n,-w_n$, e.g., in the form

$$\lambda_\nu:=\frac{1}{4}+\frac{\omega_\nu-w_\nu}{2},\lambda_0:=0,\nu\in Z,$$

"close" the Hadamard gap of indices sequences and fulfill the condition of the Kadec $1/4$ –theorem.

**Remark**: A striking generalization of the Kadec theorem is Avdonin's $1/4$ –in-the-mean theorem. We note that the Avdonin condition $\lambda_n=n+\delta_n$ is fulfilled by the sequenc$-w_n\sim n-1/\log n$.

**Remark** (KaM) p. 53 ff., (MaK), p. 46, (MoH) p. 34: the Schnirelmann density is analogous to a "relative measure", and like relative measure it is not completely additive. However, the multiplicative distribution function property can be applied to define "independent events". The related "relative statistical independent" density functions (*) are accompanied by the concepts of a "relative measure", a "mean value of a function", a "relative conditional probability" and the concept of "intersective sets".

Putting $b_n^{(1)} := \omega_n$ , $b_n^{(2)} := \frac{1}{2} - w_n$ it follows from (++)

$$2n-1 < a_n^{(1)} := \frac{1}{n}\sum_{k=1}^{2n} b_k^{(1)} < 2n+1 < a_n^{(2)} := \frac{1}{n}\sum_{k=1}^{2n} b_k^{(2)} < 2n+3 \ .$$

Putting $F^{(k)}(x) := \frac{1}{\log(\pi/2)}\sum_{p\le[x]-1} f_k\left(\frac{a_p^{(k)}}{p}\right)$ and $\pi^*(x) := \frac{1}{2}\left(F^{(1)}(x) + F^{(2)}(x)\right)$ it follows with (+)

$$\pi^*(x) \sim \frac{n}{logn} \sim \pi(x), \ n = [x].$$

In other words, the RH criterion $li(x) - \pi(x) = O(\sqrt{x}logx)$ can be reformulated into

$$[li(x) - \pi^*(x)] + [\pi^*(x) - \pi(x)] = O(\sqrt{x}logx).$$

Let $G_{2n}$ denote the number of divisions of the even integer $2n = p + q$ into a sum of two primes, where $p + q$ and $q + p$ are counted as two divisions and $\pi(2n - u) = 0$ for $u > 2(n-1)$. Then the solution of the binary Goldbach conjecture is about an appropriate asymptotics of, (LaE2),

$$H(2x) := \sum_{n=1}^{[2x]} G_{2n} = \sum_{p\le 2x} \pi(2x - p) \text{ with } H(2x) \sim \frac{1}{2}(\frac{2x}{\log(2x)})^2.$$

The approximation formula $\pi^*(x)$ enables the definition of corresponding density function in the form

$$H^*(2x) := \sum_{n=1}^{[2x]} G_{2n}^* = \sum_{p\le 2x} \pi^*(2x - p) \sim \frac{1}{2}(\frac{2x}{\log(2x)})^2 \ .$$

In contrast to the Stäckel approximation formula, (LaE2)

$$\tilde{G}_n := \frac{\pi^4}{105\zeta(3)}\frac{1}{\varphi(n)}(\frac{n}{log\ n})^2 \text{ with } \sum_{n=1}^{\left[\frac{x}{2}\right]} \tilde{G}_{2n} \sim \frac{1}{2}(\frac{x}{\log\ x})^2$$

the identical asymptotics of $H(2x)$ and $H^*(2x)$ are accompanied by appropriate error estimates of the underlying summands $G_{2n} - G_{2n}^*$.

**Remark**: in the framework of the Hardy-Littlewood circle method the number of solutions of the binary Goldbach problem equation $2n = p + q$ is equal to

$$R(n) = \int_0^1 \left[\sum_{p\le 2n} e^{-2\pi i p\alpha}\right]\left[\sum_{p\le 2n} e^{-2\pi i p\alpha}\right] e^{-2\pi i(2n)\alpha} d\alpha, \text{ since } \int_0^1 e^{-2\pi i\alpha x}dx = \begin{cases} 1 & if\ \alpha=0 \\ 0 & if\ \alpha\neq 0 \end{cases}.$$

**Remark**: the Fourier series representations of the considered functions $f_1(x)$ , $f_2(x)$ are

$$\frac{\pi}{2}log\left(tan\frac{\pi}{2}x\right) = -\sum_{n=1}^{\infty}\frac{h_n}{n}sin(2\pi nx) \in L_2^{\#}(0,1) \quad , \quad log\,\pi\,sin(\frac{\pi}{2}x) = log\frac{\pi}{2} - \sum_1^{\infty}\frac{cos\,\pi nx}{n} \in L_2^{\#}(0,2) \ .$$

**Remark**: the above permits a re-examination of Kummer's proof of Fermat's theorem for regular primes (EdH1), (KuE4), the Kummer conjecture (*) and the related contribution in (HeD). We note that the set of the larger Hurwitz quaternion integers is a Euclidian ring domain, the norm of any prime ideal is a prime number, and the Hurwitz quaternion integers ($\mathrm{HQI}$) with $24$ units are accompanied by a $\mathrm{HQI}_{even} \otimes \mathrm{HQI}_{odd}$ decomposition, (HuA) p. 23, allowing an extended Hurwitz-Schnirelmann density concept, (**), see also (BrH), (HeE).

(*) (HaH) p. 457, i.e. "prime number classes $p_1, p_3, p_5$ containing infinite primes with densities $\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right) = (\frac{3\cdot 4}{24},\frac{2\cdot 4}{24},\frac{1\cdot 4}{24})$", enabled by a proposed "(Euler; Kummer) pairing", built on pairs of odd integer $(4k-3; 4k-1)$ (where "$4k-1$" may be split into odd quadrat numbers and rest) and related $(complex;\ quaternionic)$-norms for unique prime factorizations, (StU). For prime pairs $(\mathrm{p} = 4k+1; \mathrm{q} = 4k+3)$ we note $(\varphi(\mathrm{p}) = \mathrm{p} - 1 = \mathrm{e}\cdot f_{even}; \varphi(\mathrm{q}) = \mathrm{q} - 1 = \mathrm{e}\cdot f_{odd})$ with $\mathrm{e} = 2$ and $k \ge 1$, i.e., the roots of the related Gaussian two-part $f_* -$ periods are all real resp. are all imaginary, (WeH) §77.

Related tools are the "n-Kummer (Galois) extensions", (LaS) & (MoP), and "Chevalley's lost key of idéles (ideal elements)" (**), (NeJ), (MiJ). The latter one avoids Dirichlet series density asymptotics with a Euler function $\varphi(n)$ factor, (MiJ) pp. 154 ff, which also plays a key role in the Stäckel approximation formula for the number of prime pairs $(p,q)$ with $2n = p + q$, (LaE2).

We mention Kummer's analogy to chemistry, (KuE1), where the molecules correspond to the multiplication of complex numbers, while their atom weights correspond to the prime factors.

(**) We note that the Hurwitz quaternions "basis" is in line with the cubic characters based Gaussian sums framework of the Kummer conjecture, and the "elliptic curves over Q" world. In the context of the concepts "spin", "dual turns", line geometry & kinematics, and "quaternions" we refer to Study's "movement indicator", (BlW) §58. For a related combinatorial interpretation of the Rogers-Ramanujan identities we refer to (BeB3).

**The Riemann $\zeta(s)$ function and the Riemann density function $J(x)$**

For $Re(s) > 1$ the Riemann $\zeta(s)$ function is given by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Mirroring the representation

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{\Gamma(1-s)} \int_0^{\infty} \frac{x^s}{e^x - 1} \frac{dx}{x} \text{ , } Re(s) > 1$$

leads the contour integral representation in the form

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{+\infty}^{\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} \text{ ,}$$

which is defined for all $s \in C$, except a simple pole for $s = 1$, which is the simple pole of $\Gamma(1-s)$, (EdH) 1.4.

The trivial zeros of $\zeta(s)$ are derived from the functional equation, (ApT) p. 266,

$$\zeta(1-s) = 2(2\pi)^{-s}\Gamma\left(\frac{s}{2}\right) cos\ (\frac{\pi}{2}s)\zeta(s),$$

where the zeros of $\zeta(-2n) = 0$ are the zeros of the term $cos\ (\frac{\pi}{2}s)$ with $\zeta(1-s) = 1 + 2n$.

**Remark**: This functional equation is derived from the Hurwitz formula, (ApT) p. 257, for $\sigma > 1$, resulting into

$$\zeta(1-s) = \frac{\Gamma\left(\frac{s}{2}\right)}{(2\pi)^s}\left\{e^{-\pi i s/2}\zeta(s) + e^{\pi i s/2}\zeta(s)\right\}\frac{\Gamma\left(\frac{s}{2}\right)}{(2\pi)^s} 2cos\ (\frac{\pi}{2}s)\zeta(s).$$

**Remark**: We note that the Hurwitz formula involves the periodic Zeta function, represented as the Dirichlet series

$$F(x,s) := \sum_{n \le x} \frac{e^{2\pi i n x}}{n^s} \text{ with } F(1,s) = \zeta(s).$$

**Lemma:**

i) $\lim_{s \to 1}(s-1)\zeta(s) = \gamma$, (TiE) 2.1.16

ii) $\zeta(s) - \frac{1}{s-1} = 1 - \frac{1}{2}s\{\zeta(s+1) - 1\} - \frac{s(s+1)}{2 \cdot 3}\{\zeta(s+2) - 1\} - \cdots$ , (TiE) 2.14

iii) $\zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|)$ , $\lim_{n\to\infty} \sum_{m=1}^{n}(\frac{1}{m+1} + 1 - logn) = \gamma$ , (TiE) 2.1.16

iv) $\zeta(s) = \frac{s}{s-1} + s\int_1^{\infty}([t] - t)t^{-s-1}dt$ and therefore $log|(1-s)\zeta(s)| \ll log|s| + 1$.

**Lemma**:

i) $\zeta'(0) = -\frac{1}{2}log(2\pi), -\frac{\zeta'(0)}{\zeta(0)} = -log(2\pi)$, (TiE) 2.4.5

ii) $(s-1)\zeta(s) = e^{bs}\frac{1}{\Gamma\left(1+\frac{s}{2}\right)}\prod_{\rho}\left(1 - \frac{s}{\rho}\right)e^{\frac{s}{\rho}}$, where $b := log(2\pi) - 1 - \frac{\gamma}{2}$ (TiE) 2.12.6.

For the Riemann's prime number distribution function

$$J(x) := \frac{1}{2}\left[\sum_{p^n<x}\frac{1}{n}+\sum_{p^n\leq x}\frac{1}{n}\right]$$

and the related von Mangoldt's density function

$$\psi(x) := \sum_{n\leq x}\Lambda(x)$$

the following asymptotics are valid

$$J(x)=\sum_{n\leq x}\frac{\Lambda(n)}{\log n}\approx\frac{x}{\log x}\,,\ \psi(x)\approx x.$$

Their relationships to the Riemann zeta function are given by

$$\log\zeta(s)=s\int_0^\infty x^{-s-1}J(x)dx=\int_0^\infty x^{-s}dJ(x)$$

$$-\frac{\zeta'(s)}{\zeta(s)}=s\int_0^\infty x^{-s-1}\psi(x)dx=\int_0^\infty x^{-s}d\psi(x).$$

**Lemma** (LaE1a) §89, (PrK) p. 231: For $\psi_0(x) := \frac{1}{2}\{\psi(x+0)-\psi(x-0)\}$ it holds

$$\psi_0(x)=x-\sum_\rho\frac{x^\rho}{\rho}-\frac{\zeta'}{\zeta}(0)-\frac{1}{2}\log(1-\frac{1}{x^2}),$$

where

$$\frac{1}{2}\log(1-\frac{1}{x^2})=\sum_{n=1}^\infty\frac{x^{-2n}}{-2n}\,,x^2>1\ \text{ and }\ \frac{\zeta'}{\zeta}(0)=\log(2\pi).$$

The density function $\psi_0(x)$ corresponds to $\psi(x)$ for non-prime number powers $x\neq p_0^{m_0}$, otherwise $\psi(x)-\frac{1}{2}\log p_0$. For $x>1$ it is continuous for non-integers $x>1$ and for integer non-prime number powers. It is not continuous to both sides of non-prime number powers. The same holds for the series $\sum_\rho\frac{x^\rho}{\rho}$, which is also convergent for $0<x\leq 1$.

Regarding the known asymptotic of the density function

$$\frac{1}{x}\sum_\rho\frac{x^\rho}{\rho}$$

we note that

$$\psi(x)\approx x\quad\text{iff}\quad\lim_{x\to\infty}\frac{1}{x}\sum_\rho\frac{x^\rho}{\rho}=\lim_{x\to\infty}\sum_\rho\frac{x^\rho}{\rho}\log'(\frac{x}{\rho})=0.$$

We further note the special representation ($r=1$) of the von Mangoldt function with the zeros of the Zeta function in the form

$$x\int_0^x\frac{d\psi(t)}{t}=\sum_\rho\frac{x^\rho}{1-\rho}+\sum_1^\infty\frac{x^{-2n}}{2n+1}.$$

**Remark**: Regarding the asymptotics of $\zeta(\sigma+it)$ we note

i) $\zeta(\sigma+it)=O(logt)=O(t^{\varepsilon})$ for $\sigma\geq 1$, (LaE1a) § 46

ii) $\zeta(\sigma+it)=O(logt)=O(t^{\frac{1}{2}-\sigma+\varepsilon})$ for $\sigma\leq 0$, (LaE1a) § 228

iii) $\zeta(\sigma+it)=O\left(t^{\frac{1-\sigma}{2}}logt\right)=O\left(t^{\frac{1-\sigma}{2}+\varepsilon}\right)$ for $0\leq\sigma\leq 1$, (LaE1b) § 240.

**Lemma** (Backlund's estimate of $N(T)$, EdH) 6.7: let $N(T)$ denotes the number of roots between $0<Im(s)<T$, then the relative error in the approximation

$$N(T)\sim\frac{T}{2\pi}log\left(\frac{T}{2\pi}\right)-\frac{T}{2\pi}$$

is less than a constant times $T^{-1}$ as $T\to\infty$.

**Lemma:**

i) $\frac{1}{\Gamma\left(1+\frac{z}{2}\right)}=e^{\frac{\gamma z}{2}}\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}}=\prod_{n=1}^{\infty}\left(1+\frac{z}{2n}\right)\left(1+\frac{1}{n}\right)^{-\frac{z}{2}}$ , (GrI) 8.322

ii) $e^{\Psi(z)}=x\prod_{n=1}^{\infty}\left(1+\frac{1}{x+k}\right)e^{-\frac{1}{x+k}}$, (GrI) 8.364.

**Lemma** (PaR) p. 41:

$$\Gamma(\frac{3}{2}+iv)\zeta\frac{3}{2}+iv)\in L_2(-\infty,\infty).$$

## The entire Riemann $\xi(s)$ function

**Lemma** (EdH): for the entire Zeta function

$$\xi(s) := (s-1)\frac{s}{2}\zeta(s)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}$$

it holds

i) $\xi(s) = \xi(1-s)$

ii) $\xi(s) = 4\int_1^\infty \frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx} x^{\frac{3}{4}} cosh\left[\frac{1}{2}\left(s-\frac{1}{2}\right) logx\right] \frac{dx}{x}$

iii) $\xi\left(\frac{1}{2}+it\right) = 4\int_1^\infty \frac{d\left[x^{\frac{3}{2}}\psi'(x)\right]}{dx} x^{\frac{3}{4}} cos\left(\frac{t}{2} logx\right)\frac{dx}{x}$

iv) $\xi(s) = \sum_{n=0}^\infty a_{2n}(s-\frac{1}{2})^{2n}$ , $\xi(\frac{1}{2}+it) = \sum_0^\infty (-1)^n a_{2n}\, t^{2n}$

where

$$a_{2n} := 4\int_1^\infty \frac{d[x^{3/2}\psi'(x)]}{dx} x^{3/4} \frac{(\frac{1}{2}logx)^{2n}}{(2n)!}\frac{dx}{x}.$$

v) $\xi(s) = \xi(0)\prod_\rho\left(1-\frac{s}{\rho}\right)$ where $\xi(0) = -\zeta(0) = \frac{1}{2}$

vi) $\xi\left(\frac{1}{2}+it\right) = \xi\left(\frac{1}{2}\right)\prod_{Re(\alpha_n)>0}\left(1-\frac{t^2}{\alpha_n^2}\right)$ where $\rho_n = \frac{1}{2}+i\alpha_n$ (EdH) 1.16.

The entire function $\xi(s)$ is of order $\sigma = 1$ with finite order type (as on the circle $|s| = r$ for its maximum modulus function it holds $M(r) \sim \frac{1}{2} rlogr$.

The function $\frac{1}{\Gamma(z)}$ has only zeros on the negative x-axis. It is the most prominent example where the density of the zero set is finite, while its canonical product is of maximal (i.e., infinite order) type as $\log(M(r)) \geq Crlogr$, (LeB) p. 32).

The entire function

$$\frac{1}{\Gamma\left(1+\frac{z}{2}\right)} = e^{\frac{\gamma z}{2}}\prod_{n=1}^\infty\left(1+\frac{z}{2n}\right)e^{-\frac{z}{2n}}$$

is the most prominent example where the density of the zero set is finite, while the canonical product of order one is of maximal type, i.e. $logM(r) \geq Crlogr$.

The most prominent example with „symmetrical zeros“ is given by, (LeB) p. 32

$$\sin\left(\frac{\pi}{2}s\right) = \left(\frac{\pi}{2}s\right)\prod_{n=1}^\infty\left(1-\frac{s^2}{4n^2}\right).$$

Regarding the distributions of the zeros of $\zeta(s)$ and $\xi(s)$ it holds, (InA) p. 48,

i) The zeros of $\xi(s)$ are all situated in the strip $0 < \sigma < 1$, and lie symmetrically about the lines $t = 0$ and $\sigma = 1/2$

ii) The zeros of $\zeta(s)$ are identical (in position and order of simplicity) with those of $\xi(s)$, except $\zeta(s)$has simple zero at each of the points $s = -2n$

iii) $\xi(s)$ has no zeros on the real axis.

We note the following equivalent critera for the RH:

i) $\pi(x) = Li(x) + O(\sqrt{x}\,log\,x)$

ii) $\pi(x) = Li(x) + O(x^{1/2+\varepsilon})$ , $\varepsilon > 0$, i.e. the relative error is $\frac{\pi(x)-Li(x)}{x} = O(x^{-\frac{1}{2}-\varepsilon})$ $\varepsilon > 0$

iii) $\psi(x) = x + O(\sqrt{x}\,log^2 x)$

iv) The series $\sum_{n=1}^{\infty} \mu(n)\, n^{-s}$ is convergent for $Re(s) > 1/2$ and $\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$.

Laguerre-Polya class $LP$ of functions consists of the entire functions having only real zeros with a Weierstrass factorization of the form

$$az^q e^{\alpha z-\beta z^2} \prod_k (1 - z/\alpha_k) e^{z/\alpha_k},$$

where $a, \alpha, \alpha_k$ are real, $\beta \geq 0$ and $q$ is a nonnegative integer, and $\alpha_k$ are the nonzero real numbers such that $\sum_{k=1}^{\infty} \frac{1}{\alpha_k^2} < \infty$. The subset $LP^*$ consists of all elements of order $< 2$. In this case $\beta$ is neccessarily zero.

**RH criterion** (CaD): If the function

$$\Xi(t) := \xi(1/2 + it)$$

can be realized as a convolution

$$\Xi(t) = (K * dF)(t),$$

where $K(t) \in LP^*$, i.e., is an entire function from the Laguerre-Polya class of order $< 2$, i.e.

$$cz^q e^{\alpha z} \prod_k (1 - z/\alpha_k) e^{z/\alpha_k},$$

where $c, \alpha, \alpha_k$ are real, and $q$ is a nonnegative integer, this would prove the RH.

