

# FOURIER TRANSFORMS OF POSITIVE DEFINITE KERNELS AND THE RIEMANN $\xi$ -FUNCTION

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*Dedicated to Dr. Robert D. Cushnie on the occasion of his 80<sup>th</sup> birthday.*

ABSTRACT. The purpose of this paper is to investigate the distribution of zeros of entire functions which can be represented as the Fourier transforms of certain admissible kernels. The principal results bring to light the intimate connection between the Bochner-Khinchin-Mathias theory of positive definite kernels and the generalized real Laguerre inequalities. The concavity and convexity properties of the Jacobi theta function play a prominent role throughout this work. The paper concludes with several questions and open problems.

## 1. Introduction

Today, there are no known explicit necessary and sufficient conditions that even a “nice” kernel (cf. Definition 1.2),  $K(t)$ , must satisfy in order that its Fourier transform

$$(1.1) \quad F(x) := \frac{1}{2} \int_{-\infty}^{\infty} K(t) e^{ixt} dt = \int_0^{\infty} K(t) \cos(xt) dt$$

have only real zeros (cf. [49, p. 17] and [50]). The program of investigation promulgated here is motivated, in part, by several recent results ([4, 12, 24, 32, 42–45]) and our understanding that it is desirable to discover properties of the kernel,  $K$ , which (hopefully) will lead to information about the distribution of zeros of the entire function  $F$ . The main leitmotif of this note pertains to certain inequalities, known as the *generalized Laguerre inequalities* (Section 2), which play a pivotal role in the study of functions in the Laguerre-Pólya class (cf. Definition 1.1). Notwithstanding the extensive research in this area and the impressive results dealing with the Riemann  $\xi$ -function, it is curious that to date so little progress has been made in proving some of the simplest Laguerre inequalities that  $F$  must satisfy in order that it possess only real zeros (cf. Open Problem 4.7).

An outline of this work is as follows. In the remainder of this introduction, we recall some pertinent definitions and nomenclature that will be used in the sequel. In Section 2, we review several classical and new results involving the Laguerre and the generalized real Laguerre inequalities (Theorem 2.4) and prove two important, albeit elementary, results (Propositions 2.2 and 2.3) which adumbrate some of the applications in Section 4. With the aid of the classical theorems of S. Bochner [1], A. Khinchin [33], and M. Mathias [41], we establish the positive definite character of certain canonical kernels which lead to some new classes of characteristic functions (Section 3). By extending the work of J. L. W. V. Jensen [31] and G. Pólya [49], our main results in Section 3 (cf. Theorems 3.5–3.7) establish precise relationships between certain positive definite kernels and the generalized real Laguerre inequalities. Concavity plays a prominent role throughout this paper and it is the *sine qua non* for analyzing the Jacobi theta function and related kernels. In Section 4, we apply the foregoing results and derive new necessary and sufficient conditions for the Fourier transform of the Jacobi theta function, the Riemann  $\xi$ -function, to belong to the Laguerre-Pólya class. The paper ends with several (6) open problems (Section 4).

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In the present investigation, we will adopt the following notation and nomenclature associated with real entire functions whose zeros lie in a strip. Let  $S(\tau)$  denote the closed strip of width  $2\tau$ ,  $\tau \geq 0$ , in the complex plane,  $\mathbb{C}$ , symmetric about the real axis:

$$(1.2) \quad S(\tau) = \{z \in \mathbb{C} \mid |\operatorname{Im}(z)| \leq \tau\}.$$

**Definition 1.1.** We say that a real entire function  $f$  belongs to the class  $\mathfrak{S}(\tau)$ , if  $f$  can be expressed in the form

$$(1.3) \quad f(z) = Ce^{-az^2+bz} z^m \prod_{k=1}^{\omega} (1 - z/z_k) e^{z/z_k}, \quad (0 \leq \omega \leq \infty),$$

where  $a \geq 0$ ,  $b \in \mathbb{R}$ ,  $z_k \in S(\tau) \setminus \{0\}$ ,  $\sum_{k=1}^{\infty} 1/|z_k|^2 < \infty$ . We allow functions in  $\mathfrak{S}(\tau)$  to have only finitely many zeros by letting, as usual,  $z_k = \infty$  and  $0 = 1/z_k$ ,  $k \geq k_0$ , so that the canonical product in (1.3) is a finite product. By convention, the empty product is one. If  $f \in \mathfrak{S}(\tau)$ , for some  $\tau \geq 0$ , and if  $f$  has only real zeros (i.e., if  $\tau = 0$ ), then  $f$  is said to belong to the Laguerre-Pólya class, and we write  $f \in \mathcal{L}\text{-}\mathcal{P}$ . In addition, we write  $f \in \mathcal{L}\text{-}\mathcal{P}^*$ , if  $f = pg$ , where  $g \in \mathcal{L}\text{-}\mathcal{P}$  and  $p$  is a real polynomial. Thus,  $f \in \mathcal{L}\text{-}\mathcal{P}^*$  if and only if  $f \in \mathfrak{S}(\tau)$ , for some  $\tau \geq 0$ , and  $f$  has at most finitely many non-real zeros.

The significance of the class  $\mathfrak{S}(\tau)$  in the theory of entire functions stems from the fact that  $f \in \mathfrak{S}(\tau)$  if and only if  $f$  is the uniform limit on compact sets of a sequence of real polynomials having zeros only in the strip  $S(\tau)$  (cf. [2, p. 202] and [35b, pp 373–374]). It follows from the Gauss-Lucas Theorem ([40, pp 8–22], [53, p. 71]) that this class of polynomials is closed under differentiation, and thus so is  $\mathfrak{S}(\tau)$ . For various properties and algebraic and transcendental characterizations of functions in the Laguerre-Pólya class, we refer the reader to Pólya and Schur ([52, p. 100], [51], [46, Kapitel II] or [38, Chapter VIII]).

In the sequel, we will confine our attention to special kernels which we term admissible kernels and define as follows.

**Definition 1.2.** A function  $K : \mathbb{R} \rightarrow \mathbb{R}$  is called an *admissible kernel*, if it satisfies the following properties: (i)  $K(t) \in C^\infty(\mathbb{R})$ , (ii)  $K(t) > 0$  for  $t \in \mathbb{R}$ , (iii)  $K(t) = K(-t)$  for  $t \in \mathbb{R}$ , (iv)  $K'(t) < 0$  for  $t > 0$ , and (v) for some  $\varepsilon > 0$  and  $n = 0, 1, 2, \dots$ ,

$$(1.4) \quad K^{(n)}(t) = O(\exp(-|t|^{2+\varepsilon})) \text{ as } t \rightarrow \infty.$$

Thus, the assertions that  $F(x)$  (cf. 1.1) is a real entire function readily follows if we assume that  $K(t)$  is an admissible kernel. Moreover, a calculation shows ([52, p. 269]) that  $F(x)$  is an entire function of order  $\frac{2+\varepsilon}{1+\varepsilon} < 2$ . Also, by the Riemann-Lebesgue Lemma  $F(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Observe that if we omit the requirement that  $K(t)$  is even (see, Definition 1.2 (iii)), then its transform,  $F$ , cannot have only real zeros. This claim follows from integrating by parts and invoking the Riemann-Lebesgue Lemma (cf. [49]).

