Two proofs of the RH in a nutshell

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Abstract

This note gives a summary of corresponding papers [KBr1], [KBr2], [KBr3] in

www.riemann-hypothesis.de.

The Riemann Hypothesis states that the non-trivial zeros of the Zeta function all have real part one-half. The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self-adjoint operator. All attempts failed so far to represent the Riemann duality equation in the critical stripe as convergent (!) Mellin transforms of an underlying self-adjoint integral equation relation. The constant, not vanishing Fourier terms of the Theta functions are the root cause of only formally self-adjoint invariant operator definitions with corresponding (Mellin) transform of the Riemann duality equation ((HEd) 10.3). The idea of our two proofs is built on appropriately defined (weak) self-adjoint integral equations in a distribution sense and its related Mellin transforms. The distributional approach is the prize to be paid to build convergent (!) Mellin transform integrals ((HEd) chapter 10). The Mellin transform can be seen as a Fourier transformation on the multiplicative group of positive real numbers. It corresponds to an isometry between Hilbert spaces of functions.

Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$$

enable the definitions of the norms

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \quad .$$

We give two distributional representations $\zeta_i(s)\,(i=-1,0)$ of the Zeta function ([BPe], [AZe]) as Mellin transforms of proper Hilbert space (distributional) functions $\omega_i(x) \in H_i$. They are built on the Hilbert transforms f_H, ρ_H of the Gauss-Weierstrass function $f(x) := e^{-\pi x^2}$ and the fractional part function

$$\rho(x) = x - [x] = \frac{1}{2} - \sum_{1}^{\infty}\frac{\sin 2\pi\nu x}{2\pi|\nu|} \, ,$$

given by

$$f_H(x) = 4\pi\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi$$

$$\rho_H(x) = -\log 2\sin(\pi x) = \sum_{1}^{\infty}\frac{\cos 2\pi\nu x}{\nu} \, .$$

Its convolution integral representations in the corresponding Hilbert spaces H_i enable the application of spectral analysis arguments ([GPo]) to prove that all the zeros of $\zeta_i(s)$ lie on the critical line, which is characterized by the identity $s = 1 - s$.

The distributional "functions" $\zeta_i(s)$ are identical to the Zeta function $\zeta(s)$ in a weak sense with respect to the inner products of the Hilbert spaces of functions H_i. This proves the RH in a weak sense. By standard (functional analysis) density arguments then it follows that the RH is also valid in the strong sense.

Notations

Let $H=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u\in H:=L_2^*(\Gamma)$ with $\Gamma:=S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu:=\frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$$

enable the definitions of the norms

$$\|u\|_\beta^2:=\sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \quad .$$

There is a natural representation of the Fourier decomposition

$$u(x)=\frac{a_0}{2}+\sum_1^\infty a_\nu\cos(\nu x)+\sum_1^\infty b_\nu\sin(\nu x):=\sum_{-\infty}^{\infty}u_\nu e^{i\nu x}\in L_2$$

as Laurent series description in terms of a complex variable, defined on a circle $z=e^{ix}$:

$$u(z):=\tilde{u}(z):=u(x)=\sum_{-\infty}^{\infty}u_\nu z^\nu\in H:=L_2^*(\Gamma) \quad .$$

with $$u_0:=\frac{a_0}{2}\quad,\quad u_\nu:=\frac{1}{2}(a_\nu-ib_\nu)\ ,\ c_{-\nu}:=\frac{1}{2}(a_\nu+ib_\nu)\quad,\quad \nu>0\,.$$

Then H is the space of L_2-periodic function in R.

Remark (*): From [DGa] pp.63 and [SGr] 1.441, we recall

$$\frac{1}{2\pi}\oint_{0\to2\pi}\begin{Bmatrix}\sin n\vartheta\\ \cos n\vartheta\end{Bmatrix}\cot\frac{\varphi-\vartheta}{2}d\vartheta=\begin{Bmatrix}-\cos(n\varphi)\\ \sin(n\varphi)\end{Bmatrix}\quad,\quad \frac{1}{2\pi}\oint_{0\to2\pi}\cot\frac{\varphi-\vartheta}{2}d\vartheta=0$$

resp.

$$\frac{1}{2\pi}\oint_{0\to2\pi}e^{in\varphi}\cot\frac{\varphi-\vartheta}{2}d\vartheta=\begin{cases}-ie^{in\varphi} & n=1,2,3,...\\ 0 & n=0\\ ie^{in\varphi} & n=-1,-2,..\end{cases}$$