**Lemma**: (PaR) p. 77: The even Zeta function $\Xi(z) := \xi(1/2 + iz)$ has all its zeros in the strip $|Im(z)| < 1/2$. Let $z_n$ denote its zeros, then

$$\Xi(z) = \Xi(0) \prod_{n=1}^{\infty} (1 - \frac{z^2}{z_n^2}) \text{ with } \sum_1^{\infty} \lambda_n^{-2} < \infty \text{ where } \lambda_n := |z_n|.$$

Putting

$$H(z) := \Xi(0) \prod_{n=1}^{\infty} (1 - \frac{z^2}{\lambda_n^2})$$

it holds $log\left|\frac{H(iy)}{\Xi(iy)}\right| = O(1)$ along the imaginary axis. Thus, because of

$$log\Xi(iy) = O(y) + log\Gamma\left(\frac{y}{2}\right) \sim \frac{1}{2} ylogy,\ (y > 0),$$

it follows

$$logH(iy) \sim \frac{1}{2} ylogy$$

and therefore

$$\frac{1}{logy} \int_{-y}^{y} log\left|\frac{\Xi(x)}{\Xi(0)}\right| \frac{dx}{x^2} \rightarrow -\frac{\pi}{2} \text{ as } y \rightarrow \infty.$$

**Lemma** (PaR) p. 41:

$$\Gamma(\frac{3}{2} + iv)\zeta\,\frac{3}{2} + iv) \in L_2(-\infty, \infty).$$

**Lemma** ((PaR), p. 86: Let $\Phi(z) := \prod_{n=1}^{\infty}(1 - \frac{z^2}{\pi^2\lambda_n^2})$ with $\lim_{n\to\infty}\frac{\lambda_n}{n} = 1$,

then

$$\lim_{y\to\infty}\frac{log\Phi(iy)}{y} = 1.$$

**Lemma** (LeN), p. 92: let $\Phi(z) := \prod_{n=1}^{\infty}(1 - \frac{z^2}{z_n^2})$ with $\lim_{n\to\infty}\frac{n}{z_n} = D$, where $D$ is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n - m|$. Then for $\varepsilon > 0$, as $r \to \infty$

i) $\Phi(re^{i\theta}) = O(e^{\pi D|sin\theta|r+\varepsilon r})$

ii) $\frac{1}{\Phi(re^{i\theta})} = O(e^{-\pi D|sin\theta|r+\varepsilon r})$

iii) $\frac{1}{|\Phi'(z_n)|} = O(e^{\varepsilon|z_n|})$, $n \to \infty$.

**Lemma** (LeN) p. 89: If $\lambda_n$ satisfies $\lim_{n\to\infty}\frac{n}{\lambda_n} = D$ and for some $c > 0$, $\lambda_{n+1} - \lambda_n \geq c$, and if

$$F(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\lambda_n^2}\right)$$

then as $n \to \infty$:

$$\frac{1}{F'(\lambda_n)} = O(e^{\varepsilon\lambda_n}).$$

### Riemann's method for deriving the formula for $J(x)$

The Riemann's formula for

$$J(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \log \zeta(s)\, x^s \frac{ds}{s}$$

is derived from the Fourier inverse of

$$\frac{\log \zeta(s)}{s} = \int_0^\infty J(x) x^{-s-1} dx\ , Re(s) > 1.$$

The principal term is given by the $li_1(x)$ – function, (EdH) 1.14,

$$li_1(x) = \frac{1}{2} li(x + i0) + li(x - i0) = \lim_{\varepsilon\to 0} \int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log(s-1)}{s}\right] x^s ds.$$

The tool to derive this term is given by the auxiliary function

$$H(\beta) := -\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\log(1-\frac{s}{\beta})}{s}\right] x^s\, ds$$

where $a > Re(\beta)$ and where $log(1 - \frac{s}{\beta})$ is defined for all complex numbers $\beta$ other than real numbers $\beta \leq 0$ to be $log(1-\beta) - \log(-\beta)$. Let $C^-$ denote the contour which goes over the lower semicircle from $1-\varepsilon$ to $1+\varepsilon$ and let $C^+$ denote the corresponding upper semicircle. This auxiliary function is also applied to derive all other terms.

The other two critical terms are the „oscillating" term and the Riemann error function

$$(A):\ -\sum_{Im(\rho)>0}\{Li(x^\rho) + Li(x^{1-\rho})\}\ ,\quad (B)\ :\ \int_x^\infty \frac{dt}{t(1-t^2)\log t}$$

derived from the formulas

$$(A) = \sum H(\rho) = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{\sum \log(1-\frac{s}{\rho})}{s}\right] x^s\, ds = \sum_{Im(\rho)>0}\left\{\int_{C^+} \frac{t^{\rho-1}}{logt} dt + \int_{C^-} \frac{t^\rho}{logt} dt\right\}$$

$$(B) = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty} \frac{d}{ds}\left[\frac{log\Gamma\left(1+\frac{s}{2}\right)}{s}\right] x^s\, ds = -\sum_{n=1}^\infty H(-2n) = \sum_{n=1}^\infty \int_x^\infty \frac{t^{-2n-1}dt}{logt} = \int_x^\infty \frac{(\sum_{n=1}^\infty t^{-2n})dt}{tlogt}$$

in combination with the

**Lemma**:

i) For $Re(\beta) > 0$ when $x$ is a fixed number with $x > 1$, it holds

$$\int_{C^+} \frac{t^{\beta-1}}{logt} dt = \int_0^{x^\beta} \frac{du}{logu} = Li\left(x^\beta\right) - i\pi\ , Im(\beta) > 0$$
$$\int_{C^-} \frac{t^{\beta-1}}{logt} dt = \int_0^{x^\beta} \frac{du}{logu} = Li\left(x^\beta\right) + i\pi\ , Im(\beta) < 0.$$

ii) For $Re(\beta) < 0$ when $x$ is a fixed number with $x > 1$, it holds

$$H(\beta) = -\int_x^\infty \frac{t^{\beta-1}}{logt} dt = -\int_{x^\beta}^\infty \frac{du}{logu}\ .$$

**The Kummer function $_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$**

The zeros $z_n$ of $_1F_1\left(\frac{1}{2},\frac{3}{2};\mathrm{z}\right)$ are the zeros of $_1F_1\left(1;\frac{3}{2},-\mathrm{z}\right)$ due to the relationship

$$_1F_1\left(\frac{1}{2},\frac{3}{2},z\right)=e^z\ {}_1F_1\left(1,\frac{1}{2},-z\right).$$

**Lemma** (SeA), Appendix: all the zeros $z_n$, $\nu\in Z-\{0\}$, of the function $_1F_1\left(\frac{1}{2},\frac{3}{2},z\right)$ are simple and complex valued;

i) they satisfy the asymptotic formula

$$z_n=2\pi i\nu+\left[\frac{1}{2}log2\pi|\nu|+log\sqrt{\pi}\mp i\frac{\pi}{4}\right](1-\frac{1}{4\pi i\nu})-\frac{5}{4\pi i\nu}+O\left(\frac{log|\nu|}{\nu^2}\right)),\ \nu\to\pm\infty;$$

ii) they lie in the horizontal stripes

$$(2n-1)\pi<|Im(z)|<2\pi n,\ n\in N;$$

iii) their real parts satisfy the asymptotics

$$Re(z_n)=\quad\frac{1}{2}log2\pi|\nu|\ +log\sqrt{\pi}\pm\frac{1}{16\nu}+O\left(\frac{log|\nu|}{\nu^2}\right)),\ \nu\to\pm\infty.$$

**Note** (SeA): $_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ is not of exponential type $\pi$ and cannot expressed in the form

$$\int_{-\pi}^{\pi}e^{izt}\frac{k(t)dt}{(\pi^2-t^2)^{1/2}}\ ,\ vark(t)<\infty\ ,\ k(\pm\pi\mp 0\neq 0).$$

**Note** (SeA): Only in case of $Re(c)=2Re(a)$ the function $F(z)=e^{-i\pi z}\ {}_1F_1(\mathrm{a},c,2\pi iz)$ belongs to $S_{Re(a)}$, i.e. $|F(z)|\cong|z|^{-Re(a)}e^{\pi|Im(z)|}$, where $S_0$ consists of so-called sine-type functions.

**Lemma** (BuH) p. 184: the product representations of $_1F_1\left(\frac{1}{2},\frac{3}{2};\mathrm{z}\right)$ and $_1F_1\left(1;\frac{3}{2},\mathrm{z}\right)$ are given by

$$_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)=e^{\frac{1}{3}z}\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{\frac{z}{z_n}},\quad {}_1F_1\left(1,\frac{3}{2};z\right)=e^{\frac{2}{3}z}\prod_{n=1}^{\infty}\left(1+\frac{z}{z_n}\right)e^{-\frac{z}{z_n}}.$$

**Corollary:** it holds

$$_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)\ {}_1F_1\left(1,\frac{3}{2};z\right)=\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)\left(1+\frac{z}{z_n}\right).$$

**Lemma** (SlL) p.60:

i) if $Re(z)>0$, $z\to\infty$, then $_1F_1\left(\frac{1}{2},\frac{3}{2};\mathrm{z}\right)=\frac{1}{2}\frac{e^z}{z}\left\{1+O(\frac{1}{|z|})\right\}$

ii) if $Re(z)<0$, $z\to\infty$, then $_1F_1\left(\frac{1}{2},\frac{3}{2};\mathrm{z}\right)=\frac{1}{2z}\left\{1+O\left(\frac{1}{|z|}\right)\right\}$

iii) if $Im(z)=0$, $x\to\infty$, then $_1F_1\left(\frac{1}{2};\frac{3}{2},logx\right)\sim\frac{1}{2}\frac{\mathrm{x}}{logx}$, $\ {}_1F_1\left(1;\frac{3}{2},log\frac{1}{x}\right)\sim\frac{1}{2}\frac{1}{logx}$.

### A Kummer function based entire Zeta function $\xi^*(s)$

(BrK): The entire Zeta function $\xi(s)$ is given by, (EdH) 1.8,

$$(*)\quad \xi(s) := \frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = (1-s)\zeta(s)M\left[-xf'(x)\right](s) = \xi(1-s).$$

Let $M$ and $H$ denote the Mellin resp. the Hilbert transforms. The proposed Kummer function based Zeta function theory in (BrK) is based on the following replacements

$$f(x) = e^{-\pi x^2} \quad \leftrightarrow \quad f_H(x) = \frac{2}{\sqrt{\pi}}D\left(\sqrt{\pi}x\right) = 2x\ {}_1F_1\left(1;\frac{3}{2},-\pi x^2\right)$$

$$M[f](s) = \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) \quad \leftrightarrow \quad M[f_H(x)](s) = \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right)$$

$$\psi(x^2) = \sum_{n=1}^{\infty} f(\sqrt{\pi}x) \quad \leftrightarrow \quad \psi_H(x^2) = \sum_{n=1}^{\infty} f_H(\sqrt{\pi}x).$$

It enables the definition of an alternative entire Zeta function in the form

$$\xi^*(s) := (1-s)\zeta(s)M[f_H(x)](s) = \frac{1}{2}(1-s)\zeta(s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s)$$

accompanied by the duality equation in the form

$$\xi^*(s)\,cot(\frac{\pi}{2}s) = \xi^*(1-s)\,cot(\frac{\pi}{2}(1-s),$$

i.e. the entire function $\xi^*(s)$ has the same zeros as $\xi(s)$.

**Note**: the Mellin transform $M[f_H(x)](s)$ is defined for $-1 < Re(s) < 3$.

**Lemma**:

$$\frac{1}{\Gamma\left(1+\frac{s}{2}\right)}\cdot\frac{1}{\Gamma\left(1-\frac{s}{2}\right)} = \prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)\left(1-\frac{s}{2n}\right),\quad \frac{\sin\left(\frac{\pi}{2}s\right)}{\left(\frac{\pi}{2}s\right)} = \prod_{n=1}^{\infty}\left(1+\frac{s}{2n}\right)\left(1-\frac{s}{2n}\right).$$

**Lemma**:

$$\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s) = \frac{\pi}{\cos\left(\frac{\pi}{2}s\right)}\frac{1}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\pi}{\sin\left(\frac{\pi}{2}(1-s)\right)}\frac{1}{\Gamma\left(1-\frac{s}{2}\right)}.$$

**Corollary**:

$$\zeta(s)\zeta(1-s) = \sqrt{\pi}\xi^*(s)\xi^*(1-s)\frac{\sin\left(\frac{\pi}{2}s\right)}{\frac{\pi}{2}s}\Gamma\left(1-\frac{s}{2}\right)\frac{\sin\left(\frac{\pi}{2}(1-s)\right)}{\frac{\pi}{2}(1-s)}\Gamma\left(1-\frac{1-s}{2}\right).$$

where the product formula for the left-hand side is given by the Euler product in the form

$$\zeta(s)\zeta(1-s) = \prod_p \frac{1}{1-p^{-s}}\frac{1}{1-p^{s-1}},\quad Re(s) > 1$$

resp.

$$\zeta\left(\frac{1}{2}+s\right)\zeta\left(\frac{1}{2}-s\right) = \prod_p \frac{\sqrt{p}}{\sqrt{p}-p^{-s}}\frac{\sqrt{p}}{\sqrt{p}-p^{s}}\quad, Re(s) > \frac{1}{2}.$$

**Remark** (ReH) p.145: The Haar scaling function is an universal scaling function of genus $g = 1$. Its scaling equation for rank $m$ is $\varphi(x) = \sum_{0\le k<n}\varphi(mx-k)$. This corresponds to the scaling vector $\gamma_m := \{1,1,1,1,\ldots 1\}$ of rank $k$ for a Haar matrix of rank $k$, playing a key role in the Bagchi-Beurling-Nyman RH criterion, (BaB). Its Fourier transform is given by

$$\hat{\varphi}(\xi) = F_{0,m}\left(\frac{\xi}{m}\right)\hat{\varphi}\left(\frac{\xi}{m}\right) \text{ with } F_{0,m}(\xi) := \frac{1}{m}\sum_{k=0}^{m-1}e^{2\pi ik\xi}$$

Taking the limit one finds the interesting infinite product formula

$$sinc(\pi\xi) = \lim_{m\to\infty} e^{-i\pi\xi}\prod F_{0,m}\left(\frac{\xi}{m^n}\right) = \prod(1-\left(\frac{\xi^2}{n^2}\right)) = \frac{1}{\Gamma(1+\xi)\Gamma(1-\xi)}.$$

## Kummer functions related Mellin and Laplace transforms

**Lemma 1** (Grl) 7.612:

$$\int_0^\infty t^b \ {}_1F_1(a,c,-t)\frac{dt}{t} = \Gamma(b)\frac{\Gamma(a-b)}{\Gamma(c-b)}\frac{\Gamma(c)}{\Gamma a)}\,, \quad 0 < Re(b) < Re(a).$$

For the Dawson function

$$D(x) := e^{-x^2}\int_0^x e^{u^2} du$$

the following representations are valid:

**Lemma** (Grl) 3.896), (LeN) pp. 17, 272:

i) $D(x) = \int_0^\infty e^{u^2}\sin(2xu)du$

ii) $D(x) = xe^{-x^2}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right) = x\ {}_1F_1\left(1;\frac{3}{2},-x^2\right).$

**Remark**: The Hilbert transform of the Gaussian function $f(x) \coloneqq e^{-\pi x^2}$ is related to the Dawson function by, (GaW), (KoO),

$$f_H(x) \coloneqq H[f](x) = \frac{2}{\sqrt{\pi}}D\left(\sqrt{\pi}x\right) = 2x\ {}_1F_1\left(1;\frac{3}{2},-\pi x^2\right).$$

**Remark**: We note the formula

$$\frac{\Gamma(1-s)\Gamma(s)}{\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)} = cot(\pi s) = cot\left(\frac{\pi}{2}s\right) + cot\,(\frac{\pi}{2}(1-s)).$$

Putting

$$\Omega\left(\frac{s}{2}\right) \coloneqq \frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{s}{2}\right)tan\,(\frac{\pi}{2}s)$$

one gets the

**Corollary**:

i) $M[D(x)](s) = \Omega\left(\frac{s}{2}\right)$, $\quad -1 < Re(s) < 3$

ii) $M[f_H(x)](s) = \pi^{-\frac{1+s}{2}}\Omega\left(\frac{s}{2}\right)$, $\quad -1 < Re(s) < 3.$

Proof:

i) $$\int_0^\infty x^s\left[x\ {}_1F_1\left(1;\frac{3}{2},-x^2\right)\right]\frac{dx}{x} = \frac{1}{2}\int_0^\infty x^{\frac{s+1}{2}}\ {}_1F_1\left(1;\frac{3}{2},-x\right)\frac{dx}{x} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})}$$

ii) $$M[f_H(x)](s) = 2\left[\ {}_1F_1\left(1;\frac{3}{2},-\pi x^2\right)\right](1+s) = \pi^{-\frac{1+s}{2}}M\left[\ {}_1F_1\left(1;\frac{3}{2},-x\right)\right]\left(\frac{1+s}{2}\right)$$

$$= \pi^{-\frac{1+s}{2}}\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})}.$$

For the Whittaker functions, (GrI) 9.220

$$M_{\lambda,\mu}(z) := z^{\mu+\frac{1}{2}} e^{-z/2}\ {}_1F_1\left(\mu-\lambda+\frac{1}{2}, 2\mu+1, z\right)$$

it holds

**Lemma 2** (BuH) p. 118, (EpA) p. 215, (GrI) 7.621:

$$\int_0^\infty e^{-\frac{b}{2}t} t^{\nu-1} M_{\lambda,\mu}(bt)\,dt = b^{-\nu}\frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+\lambda)}\frac{\Gamma(\lambda-\nu)\Gamma(\frac{1}{2}+\mu+\nu)}{\Gamma(\frac{1}{2}+\mu-\nu)},\ \ Re(\nu+\frac{1}{2}+\mu)>0, Re(\lambda-\nu)>0.$$

Putting $\lambda := \mu = \frac{1}{4}$, i.e.

$$M_{\frac{1}{4},\frac{1}{4}}(z) = z^{\frac{3}{4}} e^{-\frac{z}{2}}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right) = z^{\frac{3}{4}} e^{\frac{z}{2}}\ {}_1F_1\left(1,\frac{3}{2},-z\right)$$

one gets with $M(z) := z^{-\frac{1}{4}} e^{-\frac{z}{2}} M_{\frac{1}{4},\frac{1}{4}}(z) = z^{\frac{1}{2}}\ {}_1F_1\left(1,\frac{3}{2},-z\right)$ the

**Corollary:**

i) $\int_0^\infty t^{\frac{s}{2}} M(bt)\frac{dt}{t} = b^{-\frac{s}{2}}\Omega\left(\frac{s}{2}\right)$, $\quad -1 < Re(s) < 1$

ii) $\int_0^\infty t^{-\frac{s}{2}} M(bt)\frac{dt}{t} = b^{\frac{s}{2}}\Omega\left(-\frac{s}{2}\right)$, $\quad -1 < Re(s) < 1.$

Proof: Putting $\nu := \pm\frac{s}{2}-\frac{1}{4}$ in the above lemma 2 one gets

i) $\int_0^\infty e^{-\frac{b}{2}t} t^{\frac{s}{2}-\frac{1}{4}} M_{\frac{1}{4},\frac{1}{4}}(bt)\frac{dt}{t} = b^{-\frac{s}{2}+\frac{1}{4}}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)} = b^{-\frac{s}{2}+\frac{1}{4}}\Omega\left(\frac{s}{2}\right)$, $\quad -1 < Re(s) < 1$

ii) $\int_0^\infty e^{-\frac{b}{2}t} t^{-\frac{s}{2}-\frac{1}{4}} M_{\frac{1}{4},\frac{1}{4}}(bt)\frac{dt}{t} = b^{+\frac{s}{2}+\frac{1}{4}}\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+\frac{s}{2})} = b^{+\frac{s}{2}+\frac{1}{4}}\Omega\left(-\frac{s}{2}\right)$, $\ -1 < Re(s) < 1$

from which the Corollary follows with $M(bt) = (bt)^{-\frac{1}{4}} e^{-\frac{b}{2}t} M_{\frac{1}{4},\frac{1}{4}}(bt)$.