## 2. The Laguerre Inequalities

One important property, shared by all functions in  $\mathcal{L}\text{-}\mathcal{P}$ , is logarithmic concavity; that is, if  $f(x) \in \mathcal{L}\text{-}\mathcal{P}$ , then  $f(x)^2 (\log f(x))'' \leq 0$  for all  $x \in \mathbb{R}$ . In order to verify this claim, one need only to consider the derivative of the logarithmic derivative of  $f(x) \in \mathcal{L}\text{-}\mathcal{P}$  using the (Hadamard) factorization (1.3), (see, for example, [6, 7, 8]). The logarithmic concavity, in conjunction with the closure property of  $\mathcal{L}\text{-}\mathcal{P}$  under differentiation, implies that if  $f \in \mathcal{L}\text{-}\mathcal{P}$ , then  $f$  satisfies the following inequalities, known as the *Laguerre inequalities*, ([11–13, 15, 18, 22, 25, 47])

$$(2.1) \quad L_{1,p}(x; f) := (f^{(p)}(x))^2 - f^{(p-1)}(x)f^{(p+1)}(x) \geq 0, \quad p = 1, 2, 3, \dots, \quad \text{for all } x \in \mathbb{R}.$$

For the sake of simplicity of notation, we set  $L_{1,1}(x; f) := L_1(x; f) := L_1(x)$ . In the sequel, we will be primarily concerned with the case when  $p = 1$  in (2.1); that is,  $L_1(x)$ . The reason for the subscript “1” will become clear when we consider the generalized real Laguerre inequalities (see Theorem 2.4). We remark that one of the simplest manifestations of the existence of a non-real zero of an entire function  $f$ , occurs when  $f$  possesses a positive local minimum or a negative local maximum. It is this observation that motivates us to consider the Laguerre inequalities. We emphasize here that the Laguerre inequalities are only *necessary* conditions and, in general, are not sufficient for an entire function to have only real zeros. Indeed,  $f(x) := e^{-x^2}(1+x^2) \notin \mathcal{L}\text{-}\mathcal{P}$ , while a calculation shows that  $L_1(x) = 2e^{-2x^2}x^2(3+x^2) \geq 0$  for all  $x \in \mathbb{R}$ .

**Remark 2.1.** To illustrate by an example the spirit of the type of research program we are advocating here, consider again an admissible kernel  $K(t)$  and its Fourier cosine transform  $F(x)$ . Then via the change of variables,  $u = -x^2$ , we obtain the entire function

$$F_c(u) := \sum_{k=1}^{\infty} \frac{k! \beta_k}{(2k)!} \frac{u^k}{k!} := \int_0^{\infty} K(t) \cosh(t\sqrt{u}) dt, \quad \text{where } \beta_k := \int_0^{\infty} K(t) t^{2k} dt, \quad k = 0, 1, 2, \dots$$

Now set  $\gamma_k := \frac{k! \beta_k}{(2k)!}$  for  $k = 0, 1, 2, \dots$ . If  $\log(K(\sqrt{t}))$  is strictly concave for all  $t > 0$ , then we can infer that the Taylor coefficients of  $F_c(x)$  satisfy the *Turán inequalities*; that is,  $L_{1,p}(0; F_c) := (F_c^{(p)}(0))^2 - F_c^{(p-1)}(0)F_c^{(p+1)}(0) = \gamma_p^2 - \gamma_{p-1}\gamma_{p+1} \geq 0$ , for  $p = 1, 2, 3, \dots$  (see, for example, [8, 14, 16, 21]). Once again, the Turán inequalities are only necessary conditions for  $F_c$  (and whence for  $F$ ) to belong to the Laguerre-Pólya class.

Our next proposition asserts that if a real entire function  $f \in \mathfrak{S}(\tau)$ ,  $\tau = 1$ , has only real zeros in a vertical strip  $A \leq \operatorname{Re} z \leq B$ ,  $B - A > 2$ , then  $L_1(x) \geq 0$  for  $x \in [A+1, B-1] := I$ . Thus, on the interval  $I$ ,  $f$  cannot have a positive local minimum or a negative local maximum.

**Proposition 2.2.** ([22]) *Let  $f \in \mathfrak{S}(\tau)$ , where  $\tau = 1$  and suppose that  $f(0) \neq 0$ . Let  $\{x_k\}_{k=1}^{\infty}$  denote the real zeros and let  $z_j = \alpha_j + i\beta_j$ ,  $j = 1, 2, \dots, \omega$ ,  $1 \leq \omega \leq \infty$ , denote the non-real zeros of  $f$ . If there is an interval  $[A, B]$ , with  $B - A > 2$ , such that  $\alpha_j \notin [A, B]$  for all  $j \geq 1$ , then*

$$(2.2) \quad L_1(x) \geq 0 \quad \text{for all } x \in [A+1, B-1].$$

*Proof.* Using the product representation (1.3), logarithmic differentiation yields

$$(2.3) \quad L_1(x) = (f(x))^2 \left\{ 2\alpha + \sum_{k=1}^{\infty} \frac{1}{(x-x_k)^2} + 2 \sum_{j=1}^{\omega} \frac{(x-\alpha_j)^2 - \beta_j^2}{[(x-\alpha_j)^2 + \beta_j^2]^2} \right\}.$$

Since  $(x-\alpha_j)^2 - \beta_j^2 > 0$  for any  $x \in [A+1, B-1]$ , (2.3) gives the desired result (2.2).  $\square$

**Proposition 2.3.** ([22]) *Let  $g(x)$  be a real entire function and define*

$$(2.4) \quad f(x) := ((x-\alpha)^2 + \beta^2)^m g(x) \quad (\alpha \in \mathbb{R}, \beta > 0, m \in \mathbb{N}),$$

*so that  $\alpha \pm i\beta$  are two non-real zeros of order  $m$  of  $f$ . If  $g(\alpha) \neq 0$ , then*

$$(2.5) \quad L_1(\alpha; f) = -2m\beta^{4m-2}(g(\alpha))^2 + \beta^{4m}L_1(\alpha; g).$$

*Thus, there exists  $M > 0$  sufficiently small such that*

$$(2.6) \quad L_1(\alpha; f) < 0 \quad \text{for all } 0 < \beta < M.$$

*Proof.* Since  $f(\alpha) = \beta^m g(\alpha)$ , a straightforward calculation, using logarithmic differentiation, yields (2.5) and whence the desired result (2.6) follows.  $\square$

A heuristic description of Proposition 2.2 is as follows. A conjugate pair of non-real zeros  $\alpha \pm i\beta$  of  $f(x)$ , when  $\beta > 0$  is sufficiently small, forces  $L_1(\alpha; f)$  to be negative.

We consider next the so-called *generalized real Laguerre inequalities* (see, for example, [20, 25]) that are both *necessary and sufficient* for membership in the Laguerre-Pólya class.