For the integral operators

(A) $(Au)(x) := -\oint \log 2\sin\frac{x-y}{2}u(y)dy =: \oint k(x-y)u(y)dy$ and $D(A) = H = L_2^*(\Gamma)$

(H) $(Hu)(x) := [u](x) := \frac{1}{2\pi}\oint \cot\frac{x-y}{2}u(y)dt = -\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_{\varepsilon}^{\pi}[u(x+y)-u(x-y)]\cot\frac{y}{2}dy$.

the following properties are valid:

Lemma

i) The operator H is skew symmetric in the space $L_2(0,2\pi)$ (e.g. [DGa],[BPe]) and maps the space $H := L_2(0,2\pi) - R$ isometric onto itself, and it holds

$$\|Hu\| = \|u\| \text{ and } H^2 = -I \ , \ (Hu,v) = -(u,Hv) \ , \ [u'](x) = [u]'(x)$$

$$(Hu)_\nu = -isign(\nu)u_\nu \ , \ (Hu)(x) = i\sum_1^\infty [u_{-\nu}e^{-i\nu x} - u_\nu e^{i\nu x}] \in L_2 \text{ for } u \in L_2$$

ii) The operator A is symmetric in its domain $D(A)$ and the Fourier coefficients of the convolutions of both operators are

$$(Au)_\nu = k_\nu u_\nu = \frac{1}{2|\nu|}u_\nu \ , D(A) \subseteq H_A = H_{-1/2}(\Gamma) \ .$$

Remark: As a consequence there are several relationships in the context of Euler's formula (see ([ETi] 2.1), [BPe]): Let $[x]$ denote the largest integer not exceeding the real number x and let $\rho(x) := \{x\} := x - [x]$ be the fractional part (sawtooth) function of x .

i) $$\rho(x) = \{x\} = x - [x] = \frac{1}{2} - \sum_1^\infty \frac{\sin 2\pi\nu x}{2\pi|\nu|}$$

ii) $$-i\pi sign(x) = -2i\int_0^\infty \frac{\sin(tx)dt}{t} = 2\int_0^\infty \frac{\sinh(tx)dt}{t} = \left[P.v.(\frac{1}{x})\right]^\wedge \ .$$

iii) $$\sum_1^\infty \frac{\sin\nu x}{\nu} = \frac{\pi - x}{2} \ ,$$

iv) $$\sum_1^\infty \frac{\cos\nu x}{\nu} = \frac{1}{2}\log\frac{1}{2(1-\cos x)} \ , \ 0 < x < 2\pi \ .$$

Remark: The Hilbert spaces $H_{-1/2}, H_{-1}$ are characterized by

$$H_{-1/2} = \left\{\psi \| \|\psi\|_{-1/2}^2 = (A\psi,\psi)_0 < \infty\right\},\ H_{-1} = \left\{\psi \| \|\psi\|_{-1}^2 = (A\psi, A\psi)_0 < \infty\right\}.$$

In ([AZy], 5.28, 7.2, 13.11) the concept of "logarithmic", $\alpha-$capacity" of sets and convergence of Fourier series to functions with

$$\sum_1^\infty n\left[a_n^2 + b_n^2\right] < \infty$$

is given. In this context we also refer to [BRi] and the still unanswered question in it. In ([AZy]) the following two examples are provided (see also [HEd] 9.7):

i) $\lambda(x) \approx \sum_1^\infty \frac{\cos 2\pi\nu x}{\nu} = -\log 2\sin(\pi x)$ whereby $\left|\sum_1^N \frac{\cos \nu x}{\nu}\right| \le \log(\frac{1}{x}) + C$,

ii) $\lambda(x) \approx \sum_1^\infty \frac{\cos \nu x}{\nu^{1-\alpha}} \cong c_\alpha |x|^{-\alpha}$, $(x \to 0; 0 < \alpha < 1)$.

In [CBe] 8, Entry17(iv) its relationship to Ramanujan's divergent series technique is mentioned: "*Ramanujan informs us to note that*

$$\sum_1^\infty \sin(2\pi\nu x) = \frac{1}{2}\cot(\pi x) ,$$

which also is devoid of meaning" .... "may be formally established by differentiating the well known equality"

$$\sum_1^\infty \frac{\cos 2\pi\nu x}{\nu} = -\log 2\sin(\pi x) .$$

There is also a related representation of the Dirac function in the form

$$\delta(x) := \frac{1}{2\pi}\int_0^{2\pi} e^{ikx}dk = \frac{1}{\pi}\int_0^{\pi}\cos(kx)dk \in H_{-n/2-\varepsilon} \subset H_{-1} ,$$

i.e. its regularity depends from the space dimension n, but is always more regular than H_{-1}.