**Lemma**: (GrI) 7.643:

$$\int_0^\infty x^{4\nu} e^{-\frac{x^2}{2}} \sin(bx)\ {}_1F_1\left(\frac{1}{2}-2\nu, 2\nu+1, \frac{x^2}{2}\right)dx = \sqrt{\frac{\pi}{2}}\, b^{4\nu} e^{-\frac{b^2}{2}}\ {}_1F_1\left(\frac{1}{2}-2\nu, 2\nu+1, \frac{b^2}{2}\right),\ b>0,\ \ Re(\nu) < -\frac{1}{4}.$$

**Corollary**:

$$\int_0^\infty x^s e^{-\frac{x^2}{2}} \sin(bx)\ {}_1F_1\left(\frac{1-s}{2}, 1+\frac{s}{2}, \frac{x^2}{2}\right)dx = \sqrt{\frac{\pi}{2}}\, b^s e^{-\frac{b^2}{2}}\ {}_1F_1\left(\frac{1-s}{2}, 1+\frac{s}{2}, \frac{b^2}{2}\right),\ b>0, Re(s) < -1.$$

**Remark**: We note the formulas

$$\frac{\Gamma(1-s)\Gamma(s)}{\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)} = cot(\pi s) = cot\left(\frac{\pi}{2}s\right) + cot\left(\frac{\pi}{2}(1-s)\right)$$

$\frac{1}{2}cot(\pi s) = \sum_{n=1}^\infty sin(2\pi nx) \in H_{-1}^{\#}(0,1)$, (BeB) p. 200.

**Asymptotics of the zeros of degenerated Hypergeometric Functions**

(SeA): Suppose that the sequence $Z = z_n \in C$, has the asymptotics

$$(*) \quad z_n = \varphi(n) + o(1),\ n \to \pm\infty,$$

where $\varphi(n) \to \infty$. The numbering of the sequence $Z$ is said to be consistent with the asymptotics (*) via the index set $T$ if there exists a bijection $T \leftrightarrow Z$ preserving this asymptotics.

From (SeA) we recall the following two main results:

***Theorem 1.*** *Suppose that* $a, c \in Z$ *and* $a, c, c-a \in C - \{-Z_+\}$ *. Then*

1. *all the zeros of the function* ${}_1F_1(a,c;z)$ *are simple and they satisfy the asymptotic formula*

$$(**) \quad z_n = 2\pi i n + \left[(c-2a)log2\pi|n| + log\frac{\Gamma(a)}{\Gamma(c-a)} \pm i\frac{\pi}{2}(c-2)\right](1 + \frac{c-2a}{2\pi i n})$$
$$+ \frac{2a(a-c)-c}{2\pi i n} + O\left(\frac{log|n|}{n^2}\right)), \qquad n \to \pm\infty;$$

2. *the numbering of all the zeros of the function* ${}_1F_1(a,c;z)$ *is consistent with the asymptotic expansion (**) via the index set* $T = Z - \{0\}$.

***Theorem 3.***

1. *Suppose that* $1 \le a < c \le a+1$ *and* $c \ne 2$ *if* $a = 1$. *Then all zeros of* ${}_1F_1(a,c;z)$ *lie in the half-plan*

$$Re(z) < -(\sqrt{a-1} + \sqrt{1-(c-a)})^2.$$

2. *Suppose that* $1 < a \le 1$, $c \ge 1 + a$, *moreover,* $c \ne 2$ *if* $a = 1$. *Then all zeros of* ${}_1F_1(a,c;z)$ *lie in the half-plan*

$$Re(z) > (\sqrt{c-a-1} + \sqrt{1-a})^2.$$

3. *Suppose that* $0 < a \le 1$, $a < c \le 1 + a$, *moreover,* $c \ne 2$ *if* $a = 1$. *Then all zeros of* ${}_1F_1(a,c;z)$ *lie in the horizontal stripes*

$$(2n-1)\pi < |Im(z)| < 2\pi n,\ n \in N.$$

## **The Digamma function** $\Psi(x) = log'\Gamma(x)$

For the Digamma function $\Psi(x) = log'\Gamma(x)$ the following formulas are valid

i) $\Psi(x) = -\gamma - \sum_{k=0}^{\infty}(\frac{1}{x+k} - \frac{1}{1+k})$

ii) $\Psi(z) = logz + O\left(\frac{1}{|z|}\right) = log\Gamma(1+z) - log\Gamma(z) + O\left(\frac{1}{|z|}\right)$

iii) $\frac{1}{\Gamma(s)} = \Psi(s)\frac{1}{\Gamma'(s)}$ resp. $-log\Gamma(s) = log\Psi(s) - log\Gamma'(s)$.

**Lemma** (NiN) p. 99: for every $\varepsilon > 0$ there is a $R > 0$ that

$$|\Psi(x) - logx| < \varepsilon \text{ for } |x| \geq R > 0.$$

**Lemma** (NiN) p. 99, (SeP): Let $w_n$ denote the zeros of $\Psi(x)$ $n \in N_0$,

i) all zeros of $\Psi(x)$ are real; there is only one positive zero $w_0 \sim 1{,}461$, (AbM) 6.3.19

ii) all negative zeros $w_n$ of $\Psi(x)$ lie in the intervals $w_n \in (-n, 1/2 - n)$.

**Lemma** (NiN) p. 99:

i) the sequence $y_n := n + w_n$ , $n \in N$, is characterized by the relations
$\Psi(1 - w_n) = \Psi(n + 1 - y_n) = \pi\cot(\pi w_n) = \pi\cot(\pi(-n + y_n)) = \pi\cot(\pi y_n)$,
i.e.

$$y_n = \frac{1}{\pi}arctan\left(\frac{\pi}{\Psi(1-w_n)}\right)$$

ii) it holds $y_n, \frac{1}{2} - y_n \in (0, \frac{1}{2})$ and for large $n$

$$\pi\cot(\pi y_n) = logn + \delta_n \text{ with} \lim_{n\to\infty} \delta_n = 0,$$

resp.

$$y_n = \frac{1}{\pi}arctan\left(\frac{\pi}{logn+\delta_n}\right) = \frac{1}{logn} + \frac{\delta'_n}{logn}, \quad \lim_{n\to\infty} \delta'_n = 0$$

iii) $\frac{1}{2}\Psi(1 - w_n) = \frac{\pi}{2}\cot(\pi w_n) = \frac{\pi}{2}\cot(\pi y_n) = -\frac{1}{2w_n} - \frac{1}{1-w_n{}^2} + \sum_{k=0}^{\infty}(1 - s_{2k+1})w_n{}^{2k}$.

Proof: i), ii): with $\Psi(1 - x) - \Psi(\mathrm{x}) = \pi\cot(\pi\mathrm{x})$ it follows

$$\Psi(1 - w_n) = \Psi(n + 1 - y_n) = \pi\cot(\pi x_n).$$

On the other hand it holds, ($0 < y_n, 1 - y_n < 1$), $\pi\cot(\pi w_n) = \pi\cot(\pi(-n + y_n)) = \pi\cot(\pi y_n)$ and, therefore, because of $\Psi(n) \sim logn$; with the lemma above it follows

$$\Psi(1 - w_n) = \pi\cot(\pi w_n) = \pi\cot(\pi y_n) = \frac{\pi}{2}\cot(\pi w_n) - \frac{1}{2w_n} - \frac{1}{1-w_n{}^2} + \sum_{k=0}^{\infty}(1 - s_{2k+1})w_n{}^{2k}$$

and therefore

$$\frac{\pi}{2}\cot(\pi w_n) = \frac{\pi}{2}\cot(\pi y_n) = -\frac{1}{2w_n} - \frac{1}{1-w_n{}^2} + \sum_{k=0}^{\infty}(1 - s_{2k+1})w_n{}^{2k}.$$

**Corollary**: $\frac{1}{2}\Psi(1 + |w_n|) \sim \frac{1}{2|w_n|}$.

### The common denominator of the zeros of $\Psi(x)$ and ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$

**Lemma** (SeA): The imaginary part of the zeros $\{z_n\}_{n\in N}$ of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right)$ are all complex with $Re(z_n)>\frac{1}{2}$. For the absolute values of the imaginary parts of $\{z_n\}_{n\in N}$ it holds

$$n-1/2<\omega_n<n \quad \text{with } 2\pi\omega_n:=|Im(z_n)|.$$

**Lemma**: The zeros $\{w_0,w_n\}_{n\in N}$ of the Digamma function are all real and negative, except $w_0$. It holds

$$-n<w_n<1/2-n,\ 1-w_0\sim-\ 0{,}461\in(-\tfrac{1}{2},0)$$ (NiN) p. 99, (SeP).

**Lemma:** the sequences $\{-w_n,\omega_n\}$ have Snirelmann density ½.

Proof: for $k\geq 1$ it holds $\frac{1}{2}k\leq k-\frac{1}{2}<\omega_{n+k}-\omega_n<k+\frac{1}{2}$ .

**Corollary**: for $w_n\in(-n,1/2-n)$ and $\omega_n:=\in(n-1/2,n)$ it follows

$$(*)\quad |w_n-w_m|\geq\frac{1}{2}|n-m|,\ \lim_{n\to\infty}\frac{n}{w_n}=1,|\omega_n-\omega_m|\geq\frac{1}{2}|n-m|,\ \lim_{n\to\infty}\frac{n}{\omega_n}=1.$$

**Note**: The above conditions (*) are the standard prerequisite of „gaps and density theorems“ dealing with the determination of the rate of growth of analytic functions from their growth on sequences of points $a_n$, ((LeN). The more sharper Kadec condition plays the central role in the context of Paley-Wiener functions and the related theory of non-harmonic Fourier series (*):

**Lemma**: Let $a_n$, $a_{-n}:=-a_n$ be a sequence fulfilling the conditions $n-\frac{1}{2}<a_n,a_{-n}<n$. Then the „retarded“ sequence $a_\nu^*:=\frac{3a_\nu+a_{\nu+1}}{4}$ $\nu\in Z$ fulfill the Kadec condition

$$(**)\quad |\nu-a_\nu^*|\leq L<\tfrac{1}{4},\ \nu\in Z \text{ where } a_0\in\left(0,\tfrac{1}{2}\right).$$

**Kadec's Theorem** (YoR) p. 36:

If $\{\lambda_n\}_{n\in Z}$ is a sequence of real numbers for which $|\lambda_n-n|\leq L<1/4$ , then $\{e^{i\lambda_n t}\}_{n\in Z}$ satisfy the Paley-Wiener criterion and so forms a Riesz basis for $L_2\,(-\pi,\pi)$.

Kadec's theorem can be improved under „small“ displacements of the $\lambda_n$ 's.

**Corollary 1** (YoR) p. 164: Let $\{\lambda_1,\lambda_2,\lambda_3,\ldots.\}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\left\{e^{iRe(\lambda_n)t}\right\}$ is a Riesz basis for $L_2\,(-\pi,\pi)$, then so is $\left\{e^{i\lambda_n t}\right\}$.

**Lemma** (IvS): On addition of one exponential the Riesz basis a $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis of fractional Sobolev spaces $H_\beta^\#(0,1)$ of order $\beta$ with $0\leq\beta\leq1$ and $\beta\neq1/2$. Let $\{e^{2\pi i\lambda_n x}\}_{n\in Z}$ forms a Riesz basis for $L_2^\#(0,1)$. Then for each number μ, which do not belong to the spectrum $\{\lambda_n\}_{n\in Z}$, the exponential families $E_\mu^{(\beta)}=\left\{\frac{1}{(1+|2\pi\lambda_n|^\beta)}e^{2\pi i\lambda_n x}\right\}_{n\in Z}\cup\{e^{2\pi i\mu x}\}$ form a Riesz basis for the Sobolev space $H_\beta\,(0,1)$ (**).

(*) With respect to the single positive zero of the Digamma function we note the auxiliary function

$$G(w):=G(u+iv)=(w-\lambda_0)\prod_{n=1}^{\infty}\left(1-\frac{w}{\lambda_n}\right)\left(1-\frac{w}{\lambda_{-n}}\right)$$

playing a key role in the proof of the Levinson-Kadec theorem XVIII, (LeN) p. 48.

(**) We note that according to the Sobolev embedding theorem any $g\in H_\beta^\#(0,1)$ with $\beta<\frac{1}{2}$ is bounded, i.e. $|g(x)|\leq c$.

A striking generalization of the Kadec theorem was discovered by Avdonin:

**Avdonin's theorem of $\frac{1}{4}-$ in the mean**, (YoR) p. 178: let $\lambda_n = n + \delta_n$, $n = 0, \pm 1, \pm 2, \ldots$, be a separated sequence of real or complex numbers. If there exists a positive integer $N$ and a constant $d$, $0 \le d < \frac{1}{4}$, such that

$$\left|\sum_{k=mN+1}^{(m+1)N} \delta_k\right| \le dN$$

for all integers $m$, then the system $\{e^{i\lambda_n t}\}_{n\in Z}$ is a Riesz basis for $L_2^{\#}(-\pi,\pi)$.

For $\beta_m := \frac{a_m}{2}$ with $m - \frac{1}{2} < a_m < m$ it holds

$$\frac{1}{4} < \beta_m - \beta_{m-1} < \frac{3}{4}$$

i.e. the following theorem can be applied with $\Theta = \frac{1}{4}$, providing a link between the zeros of the Kummer und Digamma functions, the related non-harmonic Fourier series and the Weyl sums in the following form

$$cot\left(\frac{\pi}{8}\right) - \varepsilon < \left|\sum_{n=1}^{m} e^{-2\pi i\beta_n}\right| < \cot\left(\frac{\pi}{8}\right), \text{ for every } \varepsilon > 0.$$

**Theorem** (LaE1): let $m > 1$, $\beta_1$ real, $0 < \Theta \le \beta_2 - \beta_1 \le \beta_3 - \beta_2 \le \cdots \beta_m - \beta_{m-1} \le 1 - \Theta$, and

$$S = S_m := \left|\sum_{n=1}^{m} \sigma_n\right| \text{ with } \sigma_n := e^{-2\pi i\beta_n}.$$

Then it holds:

i) $S_m \le \cot\left(\frac{\pi}{2}\Theta\right)$;

ii) For $\Theta = 1/2$ and every positive fraction $\Theta < 1/2$ with odd nominator and odd denominator:

$$S_m = \cot\left(\frac{\pi}{2}\Theta\right);$$

iii) For all other $\Theta$ with $0 < \Theta < 1/2$ :

$$S_m < \cot\left(\frac{\pi}{2}\Theta\right);$$

iv) For all $\Theta$ with $0 < \Theta \le 1$ and every $\varepsilon > 0$:

$$S_m > cot\left(\frac{\pi}{2}\Theta\right) - \varepsilon.$$

**The Hilbert space $H_{-1/2}^{\#} \cong l_2^{-1/2}$ and Dirichlet series**

In this section we are concerned with Hilbert scales $H_\alpha^{\#} \cong l_2^\alpha$ , $\alpha \in R$, which are built on the $2\pi$ –periodic Hilbert space $L_2^{\#}(\Gamma)$ where $\Gamma \coloneqq S^1(R^2)$ denotes the boundary of the unit circle sphere. Then for $u \in L_2^{\#}(\Gamma)$ and for real $\beta \in R$, $\nu \in Z$ the Fourier coefficients

$$u_\nu \coloneqq \frac{1}{2\pi}\oint u(x)e^{i\nu x}dx$$

enable the definition of the norms

$$\|u\|_\beta^2 \coloneqq \sum_{n=-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2.$$

**Remark** (NaS): We note that

$$S(u,v) = \int_{S^1} u \cdot dg = -i\sum_{-\infty}^{\infty} \nu u_\nu v_\nu$$

defines an inner product on $l_2^{1/2} = (l_2^{-1/2})^*$; in this case the generalized Fourier coefficients $\left\{\sqrt{|\nu|}u_\nu\right\}$ are square summable.

The Dirichlet series

$$f(s) := \sum_1^\infty a_n e^{-s\log n} \quad , \quad g(s) := \sum_1^\infty b_n e^{-s\log n}$$

are linked to the (distributional) Hilbert space $H_{-1/2}^{\#} \cong l_2^{-1/2}$ by ((EdH) 9.8, (NaS))

$$\big((f,g)\big)_{-1/2} := \lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega} f(1/2+it)g(1/2-it)dt = \sum_1^\infty \frac{1}{n}a_n b_n \ .$$

**Theorem 11** (HaG): Let $s = \sigma + it$ and $\mu_n = log\lambda_n$ and the series $\sum_{n=1}^\infty a_n e^{-\mu_n s}$ convergent; then if $\sigma > 0$ the following definite integral representation is given

$$\sum_{n=1}^\infty a_n e^{-\mu_n s} = \frac{1}{\Gamma(s)}\int_0^\infty x^s\left[a_n e^{-\lambda_n s}\right]\frac{dx}{x} \ .$$

**Theorem 13** (HaG) p. 13: if the series $\sum_{n=1}^\infty a_n e^{-\lambda_n s}$ is convergent for $s = \beta + i\gamma$ and $c > 0$, $c > \beta$, $\lambda_n < \omega < \lambda_{n+1}$ then

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\, e^{\omega s}\frac{ds}{s} = \sum_{k=1}^n a_k,$$

the path of integration being the line $\sigma = Re(s) = c$. At a point of discontinuity $\omega = \lambda_n$ the integral has a value half-way between its limits on either side, but in this case the integral must regarded as being defined by its principal value. The principle value is the limit, if it exists, of

$$\frac{1}{2\pi i}\int_{c-\mathrm{iT}}^{c+\mathrm{iT}} f(s)\, e^{\omega s}\frac{ds}{s} \ ,$$

which may exist when the integral, as ordinarily defined, does not.

The theorem 13 above depends upon the following

**Lemma** (HaG): if $x$ is real, we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} e^{xs}\, e^{\omega s}\frac{ds}{s} = \begin{cases} 1, x > 0 \\ \frac{1}{2}, x = 0, \\ 0, x < 0 \end{cases}$$

it being understood that in the second case the principle value of the integral ist taken.

**Lemma** (EdH) 9.8: It holds

$$\frac{1}{2\omega}\int_{-\omega}^{\omega}|\Xi(t)|^2 dt \approx \log\omega$$

and therefore

i) $\|\Xi\|_{-1/2}^2 = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|\Xi(t)|^2 dt = \sum_{n=1}^{\infty}\frac{1}{n} = \zeta(1) = \infty$

ii) $\|\Xi\|_{-1}^2 = \sum_{n=1}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6} = \int_0^1 \frac{\log x}{x-1}dx = \left[\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}\right]^{-1}$.

**Remark**: For each positive real number $x$ the Schnirelmann density is defined for a subset $\boldsymbol{A}$ of the set of positive integers $\boldsymbol{N}$ by

$$0 \le \sigma(\boldsymbol{A}) := \inf_n \frac{A(n)}{n} \le 1 \quad \text{with} \quad A(x) := \sum_{\substack{a\in\boldsymbol{A}\\ a\le x}} 1.$$

The function $A(x)$ is called the counting function of the set $\boldsymbol{A}$. For $x > 0$ it holds

$$0 \le A(x) \le [x] \le x \quad \text{and so} \quad 0 \le \frac{A(x)}{x} \le 1.$$

In the context of the (distributional) Hilbert space $H_{-1/2}^{\#} \cong l_2^{-1/2}$ we propose the following extended Schnirelmann density $\beta \in [0,1]$ concept considering sequences $\vec{a} = (a_n)_{n\in N} \in l_2^{-\beta}$:

$$\lim_{n\to\infty}\frac{A(n)}{n} \quad \rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{n}|a_n|^2 = \|\vec{a}\|_{-\beta}^2 < \infty.$$

**Remark**: The distributional Hilbert space $H_{-1}^{\#}$ plays a key role in (BaB), where the Nyman criterion is reformulated into a purely functional analysis weighted $l^2$ –Hilbert space framework. The considered Hilbert space is about of all sequences $a = \{a_n | n \in N\}$ of complex numbers such that

$$\sum_{n=1}^{\infty}\omega_n|a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \le \omega_n \le \frac{c_2}{n^2}$$

which is isomorphic to the Hilbert space $H_{-1} \cong l_2^{-1}$.