**Theorem 2.4.** (The Generalized Real Laguerre Inequalities [20, Theorem 2.9]) *Let  $f$  denote a real entire function,  $f \not\equiv 0$ . For  $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$  and  $x \in \mathbb{R}$ , set*

$$(2.7) \quad L_n(x) := L_{n,1}(x; f) := \sum_{j=0}^{2n} \frac{(-1)^{j+n}}{(2n)!} \binom{2n}{j} f^{(j)}(x) f^{(2n-j)}(x).$$

$$(2.8) \quad \text{If } f(x) \in \mathcal{L}\text{-}\mathcal{P}, \text{ then } L_n(x) \geq 0 \text{ for all } n \in \mathbb{N}_0 \text{ and for all } x \in \mathbb{R}.$$

*Conversely, suppose that*

$$(2.9) \quad f(x) = e^{-ax^2} g(x), \quad a \geq 0, \text{ where the genus of } g(x) \text{ is 0 or 1.}$$

$$(2.10) \quad \text{If } L_n(x) \geq 0 \text{ for all } n \in \mathbb{N}_0 \text{ and for all } x \in \mathbb{R}, \text{ then } f(x) \in \mathcal{L}\text{-}\mathcal{P}.$$

**Remarks 2.5.** Observe that  $L_0(x) = f(x)^2$  and to justify the appellation “generalized Laguerre expression”, note that  $L_1(x) = f'(x)^2 - f(x)f''(x)$ . In addition, we remark that if the real entire function  $f(x)$  satisfies the generalized real Laguerre inequalities,  $L_n(x) \geq 0$  ( $n \in \mathbb{N}_0$ ,  $x \in \mathbb{R}$ ), then  $f(x)$  has only real zeros (cf. [20, p. 343]). For the sake of completeness, we mention here the following representation of  $|f(x + iy)|^2$  which can be derived by a direct calculation (see, for example, [20], [47], [49] or by using a recursion relation [6]):

$$(2.11) \quad |f(x + iy)|^2 = f(x + iy)f(x - iy) = \sum_{n=0}^{\infty} L_n(x)y^{2n}, \quad (x, y \in \mathbb{R}),$$

where  $L_n(x)$  is defined in (2.7).

**Remarks 2.6.** The action of the non-linear operators  $\{L_n\}_{n=0}^{\infty}$  taking a real entire function  $f(x)$  to  $L_n(x; f) := L_n(x)$  is given implicitly by equation (2.11). We mention here, parenthetically, a couple facts about these operators. It is known that the operators  $L_n$  satisfy a simple recursive relation [6, Theorem 2.1] and that  $L_n(x)$  is also a real entire function [6, Remark 2.4]. Interesting generalizations of these operators are given by K. Dilcher and K. B. Stolarsky [26] and D. A. Cardon [3] (see also Section 3). Recently, A. Vishnyakova and the author [25] have shown that the various sufficient conditions for a real entire function,  $f(x)$ , to belong to the Laguerre-Pólya class, expressed in terms of Laguerre-type inequalities, do not require the *a priori* assumptions about the order and type of  $f(x)$ . Thus, for instance, implication (2.10) remains valid if we omit assumption (2.9). In light of the results in [25], we can state the *complex Laguerre inequalities* as follows. Suppose  $f$ ,  $f \not\equiv 0$ , is a real entire function. Once again we do not stipulate conditions on the order and type of  $f$  [25]. Then  $f \in \mathcal{L}\text{-}\mathcal{P}$  if and only if

$$(2.12) \quad |f'(z)|^2 \geq \operatorname{Re} \left( f(z) \overline{f''(z)} \right) \quad \text{for all } z \in \mathbb{C}.$$

It may be of interest to note that the complex Laguerre expression can be also formulated in terms of two real Laguerre-type expressions [25]. Indeed, if  $f(x + iy) = U(x, y) + iV(x, y)$  is a real entire function, then a calculation shows that for all  $z = x + iy \in \mathbb{C}$ ,

$$(2.13) \quad \frac{1}{2} \frac{\partial^2}{\partial y^2} |f(x + iy)|^2 = |f'(z)|^2 - \operatorname{Re} \left( f(z) \overline{f''(z)} \right) = U_x^2 - UU_{xx} + V_x^2 - VV_{xx}.$$

### 3. Positive Definite Functions and the Laguerre Inequalities

We mention at the outset that it was M. Mathias [41] who in 1923, motivated by the results of C. Carathéodory and O. Toeplitz (cf. [55, p. 412]), first defined and studied the properties of positive definite functions. In this section, after reviewing some definitions, we will succinctly summarize a couple of classical results due to M. Mathias [41], S. Bochner [1], A. Khinchin [33] and G. Pólya [48]. Parenthetically we note that there are many excellent treatises in the literature dealing with positive definiteness and here we merely cite F. Lukacs [39], T. Kawata [34], M. Mathias [41] and J. Stewart [55], together with the original works of S. Bochner [1], A. Khinchin [33] and G. Pólya [48]. The interested reader will find 125 additional references in J. Stewart's outstanding survey article [55]. In the second part of this short section, our goal is to bring to light the connection between positive definiteness and the Laguerre inequalities.

**Definition 3.0.** ([34, p. 377]) A continuous function  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is said to be *positive definite* (or more precisely *non-negative definite*), if

$$(3.1) \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t - s) \rho(t) \overline{\rho(s)} dt ds \geq 0,$$

where  $\rho : \mathbb{R} \rightarrow \mathbb{C}$  is any measurable function with compact support.

An equivalent definition of positive definiteness ([34, p. 377]) is the following discrete formulation. A continuous function  $\varphi$  is positive definite if the Hermitian form

$$(3.2) \quad \sum_{j=1}^n \sum_{k=1}^n \varphi(x_j - x_k) \rho_j \overline{\rho_k} \geq 0 \quad \text{for every } x_1, \dots, x_n \in \mathbb{R} \text{ and } \rho_1, \dots, \rho_n \in \mathbb{C}.$$

By way illustration, we note that  $\varphi(x) = \cos x$  is positive definite, since

$$\sum_{j=1}^n \sum_{k=1}^n \cos(x_j - x_k) \rho_j \overline{\rho_k} = \left| \sum_{j=1}^n \rho_j \cos x_j \right|^2 + \left| \sum_{k=1}^n \rho_k \sin x_k \right|^2 \geq 0 \quad (x_1, \dots, x_n \in \mathbb{R}, \quad \rho_1, \dots, \rho_n \in \mathbb{C}).$$

Similarly, it is easy to check that  $e^{itx}$ , ( $t \in \mathbb{R}$ ), is positive definite; while it is not so straightforward to verify that the functions  $e^{-|x|}$ ,  $e^{-x^2}$  and  $\frac{1}{1+x^2}$  are positive definite. For the sake of clarity, we define one more term. By a *distribution function* we shall mean a non-decreasing function  $V(x)$  such that  $V(-\infty) = 0$  and  $V(+\infty) = 1$ . The Fourier-Stieltjes transform of  $V$ ,

$$(3.3) \quad f(t) = \int_{-\infty}^{\infty} e^{itx} dV(x) \quad (-\infty < t < \infty),$$

is called the *characteristic function* corresponding to the given distribution function  $V$ .

In 1932, S. Bochner proved the following celebrated theorem that bears his name.