We further note that in harmonic analysis the energy of the harmonic continuation $h = \mathrm{E}(\varphi)$ to the boundary is given by

$$[\varphi]^2 := \frac{\pi}{2}\sum_1^\infty \nu(a_\nu^2 + b_\nu^2) = \frac{1}{2}\iint |dh(z)|^2 dxdy = \frac{1}{4\pi}\int_{\partial B}\int_{\partial B} \frac{|\varphi(w) - \varphi(\zeta)|^2}{|w-\zeta|^2} ds(w)d\zeta < \infty .$$

A relationship to the Gamma function and the Euler constant is given by ([CBe] 8, entry 17(iv), ([NNi] chapter II, §33)):

i) $\log\sin\pi x = \log\frac{\pi}{\Gamma^2(x)} + \frac{2}{\pi}\sum_1^\infty (\gamma + \log(2\pi k))\frac{\sin(2\pi kx)}{k}$ for $0 < x < 1$

ii) $\gamma = \frac{1}{2} + \sum_{n=1}^\infty \int_n^\infty \cos(2\pi t)\frac{dt}{t}$.

The Gauss-Weierstrass and fractional part function

The Theta function ϑ is given by

$$\vartheta(x) := \sum_{-\infty}^{\infty} f(nx) =: 1 + \psi(x^2) = \frac{1}{x}\vartheta(\frac{1}{x}) \ ,$$

whereby f denotes the Gauss-Weierstrass density function

$$f(x) := e^{-\pi x^2} \ .$$

Lemma: We note the identity ($a > 0$)

i) $$f(x) = \frac{\pi^{-s/2}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\Gamma(\frac{s}{2})x^{-s}ds$$

ii) $$\Omega(s) := (s-1)\int_0^\infty x^s(xf'(x))d\log x = (s-1)\Gamma(1+\frac{s}{2})\pi^{-s/2} = (s-1)\int_0^\infty x^s(xf'(x))\frac{dx}{x}$$

iii) The Hilbert transform of the Gauss-Weierstrass function is given by

$$[H(f)](x) = 4\pi\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi$$

iv) It holds the identity ([CFo]):

$$[H(f)](x) = \frac{\pi^{1-s/2}}{2\pi i}\oint_C \frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})}x^{-s}ds = \frac{\pi^{1-s/2}}{2\pi i}\oint_C \tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2})x^{-s}ds$$

where one can take C also to be the critical line.

The corresponding duality relationships to the Zeta function are given by ([ETi] 2.1, [HEd] 1.6ff):

i) $$-\frac{\zeta(s)}{s} = \int_0^\infty \rho(\frac{1}{t})t^s\frac{dt}{t} = \int_0^\infty x^{-s}\rho(x)\frac{dx}{x} \quad \text{for} \quad 0 < \mathrm{Re}(s) < 1$$

ii) $$\xi(s) := \zeta(s)\Omega(s) = \xi(1-s) \quad \text{for all complex} \quad s \in C \ .$$

Rational: We refer to [HEd] 10.1, 10.3, 10.5, [ETi]2.11: The constant, non-vanishing Fourier terms of

$$\rho(x)=\frac{1}{2}-\sum_{1}^{\infty}\frac{\sin 2\pi\nu x}{2\pi|\nu|} \quad , \quad \vartheta(x)=1+2\sum_{n=1}^{\infty}f(nx)$$

jeopardize the application of the Müntz formula to build the Riemann duality equation as transforms of a self-adjoint integral operator. The alternatively defined density function $\tilde{f}(x):=xf'(x)$ with its corresponding Mellin transform

$$\Gamma(1+\frac{s}{2})\pi^{-s/2}=\int_{0}^{\infty}x^{s}(xf'(x))\frac{dx}{x}$$

is the today's basic "trick/idea" to overcome the conceptual problem of the non-vanishing Fourier term (just by differentiating). Its consequence of a reduced regularity of f' is balanced/leveraged by a corresponding multiplication with the factor x again. This then rebuilds the original structural properties of f (but just enabled by the special exponential of f, only). The (too high!) prize to be paid is the loss of the essential Theta function property. Our alternative solution concept is applying as alternative density function just the Hilbert transform of f (which doesn't change the regularity (in a L_2-sense), keeps the Theta function property, while leading to a vanishing Fourier term at the same time), i.e. the alternative concept is about a replacement of