**Remark**: for $\gamma := \{11,1,1,\ldots\ldots\}$ then it holds

$$\|\gamma\|_{-1}^2 = \sum_1^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{i.e. } \gamma \in l_2^{-1}.$$

The **Bagchi-Beurling-Nyman RH criterion**, (BaB): Let

$$\gamma_k := \left\{\rho(\tfrac{n}{k})\middle| n = 1,2,3,\ldots.\right\} \text{ for } k = 1,2,3,\ldots$$

and $\Gamma_k$ be the closed linear span of $\gamma_k$. Then the Nyman criterion states that the following statements are equivalent:

The Riemann Hypothesis is true if and only if $\gamma \in \bar{\Gamma}_k$.

**Remark**: The Riemann Hypothesis states that $\zeta(s) \neq 0$ for all $s = \sigma + it$ with $1/2 < \sigma < 1$, i.e.

$$\frac{1}{\zeta(s)}, \frac{\zeta'(s)}{s\zeta(s)}$$

has no poles in case of $1/2 < \sigma < 1$.

### $\zeta(s)$ and a Dirichlet series in the form $\zeta_h^*(s) := \sum_{n=1}^{\infty} \frac{h_n}{n} n^{-s}$

**Lemma** (Grl) 4.224:

$$\int_0^\infty log^2 \sin(x)\, dx = \frac{\pi}{2}\left[log^2 2 + \frac{\pi^2}{12}\right].$$

**Lemma** (Grl) 3.761:

$$\int_0^\infty y^s\, sin\, y \frac{dy}{y} = \Gamma(s)\, sin(\frac{\pi}{2}s),\ 0 < |Re(s)| < 1$$

$$\int_0^\infty y^s\, cos\, y \frac{dy}{y} = \Gamma(s)\, cos(\frac{\pi}{2}s),\ 0 < Re(s) < 1.$$

**Lemma**: let $M$ denote the Mellin transform operator. Then it holds

$$M\left[xh'\right](s) = -sM[h](s).$$

**Lemma** (TiE) II, BeB) (17.12/13): The Fourier series representations of the fractional part function $\rho(x) := \{x\} := x - [x]$ and its related Hilbert transform $\rho_H(x) := H[\rho](x)$ are given by

i) $\rho(x) = \frac{1}{2} - \sum_1^\infty \frac{sin\, 2\pi\nu x}{\pi\nu} \in L_2^\#(0,1)\, , \hat{\rho}(0) \neq 0$

ii) $\rho_H(x) = \sum_1^\infty \frac{cos\, 2\pi\nu x}{\pi\nu} = -\frac{1}{\pi} log\, 2\, sin(\pi x) \in L_2^\#(0,1)\ ,\ \hat{\rho}_H(0) = 0.$

**Corollary:** In a distributional sense it holds

i) $\rho'(x) = -2\sum_1^\infty cos\, 2\,\pi\nu x \in H_{-1}^\#(0,1)$

ii) $\rho_H'(x) = -2\sum_1^\infty sin\, 2\,\pi\nu x = -\, cot(\,\pi x) \in H_{-1}^\#(0,1).$

**Lemma** (TiE) II:The following Mellin transform representations are valid

i) $\zeta(s) = -s\int_0^\infty x^{-s}\rho(x)\frac{dx}{x}\,,\qquad 0 < Re(s) < 1$

ii) $\zeta(s) = -s\int_0^\infty x^{-s}\left[\rho(x) - \frac{1}{2}\right]\frac{dx}{x}\,,\quad -1 < Re(s) < 0.$

**Remark**: The following lemma is valid primarily for $-1 < Re(s) < 0$. However, the right-hand side is analytic for all values such that $Re(s) < 0$.

**Lemma** (TiE) II: for $-1 < Re(s) < 0$ it holds

$$\zeta(s) = -\frac{s}{2}M\left[\rho(x) - \frac{1}{2}\right](-s) = \frac{1}{2}M[\rho'(x)](-s) = \frac{s}{2\pi}(2\pi)^s\Gamma(1-s)\, sin(\frac{\pi}{2}s)\zeta(1-s).$$

Proof: for $1 < Re(1-s) < 2$ the Dirichlet series $\sum_1^\infty \frac{\nu^s}{\nu}$ is convergent; as $\rho(x) \in L_2^\#(0,1)$ term-by-term integration is justified in the following, i.e., it holds

$$\zeta(s) = \frac{s}{2\pi}\sum_1^\infty \frac{1}{\nu}\int_0^\infty x^{-s}\, sin\, 2\,\pi\nu x\frac{dx}{x} = \frac{s}{2\pi}\sum_1^\infty \frac{(2\pi\nu)^s}{\nu}\int_0^\infty y^{-s}\, sin\, y\frac{dy}{y}$$

$$= \frac{s}{2\pi}(2\pi)^s[-\Gamma(-s)]\, sin(\frac{\pi}{2}s)\zeta(1-s)$$

$$= \frac{s}{2\pi}(2\pi)^s\Gamma(1-s)\, sin(\frac{\pi}{2}s)\zeta(1-s).$$

Let

$$h_n := \sum_{k=1}^{n} \frac{1}{2k-1} = H_{2n} - \frac{1}{2} H_n,\ n \in N, \text{ where } \ H_n := \sum_{k=1}^{n} \frac{1}{k}$$

denote the harmonic numbers. Putting

$$T(x) := -\frac{\pi}{2} log\left(tan \frac{\pi}{2} x\right)$$

it holds

**Lemma** (EIL) (EIL1):

i) $\sum_{k=1}^{n} (\frac{h_n}{n})^2 < \infty$

ii) $\sum_{n=1}^{\infty} \frac{h_n}{n} \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} (\sum_{n=1}^{\infty} \frac{h_n}{n} e^{-nx})\, dx$ for $Re(s) > 0$

iii) $T(x) = \sum_{n=1}^{\infty} \frac{h_n}{n} sin(2\pi nx) \in L_2(0,1)$

iv) $T'(x) = -\frac{\pi^2}{2 sin \pi x} = -\frac{\pi}{2\sin(\frac{\pi}{2}x)} \frac{\pi}{2\cos(\frac{\pi}{2}x)} \in H_{-1}^{\#}(0,1)$.

Putting

$$\zeta_h^*(s) := \sum_{n=1}^{\infty} \frac{h_n}{n} n^{-s}$$

$$\zeta^*(s) := \frac{1}{2} M[\rho_H'(x)](1-s)$$

$$\eta(s) := (2\pi)^{s-1} \Gamma(1-s)\, cos(\frac{\pi}{2} s)$$

one gets

**Lemma:** for $-1 < Re(s) < 0$ it holds

$$M[T(x)](-s) = (2\pi)^s \Gamma(s)\, sin(\frac{\pi}{2} s) \zeta_h^*(-s).$$

Proof:

$$M[T(x)](-s) = \int_0^\infty x^{-s} (\sum_{n=1}^{\infty} \frac{h_n}{n} sin(2\pi nx) \frac{dx}{x}$$
$$= \sum_{n=1}^{\infty} \frac{h_n}{n} \int_0^\infty x^{-s} sin(2\pi nx) \frac{dx}{x}$$
$$= \sum_{n=1}^{\infty} \frac{h_n}{n} (2\pi n)^s \int_0^\infty y^{-s} sin\, y \frac{dy}{y}$$
$$= (2\pi)^s [\sum_{n=1}^{\infty} h_n n^{s-1}] \Gamma(-s)\, sin(\frac{\pi}{2}(-s)).$$

**Corollary:** for $0 < Re(s) < 1$ it holds

$$M[T(x)](1-s) = \eta(s) \zeta_h^*(1-s)$$

**Corollary**: for $0 < Re(s) < 1$ it holds formally only

$$\zeta^*(s) = \frac{1}{2} M[\rho_H'(x)](1-s) = \eta(s)\zeta(1-s).$$

**Remark** (PeB) I §15, II §12: the $\zeta^*(s)$ function can be interpreted as a complex-valued $H_{-1}^{\#}(0,1)$ based distributional integral representation of $\zeta(s)$ where $\zeta_h^*(s)$ is a complex-valued $L_2^{\#}(0,1)$ based approximation distributional integral representation of $\zeta^*(s)$

**Remark** (EsR) pp. 105/139: for the functions $log\Gamma(\lambda)$ there exists a moment asymptotic representation in a distributional space of „less than exponential asymptotics"; for the Hilbert transform operator there exists an asymptotic representation in a distributional space.

**Remark** (VlV) p.189 ff.: the theory of Tauberian theorems for generalized functions provides quasi-asymptotics of the fundamental solutions of distributional convolution equations.

The functions $-\frac{\pi}{2}log\left(tan\frac{\pi}{2}x\right)$ and $-\frac{1}{\pi}log\, 2\, sin(\pi x)$ are related to the alternatively proposed entire Zeta function $\xi^*(s)=\frac{1}{2}(1-s)\zeta(s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\, tan(\frac{\pi}{2}s)$ by the formula

**Lemma**:

$$\Gamma(\frac{s}{2})\, tan(\frac{\pi}{2}s)=\frac{\pi}{\cos\left(\frac{\pi}{2}s\right)}\frac{1}{\Gamma\left(1-\frac{s}{2}\right)}=\frac{\pi}{\sin\left(\frac{\pi}{2}(1-s)\right)}\frac{1}{\Gamma\left(1-\frac{s}{2}\right)}.$$

The lemmata leads to the following „telescope" product representation

**Corollary**:

$$\zeta(s)\zeta(1-s)=4\sqrt{\pi}\frac{\xi^*(s)}{\pi(1-s)}\frac{\xi^*(1-s)}{\pi s}\sin\left(\frac{\pi}{2}s\right)\Gamma\left(1-\frac{s}{2}\right)\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma\left(1-\frac{1-s}{2}\right).$$

Proof:

$$\frac{1}{\Gamma(\frac{s}{2})\, tan(\frac{\pi}{2}s)}=\frac{1}{\pi}\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma\left(1-\frac{s}{2}\right)$$

$$(1-s)\zeta(s)=2\pi^{\frac{s}{2}}\frac{\xi^*(s)}{\Gamma(\frac{s}{2})\, tan(\frac{\pi}{2}s)}=2\pi^{\frac{s}{2}}\xi^*(s)\frac{1}{\pi}\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma\left(1-\frac{s}{2}\right)$$

$$s\zeta(1-s)=2\pi^{\frac{1-s}{2}}\frac{\xi^*(1-s)}{\Gamma(\frac{1-s}{2})\, tan(\frac{\pi}{2}(-s))}=2\pi^{\frac{1-s}{2}}\xi^*(1-s)\frac{1}{\pi}\sin\left(\frac{\pi}{2}s\right)\Gamma\left(1-\frac{1-s}{2}\right).$$

**Remark**: the telescope" product representation enables the application of the auxiliary function, (EdH) 1.14,

$$H(\beta):=-\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log(1-\frac{s}{\beta})}{s}\right]x^s\, ds$$

resulting into formulas in the form $Li(x^\beta)\pm i\pi$ for $Re(\beta)>0$.

**Remark**: Taking the $log$-function of the term $(1-s)\zeta(s)$ leads to the Gaussian density functions $li(x)$ resp. the Riemann density function $J(x)$, (EdH) 1.12 ff.

The related density functions of $\sin\left(\frac{\pi}{2}s\right)\Gamma\left(1-\frac{s}{2}\right)$ resp. $\sin\left(\frac{\pi}{2}(1-s)\right)\Gamma\left(1-\frac{s}{2}\right)$ govern the densities of the non-trivial zeros of $\xi(s)$ with positive resp. negative imginary parts.

Taking the $log$-function of the terms $\frac{\pi}{2}tan(\frac{\pi}{2}s)$ and $\frac{1}{\pi}\sin\left(\pi\frac{s}{2}\right)$ provides two additional density functions $t(x)$ and $s(x)$.

**Remark** (GrI) 4.224, 4.227:

i) $\int_0^1 log\,(\pi sin(\frac{\pi}{2}x))dx = log\frac{\pi}{2}$

ii) $\int_0^1 log(\frac{\pi}{2}tan\left(\frac{\pi}{2}x\right))dx = log\frac{\pi}{2}$.

**Remark** (GrI) 4.227: the product factors $\pi$, $\frac{\pi}{2}$ of $sin\left(\frac{\pi}{2}x\right)$, $tan\left(\frac{\pi}{2}x\right)$ are essential, as it holds

$$\int_0^1 log(sin\left(\frac{\pi}{2}x\right))dx < 0, \quad \int_0^1 log(tan\left(\frac{\pi}{2}x\right))dx = 0\,.$$

**Lemma** (PoG1): Let $f(t)$ Riemann integrable in the domain $[0,1]$, and $0 < q_1 \leq q_2 \leq q_3 \leq \cdots$. a divergent sequence of positive numbers. Let $w(x)$ a positive, non decreasing function with $\lim_{n\to\infty}\frac{w(\beta x)}{w(\alpha x)} = 1$ with $\alpha, \beta$ positive numbers. Then

$$\lim_{n\to\infty}\frac{w(n)}{n}\sum_{q\leq n} f(\frac{q}{n}) = \int_0^1 f(t)dt.$$

**Examples**:

$$f(t) = \rho\left(\frac{1}{t}\right) + \gamma, f(t) = \; log(\frac{\pi}{2}tan\left(\frac{\pi}{2}t\right), w(x) = log(x), w(x) = \log\left(a\cdot tan(x)\right), a > 1.$$

**Remark**: The Landau-type function $w(x)$ puts the spot on automodel (or regular varying) functions equipped with an order of automodelity, providing Tauberian theorems for generalized Functions (VlV) p. 56.

**Remark**: A quick distributional way to prove the PNT is provided in (ViJ).

With respect to the proposed approximating zeta function Dirichlet series

$$\zeta_h^*(s) := \sum_{n=1}^{\infty}\frac{h_n}{n}n^{-s}$$

we note the

**Lemma** (LaE1a) §86: Let $a > 0$ $\sum_{n=1}^{\infty}\frac{|b_n|}{n^s}$ convergent, i.e. $D(s) = \sum_{n=1}^{\infty}\frac{b_n}{n^s}$ is an absolute convergent Dirichlet series for $a := Re(s)$. Putting

$$f(x) := \begin{cases} \sum_{n=1}^{x} b_n & for \quad x \quad not\ integer \\ \sum_{n=1}^{x} b_n - b_x/2 & for \quad x \quad integer \end{cases}$$

Then it holds

$$f(x) = \lim_{n\to\infty}\frac{1}{2\pi i}\int_{a-it}^{a+it} D(s)x^s\frac{ds}{s} \quad \text{resp.} \quad \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} D(s)x^s\frac{ds}{s} = \sum_{n=1}^{x} b_n \log\,(\frac{x}{n}).$$

**Remark**: $D(s)$ does not need to be regular; only for $a < Re(s)$ this follows from the pre-requisites.

### An alternative contour integral represention of $\zeta(s)$ and a formula for $\zeta\left(\frac{1}{2}\right)$

Riemann's contour integral representation

$$\zeta(s)=\frac{\Gamma(1-s)}{2\pi i}\int_{+\infty}^{\infty}\frac{(-x)^s}{e^x-1}\frac{dx}{x}$$

is derived from the integral representation of the $\zeta(s)$ –Dirichlet series

$$\zeta(s)\Gamma(1-s)=\Gamma(1-s)\sum_{n=1}^{\infty}\frac{1}{n^s}=\int_0^{\infty}\frac{x^s}{e^x-1}\frac{dx}{x}\;,\; Re(s)>1.$$

For the function $M(z):=z^{-\frac{1}{4}}e^{-\frac{z}{2}}M_{\frac{1}{4},\frac{1}{4}}(z)=z^{\frac{1}{2}}\,{}_1F_1\left(1,\frac{3}{2},-z\right)$ the one-side Laplace transform is given by

$$\int_0^{\infty}t^{\frac{s}{2}}M(nt)\frac{dt}{t}=n^{-\frac{s}{2}}\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right), \quad -1<Re(s)<1$$

enabling a corresponding contour integral representation.

Putting $f(x):=e^{-\pi x^2}$ the entire Zeta function $\xi(s)$ is given by, (EdH) 1.8,

$$\xi(s):=\frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s)=(1-s)\zeta(s)M[-xf'(x)](s)=\xi(1-s).$$

The Mellin transform of the Hilbert transformed Gaussian function $f_H(x)$ is given by

$$M[f_H(x)](s)=2\left[\,{}_1F_1\left(1;\frac{3}{2},-\pi x^2\right)\right](1+s)=\frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right), \quad -1<Re(s)<3.$$

enabling an alternatively entire Zeta function $\xi^*(s)$ (which is in line with the alternative contour integral representation) defined by

$$\xi^*(s):=(1-s)\zeta^*(s)M[f_H(x)](s)=\frac{1}{2}(1-s)\zeta(s)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s).$$

The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self-adjoint operator (*). Its "Unified Field Theory" related counterpart is given by the Berry-Keating conjecture (BeM). A proof of this conjecture is basically about defining a symmetric operator and a corresponding appropriate domain enabling an unbounded and self-adjoint Friedrichs extension. The claim is, that the alternative contour integral representation provides the appropriate symmetric operator accompanied by an appropriate distributional Hilbert space domain enabling an (Friedrichs extended) self-adjoint Friedrichs extension, which is unbounded with respect to the $L_\infty$ – norm.

(*) (EdH) section 10: Let $V$ be the vector space of all complex-valued functions on $R^+$ with the inner product $(u,v):=\int_0^{\infty}u(x)v(x)dx$.By $I:v(x)\mapsto\int_0^{\infty}v(ux)F(u)du$ an integral operator $\mathrm{I}:V\to V$ is defined. An operator is said to be invariant if it commutes with all translation operators $T_u:v(x)\mapsto v(ux)$. The transform of an invariant operator is the function whose domain is the set of complex numbers $s$ such that the function $v(x):=x^{-s}$ lies in the domain of the operator and whose value for such an $s$ is the factor by which the operator multiplies $v(x):=x^{-s}$ . Thus e.g. the Zeta function $\zeta(s)$ for $Re(s)>1$ is the transform of the summation operator $v(x)\mapsto\sum_1^{\infty}v(nx)$. When defining the adjoint of an invariant operator on $V$ the inner product is defined on a rather small subset of $V$, whenever both side of $(Lu,v)=(u,L^*v)$ are defined.

There is an only formally valid representation of Riemann's duality equation as transform of an integral operator $I:v(x)\mapsto\int_0^{\infty}v(ux)G(u)du$ *in the form* $\int_0^{\infty}x^{1-s}G(x)\frac{dx}{x}=\frac{2\xi(s)}{s(s-1)}$ *. But the operator* $I$ *has no transform at all, as the integral does not converge for any* $s$*. The integral would converge at ∞ if the constant term* $f(0)=\hat{f}(0)=1$ *is absent. If one would find an integral operator in the form* $I$*satisfying the same functional equation than G does and if* $\int_0^{\infty}x^{1-s}G(x)\frac{dx}{x}$ *converges and* $\int_0^{\infty}x^{1-s}G(x)\frac{dx}{x}=\int_0^{\infty}x^{s}G(x)\frac{dx}{x}$ *then this operator would be self-adjoint in the sense of above.*

For a fixed $q \geq 0$ let $\chi(n)$ denote the arithmetical function known as a character modulo $q$ (for $q = 1, L(s,\chi) = \zeta(s)$, if $(a,q) > 1$ then $\chi((a) = 0)$. Then the Hurwitz Zeta function is the Dirichlet series defined by, (IvA) 1.8,

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \quad (1-\chi(p)p^{-s})^{-1} \text{ for } Re(s) > 1$$

**Remark**: For $q = 1$ it holds

$$L(s,\chi_1) = \zeta(s) \prod_p (1-p^{-s})^{-1} = \prod_p (1-p^{-s})^{-1} \prod_{p|q} (1-p^{-s})^{-1}.$$

Thus $L(s,\chi_1)$ has a first-order pole at $s = 1$ just like $\zeta(s)$ and it behaves similar as $\zeta(s)$ in many other ways, while $L(s,\chi)$ for $\chi \neq \chi_1$ is regular for $Re(s) > 0$.