**Theorem 3.1.** ([1], [39, p. 71]) *A continuous function,  $f(t)$ , with  $f(0) = 1$ , is a characteristic function if and only if  $f(t)$  is positive definite.*

We remark that since  $e^{itx}$ , ( $t \in \mathbb{R}$ ) is positive definite, it is easy to show that a characteristic function is positive definite. The converse implication is the difficult part of Theorem 3.1 (see, for example, T. Kawata [34, p. 377] or E. Lukacs [39, p. 71]). For our purposes the following version of the Khinchin's criterion [33] for a characteristic function will suffice (see also E. Lukacs [39, Theorems 4.2.4 and 4.2.5]).

**Theorem 3.2.** ([34, p. 387]) *A function of the form*

$$(3.4) \quad f(t) = \frac{1}{c} \int_{-\infty}^{\infty} \varphi(x+t) \overline{\varphi(x)} dx,$$

where  $\varphi(x)$  is any function in  $L^2(\mathbb{R})$  with  $\|\varphi\|_2 = c > 0$ , or the local uniform limit of such functions, is a characteristic function. The converse is also true.

Theorem 3.2 implies Mathias's result [41, Satz 15] which may be stated as follows. If  $\varphi \in L^2(\mathbb{R})$ , then the function

$$(3.5) \quad f(t) = \int_{-\infty}^{\infty} \varphi(s+t) \overline{\varphi(s-t)} ds,$$

is positive definite. We remark that if  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is an admissible kernel, then  $\varphi$  is a bounded integrable function. Moreover, it is not difficult to demonstrate that  $\varphi$  satisfies the conditions of Fourier's inversion theorem (cf. [56, Pringsheim's theorem, p. 16]). Thus, with the terminology adopted here we can express Mathias's main theorem (cf. [41, Hauptsatz, p. 108] or [55, p. 412]) in the following form.

**Theorem 3.3.** *Let  $\varphi$  be an admissible kernel and let*

$$(3.6) \quad f(t) := \int_{-\infty}^{\infty} \varphi(x) \cos(xt) dx.$$

*Then  $\varphi$  is positive definite if and only if  $f(t) \geq 0$  for all  $t \in \mathbb{R}$ .*

The above necessary and sufficient conditions for a characteristic function are, in general, not readily applicable in order to determine whether a given function is a characteristic function. There is, however, a beautiful and simple criterion due to Pólya [48] (see also Lukacs [39, p. 85]).

**Theorem 3.4.** (Pólya's criterion) *Suppose that  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous and satisfies the following conditions: (i)  $f(0) = 1$ , (ii)  $f(-t) = f(t)$ , (iii)  $f$  is convex for  $t > 0$  and (iv)  $\lim_{t \rightarrow \infty} f(t) = 0$ . Then  $f(t)$  is the characteristic function of an absolutely continuous distribution function  $V(x)$ .*

Thus, Pólya's criterion provides a sufficient condition for a continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}$  to be a characteristic function. There is however a caveat in order: our admissible kernels do not satisfy the convexity hypothesis of Theorem 3.4. Preliminaries aside, we will now relate positive definiteness to the various Laguerre-type inequalities presented in Section 2. Our first result in this direction shows that if  $\varphi(t)$  is an admissible kernel such that  $\log \varphi(t)$  is strictly concave (i.e.,  $d^2/dt^2 \log \varphi(t) < 0$  for  $t > 0$ ), then for each  $n \in \mathbb{N} \cup 0$  we can associate with  $\varphi(t)$  a (canonical) kernel  $K_n$  which is also an admissible kernel.

**Theorem 3.5.** *Let  $\varphi(t)$  be an admissible kernel. If  $\log \varphi(t)$  is strictly concave for  $t > 0$ , then for each non-negative integer  $n$ , the associated kernel*

$$(3.7) \quad K_n(t) := \int_{-\infty}^{\infty} \varphi(s+t) \varphi(s-t) s^{2n} ds \quad (n = 0, 1, 2, \dots),$$

is also an admissible kernel.

*Proof.* Fix a non-negative integer  $n$ . Consulting Definition 1.2, we readily deduce that  $K_n(t)$  satisfies the properties (i), (ii), (iii) and (v) of Definition 1.2. Thus, it remains to show that  $K_n'(t) < 0$  for  $t > 0$ . Invoking Leibniz's rule to justify the differentiation under the integral, we have

$$(3.8) \quad K_n'(t) = 2 \int_0^\infty [\varphi'(t+s)\varphi(t-s) + \varphi(t+s)\varphi'(t-s)] s^{2n} ds.$$

Next, we fix  $t > 0$  and consider the intervals of integration  $I_1 := (0, t)$  and  $I_2 := (t, \infty)$ . Since  $\log \varphi(t)$  is strictly concave for  $t > 0$ ,  $\frac{\varphi'(t)}{\varphi(t)}$  is strictly decreasing for  $t > 0$ . Hence, for  $s > 0$ , we claim that

$$(3.9) \quad \frac{\varphi'(t+s)}{\varphi(t+s)} < -\frac{\varphi'(t-s)}{\varphi(t-s)}.$$

If  $s \in I_1$ , then  $0 < s < t$  and  $-\varphi'(t-s) > 0$ . Since  $\varphi'(t+s) < 0$ , we see that (3.9) holds. On the other hand, if  $s \in I_2$ , then  $t-s < 0$ . Since  $\varphi(t)$  is an even function,  $\varphi(t-s) = \varphi(s-t)$ . Also,  $0 < s-t < s+t$ , and thus, (3.9) holds, since

$$\frac{\varphi'(t+s)}{\varphi(t+s)} < \frac{\varphi'(s-t)}{\varphi(s-t)} = -\frac{\varphi'(t-s)}{\varphi(t-s)}.$$

□

Following Pólya's work involving Jensen's *Nachlass* ([49, pp 278–308]), we next establish an important relationship between a given strictly logarithmically concave admissible kernel and the associated admissible kernel  $K_n(t)$  defined in (3.7).

**Lemma 3.6.** *If  $\varphi(t)$  is a strictly logarithmically concave admissible kernel for  $t > 0$ , then*

$$(3.10) \quad \int_{-\infty}^\infty \int_{-\infty}^\infty \varphi(t)\varphi(s)e^{ix(s+t)}(s-t)^{2n} dt ds = 2 \cdot 2^{2n} \int_{-\infty}^\infty K_n(v) \cos(2xv) dv, \quad (n = 0, 1, 2, \dots)$$

where  $K_n$  is the associated admissible kernel defined by (3.7).