$$xf'(x)\rightarrow f_{H}(x):=[H(f)](x)=4\pi\int_{0}^{\infty}f(\xi)\sin(2\pi\xi x)d\xi \ ,$$

$$\Omega(s)=(s-1)\int_{0}^{\infty}x^{s}(xf'(x))\frac{dx}{x}\rightarrow\Omega_{0}(s):=\int_{0}^{\infty}x^{s}f_{H}(x)\frac{dx}{x}=\pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2}) \ .$$

The (only) prize to be paid in this case is, that the duality relationship is "only" valid in a weak sense as the Hilbert transform creates L_2-function only, but at the same time it transforms L_2-function to L_2-function.

Fortunately weak solutions are equivalent to strong solutions in case of appropriate regularity assumptions, especially for complex valued functions with same spectrum, i.e. same set of zeros.

A special addition on the positive side, is, that it ends up with a new (only in a distributional sense valid) duality equation in the form (see lemma above and **D2** below):

$$\xi_{0}(s):=\zeta(s)\Omega_{0}(s)=2\int_{0}^{\infty}x^{s}\left[\sum_{n=1}^{\infty}f_{H}(nx)\right]\frac{dx}{x}=\xi_{0}(1-s)$$

On the additional positive side this enables leveraging with results and propositions of Ramanujan, which are still until these days "described" in a form like e.g. (see above):

([CBe] 8, Entry17(iv): "*Ramanujan informs us to note that …… which also is devoid of meaning" .... "may be formally established by differentiating the well known equality" ….*

Two $H_{-1}, H_0 -$ related distributional Zeta functions

as transforms of self-adjoint integral operators

For $u, v \in L_2(0,2\pi)$ it holds $Hu, Hv \in L_2(0,2\pi)$, $(Hu,v) = -(u,Hv)$, $(Hu)_{\nu=0} = (Hv)_{\nu=0} = 0$ and

$(u,\lambda) = (v,\lambda)$ for all $\lambda \in H$ $\Leftrightarrow$ $(Hu,\mu) = (Hv,\mu)$ for all $\lambda = H\mu \in H, \mu \in H$.

Let $\omega_{-1}(x) := H\rho(x) = -\log 2\sin(\pi x)$ resp. $\omega_0(x) := H\vartheta(x)$ denote the (periodical) Hilbert transforms of the fractional part function resp. of the Theta function. Let $\bar{g}$ be defined by

$$\bar{g}(x) := \frac{1}{x} g(\frac{1}{x}) \cdot$$

Then with respect to the inner products of the Hilbert spaces H_{-1}, H_0 the integral "density" functions $\omega_i(x)$ for $i = -1,0$ fulfill (in the following weak sense) the Theta function property (remark (*), [BPe] examples 9.9-9.11, 9.13, 11.12, Corollary 11.9, §12, [KBr1], [KBr2]):

$$(\omega_i, \lambda)_i = (\bar{\omega}_i, \lambda)_i \quad \text{for all} \quad \lambda \in H_i \ .$$

The Theta function property is equivalent to a Riemann duality type equation for the corresponding Mellin transformed holomorphic functions $\zeta_i, s =: \zeta_i(s)$ ([HHa]), if the corresponding integrals are all convergent. In the sense of the definition HF above, this is given in the Hilbert space frameworks H_{-1}, H_0:

As a consequence the "functions" $\zeta_i, s =: \zeta_i(s)$ fulfill in a weak (distributional) sense in the critical stripe the duality equations

(D1) $$\zeta_{-1}(s)\Omega_{-1}(s) = \int_0^\infty \omega_{-1}(\frac{1}{t})t^s \frac{dt}{t} = \int_0^\infty x^{-s}\omega_{-1}(x)\frac{dx}{x} = \zeta_{-1}(1-s)\Omega_{-1}(s)$$

(D2) $$\zeta_0(s)\Omega_0(s) = \int_0^\infty \omega_0(\frac{1}{t})t^s \frac{dt}{t} = \int_0^\infty x^{-s}\omega_0(x)\frac{dx}{x} = \zeta_0(1-s)\Omega_0(1-s) \quad .$$

At the same time the Hilbert space frameworks provide a self-adjoint integral function representation of the distributional Zeta functions along the "symmetry" line within the critical stripe, defined by the identity $s = 1-s$.