The Generalized Riemann Hypothesis (GRH) states that all non-trivial zeros of all Dirichlet $L$ –functions have real part equal to $1/2$.

**Remark**: There are also other generalized Zeta function, e.g., the Lerch, the Epstein or the Dedekind Zeta functions, as well as Zeta functions associated with cusp forms, (IvA) 11.8.

Ramanujan's following formula may be regarded as a representation for $\zeta\left(\frac{1}{2}\right)$, it also can be viewed as an identity for an infinite sum of theta functions.

**Entry 8.3.1** (BeB2) p. 191: let $\alpha$ and $\beta$ and be positive numbers such that $\alpha\beta = 4\pi^3$. Then

$$\sum_{n=1}^{\infty} \frac{1}{e^{n^2\alpha}-1} = \frac{\pi^2}{6\alpha} + \frac{1}{4} + \frac{\sqrt{\beta}}{4\pi}\left\{\zeta\left(\frac{1}{2}\right) + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \frac{cos\sqrt{n\beta} - sin\sqrt{n\beta} - e^{-\sqrt{n\beta}}}{cosh\sqrt{n\beta} - cos\sqrt{n\beta}}\right\}.$$

**Remark** (BeB1) p. 225: Ramanujan also recorded a Quasi-theta product formula using hyperbolic functions and elliptic functions. The equivalent formulation in terms of hyperbolic series is given by the identity

$$\alpha \sum_{n=1}^{\infty} \frac{sinh\{(2n+1)\alpha/2\}}{(2n+1)cosh^2(\frac{(2n+1)\alpha}{2})} + \pi \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)cosh^2(\frac{(2n+1)\beta}{2})}$$

where $\alpha$ and $\beta$ are complex numbers with nonzero real part and with $\alpha\beta = \pi^2$.

**Exponential sums related inequalities for almost periodic functions**

**Remark** (ReH) p.145: The Haar (wavelet) scaling function is an universal scaling function of genus $g = 1$. Its scaling equation for rank $m$ is

$$\varphi(x) = \sum_{0\le k<n} \varphi(mx-k).$$

This corresponds to the scaling vector $\gamma_m := \{1,1,1,1,\ldots 1\}$ of rank $k$ for a Haar matrix of rank $k$ playing a key role in the Bagchi-Beurling-Nyman RH criterion, (BaB).

The Fourier transform of the Haar scaling function is given by

$$\hat{\varphi}(\xi) = F_{0,m}\left(\frac{\xi}{m}\right)\hat{\varphi}\left(\frac{\xi}{m}\right) \quad \text{with} \quad F_{0,m}(\xi) := \frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi i k\xi}$$

Taking the limit one finds the interesting infinite product formula

$$sinc(\pi\xi) = \lim_{m\to\infty} e^{-i\pi\xi} \prod F_{0,m}\left(\frac{\xi}{m^n}\right) = \prod(1-\left(\frac{\xi^2}{n^2}\right)) = \frac{1}{\Gamma(1+\xi)\Gamma(1-\xi)}.$$

**Remark** (ReH) p. 142: the Haar scaling function is defined by

$$\varphi(x) := \begin{cases} 1 & if \quad 0\le x\le 1 \\ 0 & if \ otherwise \end{cases}.$$

It is a specific compactly supported solution of the corresponding scaling equation

$$(*) \quad \varphi(x) = \varphi(2x) + \varphi(2x-1).$$

The compactly supported solutions of (*) differ only a the endpoints of $(0,1)$.

**Theorem 15** (VaJ): Let $\lambda_1, \lambda_2, \ldots., \lambda_N$ be real numbers and

$$f(x) := \sum_{n=1}^{N} a(n) e(\lambda_n x) = \sum_{n=1}^{N} a(n) e^{2\pi i \lambda_n x}.$$

If $|\lambda_n| \ge \delta > 0$ for $n = 1,2,\ldots.N$, then

$$sup_x|f(x)| \le (4\delta)^{-1} sup_\omega |f'(\omega)|.$$

If, additionally, $f(x)$ is real valued, then

$$sup_x|f(x)| \le (2\delta)^{-1} sup_\omega |f'(\omega)|.$$

Moreover, the constants $(4\delta)^{-1}$ and $(2\delta)^{-1}$ are asymptotically best possible as $N \to \infty$.

A general form of Hilbert's inequality was first obtained by Montgomery and Vaughan:

**Theorem 16** (VaJ): Let $\lambda_1, \lambda_2, \ldots., \lambda_N$ be real numbers satisfying $|\lambda_m - \lambda_n| \ge \delta > 0$ whenever $m \ne n$, and let $a(1), \ldots a(n)$ be arbitrary complex numbers. Then

$$\left|\sum_{\substack{m=1 \\ m\ne n}}^{N} \sum_{\substack{n=1 \\ n\ne n}}^{N} \frac{a(m)\overline{a(n)}}{\lambda_m - \lambda_n}\right| \le \frac{\pi}{\delta}\sum_{n=1}^{N} |a(n)|^2.$$

**Theorem A.5** (GrS) p. 115: Let $N$ and $T$ be positive real numbers, and let $a_n$ be a sequence of complex numbers with $a_n = 0$ if $n \le N$ or $n > 2N$. Suppose $g$ is a real-valued function with $|g(m) - g(n)| > \delta$ whenever $N < m, n \le 2N$ and $m \ne n$. Then

$$(T - \delta^{-1})\sum_n |a_n|^2 \le \int_T^{2T} |\sum_n a_n e(tg(n)|^2 dt \le (T + \delta^{-1})\sum_n |a_n|^2.$$

Turán's first main theorem is concerned with lower bounds of exponential sums. From this it follows

**Fabry's Gap Theorem** (MoH) p. 89: Suppose that $T(x)$ is an exponential polynomial of $N$ terms and period $1$, say

$$T(x) = \sum_{n=1}^{N} b_n e(\lambda_n x)$$

where the $\lambda_n$ are integers. Let $I$ be a closed arc on the circle group $\boldsymbol{T}$, and let $L$ denote the length of $I$. Then

$$\max_{x \in I} |T(x)| \geq (\frac{L}{2e})^{N-1} \max_{x \in \boldsymbol{T}} |T(x)|.$$

**Remark**: The striking feature of this bound is that it does not depend on the size of the $\lambda_n$.

The Beurling function

$$B(z) := (\frac{\sin\pi z}{\pi})^2 \{\sum_{n=0}^{\infty} (z-n)^{-2} - \sum_{m=-\infty}^{-1} (z-m)^{-2} - 2z^{-1}\}$$

is an entire function of exponential type $2\pi$ satisfying a simple and useful extremal property (*).

**Remark** (CaJ1): A. Selberg used the function $B(z)$ to obtain a sharp form of the large sieve inequality.

The Beurling function could be used to construct trigonometric polynomial approximation to

$$\psi(x) := \rho(x) - \frac{1}{2} = x - [x] - \frac{1}{2} = -\sum_{1}^{\infty} \frac{\sin 2\pi\nu x}{\pi\nu}.$$

Putting

$$H(z) := (\frac{\sin\pi z}{\pi})^2 \left\{\sum_{n=-\infty}^{\infty} \frac{sgn(n)}{(z-n)^2} + \frac{2}{z}\right\} \quad , J(x) := \frac{1}{2} H'(z)$$

where

$$\hat{J}(t) = \begin{cases} 1 & if\ t = 0 \\ \pi t(1-|t|)\cot(\pi t) + |t|, & if\ 0 < |t| < 1 \\ 0 & if\ \geq 1 \end{cases}$$

it holds

**Theorem A.6** (GrS): The trigonometric polynomial

$$\psi^*(x) = -\sum_{1 \leq |n| \leq N} \frac{1}{2\pi i n} \hat{J}_{N+1}(n) e(nx)$$

satisfies

$$|\psi(x) - \psi^*(x)| \leq \frac{1}{N+1} \sum_{1 \leq |n| \leq N} (1 - \frac{|n|}{N+1}) e(nx).$$

**Theorem A.5** (GrS) p. 115: Let $N$ and $T$ be positive real numbers, and let $a_n$ be a sequence of complex numbers with $a_n = 0$ if $n \leq N$ or $n > 2N$. Suppose $g$ is a real-valued function with $|g(m) - g(n)| > \delta$ whenever $N < m, n \leq 2N$ and $m \neq n$. Then

$$(T - \delta^{-1}) \sum_n |a_n|^2 \leq \int_T^{2T} |\sum_n a_n e(tg(n)|^2 dt \leq (T + \delta^{-1}) \sum_n |a_n|^2.$$

(*) **Lemma** (GrS), (VaJ): It holds

i) $B(z) \geq sgn(x)$ for all real $x$

ii) if $F(z)$ is any entire function of exponential type $2\pi$ satisfying $F(x) \geq sgn(x)$ for all real $x$, then
$$\int_{-\infty}^{\infty} F(x) - sgn(x) dx \geq 1$$

iii) $F(x) = sgn(x)$ if and only $F(z) = B(z)$.

**Riesz basis systems** $\left\{e^{i\lambda_n t}\right\}_{n\in N}, \left\{e^{i\lambda_\nu t}\right\}_{\nu\in Z}$

The core theorem of non-harmonic Fourier series theory is based on the Levinson-Kadec theorems (e.g. (LeN) p. 48, (PaR) p. 113.

**Levinson theorem XVIII**, (LeN) p. 48: If $\{\lambda_n\}$ is a sequence and $L$ a constant such that $|\lambda_n - n| \leq L < 1/4$, then the system $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$ is close in $L_2^{\#}(-\pi,\pi)$ (i.e. from $\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0$ it follows that $f(x)$ is identically zero) and possesses a unique biorthogonal set $\{h_n(x)\}$ such that for any $f \in L_2^{\#}(-\pi,\pi)$ the series

$$\sum_{-\infty}^{\infty} \frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi - e^{i\lambda_n x}\int_{-\pi}^{\pi} f(\xi)h_n(\xi)d\xi$$

converges uniformly to zero over the interval $[-\pi+\delta, \pi-\delta]$ for any $\delta > 0$. Moreover the difference of weighted sums (Riesz, Abel, and so on) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over $[-\pi+\delta, \pi-\delta]$.

The main ingredients to prove the Levinson theorem are the following two lemmata concerning the function

$$G(w) := G(u+iv) = (w-\lambda_0)\prod_{n=1}^{\infty}\left(1-\frac{w}{\lambda_n}\right)\left(1-\frac{w}{\lambda_{-n}}\right).$$

**Lemma A** (LeN) p. 55: If $\{\lambda_n\}$ fulfills the Kadec condition, then for different constants $c$ it holds

i) $|G(w)| < c(|w|+1)e^{\pi|v|}$

ii) $|G(w)| > c|v|(|w|+1)^{-2}e^{\pi|v|}$

iii) $|G1/2+iv)| > c$.

**Lemma B** (LeN) p. 57: The functions $h_n(\xi)$ defined by

$$h_n(\xi) := \int_{-\infty}^{\infty}\frac{G(u)}{(u-\lambda_n)G'(u)}e^{-iu\xi}du$$

form a sequence of biorthogonal to $\left\{e^{i\lambda_n x}\right\}$ over $(-\pi,\pi)$.

**Kadec\`s $\frac{1}{4}$ – Theorem** (YoR) p. 36: If $\{\lambda_n\}_{n\in Z}$ is a sequence of real numbers for which $|\lambda_n - n| \leq L < 1/4$ , then $\left\{e^{i\lambda_n t}\right\}_{n\in Z}$ satisfy the Paley-Wiener criterion and so forms a Riesz basis for $L_2\,(-\pi,\pi)$.

**Lemma** (LeN) p. 48, (YoR) p. 100: If $\{\lambda_n\}$ is a sequence and $L$ a constant such that

$$(*) \quad |\lambda_n - n| \leq L < 1/4,$$

then the system $\left\{e^{i\lambda_n x}\right\}_{n\in Z}$ is close in $L_2^{\#}(-\pi,\pi)$ (i.e. from $\int_{-\pi}^{\pi} f(x)e^{-inx}dx = 0$ it follows that $f(x)$ is identically zero) and possesses a unique biorthogonal set $\{h_n(x)\}$ such that for any $f \in L_2^{\#}(-\pi,\pi)$ the series

$$\sum_{-\infty}^{\infty} \frac{e^{inx}}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi - e^{i\lambda_n x}\int_{-\pi}^{\pi} f(\xi)h_n(\xi)d\xi$$

converges uniformly to zero over the interval $[-\pi+\delta, \pi-\delta]$ for any $\delta > 0$. Moreover the difference of weighted sums (Riesz, Abel, and so on) of the non-harmonic and ordinary Fourier series also converges uniformly to zero over $[-\pi+\delta, \pi-\delta]$.

**Remark** (YoR) p. 36: Kadec's $\frac{1}{4}$ –theorem shows that the system $\{e^{i\lambda_n x}\}_{n\in Z}$ constitutes a Riesz basis for $L_2^\#(-\pi,\pi)$ whenever every $\mu_n$ is real and

$$|\lambda_n - n| \le L < 1/4,$$

but need not constitute a basis when $L = 1/4$.

**Lemma**, (YoR) p. 109: The system $\{e^{i\lambda_n t}\}$ is minimal in $L_2\,(-\pi,\pi)$ if and only if there exists a nontrivial function $f(z)$ of exponential type at most $\pi$, zeros at every $\lambda_n$ , and such that

$$\int_{-\infty}^{\infty} \frac{|f(x)|^2}{1+x^2}\,dx < \infty.$$

A sequence of vectors in a separable Hilbert space is called complete, if its linear span is dense in the Hilbert space, resp. if the zero vector alone is perpendicular to every basis vector. A characterization of an orthogonal Schauder bases of a separable Hilbert space is that they are complete orthogonal sequences.

A complete sequence of vectors in a separable Hilbert space is a Riesz basis if and only if its moment space is equal to $l^2$ , (YoR) p. 142.

Regarding the stability of the class of Riesz bases $\{e^{i\lambda_n t}\}$ in $L_2\,(-\pi,\pi)$ Kadec's theorem can be dramatically improved, first under „small" displacements of the $\lambda_n$ 's and then under more general „vertical" displacements, (YoR) pp. 160 ff.

> **Corollary 1** (YoR) p. 164: Let $\{\lambda_1, \lambda_2, \lambda_3, \ldots.\}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\{e^{iRe(\lambda_n)t}\}$ is a Riesz basis for $L_2\,(-\pi,\pi)$, then so is $\{e^{i\lambda_n t}\}$.
>
> **Corollary 2** (YoR) p. 164: if $\{\lambda_n\}_{n\in Z}$ be a sequence of scalars for which $sup_n|Re(\lambda_n) - n| < 1/4$ and $sup_n|Im(\lambda_n)| < \infty$, then the system $\{e^{i\lambda_n x}\}_{n\in Z}$ is a Riesz basis for $L_2^\#(-\pi,\pi)$.

**Theorem 13** (YoR) p. 160: If the system $\{e^{i\lambda_n x}\}_{n\in N}$ is a frame in $L_2^\#(-\pi,\pi)$, then there is a positive constant $L$ with the property that $\{e^{i\mu_n x}\}_{n\in N}$ is also a frame in $L_2^\#(-\pi,\pi)$ whenever

$$|\lambda_n - \mu_n| \le L \text{ for every } n.$$

**Corollary** (YoR) p. 161: If the system $\{e^{i\lambda_n x}\}_{n\in N}$ is a Riesz basis for $L_2^\#(-\pi,\pi)$ then there is a positive constant $L$ with the property that $\{e^{i\mu_n x}\}_{n\in N}$ is also a Riesz basis for $L_2^\#(-\pi,\pi)$ whenever

$$|\lambda_n - \mu_n| \le L \text{ for every } n.$$

**Theorem 14** (YoR) p. 161: let the sequence $\{\lambda_n\}_{n\in N}$ be a sequence of points lying in a strip parallel to the real axis. If the system $\{e^{iRe(\lambda_n)x}\}_{n\in N}$ is a frame in $L_2^\#(-\pi,\pi)$, then so is $\{e^{i\lambda_n x}\}_{n\in N}$.

A striking generalization of the Kadec theorem was discovered by Avdonin:

**Avdonin's theorem of $\frac{1}{4}$ – in the mean**, (YoR) p. 178: let $\lambda_n = n + \delta_n$, $n = 0, \pm 1, \pm 2, \ldots.$, be a separated sequence of real or complex numbers. If there exists a positive integer $N$ and a constant $d$, $0 \le d < \frac{1}{4}$, such that

$$\left|\sum_{k=mN+1}^{(m+1)N} \delta_k\right| \le dN$$

for all integers $m$, then the system $\{e^{i\lambda_n t}\}_{n\in Z}$ is a Riesz basis for $L_2^\#(-\pi,\pi)$.

### The Paley-Wiener (separable Hilbert) space $PW$

The Paley-Wiener (separable Hilbert) space $PW$ is the totality of all entire functions of exponential type at most $\pi$ (i.e., $|f(z)| \leq e^{\pi|z|}$) that are square integrable on the real axis, i.e. it holds

$$|f(x+iy)| \leq e^{\pi|y|}\|f\|.$$

It is equipped with the inner product

$$(f,g) = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx.$$

**Lemma** (YoR) p. 90: The Paley-Wiener space $PW$ is isometrically isomorph to $L_2\,(-\pi,\pi)$. Every function $f \in PW$ can be recaptured from its values at the integers, which is achieved by the cardinal series representation of $f$.

A sequence of real or complex numbers $\{\lambda_n\}_{n\in N}$ is said to be an interpolating sequence for $PW$ if the set of all sequences $\{f(\lambda_n)\}_{n\in N}$ where $f$ ranges over $PW$, coincides with $l^2$ .

If, in addition, the system $\{e^{i\lambda_n t}\}_{n\in N}$ is complete in $L_2^{\#}(\Gamma)$, then $f(\lambda_n) = c_n$ has exactly one solution, provided $c_n \in l^2$, and in this case we shall call $\{\lambda_n\}_{n\in N}$ a complete interpolating sequence.

A complete interpolation sequence is „maximal" in the sense that it is not contained in any larger interpolating sequence, and the converse is also true, (YoR) p. 142.