*Proof.* (A sketch.) Consider the entire function

$$(3.11) \quad F(x) := \int_{-\infty}^\infty e^{itx} \varphi(t) dt.$$

Then, since both  $\varphi(t)$  and  $K_n(t)$  are admissible kernels (Theorem 3.5), the following calculations are valid:

$$(3.12) \quad \begin{aligned} |F(x+iy)|^2 &= F(x+iy)F(x-iy) \\ &= \int_{-\infty}^\infty \int_{-\infty}^\infty \varphi(t)\varphi(s)e^{ix(s+t)}e^{-(s-t)y} dt ds \\ &= \sum_{n=0}^\infty \frac{y^{2n}}{(2n)!} \int_{-\infty}^\infty \int_{-\infty}^\infty \varphi(t)\varphi(s)e^{ix(s+t)}(s-t)^{2n} dt ds. \end{aligned}$$

Next, we use (i) Euler's formula  $e^{ix(s+t)} = \cos(x(s+t)) + i \sin(x(s+t))$ , (ii) the fact that the odd functions integrate to zero and (iii) the absolute value of the Jacobian of the transformation,  $s \rightarrow u+v$  and  $t \rightarrow u-v$ ,

is 2. Accordingly, we obtain

$$\begin{aligned}
& \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s)e^{ix(s+t)}(s-t)^{2n} dt ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s) \cos(x(s+t))(s-t)^{2n} dt ds \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(t)\varphi(s) \cos(x(s-t))(s+t)^{2n} dt ds \\
&= 2 \cdot 2^{2n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi(u+v)\varphi(u-v) \cos(2xv) (u)^{2n} du dv \\
(3.13) \quad &= 2 \cdot 2^{2n} \int_{-\infty}^{\infty} K_n(v) \cos(2xv) dv.
\end{aligned}$$

□

**Theorem 3.7.** *Let  $\varphi(t)$  be a strictly logarithmically concave (for  $t > 0$ ) admissible kernel and let  $K_n$  ( $n = 0, 1, 2, \dots$ ) denote the associated admissible kernel defined by (3.7). Let  $F(x) := \int_{-\infty}^{\infty} e^{itx} \varphi(t) dt$ . Then,*

$$(3.14) \quad L_n(x) := L_n(x; F) := \frac{2 \cdot 2^{2n}}{(2n)!} \int_{-\infty}^{\infty} K_n(t) \cos(2xt) dt, \quad (n = 0, 1, 2, \dots),$$

where  $L_n(x)$  is the generalized real Laguerre expression (cf. (2.7) of Theorem 2.4) for the entire function  $F$ . Moreover,  $F \in \mathcal{L}\text{-}\mathcal{P}$  if and only if  $K_n$  is a positive definite kernel for all  $n = 0, 1, 2, \dots$ .

*Proof.* We recall from Section 2 (see (2.11) of Remark 2.5) that the Taylor coefficient of  $y^{2n}$  in the expansion  $|F(x+iy)|^2$  is precisely  $L_n(x)$ . Hence, (3.14) follows from (3.12) and (3.13). Next, by Theorem 2.4,  $F \in \mathcal{L}\text{-}\mathcal{P}$  if and only if  $L_n(x) \geq 0$  for  $n = 0, 1, 2, \dots$  and for all  $x \in \mathbb{R}$ . By Theorem 3.3 and (3.14),  $L_n(x) \geq 0$  for  $n = 0, 1, 2, \dots$ , and for all  $x \in \mathbb{R}$ , if and only if the kernel  $K_n$  ( $n = 0, 1, 2, \dots$ ) is positive definite. □

**Remarks 3.8.** (a) In [41, Satz 15], Mathias has proved that the kernel  $K_0(t) = \int_{-\infty}^{\infty} \varphi(s+t)\varphi(s-t) ds$  is positive definite. This is clear in our setting, since  $L_0(t) = |F(t)|^2$ . The case when  $n = 1$ ; that is,

$$K_1(t) = \int_{-\infty}^{\infty} \varphi(s+t)\varphi(s-t) s^2 ds \quad \text{and} \quad L_1(x) = (F'(x))^2 - F(x)F''(x) = 4 \int_{-\infty}^{\infty} K_1(t) \cos(2xt) dt,$$

appears to be much more difficult. (b) The desideratum to characterize Fourier transforms in the Laguerre-Pólya class, in terms of the indicated kernels, is achieved by Theorem 3.7. However, the elusive nature of positive definiteness certainly remains as an issue. (c) It may be noteworthy to remark that in conjunction with the Bochner and Khinchin results (cf. Theorems 3.1 and 3.2), our Theorem 3.7 gives rise to new families of characteristic functions when the kernels  $K_n$  associated with functions  $F \in \mathcal{L}\text{-}\mathcal{P}$  are appropriately normalized.

**Open Problem 3.9.** Characterize the logarithmically concave admissible kernels  $\varphi(t)$  such that the associated admissible kernel  $K_1(t)$  is positive definite.

Striving for simplicity, we propose here another, direct, approach for showing that  $\int_0^{\infty} K_1(t) \cos(xt) dt \geq 0$  for all  $x \in \mathbb{R}$ .

**Proposition 3.10.** *Let  $F(x) := \int_0^{\infty} \varphi(t) \cos xt dt$  and set  $K_1(t) := \int_0^{\infty} \varphi(s+t)\varphi(s-t) s^2 ds$ . Let  $\overline{G}(t) := \int_t^{\infty} K_1(u) du$  and  $A := \overline{G}(0)$ . Then  $K_1(t)$  is positive definite if and only if*

$$(3.15) \quad \int_0^{\infty} \overline{G}(t) \sin xt dt \leq \frac{A}{x} \quad \text{for all } x \neq 0.$$



**Remark 3.11.** Before we prove Proposition 3.10, we recall that Pólya's argument [48] shows that in general, the non-negativity of the Fourier sine transform is easier to demonstrate than that of the Fourier cosine transform. Indeed, consider the function  $\overline{G}(t)$  defined in Proposition 3.10. Then for each fixed  $x > 0$ ,

$$\begin{aligned} I(x) &:= \int_0^\infty \overline{G}(t) \sin xt \, dt = \sum_{k=0}^\infty \int_{\pi k/x}^{\pi(k+1)/x} \overline{G}(t) \sin xt \, dt && \left( t = s + \frac{\pi k}{x} \right) \\ &= \sum_{k=0}^\infty \int_0^{\pi/x} \overline{G}\left(s + \frac{\pi k}{x}\right) \sin(xs + \pi k) \, ds \\ &= \sum_{k=0}^\infty (-1)^k \int_0^{\pi/x} \overline{G}\left(s + \frac{\pi k}{x}\right) \sin(xs) \, ds. \end{aligned}$$

Since  $\overline{G}(s) > 0$ ,  $\overline{G}'(s) < 0$  ( $s > 0$ ), and  $\overline{G}(s) \rightarrow 0$  as  $s \rightarrow \infty$ , it follows from the alternating series test that  $I(x) > 0$  for  $x > 0$ .

*Proof of Proposition 3.10.* Integration by parts yields,

$$\begin{aligned} \int_0^\infty K_1(t) \cos(xt) \, dt &= \int_0^\infty K_1(u) \, du - x \int_0^\infty \left( \int_t^\infty K_1(u) \, du \right) \sin xt \, dt \\ &= A - \int_0^\infty \overline{G}(t) \sin xt \, dt \end{aligned}$$

and whence inequality (3.15) follows if and only if  $K_1(t)$  is positive definite.  $\square$

We conclude this section with a concrete example which demonstrates that, if  $K_n$  is positive definite, then in general,  $K_{n+1}$  need not be positive definite. There are several ways we can illustrate this fact. The kernel we will use is a Gaussian,  $e^{-t^2}$ , times a polynomial and therefore it will not satisfy condition (v) of Definition 1.2. Nevertheless, our choice facilitates the exact evaluation of the required integrals. The calculations are sufficiently involved, albeit elementary, to warrant the use of a computer.