As a consequence it holds:

Proposition: The complex functions $\zeta_i(s)$ defined by (D1), (D2) are identical to the Zeta function in a weak sense with respect to the norms of the Hilbert spaces H_{-1}, H_0 . At the same time they are transforms of convolution integrals and "symmetric" with respect to $s = 1-s$. Therefore all its zeros lie on the critical line. *This proves the Riemann Hypothesis* as by definition the zeros of the Zeta function are (in a distributional sense) the same as those of $\zeta_i(s)$. Therefore, by density arguments it follows, that this is also valid in a strong sense.

Appendix

Holomorphic function in the distributional sense

Definition (HF): ([BPe] §15, 16, Distribution valued and boundary values of holomorphic functions) Let $z \to g_z$ be a function defined on an open subset $U \subset C$ with values in the distribution space. Then g_z is called a holomorphic in $U \subset C$ (or $g(z) := g_z$ is called holomorphic in $U \subset C$ in the distribution sense, if for each $\varphi \in C_c^\infty$ the function $z \to (g_s, \varphi)$ is holomorphic in $U \subset C$ in the usual sense.

Remark: ([BPe]) In the one-dimensional case any complex-analytical function, as any distribution f on R, can be realized as the "jump" across the real axis of the corresponding in $C - R$ holomorphic Cauchy integral function

$$F(x) := \frac{1}{2\pi i} \oint \frac{f(t)dt}{t - x} ,$$

given by

$$(f, \varphi) = \lim_{y \to 0^1} \int_{-\infty}^{\infty} F(x + iy) - F(x - iy))\varphi(x)dx \quad .$$

In the one-dimension case the Riesz operator is identical with Hilbert transform ([BPe] 2.9), that is a Cauchy principle-valued function, expressed in the form

(*) $$(R_1 u)(x) := (Hu)(x) := \lim \frac{1}{\pi} \oint_{|x-y|>\varepsilon} \frac{u(y)}{x - y} dy = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{x - y} dy \qquad \text{for } \varepsilon \to 0 ,$$

whereby the Fourier coefficients are given by

$$(Hu)_\nu = -i \operatorname{sgn}(\nu) u_\nu ,$$

i.e. the Hilbert transform is a classical pseudo-differential operator ([BPe] 3.6) with symbol $i \operatorname{sgn}(s)$.

The principle value $P.v.(1/x)$ of the not locally integrable function $1/x$ is the distribution g defined by ([BPe] 1.7)

$$(g, \varphi) := \lim \int_{x \geq \varepsilon} \varphi(x) \frac{dx}{x} = \int_{-\infty}^{\infty} \log|x| \varphi'(x) dx \qquad \text{for each } \varphi \in C_c^\infty .$$

The relationship of this specific principle value to the Fourier and Hilbert transforms is given by ([BPe] 2.9)

$$\left[P.v.(\frac{1}{x}) \right]^\wedge = -i\pi \operatorname{sgn}(s) \quad \text{and} \quad \left[P.v.(\frac{1}{x}) \right]^{\wedge\wedge} = -2\pi P.v.(\frac{1}{x}) .$$

The Hilbert transform of the function $\sin(\omega t)$ is given by $\cos(\omega t)$. This gives a $\pm \frac{\pi}{2}$ _ phase-shift operator, which is another basic property of the Hilbert transform. It can be used to remove the not needed negative frequency axis.

There is an obvious relationship between the Mellin and the Laplace transform:

$$F(s)=\int_0^\infty f(t)t^{s-1}dt=\int_{-\infty}^{-\infty} f(e^{-x})e^{-sx}dx=-\int_0^1 f(\frac{1}{y})\left[\log(\frac{1}{y})\right]^{s-1}\frac{dy}{y}\ .$$

If $f\in L_1(R^n)$ has compact support the Laplace transform of f is the entire function F defined by ([BPe] §12)

$$F(s)=\int e^{-i\langle s,x\rangle}f(x)dx=f(\eta-i\xi)\ .$$

This definition extends immediately to distributions with compact support. If $f\in E'(R^n)$ we define the Laplace transform F of f by

$$F(s)=\left\langle f,e^{-i\langle s,\circ\rangle}\right\rangle\ .$$

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