Putting

$$G(z) := z\prod_{k=0\to\infty}\left(1-\frac{z^2}{\lambda_n^2}\right) \text{ and } G_n(z) := \frac{G(z)}{G'(\lambda_{\boldsymbol{n}})(z-\lambda_{\boldsymbol{n}})}$$

then $G_n(z)$ belongs to the Paley-Wiener space $PW$ and $g_n(t)$ is the inverse Fourier transform of $G_n(z)$, i.e. for almost all $t \in [-\pi,\pi]$,

$$g_n(t) := \int_{-\infty}^{\infty} G_n(z)e^{ixt}dt.$$

The exponentials $e^{i\lambda_n t}$ are transformed into the reproducing functions $K_n(z) = \frac{\sin\,\pi(z-\lambda_n)}{\pi(z-\lambda_n)}$ , $g_n(t)$ is transformed into $G_n(z)$, while the moment problem itself becomes

$$f(\lambda_n) = 0,\ n = 0, \pm 1, \pm 2, \pm 3, \ldots..,$$

since $f(\lambda_n) = (f, K_n)$. Here $c_n \in l^2$ and $f \in PW$ is to be found. By taking the Fourier transform of $\{e^{int}\}_{n\in Z}$, we see that the set of functions

$$\left\{\frac{\sin\,\pi(z-n)}{\pi(z-n)}\right\}_{n\in Z}$$

forms an orthogonal basis for $PW$. Accordingly every function $f$ in $PW$ has an unique expansion of the form

$$f(z) = \sum_{-\infty}^{\infty} c_n \frac{\sin\,\pi(z-n)}{\pi(z-n)} \text{ with } \sum_{-\infty}^{\infty}|c_n|^2 < \infty.$$

The convergence of the series is understood to be in the metric of $PW$. But convergence in $PW$ implies uniform convergence in each horizontal strip. This is an immediate consequence of the following useful estimate, (YoR) p. 90:

$$|f(x+iy)| \leq e^{\pi|y|}\|f\|.$$

## Some useful formulas and references

**Lemma** (Grl) 1.431:

i) $sinx = x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2\pi^2}\right)$

ii) $sinhx = x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2\pi^2}\right)$

iii) $cosx = \prod_{k=0}^{\infty}\left(1-\frac{4x^2}{(2k+1)^2\pi^2}\right)$

iv) $coshx = \prod_{k=0}^{\infty}\left(1+\frac{4x^2}{(2k+1)^2\pi^2}\right)$.

**Remark** (EdH) 1.10: The infinite product $\prod_{n=1}^{\infty}(1+a_n)$ is absolutely convergent if and only if the series $\sum|a_n|$ converges.

**Lemma** (Grl) 1.441:

i) $\sum_{k=1}^{\infty}\frac{sin\,(kx)}{k} = \frac{\pi-x}{2},\ 0<x<2\pi$

ii) $\sum_{k=1}^{\infty}\frac{\cos\,(kx)}{k} = \frac{1}{2}log\frac{1}{2(1-cosx)} = log\frac{1}{2\left|\sin\,(\frac{x}{2})\right|},\ 0<x<2\pi$

iii) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin\,(kx)}{k} = \frac{x}{2},\ -\pi<x<\pi$

iv) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\cos\,(kx)}{k} = \log\,(2\cos\frac{x}{2}),\ -\pi<x<\pi.$

**Lemma** (Grl) 1.442:

i) $\sum_{k=1}^{\infty}\frac{sin\,(2k-1)x}{2k-1} = \frac{\pi}{4},\ 0<x<\pi$

ii) $\sum_{k=1}^{\infty}\frac{cos\,(2k-1)x}{2k-1} = \frac{1}{2}log\,(cot\frac{x}{2}),\ 0<x<\pi$

iii) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{sin\,(2k-1)x}{2k-1} = \frac{1}{2}log\,(tan\,(\frac{\pi}{4}+\frac{x}{2}),\ -\frac{\pi}{2}<x<\frac{\pi}{2}$

iv) $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{cos\,(2k-1)x}{2k-1} = -\frac{\pi}{4},\ \frac{\pi}{2}<x<\pi.$

**Lemma** (Grl) 3.896, 3.952:

i) $\int_0^{\infty}e^{-ax^2}\,sin(bx)\,dx = \frac{b}{2a}e^{-\frac{b^2}{4a}}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2},\frac{b^2}{4a}\right) = \frac{b}{2a}\ {}_1F_1\left(1,\frac{3}{2},-\frac{b^2}{4a}\right)$

ii) $\int_0^{\infty}x^{\mu-1}e^{-\beta x^2}\,sin(\gamma x)\,dx = \frac{\gamma}{2\beta^{\frac{\mu+1}{2}}}e^{-\frac{\gamma^2}{4\beta}}\Gamma(\frac{1+\mu}{2})\ {}_1F_1\left(1-\frac{\mu}{2},\frac{3}{2},\frac{\gamma^2}{4\beta}\right).$

**Lemma** (Grl) 3.761: For $a>0$ it holds

i) $\int_0^{\infty}x^{\mu-1}\,sin(\,ax)dx = \frac{\Gamma(\mu)}{a^{\mu}}sin(\frac{\pi}{2}\mu)\ ,\ 0<|Re(\mu)|<1$

ii) $\int_0^{\infty}x^{\mu-1}\,cos(\,ax)dx = \frac{\Gamma(\mu)}{a^{\mu}}cos(\frac{\pi}{2}\mu),\ \ 0<Re(\mu)<1$

iii) $\int_0^{1}x^{\mu-1}\,sin(\,ax)dx = \frac{-i}{2\mu}\left[\,{}_1F_1(\mu,\mu+1,ia) - {}_1F_1(\mu,\mu+1,-ia)\right],\ -1<Re(\mu)$

iv) $\int_0^{1}x^{\mu-1}\,cos(\,ax)dx = \frac{1}{2\mu}\left[\,{}_1F_1(\mu,\mu+1,ia) + {}_1F_1(\mu,\mu+1,-ia)\right],\ \ 0<Re(\mu).$

**Corollary**:

i) $\int_0^\infty x^{s-1}\cos(nx)dx = \frac{\Gamma(s)}{n^s}\cos(\frac{\pi}{2}s),\ 0 < Re(s) < 1$

ii) $\int_0^1 x^{s/2-1}\cos(2\pi i x)dx = \frac{1}{s}\left[ {}_1F_1\left(\frac{s}{2}, 1+\frac{s}{2}, 2\pi n\right) + {}_1F_1\left(\frac{s}{2}, 1+\frac{s}{2}, -2\pi n\right)\right],\ Re(s) > 0.$

**Corollary**: For $a > 0$ it holds

i) $\int_0^\infty x^{\mu-1}e^{iax}dx = \frac{\Gamma(\mu)}{a^\mu}e^{ia\frac{\pi}{2}\mu}, \qquad 0 < Re(\mu) < 1$

ii) $\int_0^1 x^{\mu-1}e^{iax}dx = \frac{1}{\mu}\ {}_1F_1(\mu, \mu+1, ia), \qquad Re(\mu) > 0.$

**Lemma** (Grl) 8.353: for $\Gamma(\alpha, x) := \int_0^\infty e^{-t}t^{-\alpha-1}dt$ it holds

$$\Gamma(\alpha, xy) = y^\alpha e^{-xy}\int_0^\infty e^{-ty}(t+x)^{\alpha-1}dt,\ Re(y) > 0,\ x > 0,\ Re(\alpha) > 1.$$

**Lemma** (Grl) 6.246: For $a > 0$ it holds

i) $\int_0^\infty x^{\mu-1}si(ax)dx = -\frac{1}{\mu}\frac{\Gamma(\mu)}{a^\mu}\sin(\frac{\pi}{2}\mu),\ 0 < Re(\mu) < 1,$ with $si(x) := -\int_{-x}^\infty \frac{\sin(t)}{t}dt$

ii) $\int_0^\infty x^{\mu-1}ci(ax)dx = -\frac{1}{\mu}\frac{\Gamma(\mu)}{a^\mu}\cos(\frac{\pi}{2}\mu),\ 0 < Re(\mu) < 1,$ with $ci(x) := -\int_{-x}^\infty \frac{\cos(t)}{t}dt.$

**Lemma** (Grl) 8.334:

i) $\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right) = \frac{\pi}{\cos(\pi s)}$

ii) $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$

iii) $\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{1-s}{2}\right) = \frac{\pi}{\cos(\frac{\pi}{2}s)},\ \Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right) = \frac{\pi}{\sin(\frac{\pi}{2}s)}$

iv) $\frac{\Gamma(1-s)\Gamma(s)}{\Gamma(\frac{1}{2}+s)\Gamma(\frac{1}{2}-s)} = \cot(\pi s) = \cot\left(\frac{\pi}{2}s\right) + \cot\left(\frac{\pi}{2}(1-s)\right)$

v) $\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} = \Gamma(\frac{s}{2})\tan\left(\frac{\pi}{2}s\right),\ \frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1+\frac{s}{2})} = -\Gamma\left(-\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right)$

vi) $\frac{\pi}{2}\log'(\tan(\frac{\pi}{2}s)) = \frac{\pi}{\sin(\pi s)} = \Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it).$

**Lemma** (Grl) 8.328:

i) $\lim_{|y|\to\infty}|\Gamma(x+iy|e^{\frac{\pi}{2}|y|}|y|^{\frac{1}{2}-x} = \sqrt{2\pi},\ x, y \in R$

ii) $\lim_{|z|\to\infty}\frac{\Gamma(z+a)}{\Gamma(z)}e^{-a\log z} = 1.$

**Remark**: In the neighborhood of $x \approx 1$ it holds $\frac{\pi}{2}\tan\frac{\pi}{2}x = \frac{1}{1-x} + O(|1-x|)$. This indicates a circular density function on the half-circle in the form

$$\log\frac{1}{x} \quad \to \quad \log\frac{\pi}{2}(\tan\frac{\pi}{2}(1-x)) = \log\frac{\pi}{2}\cot(\frac{\pi}{2}x).$$

**Lemma** (InA) p. 57:

$$log\Gamma(z+\alpha) = \left(z+\alpha-\frac{1}{2}\right)logz + \frac{1}{2}log2\pi + O(\frac{1}{|z|})$$

uniformly in any fixed angle $|\arg(z)| \leq \pi - \delta < \pi$ and any bounded range of $\alpha$, as $|z| \to \infty$; and

$$log\Gamma'(z) = logz - \frac{1}{2z} + O(\frac{1}{|z|}).$$

**Lemma** (PaR) pp. 31, 39:

$$\left|\Gamma\left(\frac{1}{2}+iy\right)\right| = \sqrt{\left|\Gamma\left(\frac{1}{2}+iy\right)\right|\left|\Gamma\left(\frac{1}{2}-iy\right)\right|} = \sqrt{\frac{\pi}{\cosh(\pi y)}} \sim \sqrt{2\pi}e^{-\frac{\pi}{2}|y|} \sim \sqrt{\frac{\pi}{2}}\frac{1}{cosh\frac{\pi}{2}y}$$

i.e.

$$\Gamma\left(\frac{1}{2}+iy\right) = \int_{-\infty}^{\infty} e^{-e^{u}} e^{u(\frac{1}{2}+iy)} du \sim O(e^{-\frac{\pi}{2}|y|}).$$

A more general approximation formula is given

**Lemma** (GrI) 8.328,

$$\lim_{|y|\to\infty} |\Gamma(x+iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2}-x} = \sqrt{2\pi}, \quad x, y \text{ are real.}$$

**Lemma** (Theorem 20, (PaR) p. 128): Let $f(z)$ be a measureable function for which

$$\frac{1}{2T}\int_{-T}^{T}|f(x)|^2 dx$$

is bounded in $T$. Then

$$\int_{-\infty}^{\infty}\frac{|f(x)|^2}{1+x^2}dx < \infty.$$

As Hardy-Littlewood proved, (EdH) p. 201,

$$\frac{1}{2T}\int_{-T}^{T}\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2 dx \sim logT$$

it follows the

$$\int_{-\infty}^{\infty}\frac{\left|\zeta\left(\frac{1}{2}+ix\right)\right|^2}{1+x^2}dx < \infty.$$

Remark: Regarding the density $\frac{dx}{x}$ we mention the relation to the Bessel functions, (WaG) p. 514,

$$dlogx = \frac{dx}{x} = \frac{\pi}{2}[J_0^2(x) + Y_0^2(x)]d\theta,\ \theta(x) := \arctan(\frac{Y_0(x)}{J_0(x)}).$$

The asymptotics of the sum of the Dawson function $\mathrm{D}(x) := e^{-x^2}\int_0^{\infty} e^{u^2}du$ and the Digamma function $\Psi(x)$ is related to the $logx$ function by

$$\Psi(x) + D(x) \cong (logx - \frac{1}{2x}) + \frac{1}{2x} \cong logx.$$

**Lemma** (GrI) 1.421:

$$\cot(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{k=1}^{\infty}\frac{1}{x^2+k^2} = \frac{1}{\pi x} + \frac{2x}{\pi}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\frac{1}{k(x-k)}.$$

**Lemma** (GrI) 8.364:

$$log\Psi(z) = logx + \sum_{k=1}^{\infty} log\left(1 + \frac{1}{x+k}\right) e^{-\frac{1}{x+k}}.$$

**Lemma** (NiN) p. 38: let $s_1 := \gamma$ and $s_n := \frac{1}{1^n} + \frac{1}{2^n} + \frac{1}{3^n} + \cdots$ for $n \geq 2$, then

$$\Psi(1+z) = \frac{1}{2x} - \frac{\pi}{2} cot(\pi x) - \frac{1}{1-x^2} + \sum_{k=0}^{\infty}(1 - s_{2k+1})x^{2k}.$$

**Lemma** (TiE) 2.15: for $0 < \sigma < 1$

$$\Psi(1+x) - logx = -\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \zeta(1-s)\frac{\pi}{sin\,(\pi s)} x^{-s} ds.$$

**Lemma** (GrI) 6.469:

$$\int_0^1 \Psi(x)\, sin(\pi x)\, sin(n\pi x)\, dx = \begin{cases} 0 & n \quad even \\ \frac{1}{2} log\frac{n-1}{n+1} & n \quad odd \end{cases}.$$

Let $M[h](s)$ denote the Mellin transform operator

$$M[h](s) = \int_0^{\infty} x^s h(x) \frac{dx}{x}.$$

**Lemma**:

i) $M[-xh'](s) = sM[h](s)$

ii) $M[(xh)'](s) = (1-s)M[h](s)$.

**Conjecture** (Frank Morgan's math chat): Suppose that there is a nice probability function $P(x)$ that a large integer $x$ is prime. As $x$ increases by $\Delta x = 1$, the new potential divisor $x$ is prime with probability $P(x)$ and divides future numbers with probability $1/x$. Hence $P$ gets multiplied by $(1 - P/x)$, so that $\Delta P = (1 - P/x)P - P = -P^2/x$, or roughly $-xP' = P^2$ resp.

$$-x\frac{P(x)'}{P(x)} = P(x).$$

The general solution to this equation is $P(x) = \frac{1}{c+logx} \sim \frac{1}{logx}$.

**Lemma** (GrI) 7.612:

$$\int_0^{\infty} t^b\ {}_1F_1(a,c,-t)\frac{dt}{t} = \Gamma(b)\frac{\Gamma(a-b)}{\Gamma(c-b)}\frac{\Gamma(c)}{\Gamma a)}, \quad 0 < Re(b) < Re(a).$$

**Remark** (SlL) p. 43: $\int_0^{\infty} e^{-bt}t^{\nu}\, dt = \Gamma(1+\nu)b^{-1-\nu}$.

**Lemma** (GrI) 7.612:

i) $M\left[\ {}_1F_1\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right) = \int_0^{\infty} x^{s/2}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2},-x\right)\frac{dx}{x} = \frac{\Gamma(s/2)}{1-s}, 0 < Re(s) < 1$

ii) $M\left[x\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right) = -\frac{s}{2}\frac{\Gamma(s/2)}{1-s}, 0 < Re(s) < 1$

iii) $M\left[\ {}_1F_1\left(\frac{1}{2},\frac{3}{2},-x\right) + x\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right) = \frac{\Gamma\left(\frac{s}{2}\right)}{1-s}\left(1-\frac{s}{2}\right), 0 < Re(s) < 1$

iv) $\Gamma\left(\frac{s}{2}\right) = \frac{1-s}{1-\frac{s}{2}} M\left[\ {}_1F_1\left(\frac{1}{2},\frac{3}{2},-x\right) + x\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2},-x\right)\right]\left(\frac{s}{2}\right), 0 < Re(s) < 1.$

**Lemma** (Grl) 9.212:

$$ {}_1F_1(a,c,z) = e^z \; {}_1F_1(c-a,c,-z). $$

For the Dawson function

$$ D(x) := e^{-x^2}\int_0^x e^{u^2}du $$

the following relations are valid,

**Lemma** (Grl) 3.896), (LeN) pp. 17, 272:

i) $D(x) = \int_0^\infty e^{u^2}\sin(2xu)\,du$

ii) $D(x) = xe^{-x^2}\; {}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right) = x\; {}_1F_1\left(1;\frac{3}{2},-x^2\right).$

**Lemma**: (Grl) 7.612: The Mellin transform of the Dawson function is given by

$$ \int_0^\infty x^s x\; {}_1F_1\left(1;\frac{3}{2},-x^2\right)\frac{dx}{x} = \frac{1}{2}\int_0^\infty x^{\frac{s+1}{2}}\; {}_1F_1\left(1;\frac{3}{2},-x\right)\frac{dx}{x} = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})},\; -1 < Re(s) < 3. $$

$$ M[D(x)](s) = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} = \frac{\sqrt{\pi}}{2}\Gamma\left(\frac{s}{2}\right)tan\,(\frac{\pi}{2}s). $$

**Lemma**: (Grl) 7.643:

$$ \int_0^\infty x^{4\nu}e^{-\frac{x^2}{2}}\sin(bx)\; {}_1F_1\left(\frac{1}{2}-2\nu,2\nu+1,\frac{x^2}{2}\right)dx = \sqrt{\frac{\pi}{2}}b^{4\nu}e^{-\frac{b^2}{2}}\; {}_1F_1\left(\frac{1}{2}-2\nu,2\nu+1,\frac{b^2}{2}\right),\; b>0, Re(\nu) < -\frac{1}{4}. $$

The Hilbert transform is defined by the Cauchy principle value integral,

$$ H[g(t)](x) := \frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{g(t)}{x-t}dt. $$

The Hilbert transform of the Gaussian function $f(x) \coloneqq e^{-\pi x^2}$ is related to the Dawson function by, (GaW), (KoO),

$$ f_H(x) \coloneqq H[f](x) = \frac{2}{\sqrt{\pi}}D\left(\sqrt{\pi}x\right) = 2x\; {}_1F_1\left(1;\frac{3}{2},-\pi x^2\right). $$

**Lemma**: The Mellin transform of the Hilbert transformed Gaussian function $f_H(x)$ is given by

$$ M[f_H(x)](s) = 2\left[\; {}_1F_1\left(1;\frac{3}{2},-\pi x^2\right)\right](1+s) = \pi^{-\frac{1+s}{2}}M\left[\; {}_1F_1\left(1;\frac{3}{2},-x\right)\right]\left(\frac{1+s}{2}\right) $$

$$ = \pi^{-\frac{1+s}{2}}\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} = \frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)tan\,(\frac{\pi}{2}s), \qquad -1 < Re(s) < 3. $$

**Remark**: The considered Kummer functions are linked to the Whittaker function, (Grl) 9.220

$$ M_{\lambda,\mu}(z) \coloneqq z^{\mu+\frac{1}{2}}e^{-z/2}\; {}_1F_1\left(\mu-\lambda+\frac{1}{2},2\mu+1,z\right) $$

by

$$ M_{\frac{1}{4},\frac{1}{4}}(z) = z^{\frac{3}{4}}e^{-\frac{z}{2}}\; {}_1F_1\left(\frac{1}{2},\frac{3}{2},z\right) = z^{\frac{3}{4}}e^{\frac{z}{2}}\; {}_1F_1\left(1,\frac{3}{2},-z\right), $$

$$ M_{-\frac{1}{4},\frac{1}{4}}(z) = z^{\frac{3}{4}}e^{-\frac{z}{2}}\; {}_1F_1\left(1,\frac{3}{2},z\right) = z^{\frac{3}{4}}e^{\frac{z}{2}}\; {}_1F_1\left(\frac{1}{2},\frac{3}{2},-z\right). $$

**Lemma** (ErA) p. 215: for $Re(\mu+\nu)>-\frac{1}{2}$ and $Re(p)>\frac{1}{2}Re(a)$ it holds

$$\int_0^\infty e^{-\mathrm{pt}}t^{\nu-1}M_{\lambda,\mu}(at)\,dt = a^{\mu+\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\mu+\nu)}{(p+\frac{a}{2})^{\frac{1}{2}+\mu+\nu}}\;{}_2F_1\left(\frac{1}{2}+\mu+\nu,\frac{1}{2}+\mu-\lambda,2\mu+1,\frac{a}{p+\frac{a}{2}}\right).$$