**Example 3.12.** Let  $\varphi(t) := e^{-t^2}(15 + t^2 + t^4)$ . Then it is easy to confirm that  $\varphi(t)$  satisfies conditions (i)–(iv) (but not (v)) of Definition 1.2. In addition,  $\log(\varphi(t))$  is strictly concave for  $t > 0$ . In the subsequent calculations, we will denote by  $c_j$ ,  $j \geq 1$ , a positive constant whose exact value is irrelevant. Then  $F(x) = \int_{-\infty}^\infty \varphi(t) \cos xt \, dt = c_1 e^{-x^2/4}(260 - 16x^2 + x^4)$ . Since  $F(x) > 0$ ,  $F$  has 4 non-real zeros (i.e.,  $F \notin \mathcal{L}\text{-}\mathcal{P}$ ) and whence by Theorem 3.7 at least one of the kernels  $K_n$  (cf. (3.7)) fails to be positive definite. Since  $\int_{-\infty}^\infty K_1(t) \cos 2xt \, dt = c_2 e^{-x^2/2}(84240 - 13536x^2 + 712x^4 - 24x^6 + x^8) > 0$  for all  $x \in \mathbb{R}$ ,  $K_1$  is (strictly) positive definite. On the other hand,  $\int_{-\infty}^\infty K_2(t) \cos 2xt \, dt = c_3 e^{-x^2/2}(107088 - 18496x^2 + 696x^4 - 16x^6 + x^8)$  has 4 simple real zeros and consequently  $K_2$  is *not* positive definite.

#### 4. Scholia: the Jacobi Theta Function and the Riemann $\xi$ -function

The purpose of this section is three-fold: (i) to investigate the properties of the Jacobi theta function (cf. (4.2)) and related kernels, (ii) apply the results of Section 3 (Theorem 3.5 and Theorem 3.7) and provide new necessary and sufficient conditions for  $H(x) := \xi(x/2)/8 \in \mathcal{L}\text{-}\mathcal{P}$  (cf. (4.1)), and (iii) formulate some open problems involving kernels associated with the Jacobi theta function.

By way of background information, we commence with Riemann's definition of his  $\xi$ -function ([49, p. 10]); that is,

$$\xi(iz) := \frac{1}{2} \left( z^2 - \frac{1}{4} \right) \pi^{-z/2-1/4} \Gamma\left(\frac{z}{2} + \frac{1}{4}\right) \zeta\left(z + \frac{1}{2}\right),$$

Then it is known ([49 p. 11]), [57, p. 255] or [52, p. 286]) that  $\xi(x)$  admits the integral representation of the form

$$(4.1) \quad H(x) := \frac{1}{8}\xi\left(\frac{x}{2}\right) := \int_0^\infty \Phi(t) \cos(xt) dt,$$

where the *Jacobi theta function*, (without the usual factor 4) is defined as

$$(4.2) \quad \Phi(t) := \sum_{n=1}^{\infty} \pi n^2 (2\pi n^2 e^{4t} - 3) \exp(5t - \pi n^2 e^{4t}).$$

The Riemann Hypothesis is equivalent to the statement that all the zeros of  $H(x)$  are real (cf. [57, p. 255]). We also recall that  $H(x)$  is an entire function of order one ([57, p. 16]) of maximal type (cf. [19, Appendix A]). Thus, with the above nomenclature (cf. Section 1) the Riemann Hypothesis is true if and only if  $H \in \mathcal{L}\text{-}\mathcal{P}$ . It is also known ([30, p. 7]) that all the zeros of  $H$  lie in the *interior* of the strip  $S(1)$ , so that  $H(x) \in \mathfrak{G}(\tau)$ , with  $\tau = 1$  and that  $H(x)$  has an infinite number of real zeros [57, p. 256]. Before we begin with a synopsis of results, we emphasize that the *raison d'être* for investigating the kernel  $\Phi$  is that there is an intimate connection (the precise meaning of which is yet unknown) between the properties of  $\Phi$  and the distribution of the zeros of its Fourier transform  $H(x)$  (cf. (4.1)).

**Theorem 4.1.** ([16, Theorem A]) *Consider the function  $\Phi$  of (4.2) and set*

$$\Phi(t) = \sum_{n=1}^{\infty} a_n(t), \quad \text{where } a_n(t) := \pi n^2 (2\pi n^2 e^{4t} - 3) \exp(5t - \pi n^2 e^{4t}) \quad (n = 1, 2, \dots).$$

*Then, the following are valid:*

- (i) *for each  $n \geq 1$ ,  $a_n(t) > 0$  for all  $t \geq 0$ , so that  $\Phi(t) > 0$  for all  $t \geq 0$ ;*
- (ii)  *$\Phi(z)$  is analytic in the strip  $-\pi/8 < \text{Im } z < \pi/8$ ;*
- (iii)  *$\Phi$  is an even function, so that  $\Phi^{(2m+1)}(0) = 0$  ( $m = 0, 1, \dots$ );*
- (iv) *for any  $\varepsilon > 0$ ,  $\lim_{t \rightarrow \infty} \Phi^{(n)}(t) \exp[(\pi - \varepsilon)e^{4t}] = 0$ ;*
- (v)  *$\Phi'(t) < 0$  for all  $t > 0$ .*

The proofs of statements (i) – (iv) can be found in G. Pólya [49], whereas the proof of (v) is in A. Wintner [58] (see also Spira [54]).  $\square$

In order to indicate the significance of the next theorem, we consider the Taylor series of  $H(x)$  about the origin

$$(4.3) \quad H(z) = \sum_{k=0}^{\infty} \frac{(-1)^k b_k}{(2k)!} z^{2k}, \quad \text{where } b_k := \int_0^\infty t^{2k} \Phi(t) dt \quad (k = 0, 1, 2, \dots).$$

The change of variable,  $z^2 = -x$  in (4.3), yields the entire function

$$(4.4) \quad F(x) := \sum_{k=0}^{\infty} \frac{\gamma_k}{k!} x^k, \quad \text{where } \gamma_k := \frac{k! b_k}{(2k)!} > 0 \quad (k = 0, 1, 2, \dots).$$

Then it is easy to see that  $F(x)$  is an entire function of order  $\frac{1}{2}$  and that the Riemann Hypothesis is equivalent to the statement that all the zeros of  $F(x)$  are real and negative. Now it is known (Pólya and Schur [51]) that a *necessary condition* for  $F(x)$  to have only real zeros is that the moments  $b_k$  (in (4.3)) satisfy the *Turán inequalities*; that is,

$$(4.5) \quad b_k^2 - \frac{2k-1}{2k+1} b_{k-1} b_{k+1} \geq 0 \quad \text{or equivalently } T_k := \gamma_k^2 - \gamma_{k-1} \gamma_{k+1} \geq 0 \quad (k = 1, 2, 3, \dots).$$

These inequalities have been established (cf. [16], for  $m \geq 2$ , and [21]) as a consequence of either one of the two concavity properties ((a) or (b)) of  $\Phi$  stated in the following theorem. (For related interesting results see also D. K. Dimitrov and F. R. Lucas [29] and D. K. Dimitrov [27], [28]).

**Theorem 4.2.** *Let  $\Phi$  be defined by (4.1). Then  $\Phi$  satisfies the following concavity properties.*

(a) ([16, Proposition 2.1]) *If*

$$K_{\Phi}(t) := \int_t^{\infty} \Phi(\sqrt{u}) du \quad (t \geq 0),$$

*then  $\log K_{\Phi}(t)$  is strictly concave for  $t > 0$ ; that is,  $\frac{d^2}{dt^2} \log K_{\Phi}(t) < 0$  for  $t > 0$ .*

(b) ([21, Theorem 2.1]) *The function  $\log \Phi(\sqrt{t})$  is strictly concave for  $t > 0$ .  $\square$*

**Remarks 4.3.** (a) A calculation shows that  $\log \Phi(\sqrt{t})$  is strictly concave for  $t > 0$  if and only if  $g(t) := t[(\Phi'(t))^2 - \Phi(t)\Phi''(t)] + \Phi(t)\Phi'(t) > 0$  for  $t > 0$ . Since  $\Phi(t) > 0$  and  $\Phi'(t) < 0$  for  $t > 0$ , it is easy to check that the inequality  $g(t) > 0$  is stronger than the assertion that  $\log(\Phi(t))$  is strictly concave for  $t > 0$ . Indeed, the inequality  $\Phi'(t)^2 - \Phi(t)\Phi''(t) > 0$  does not imply, in general, the Turán inequalities (4.5) (see, for example, [5, Example 3.4]).

(b) Since  $\Phi(t) > 0$  and  $\Phi'(t) < 0$  for  $t > 0$ , we can also demonstrate that the “average value” of  $H(x)$ , the Fourier cosine transform of  $\Phi$  (cf. (4.1)), is positive. Indeed, for  $t > 0$ ,

$$\int_0^t H(u) du = \int_0^{\infty} \Phi(x) \left( \int_0^t \cos xu du \right) dx = \int_0^{\infty} \Phi(x) \frac{\sin xt}{x} dx > 0,$$

where the last inequality can be established using the method of proof presented in Remark 3.11. We pause for a moment, and append here yet another convexity result involving  $\Phi$ .

**Theorem 4.4.** ([11, pp 43–44]) *The function  $\Phi(\sqrt{t})$  is strictly convex for  $t > 0$ ; (that is,  $\frac{d^2}{dt^2} \Phi(\sqrt{t}) > 0$  for  $t > 0$ ) and hence*

$$\int_0^{\infty} \Phi(\sqrt{t}) \cos xt dt > 0 \quad \text{for all } x \in \mathbb{R}.$$

Having reviewed some of the salient properties of the Jacobi theta function, we are now in position to apply the results of Section 3.

**Theorem 4.5.** *The Jacobi theta function,  $\Phi(t)$ , is a strictly logarithmically concave admissible kernel. Moreover, the associated kernel*

$$(4.6) \quad K_n(t) := K_n(t; \Phi) := \int_{-\infty}^{\infty} \Phi(s+t)\Phi(s-t)s^{2n} ds \quad (n = 0, 1, 2, \dots),$$

*is also an admissible kernel.*

*Proof.* By Theorem 4.1,  $\Phi$  is an admissible kernel. Now, it follows from Theorem 4.2 and Remarks 4.3 (a) that  $\log \Phi(t)$  is strictly concave for  $t > 0$ . Thus, by Theorem 3.5, for each non-negative integer  $n$ , the associated kernel  $K_n(t) := K_n(t; \Phi)$  is also an admissible kernel.  $\square$

Finally, with the aid of Lemma 3.6, Theorem 3.5 and Theorem 3.7, we obtain the following equivalent formulation of the Riemann Hypothesis.

**Theorem 4.6.** *Let  $K_n := K_n(t; \Phi)$  ( $n = 0, 1, 2, \dots$ ) denote the associated admissible kernel defined by (4.6). Let  $H(x) := \int_0^{\infty} \Phi(t) \cos xt dt$ . Then, for  $n = 0, 1, 2, \dots$ ,*

$$(4.7) \quad L_n(x) := L_n(x; H) := \frac{2 \cdot 2^{2n}}{(2n)!} \int_{-\infty}^{\infty} K_n(t) \cos(2xt) dt,$$

where  $L_n(x)$  is the generalized real Laguerre expression (cf. (2.7) of Theorem 2.4) for the entire function  $H$ . Moreover,  $H \in \mathcal{L}\text{-}\mathcal{P}$  if and only if  $K_n$  is a positive definite admissible kernel for all  $n = 0, 1, 2, \dots$ .

At this juncture, we are obliged to expose our ignorance and state the following tantalizing open problem.

**Open Problem 4.7.** (One of the simplest Laguerre inequalities for the Riemann  $\xi$ -function.) Let  $\Phi$  denote the Jacobi theta function and let  $H(x) := \xi(x/2)/8 = \int_0^\infty \Phi(t) \cos xt \, dt$ . Then, is it true that

$$(4.8) \quad L_1(x) = (H'(x))^2 - H(x)H''(x) \geq 0 \quad \text{for all } x \in \mathbb{R}?$$

**Remark 4.8.** The verification of the special Laguerre inequality (4.8) itself would be significant. If we could prove that  $L_1(x) > 0$  for all real  $x$ , then it would follow that all the real zeros of  $H$  are *simple*. Of course, should inequality (4.8) fail to hold for some  $x_0$ , then the Riemann Hypothesis would be false. Now it follows from the numerical results of van de Lune, te Riele, and Winter [37] that the zeros of  $H(x)$  are real and simple for  $|x| < 1.09 \cdots \times 10^9$  and whence, by Proposition 2.2,  $L_1(x) > 0$  for  $|x| < 1.09 \cdots \times 10^9$ .

Open Problem 4.7 need not be construed as an insurmountable barrier for further research. Indeed, in the interest of new investigations, we propose here a variant of the *Pólyaesque* approach: namely, if you cannot solve a problem change it (for example, generalize it). In this spirit, we mention that in the study of the distribution of zeros of entire functions  $f(x) \in \mathfrak{S}(\tau)$  (of order  $< 2$ ) under the action of the operator  $e^{-tD^2}$ , ( $D := d/dx$ ) there is a simple heuristic principle formulated by Pólya. If  $t > 0$ , then under the action of  $e^{-tD^2}$  the zeros of  $f(x)$  tend to be “attracted” to the real axis, while under the action of  $e^{tD^2}$  the zeros of  $f$  tend to be repelled by the real axis. Guided by this principle, we apply  $e^{-tD^2}$  to the Riemann  $\xi$ -function (see (4.1)). For convenience and to adhere to the notation employed in the papers cited below, we set  $H(x) := H_0(x) := \xi(x/2)/8$ . Let

$$(4.9) \quad H_t(x) = e^{-tD^2} H(x) = \int_0^\infty e^{ts^2} \Phi(s) \cos(xs) \, ds \quad \left( t \in \mathbb{R}; x \in \mathbb{C}, D := \frac{d}{dx} \right).$$