**Remark** (AbM) p. 559:

$${}_2F_1(a,b,c,z)=\frac{1}{2\pi i}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\int_{-i\infty}^{i\infty}\frac{\Gamma(a+s)\Gamma(b+s)}{\Gamma(c+s)}\Gamma(-s)(-z)^s ds.$$

**Remark** (BuH) p. 118: When $Re(\lambda-\nu)>0$ it is permissible to put $p=1/2$. With

$${}_2F_1(a,b,c,1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

one thus obtains

**Lemma** (BuH) p. 118, (EpA) p. 215, (GrI) 7.621:

$$\int_0^\infty e^{-\frac{b}{2}t}t^{\nu-1}M_{\lambda,\mu}(bt)\,dt = b^{-\nu}\frac{\Gamma(1+2\mu)}{\Gamma(\frac{1}{2}+\mu+\lambda)}\frac{\Gamma(\lambda-\nu)\Gamma(\frac{1}{2}+\mu+\nu)}{\Gamma(\frac{1}{2}+\mu-\nu)},\quad Re(\nu+\frac{1}{2}+\mu)>0, Re(\lambda-\nu)>0.$$

**Remark**: We note the formula

$$\frac{\Gamma(1-s)\Gamma(s)}{\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-s\right)}=cot(\pi s)=cot\left(\frac{\pi}{2}s\right)+cot\left(\frac{\pi}{2}(1-s)\right).$$

Putting $\nu:=\pm\frac{s}{2}-\frac{1}{4}$, $\lambda:=\mu=\frac{1}{4}$ and

$$\Omega\left(\frac{s}{2}\right):=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}=\frac{\sqrt{\pi}}{2}\Gamma\left(\frac{s}{2}\right)\tan\left(\frac{\pi}{2}s\right)$$

one gets

**Corollary:**

iii) $$\int_0^\infty e^{-\frac{b}{2}t}t^{\frac{s}{2}-\frac{1}{4}}M_{\frac{1}{4},\frac{1}{4}}(bt)\frac{dt}{t}=b^{-\frac{s}{2}+\frac{1}{4}}\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}=b^{-\frac{s}{2}+\frac{1}{4}}\Omega\left(\frac{s}{2}\right),\quad -1<Re(s)<1$$

iv) $$\int_0^\infty e^{-\frac{b}{2}t}t^{-\frac{s}{2}-\frac{1}{4}}M_{\frac{1}{4},\frac{1}{4}}(bt)\frac{dt}{t}=b^{+\frac{s}{2}+\frac{1}{4}}\frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+\frac{s}{2})}=b^{+\frac{s}{2}+\frac{1}{4}}\Omega\left(-\frac{s}{2}\right),\ -1<Re(s)<1.$$

From the above we recall that the Mellin transform of the Hilbert transformed Gaussian function $f_H(x)$ is given by

$$M[f_H(x)](s)=2\left[{}_1F_1\left(1;\frac{3}{2},-\pi x^2\right)\right](1+s)=\pi^{-\frac{1+s}{2}}M\left[{}_1F_1\left(1;\frac{3}{2},-x\right)\right]\left(\frac{1+s}{2}\right)$$
$$=\frac{1}{2}\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)tan\left(\frac{\pi}{2}s\right)=\pi^{-\frac{1+s}{2}}\Omega\left(\frac{s}{2}\right),\qquad -1<Re(s)<3.$$

A relationship of the Whittaker functions to the probability integral $\Phi(x):=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is given by

**Lemma** (GrI) 7.672: with it holds

$$\int_0^\infty x^{-\frac{1}{2}}e^{-\frac{x^2}{2}}M_{\frac{1}{2}\nu-\frac{1}{4},\frac{1}{2}\nu+\frac{1}{4}}(x^2)J_0(xy)dx=(2\nu+1)2^{-\nu}y^{\nu-1}(1-\Phi(\tfrac{y}{2}),\ y>0, Re(\nu)>-\frac{1}{2}$$

$$\int_0^\infty x^{-1}e^{-\frac{x^2}{2}}M_{\frac{1}{2}\nu+\frac{1}{2},\frac{1}{2}\nu+\frac{1}{2}}(x^2)J_0(xy)dx=\frac{\Gamma(\nu+2)y^\nu}{\Gamma\left(\nu+\frac{3}{2}\right)2^\nu}(1-\Phi(\tfrac{y}{2}),\ y>0, Re(\nu)>-1.$$

**Lemma** (GrI) 7.693:

$$\int_{-i\infty}^{i\infty}\Gamma(\tfrac{1}{2}+\nu+\mu+x)\Gamma(\tfrac{1}{2}+\nu+\mu-x)\Gamma(\tfrac{1}{2}+\nu-\mu+x)\Gamma(\tfrac{1}{2}+\nu-\mu-x)M_{\mu+ix,\nu}(a)M_{\mu-ix,\nu}(b)\,dx =$$

$$= 2\pi\frac{(ab)^{\nu+\frac{1}{2}}}{(a+b)^{2\nu+1}}\frac{\Gamma^2(2\nu+1)\Gamma(2\nu+2\mu+1)\Gamma(2\nu-2\mu+1)}{\Gamma(4\nu+2)}M_{2\mu,2\nu+1}(a+b),\quad Re(\nu)>|Re(\mu)|-\frac{1}{2}\,.$$

**Theorem 11** (HaG) p. 11: let $\mu_n = log\,\lambda_n$. Then

$$\sum_{n=1}^{\infty} a_n e^{-\mu_n s} = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}\left(\sum_{n=1}^{\infty} a_n e^{-\lambda_n x}\right)dx$$

if $Re(s)>0$, and the series on the left-hand side is convergent.

Let

$$h_n \coloneqq \sum_{k=1}^{n}\frac{1}{2k-1} = H_{2n}-\frac{1}{2}H_n,\ n\in N, \text{ where } H_n \coloneqq \sum_{k=1}^{n}\frac{1}{k}$$

denote the harmonic numbers and $c_n := \frac{h_n}{n}$.

**Lemma** (ElL): for $Re(s)>0$ the following formula is valid

$$\sum_{n=1}^{\infty}\frac{h_n}{n}\frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} x^{s-1}\left(\sum_{n=1}^{\infty}\frac{h_n}{n}e^{-nx}\right)dx$$

as it holds $\sum_{k=1}^{n} c_n^2 < \infty$.

**Lemma** (KaM), (ZyA) p. 164: Let $\{n_k\}$ be a sequence of (integer) indices satisfying the “Hadamard” gap” condition, i.e.

$$\frac{n_{k+1}}{n_k} > q > 1.$$

Then the trigonometric gap series

$$\sum_{k=1}^{\infty} c_k \sin(2\pi n_k x)$$

converges almost everywhere, if and only if,

$$\sum_{k=1}^{\infty} c_k^2 < \infty.$$

Let $h_n \coloneqq \sum_{k=1}^{n}\frac{1}{2k-1} = H_{2n}-\frac{1}{2}H_n$, $n\in N$, where $H_n \coloneqq \sum_{k=1}^{n}\frac{1}{k}$ denote the harmonic numbers.

**Lemma** (ElL), (ElL1): for $g(x) \coloneqq -\frac{\pi}{2}log\left(tan\frac{\pi}{2}x\right)$ it holds

i) $g(x)\in H_{odd} \coloneqq \left\{f\in L_2(0,1) \mid f\left(\frac{\pi}{2}(1-x)\right) = -f(\frac{\pi}{2}x)\right\}$

ii) $\int_0^1 g(x)\,sin(2\pi nx)\,dx = \frac{h_n}{n}$

iii) $\int_0^1 g(x)\,cos(k\pi x)\,dx = \begin{cases}\frac{\pi}{2k} & k\ odd\\ 0 & k\ even\end{cases}$

*iv)* $g(x) = 2\sum_{n=1}^{\infty}\frac{h_n}{n}sin(2\pi nx)\in L_2(0,1)$

v) $\sum_{n=1}^{\infty}\frac{h_n}{n}e^{-nx} = \frac{1}{4}log^2(tanh\left(\frac{x}{4}\right))$

vi) $\int_0^1 g^2(x)dx = 2\sum_{n=1}^{\infty}\frac{{h_n}^2}{n^2} = \frac{\pi^4}{16}$

vii) $log'\left(tan\frac{\pi}{2}x\right) = \frac{\pi}{sin\pi x}\in H_{-1}^{\#}(0,1).$

**Theorem** (LaE1): let $m > 1$, $\beta_1$ real, $0 < \Theta \leq \beta_2 - \beta_1 \leq \beta_3 - \beta_2 \leq \cdots \beta_m - \beta_{m-1} \leq 1 - \Theta$, and

$$S = S_m := |\sum_{n=1}^{m} \sigma_n| \text{ with } \sigma_n := e^{-2\pi i \beta_n}.$$

Then it holds:

v) $S_m \leq \cot(\frac{\pi}{2}\Theta)$;

vi) For $\Theta = 1/2$ and every positive fraction $\Theta < 1/2$ with odd nominator and odd denominator:

$$S_m = \cot(\frac{\pi}{2}\Theta);$$

vii) For all other $\Theta$ with $0 < \Theta < 1/2$ :

$$S_m < \cot(\frac{\pi}{2}\Theta);$$

viii) For all $\Theta$ with $0 < \Theta \leq 1$ and every $\varepsilon > 0$:

$$S_m > cot\left(\frac{\pi}{2}\Theta\right) - \varepsilon.$$

**Landau theorem** (PoG1): Let $q_n$ denote a divergent sequence of positive numbers $0 < q_1 \leq q_2 \leq q_3 \leq \cdots$ $\lim_{n\to\infty} q_n = \infty$, $\tau(x)$ the corresponding counting function of the numbers of $q_n$ less than $\leq x$ and $w(x)$ a positive, non decreasing function with

$$\lim_{n\to\infty} \frac{w(2x)}{w(x)} = 1, \text{ and } \lim_{n\to\infty} \frac{\tau(x)w(x)}{x} = 1.$$

Then

$$\lim_{n\to\infty} \frac{1}{\tau(x)} \sum_{q\leq x} \rho(\frac{x}{q}) = 1 - \gamma\ .$$

**Generalized Landau Theorem**, (PoG1): Let $w(x)$ a positive, non decreasing function with $\lim_{n\to\infty} \frac{w(\beta x)}{w(\alpha x)} = 1$ with $\alpha, \beta$ positive numbers. Then

$$\lim_{n\to\infty} \frac{w(x)}{x} \sum_{n\leq x} f(\frac{x}{n}) = \int_0^1 f(t)dt.$$

**Lemma** (LaE1a) §86: Let $a > 0$ $\sum_{n=1}^{\infty} \frac{|b_n|}{n^s}$ convergent, i.e. $D(s) = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$ is an absolute convergent Dirichlet series for $a := Re(s)$. Putting

$$f(x) := \begin{cases} \sum_{n=1}^{x} b_n & for \quad x \quad not\ integer \\ \sum_{n=1}^{x} b_n - b_x/2 & for \quad x \quad integer \end{cases}$$

Then it holds

$$f(x) = \lim_{n\to\infty} \frac{1}{2\pi i} \int_{a-it}^{a+it} D(s) x^s \frac{ds}{s} \quad \text{resp.} \quad \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{D(s)}{s} x^s \frac{ds}{s} = \sum_{n=1}^{x} b_n \log(\frac{x}{n}).$$

**Remark**: $D(s)$ does not need to be regular; only for $a < Re(s)$ this follows from the pre-requisites.

For each positive real number $x$ the Schnirelmann density is defined for a subset $\boldsymbol{A}$ of the set of positive integers $\boldsymbol{N}$ by

$$0 \leq \sigma(\boldsymbol{A}) := inf_n \frac{A(n)}{n} \leq 1 \quad \text{with} \quad A(x) := \sum_{\substack{a \in \boldsymbol{A} \\ a \leq x}} 1.$$

The function $A(x)$ is called the counting function of the set $\boldsymbol{A}$. For $x > 0$ it holds

$$0 \leq A(x) \leq [x] \leq x$$

and so

$$0 \leq \frac{A(x)}{x} \leq 1.$$

It also holds $A(n) \geq kn$ for $\sigma(\boldsymbol{A}) = k$; $A(1) = 0$ (and therefore $\sigma(\boldsymbol{A}) = 0$), if $1$ is not an element of $\boldsymbol{A}$; and $A(n) = n$ (and therefore $\sigma(\boldsymbol{A}) = 1$), if $\boldsymbol{A} = \boldsymbol{N}$. In other words

- If the integer „1“ is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $= 0$
- If the integer "2" is not an element of $\boldsymbol{A}$, the Snirelmann density of $\boldsymbol{A}$ is $\leq \frac{1}{2}$

  if $\boldsymbol{A} = \boldsymbol{N}$, the Snirelmann density of $\boldsymbol{A}$ is $= 1$.

The Schnirelmann-Goldbach theorem states, that the set $\boldsymbol{A} := \{0,1\} \cup \{p+q; p,q\ prim\}$ has positive Schnirelmann density. In case of a Schnirelmann density ½ the binary Goldbach conjecture would be proven.

**Remark** (StE): Algebraic resp. transcendental field extensions are the main tool of the theory of transcendental numbers. Algebraic closed field extensions are concerned with fractional rational functions. For algebraic number fields the Galois theory is applicable, which is based on the assumption that every equation with vanishing discriminant must be reducible. The main tool for transcendental field extensions are transcendental bases accompanied with infinite extensions. The transcendental bases basically corresponds to the (Schauder!) bases of vector spaces. The Steinitz theory basically states that each extension can be decomposed into an algebraic and a purely transcendental extension. The link to the Hilbert space based proposed UFT and the underlying concept of Quaternionic inner product spaces V, accompanied by a non-degenated indefinite inner product on V/V(0), where V(0) denotes the isotropic part of V, is provided in (AlD).

**Remark**: the theory of algebraic integrals containing the square root of functions of 1$^{st}$ or 2$^{nd}$ order are built on trigonometric functions and the irrelated circle functions. The theory of algebraic integrals containing the square root of functions of 3$^{rd}$ or 4$^{th}$ order (e.g. the Fagnano-integral to calculate the value of $\Gamma(\frac{1}{4})$ are built on elliptic functions and their related elliptic integrals. The conceptual new property, which is unique in the whole area of elementary functions, is the double-periodicity of the elliptic functions. This property, in combination with the change from the set of zeros of the trigonometric functions to the set of the $\pm z_n$ zeros of the Kummer function (accompanied by a change from harmonic Fourier series to non-harmonic Fourier series, cohomology groups interpreted as homology groups of negative order, (NeJ), and the $H_{1/2}$ space as first cohomology, (NaS)), puts the spot on the concept of the $n$-Kummer-Galois extension of the field containing a primitive nth root of unity and the Kummer's prime number irregularity theorem in the context of cyclotomic fields, zeta-function values and the Kummer pairing. Kummer's prime number irregularity theorem basically reduces the problem of deciding whether a prime $p$ is irregular to a question of arithmetic modulo p, (CoJ).

**Remark** (CoG), (RiP): A pair $(p, 2k)$ is called an irregular pair, if $2 \leq 2k \leq p-3$ and $p$ divides the nominator of the Bernoulli number $B_{2k}$. Kummer guessed that there are infinite regular primes. Jensen proved 1915 the existence of infinite numbers of irregular primes. Under certain assumptions Siegel proved 1964 that the density of regular primes is $\frac{1}{\sqrt{e}} \sim 61\%$.

## Auxiliary theorems

**Lemma** (InA) p. 56: The entire function $\xi(z)$ is of order $\sigma = 1$ with finite order type, as on the circle $|z| = r$ for its maximum modulus function $M(r)$ it holds

$$\frac{1}{2} r log r - A_1 r < log M(r) < \frac{1}{2} r log r + A_2 r.$$

**Lemma**: The Hadamard product formulae of the considered entire functions are given by

i) $\frac{1}{\Gamma(z/2)} = \frac{s}{2} e^{\gamma z/2} \prod_{n=1}^{\infty} (1 + \frac{z}{2n})\, e^{-z/2n}$, (LeB) p. 32

ii) $\sin(\frac{\pi}{2} z) = \frac{\pi}{2} z \prod_{n=1}^{\infty} (1 - \frac{z^2}{4n^2}) = \frac{\pi}{2} z \prod_{n=-\infty, n \neq 0}^{\infty} (1 - \frac{z}{2n})\, e^{z/2n}$, (LeB) p. 32

iii) $\xi(z) = \frac{1}{2} e^{Bz} \prod_{\rho} \left(1 - \frac{z}{\rho}\right) e^{\frac{z}{\rho}}$ where $B := log\, 2 + \frac{log\, \pi}{2} - 1 - \frac{\gamma}{2} \approx -0{,}023..$, ((InA) p. 58

iv) ${}_1F_1\left(1, \frac{3}{2}; z\right) = e^{\frac{1}{3} z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}$, ${}_1F_1\left(\frac{1}{2}, \frac{3}{2}; z\right) = e^{\frac{3}{2} z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{\frac{z}{z_n}}$ , (BuH) p.184.

**Lemma**:

i) $\frac{1}{2}(e^{\frac{\pi}{2}|y|} - e^{\frac{\pi}{2}|y|}) \leq \left|\sin(\frac{\pi}{2} z)\right| < e^{\frac{\pi}{2}|y|}$ resp. $\frac{1}{2}(e^{\frac{\pi}{2} r} - e^{\frac{\pi}{2} r}) \leq M(r) < e^{\frac{\pi}{2} r}$,

i.e. $log M(r) \sim \frac{\pi}{2} r$ with the (finite) order type $\sigma_f = \frac{\pi}{2}$

ii) $\frac{1}{\Gamma\left(\frac{1}{2} + iy\right)} \sim O(e^{\frac{\pi}{2}|y|})$ (PaR) p. 39

iii) $\lim_{|y| \to \infty} |\Gamma(x + iy)| e^{\frac{\pi}{2}|y|} |y|^{\frac{1}{2} - x} = \sqrt{2\pi}$, for real $x, y$, (GrI) 8.328

iv) $log \frac{1}{\Gamma(z)} = -z\big(1 + o(1)\big) log z$ , $|\varphi| < \pi$ (LeB) p. 32

v) $log \frac{1}{|\Gamma(z)|} = -z\big(1 + o(1)\big)(cos\varphi) \cdot r \cdot log r$, $z = e^{i\varphi}$, $\frac{\pi}{2} < |\varphi| < \pi$,

and therefore

$$log M(r) \geq C \cdot r \cdot log r,$$

i.e. $1/\Gamma(z)$ is of maximal order type, while the upper density of the zero set is finite.

**Remark**: The „root of evil“ of the maximal order type of $1/\Gamma(z)$ is the absence of a symmetry in the distribution of its zeros, (LeB) p. 32.

## Gap and Density Theorems

**Lemma** ((LeN), p. 89: For $\Phi(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2})$ with $\lim_{n\to\infty}\frac{n}{z_n} = D$, where $D$ is real, and such that for some $d > 0$, $\left|re^{i\theta} - z_n\right| \geq \frac{1}{8}d$. Then for $\varepsilon > 0$, as $r \to \infty$

iv) $\Phi\left(re^{i\theta}\right) = O(e^{\pi D|sin\theta|r+\varepsilon r})$

v) $\frac{1}{\Phi\left(re^{i\theta}\right)} = O(e^{-\pi D|sin\theta|r+\varepsilon r})$

vi) $\frac{1}{\Phi'(z_n)} = O(e^{\varepsilon|z_n|})$, $n \to \infty$.