In 1950, de Bruijn [2] established that (i)  $H_t(x)$  has only real zeros for  $t \geq 1/2$  (this is a consequence of the fact that  $H \in \mathfrak{S}(\tau)$ , with  $\tau = 1$ , and that  $\cos(tD)H \in \mathcal{L}\text{-}\mathcal{P}$  for all  $t \geq 1$ ) and (ii) if  $H_t(x)$  has only real zeros for some real  $t$ , then  $H_{t'}(x)$  also has only real zeros for any  $t' \geq t$ . Subsequently, C. M. Newman [42] showed, in 1976, that there is a real constant  $\Lambda$ , which satisfies  $-\infty < \Lambda \leq 1/2$ , such that  $H_t$  has only real zeros if and only if  $t \geq \Lambda$ . This constant  $\Lambda$  is now called the *de Bruijn-Newman constant* in the literature, and the Riemann Hypothesis is equivalent to the statement that  $\Lambda \leq 0$ . Recently, A.M. Odlyzko, W. Smith, R. S. Varga and the author [17] have shown that  $-5.895 \cdot 10^{-9} < \Lambda$  (see also [23]).

Differentiation under the integral sign in equation (4.9) (which can be readily justified by Leibniz’s rule, see also [9]) shows that  $H_t(x)$  satisfies the backward heat equation:

$$(4.10) \quad \frac{\partial(H_t(x))}{\partial t} = -\frac{\partial^2 H_t(x)}{\partial x^2}.$$

This observation is the key ingredient in the proof of the following proposition.

**Proposition 4.9.** ([18, Proposition 1]) *Suppose that  $H_{t_0}$  has a multiple real zero. Then  $t_0 \leq \Lambda$ . In particular, if  $t > \Lambda$ , then the zeros of  $H_t$  are real and simple.*

We next consider two open problems involving  $H_\lambda(x)$  and the “new” kernels  $\Phi_\lambda(t) := e^{\lambda t^2} \Phi(t)$  when (i)  $\lambda < 0$  and when (ii)  $\lambda > 0$ .

**Open Problem 4.10.** Fix  $\lambda < 0$ . Using the theory of positive definite kernels (see Section 3) show that for some non-negative integer  $n$ , the kernel

$$(4.11) \quad K_n(t) := K_n(t; \Phi_\lambda) := \int_{-\infty}^\infty \Phi_\lambda(s+t) \Phi_\lambda(s-t) s^{2n} \, ds, \quad \text{is not positive definite.}$$

Secondly, assume that  $\lambda > 0$ . In this case, the factor  $e^{\lambda s^2}$  under the integral sign (cf. (4.9)) is an example of a function that Pólya termed an *universal factor* (see the beautiful papers by Pólya [50] and de Bruijn [2]). Universal factors preserve the Laguerre - Pólya class. In 2009, H. Ki, Y.-O. Kim and J. Lee [36] proved that for every fixed  $\lambda > 0$  all but a finite number of the zeros of  $H_\lambda$  are real and simple. Thus, in particular, if  $\lambda > 0$ , then  $H_\lambda \in \mathcal{L}\text{-}\mathcal{P}^*$  (see Definition 1.1). Now, in 1987, T. Craven, W. Smith and the author proved the Pólya-Wiman Conjecture [10] (for a more elegant proof see H. Ki and Y.-O. Kim [35]); namely, if  $f(x) \in \mathcal{L}\text{-}\mathcal{P}^*$ , then there is a positive integer  $m_0$  such that  $f^{(m)}(x) \in \mathcal{L}\text{-}\mathcal{P}$  for all  $m \geq m_0$ . Therefore, it follows from the aforementioned results that for each fixed  $\lambda > 0$ , there is a positive integer  $m_0 = m_0(\lambda)$  such that for  $2m \geq m_0$  (we work with an even integer so that the new kernel is also even)

$$(4.12) \quad H_\lambda^{(2m)}(x) = \frac{d^{2m}}{dx^{2m}} e^{-\lambda D^2} H(x) = \int_0^\infty s^{2m} e^{\lambda s^2} \Phi(s) \cos(xs) ds \in \mathcal{L}\text{-}\mathcal{P}.$$

Observe that the new kernel  $s^{2m}\Phi_\lambda(s) = s^{2m}e^{\lambda s^2}\Phi(s)$ , ( $s > 0$ ), is not monotone decreasing, it is not logarithmically concave and it tends to 0 (as  $s \rightarrow \infty$ ) a “little” slower than  $\Phi$ .

**Open Problem 4.11.** With the above notation and assumptions, is the kernel

$$K_1(t; \Phi_\lambda, m) := \int_{-\infty}^\infty \Phi_\lambda(s+t)\Phi_\lambda(s-t)(s^2-t^2)^m s^2 ds, \quad \text{positive definite?}$$

We conclude this paper with three additional open problems.

**Open Problem 4.12.** Characterize the admissible kernels whose Fourier transforms have all their zeros located in the strip  $S(1)$ .

**Open Problem 4.13.** ([5, Conjecture 2.5]) Show that the derivatives of the Jacobi theta function,  $\Phi(t)$ , are (strictly) log-concave on  $\mathbb{R}$ ; that is, for each  $n \in \mathbb{N}$ ,

$$(4.13) \quad J_n(t) := (\Phi^{(n)}(t))^2 - \Phi^{(n-1)}(t)\Phi^{(n+1)}(t) > 0 \quad \text{for } t \in \mathbb{R}.$$

Since  $\Phi(t)$  is an even function (cf. Theorem 4.1),  $J_n(t)$  is even and whence it suffices to establish (4.13) for  $t \geq 0$ .

Consider again the entire function  $F$  (cf. (4.4)) related to the Riemann  $\xi$ -function:  $F(x) := \sum_{k=0}^\infty \frac{k! \gamma_k}{(2k)!} \frac{x^k}{k!}$ , where  $\gamma_k := \frac{k! b_k}{(2k)!}$ , ( $k = 0, 1, 2, \dots$ ). Let  $T_k := \gamma_k^2 - \gamma_{k-1}\gamma_{k+1} \geq 0$ , ( $k = 1, 2, 3, \dots$ ), and  $E_k := T_k^2 - T_{k-1}T_{k+1}$  for  $k = 2, 3, 4, \dots$ . Then a necessary condition for the Riemann Hypothesis to hold is that the *double Turán inequalities* should hold; i.e.,  $E_k \geq 0$  for  $k = 2, 3, 4, \dots$ . In [14, Theorem 2.4], we derived a concavity condition (for an admissible kernel) which implies the double Turán inequalities (see also [27, 28]). Thus, an affirmative answer to the following conjecture, will establish yet another necessary condition for the validity of the Riemann Hypothesis.

**Open Problem 4.14.** [14, Problem 3.3] (A new concavity condition of  $\Phi(t)$ .) Let  $s(t) := \Phi(\sqrt{t})$  and set  $f(t) := s'(t)^2 - s(t)s''(t)$ . By Theorem 4.2 (b),  $f(t) > 0$  for  $t > 0$ . Then we conjecture that

$$\frac{d^2}{dt^2} \log f(t) < 0 \quad \text{for } t > 0.$$

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