This lemma is contained in

**Lemma** ((LeN), p. 92: Let $\Phi(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2})$ with $\lim_{n\to\infty}\frac{n}{z_n} = D$, where $D$ is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n-m|$. Then for $\varepsilon > 0$, as $r \to \infty$

vii) $\Phi\left(re^{i\theta}\right) = O(e^{\pi D|sin\theta|r+\varepsilon r})$

viii) $\frac{1}{\Phi\left(re^{i\theta}\right)} = O(e^{-\pi D|sin\theta|r+\varepsilon r})$

ix) $\frac{1}{|\Phi'(z_n)|} = O(e^{\varepsilon|z_n|})$, $n \to \infty$.

**Lemma** ((PaR), p. 86: Let $F(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{\pi^2\lambda_n^2})$ with $\lim_{n\to\infty}\frac{\lambda_n}{n} = 1$, then

$$\lim_{y\to\infty}\frac{logF(iy)}{y} = 1.$$

**Theorem XXIX** ((PaR), p. 86: Let $F(z) \coloneqq \prod_{n=1}^{\infty}(1-\frac{z^2}{\pi^2\lambda_n^2})$ with $\lim_{n\to\infty}\frac{\lambda_n}{n} = 1$ belong to $L_2(-\infty,\infty)$. Then the set of functions $\{e^{\pm i\lambda_n x}\}$ cannot be closed $L_2$ over $(-\pi,\pi)$.

Again, let $zF(z)$ belong to $L_2(-\infty,\infty)$. Then the set of functions $\{1, e^{\pm i\lambda_n x}\}$ cannot be closed $L_2$ over $(-\pi,\pi)$. In either case, a finite number of functions of the set may be replaced by other functions of the form $e^{i\lambda x}$ to the same number.

**Lemma** (LeN) p. 89: Let $f(z) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n z}$ where $\lim_{n\to\infty}\frac{n}{\lambda_n} = D$ and for some $c > 0$

$$\lambda_{n+1} - \lambda_n \geq c.$$

Then on the abscissa of convergence there is at least one singularity in every interval of length exceeding $\lambda_n 2\pi D$.

**Lemma** (LeN) p. 89: If $\lambda_n$ satisfies $\lim_{n\to\infty}\frac{n}{\lambda_n} = D$ and for some $c > 0$, $\lambda_{n+1} - \lambda_n \geq c$, and if

$$F(z) = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)$$

then as $n \to \infty$:

$$\frac{1}{F'(\lambda_n)} = O(e^{\varepsilon\lambda_n}).$$

**Lemma** ((LeN), p. 92: Let $\Phi(z) := \prod_{n=1}^{\infty}(1-\frac{z^2}{z_n^2})$ with $\lim_{n\to\infty}\frac{n}{z_n} = D$, where $D$ is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n-m|$. Then for $\varepsilon > 0$, as $r \to \infty$

i) $\Phi(re^{i\theta}) = O(e^{\pi D|sin\theta|r+\varepsilon r})$

ii) $\frac{1}{\Phi(re^{i\theta})} = O(e^{-\pi D|sin\theta|r+\varepsilon r})$

iii) $\frac{1}{|\Phi'(z_n)|} = O(e^{\varepsilon|z_n|}), n \to \infty.$

**Lemma** (LeN) p. 108: Let $\Phi(z)$ be analytic and of exponential type in the half-plane $|ph(z)| \leq \frac{\pi}{2}$ and let $\Phi(iy) = O(e^{\pi L|y|})$. If $\{\lambda_n\}$ be an increasing positive sequence such that $\lim_{n\to\to\infty}\frac{n}{\lambda_n} = D$ , $\lambda_{n+1} - \lambda_n \geq d > 0$. Let $\Lambda(u)$ be the number of $\lambda_n < u$. If $\int_1^{\infty}\frac{\Lambda(y)-Ly}{y^2}dy = \infty$ and if $\Lambda(y) + t(y) > Ly$ for some positive $t(y)$ satisfying $\int_1^{\infty}\frac{t(y)}{y^2}dy < \infty$, then

$$\limsup_{n\to\infty}\frac{log|\boldsymbol{\Phi}(\lambda_n)|}{\lambda_n} = \limsup_{x\to\infty}\frac{log|\boldsymbol{\Phi}(x)|}{x}.$$

The following lemma show, roughly stated, that the rate of growth of an analytic function (in this case of order one) along a line can be determined by a sufficiently dense sequence of points on the line.

**Lemma** ((LeN), p. 100: Let $\Phi(z)$ be analytic in some sector $|\arg(z)| \leq \alpha$. Suppose

$$\limsup_{n\to\infty}(\frac{log|\Phi(re^{i\theta})|}{r}) \leq acos\theta + b|sin\theta|, \theta \leq \alpha < \frac{\pi}{2}.$$

Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty}\frac{n}{z_n} = D$, where $D$ is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n-m|$. If $b < \pi D$, then

$$\limsup_{n\to\infty}(\frac{log|\Phi(z_n)|}{|z_n|}) = \limsup_{n\to\infty}(\frac{log|\Phi(r)|}{r}).$$

A special case of this lemma is the

**Lemma** ((LeN), p. 101: Let $\Phi(z)$ be analytic and of finite exponential type in the half-plane $|\arg(z)| \leq \frac{\pi}{2}$. Let

$$\limsup_{n\to\infty}\left(\frac{log|\Phi(iy)|}{y}\right) = \pi L.$$

Let $\{z_n\}$ be a sequence of complex numbers such that $\lim_{n\to\infty}\frac{n}{z_n} = D$, where $D$ is real, and such that for some $d > 0$, $|z_n - z_m| \geq d|n-m|$, then $L < D$ implies

$$(*) \quad \limsup_{n\to\infty}(\frac{log|\Phi(z_n)|}{|z_n|}) = \limsup_{n\to\infty}(\frac{log|\Phi(r)|}{r}).$$

**Remark**: If (*) does not hold there exists a $c$ such that

$$\limsup_{n\to\infty}(\frac{log|\Phi(z_n)|}{|z_n|}) < c < a := \limsup_{n\to\infty}(\frac{log|\Phi(r)|}{r}).$$

**Remark** (YoR) p. 57: Let $f(z)$ be an entire function; then $f(z)$ and $f'(z)$ are of the same order. If $f(z)$ has at least one zero, but is not identically zero, then

$$\lambda = \limsup_{n\to\infty}\frac{\log(n(r))}{logr}$$

where $n(r)$ is the number of zeros in the closed disk $|z| \leq r$ and $\lambda$ denotes the exponent of convergence of the sequence of zeros. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is of order $\sigma$, then

$$\rho = \limsup_{n\to\infty}\frac{nlogn}{log(\frac{1}{|a_n|})}.$$

## The Hardy space $H^2(D)$ and the Besicovitch space

(PaJ) p. 4: The Hardy space $H^2(D)$ is defined as the space of all analytic functions on the disk $D = \{z \in C \,||z| < 1\}$ for which the norm

$$|||f|||_2^2 := sup_{r<1}\left\{\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\varphi}\right|^2 d\varphi\right\}$$

is finite. The radial limit (function) $\tilde{f}(e^{i\varphi}) = \lim_{r\to 1} f(e^{i\varphi})$ on $\Gamma = S^1(R^2)$ exist almost everywhere with $|||f|||_2^2 = \frac{1}{2\pi}\int_0^{2\pi}\left|f(e^{i\varphi})\right|^2 d\varphi$.

**Lemma**: Let (YoR) p. 12: Let $X$ denote the vector space of all finite linear combinations of functions of the form $e^{i\lambda x}$, $(-\infty < t < \infty)$, where the parameter $\lambda$ is real. An inner product in $X$ is defined by

$$((f,g)) := \lim_{\omega\to\infty}\frac{1}{2\omega}\int_{-\omega}^{\omega} f(t)\overline{g(t)}dt.$$

When $X$ is closed by means of the metric generated by this inner product, we obtain a certain Hilbert space $B^2$ ($B$ is for Besicovitch). The continuum of elements $e^{i\lambda x}$ forms a complete orthogonal subset of $B^2$. The Hilbert space $B^2$ contains the important class of Bohr almost periodic functions. Those functions are obtained by adding to $X$ the limits of sequences of function in $X$ that are uniformly convergent on the entire real line.

The Hardy space $H^2$ can also defined as the subspace of those $L_2(\Gamma)$ functions for which the negative Fourier coefficients vanish, that is

$$\hat{f}(n) = f_n = \frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(e^{i\varphi})e^{-in\varphi}d\varphi < 0,\, \nu < 0.$$

Then a function $\tilde{f}$ with $\tilde{f}(e^{i\varphi}) \sim \sum_{n=0}^{\infty} f_n e^{i\varphi}$ can be naturally identified with the power series $f(z) = \sum_{n=0}^{\infty} f_n z^n$, defining an analytical function $f$ in $D$.

In other words, the Hardy space $H^2$ is a Hilbert space, being a closed subspace of the Hilbert space $H := L_2^*(\Gamma)$ and the orthogonal projection $P_{H^2}: L_2^*(\Gamma) \to H^2$ is defined by

$$P_{H^2}:\ \sum_{n=-\infty}^{\infty} a_n e^{i\varphi} \ \to\ \sum_{n=0}^{\infty} a_n e^{i\varphi}.$$

As there is an isometric isomorphism between $L_2^*(\Gamma)$ and $l^2\,(Z)$, and as the sequence space $l^2\,(Z_+)$ maps to the Hardy space $H^2$, one may regard $l^2\,(Z_+)$ as embedding into $l^2\,(Z)$ as the subspace of all $(a_n)_{n=-\infty}^{\infty}$ with $a_n = 0$ for $n < 0$ .

**Remark**: For $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2(D)$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n \in H^2(D)$ it holds $\frac{1}{2\pi}\int_0^{2\pi}\left|f(re^{i\varphi}\right|^2 d\varphi = \sum_{n=0}^{\infty}|a_n|^2 r^{2n}$ for $0 \le r < 1$ and

$$((f,g)) := \sum_{n=0}^{\infty} a_n \bar{b}_n$$

defines an inner product of $H^2(D)$ with $|||f|||_2^2 := (((f,f))$.

**Remark**: The mapping $z \to 1 - \frac{1}{z}$ takes the right half plane $Re(z) > 1/2$ to the interior of the unit circle $D := \{z||z| < 1\}$ in the complex $z$-plane and maps the critical line $Re(z) = 1/2$ onto the unit circle

Proof:

$$\left|1 - \frac{1}{z}\right|^2 = \left|\frac{z-1}{z}\right|^2 = \frac{z-1}{z}\cdot\frac{\bar{z}-1}{\bar{z}} = \frac{|z|^2 - 2Re(z) + 1}{|z|^2} < \frac{|z|^2 - 1 + 1}{|z|^2} < 1.$$

## Relations of the Besicovitch space to the Dirichlet series

**Lemma** (LaE1b) §228: let $0 < \beta < 1$, and let $g(s) := \sum_1^\infty a_n e^{-s \log n}$ be absolute convergent for $Re(s) = \sigma = \gamma$. Then

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \zeta(\beta + it) g(\gamma - it) dt = \sum_1^\infty a_n \frac{\log n}{n^{\beta+\gamma}},$$

i.e. it especially holds

$$(*) \lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \zeta(\beta + it) \zeta(\gamma - it) dt = \sum_1^\infty \frac{1}{n^{\beta+\gamma}} = \zeta(\beta + \gamma).$$

The latter formula (*) can be generalized by

**Lemma** (LaE1b) §228: let $-1 < \beta, \gamma$ with $\beta + \gamma = 1$, $\beta > 1, \gamma > 1$ or $\beta < 1, \gamma < 1$. Then

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \zeta(\beta + it) g(\gamma - it) dt = \zeta(\beta + \gamma).$$

**Lemma**:

$$\frac{1}{2\pi} \int_0^\infty \left| \Gamma(\frac{1}{2} + it \right|^2 dx = \frac{1}{2}.$$

The most general results concerning the average values of $(|\zeta(s)|^2$ are provided in (LaE1b) §228:

**Lemma**: In the sense that the relative error approaches zero as $\omega \to \infty$

$$\frac{1}{2\omega} \int_{-\omega}^{\omega} (\left| \zeta(\frac{1}{2} + it) \right|^2 dt \sim \log\omega.$$

Regarding Dirichlet series we recall from (TiE) p. 138:

**Lemma:** Let $f(s) := \sum_1^\infty a_n e^{-s \log n}$, $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be absolute convergent for $Re(s) > 1/2$. Then for $\alpha > 1/2$

$$\langle f, g \rangle_{-\alpha} = \lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\sigma + it) g(\sigma - it) dt = \sum_1^\infty \frac{1}{n^{2\alpha}} a_n b_n,$$

i.e. $f, g \in H_{-\alpha}^{\#} \cong l_2^{-\alpha}$ for $\alpha > 1/2$.

The generalization is provided by the "main theorem" from (LaE1b) §225:

**Theorem 37**: Let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$ and $\frac{\beta - \alpha_1}{\varepsilon_1} + \frac{\gamma - \alpha_2}{\varepsilon_2} > 1$ it holds

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\gamma - it) dt = \sum_1^\infty a_n b_n \, e^{-\lambda_n (\beta + \gamma)}.$$

Putting $\alpha = \alpha_1, = \alpha_2$ and $\varepsilon = \varepsilon_1 = \varepsilon_2$ Theorem 37 leads to

**Theorem 38**: Let the series $f(s) := \sum_1^\infty a_n e^{-s \log n}$ and $g(s) := \sum_1^\infty b_n e^{-s \log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta, \gamma > \alpha$, $(\beta - \alpha) + (\gamma - \alpha) > \varepsilon$ it holds

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} f(\beta + it) g(\gamma - it) dt = \sum_1^\infty a_n b_n \, e^{-\lambda_n (\beta + \gamma)}.$$

Putting

$$l := \lim\sup\left(\frac{logn}{\lambda_n}\right) \text{ (chosing } \varepsilon_1 = \varepsilon_2 \coloneqq l)$$

Theorem 37 leads to

**Theorem 39**: Let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ be convergent for $s > \alpha_1$, and absolute convergent for $s > \alpha_1 + \varepsilon_1$ with $\varepsilon_1 > 0$. Let the series $g(s) := \sum_1^\infty b_n e^{-s\log n}$ be convergent for $s > \alpha_2$, and absolute convergent for $s > \alpha_2 + \varepsilon_2$ with $\varepsilon_2 > 0$. Then for $\beta > \alpha_1, \gamma > \alpha_2$, $(\beta - \alpha_1) + (\gamma - \alpha_2) > l$

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta + it)g(\gamma - it)dt = \sum_1^\infty a_n b_n\, e^{-\lambda_n(\beta+\gamma)}.$$

chosing $\varepsilon_1 = \varepsilon_2 \coloneqq \alpha, \beta = \gamma$ leads to

**Theorem 40**: Let $l := \lim\sup\left(\frac{logn}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ and $g(s) := \sum_1^\infty b_n e^{-s\log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\beta + it)g(\beta - it)dt = \sum_1^\infty a_n b_n\, e^{-2\lambda_n\beta}.$$

**Theorem 41**: Let $l := \lim\sup\left(\frac{logn}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ be convergent for $s > \alpha$, and absolute convergent for $s > \alpha + \varepsilon$ with $\varepsilon > 0$. Then for $\beta > \alpha + \varepsilon/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f|(\beta + it)|^2 dt = \sum_1^\infty |a_n|^2\, e^{-2\lambda_n\beta}$$

**Theorem 42**: Let $l := \lim\sup\left(\frac{logn}{\lambda_n}\right)$ positive and finite and let the series $f(s) := \sum_1^\infty a_n e^{-s\log n}$ be convergent for $s > \alpha$. Then for $\beta > \alpha + l/2$

$$\lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f|(\beta + it)|^2\, dt = \sum_1^\infty |a_n|^2\, e^{-2\lambda_n\beta}.$$

**Lemma** (ApT) p. 188: let $f(s) := \sum_1^\infty a_n e^{-s\lambda_n}$ absolute convergent for $\sigma > \sigma_a$ then

$$\frac{1}{2\omega}\int_{-\omega}^{\omega} e^{\lambda(\sigma+it)} f(\sigma + it)dt = \begin{cases} a_n & if\ \lambda = \lambda(n) \\ 0 & if\ \lambda \neq \lambda(n) \end{cases}.$$

**Lemma** (ApT) p. 188: let $\mu_n \coloneqq e^{\lambda_n}$, then $g(s) := \sum_1^\infty a_n e^{-s\mu_n}$ is absolute convergent for $\sigma > 0$; if $\sigma > \sigma_a$ then

$$\Gamma(s)f(s) = \int_0^\infty g(t)t^{s-1}dt$$

which is an extension of the classical formula

$$\zeta(s)\Gamma(s) = \int_0^\infty \frac{1}{e^t - 1} t^{s-1} dt.$$

## Entire functions and different types of same finite order

Extracts from (LeN) pp. 4, 32, (InA), (PaR)

An entire function is of finite order, if $M_f(r) := \max_{|z|=r}|f(z)| \overset{as}{<} e^{r^k}$ for some constant $k > 0$. The order of growth of an entire function is the greatest lower bound of these values of $k$. As entire functions with same order can grow differently, there are different types of orders, grouped into minimal, normal (mean), and infinite types. The entire function $f(z)$ is said to have a finite type, if for some $A > 0$ the inequality $M_f(r) := \max_{|z|=r}|f(z)| \overset{as}{<} e^{Ar^k}$ is fulfilled. The greatest lower bound for those values of is called the type $\sigma = \sigma_f$ with respect to the order $k$. If $\sigma_f = 0$ the type is called minimal, if $0 < \sigma_f < \infty$ the type is called normal (or mean), if $\sigma_f$ is infinite, the type is called maximal. Entire functions of order $k = 1$ and normal type $\sigma = \sigma_f$ are called entire functions of exponential type $\sigma = \sigma_f$. For an entire function $f(z)$ of minimal type with respect to an order $k$ it holds $log|f(z)| = o(|z|^k)$, $|z| \to \infty$, (LeB) p 90.

Examples:

i) $\sin(Az)$ is of order $k = 1$ and type $\sigma = |A|$, which mean that it is an entire function of exponential type $|A|$

ii) $\frac{\mathrm{Sin}(\sqrt{z})}{\sqrt{z}}$ is of order $k = \frac{1}{2}$ and type $1$

iii) $\frac{1}{\Gamma(z)}$ is of maximal order type, as $logM(r) \geq Crlogr$.

The function $\frac{1}{\Gamma(z)}$ is most prominent example where the density of the zero set is finite, while the canonical product is of maximal order type as $logM(r) \geq Crlogr$.

**Remark**: We note that the canonical product $\prod_{n=1}^{\infty}(1 - \frac{z}{a_n})$ is of exponential type zero, and so cannot be bounded along the real axis (applying Bernstein‘ inequality), (YoR) p. 118.

**Remark**: Sequences $y_n$ fulfilling the Kadec condition $|n - y_n| \leq L < \frac{1}{4}$ play a central role in the theory of non-harmonic Fourier series and its isometric connection with the Paley-Wiener space. The corresponding Paley-Wiener function is given by

$$g(z) := \prod_{n=1}^{\infty}(1 - \frac{z^2}{y_n^2}).$$

By making a change of variables $z^2 = \omega$, one obtains an entire function of non-integer order $\rho = 1/2$, LeB) p. 89.

**Theorem 17** (InA) p. 56: If $M(r)$ is the maximum of $|\xi(s)|$ on the circle $|s| = r$, then

$$M(r) \sim \frac{1}{2} rlogr.$$

**Theorem XXIV** (PaR) p. 77: If the $y_n$ are real and positive, if the series $\sum \frac{1}{y_n^2}$ converges, and if $\varphi(z) := \prod_{n=1}^{\infty}(1 - \frac{z^2}{y_n^2})$, then the statements $log\varphi(iy) \sim \pi A|y|log|y|$, as $y \to \infty$ and $\int_{-\pi}^{\pi} \frac{log|\varphi(x)|}{x^2} dx \sim \pi^2 A \cdot log|y|$ are completely equivalent.

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