A κ-Krein space $H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$ based mechanical and dynamical quanta energy field model

Supporting mathematics

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The Hilbert scale $\{H_\alpha | \alpha \in R\}$

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$ and the norm $\|..\|$. Let A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A).$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have :

there is a countable sequence $\{\lambda_i, \phi_i\}$ with

$$A\phi_i = \lambda_i \phi_i \ , \ (\phi_i, \phi_k) = \delta_{i,k} \quad \text{and} \quad \lim_{i\to\infty} \lambda_i \to \infty$$

any $x \in H$ is represented by

$$(*) \quad x = \sum_{i=1}^{\infty} (x, \phi_i)\, \phi_i \ \text{ and } \ \|x\|^2 = \sum_1^{\infty} (x, \phi_i)^2.$$

Similarly one can define the spaces H_α, where the case $\alpha < 0$ is related to the theory of distributions. They consist of those elements $x \in H$ with scalar product

$$(x, y)_\alpha = \sum_i^{\infty} \lambda_i^\alpha (x, \phi_i)\,(y, \phi_i) = \sum_i^{\infty} \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x, x)_\alpha .$$

Because of the Parseval identity we have especially

$$(x, y)_0 = (x, y)$$

and because of (*) it holds

$$\|x\|_2^2 = (Ax, Ax)_0 \, , H_2 = D(A).$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices, (NiJ), (NiJ1):

Lemma :

i) Let $\alpha < \beta$. Then $\|x\|_\alpha \leq \|x\|_\beta$ for $x \in H_\beta$ and the embedding $H_\beta \to H_\alpha$ is compact.

ii) Let $\alpha < \beta < \gamma$. Then $\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu$ for $x \in H_\gamma$ with $\mu = \frac{\gamma-\beta}{\gamma-\alpha}$ and $\nu = \frac{\beta-\alpha}{\gamma-\alpha}$.

iii) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

iv) $\|x - y\|_\alpha \leq t^{\beta-\alpha}\|x\|_\beta$

v) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

vi) $\|y\|_\gamma \leq t^{-(\gamma-\beta)}\|x\|_\beta$

Lemma:

i) Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

ii) $\|x - y\|_\rho \leq t^{\beta-\rho}\|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

iii) $\|y\|_\sigma \leq t^{-(\sigma-\beta)}\|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Eigen-functions and Eigen-differentials

Let H be a (infinite dimensional) Hilbert space with inner product $(.,.)$, the norm $\|\cdot\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1} is compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$

$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \phi_\mu(\phi_\mu,*)d\mu \quad , \quad \mu \in [\lambda_0, \infty) \, ,$$

i.e.

$$dE_\lambda = \phi_\lambda(\phi_\lambda,*)d\lambda \, .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ. The set E_λ fulfills the following properties:

E_λ is a projection operator for all $\lambda \in R$

for $\lambda \leq \mu$ it follows $E_\lambda \leq E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

$\lim\limits_{\lambda \to -\infty} E_\lambda = 0$ and $\lim\limits_{\lambda \to \infty} E_\lambda = Id$

$\lim\limits_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f:[a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f: H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) \, .$$

Symbolically one writes $A = \int_{-\infty}^{\infty} \lambda dE_\lambda$. Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \;\; , \;\; d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax,y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \qquad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2x,y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \, , \;\; \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \, .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function. It holds the following eigen-pair relations

$$A\phi_i = \lambda_i \phi_i \quad A\phi_\lambda = \lambda\phi_\lambda \quad \|\phi_\lambda\|^2 = \infty \, , (\phi_\lambda, \phi_\mu) = \delta(\phi_\lambda - \phi_\mu).$$

The ϕ_λ are not elements of the Hilbert space. The so-called eigen-differentials, which play a key role in quantum mechanics, are built as superposition of such eigen-functions.

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\phi_\lambda(x) = \lambda\phi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\phi_\lambda(x)$ it results into elements of H. The practical way to find eigen-differentials is looking for solutions of a distribution equation.

The extended Hilbert space $H_{\alpha.(\tau)}$
(NiJ), (NiJ1)

The extended Hilbert space $H_{\alpha.(\tau)}$ is defined by the following inner product resp. norm

$$(x,y)_{(\tau)} = \sum_{i=1} e^{-\sqrt{\lambda_i}\tau}(x,\phi_i)(y,\phi_i), \;\; \|x\|^2_{(\tau)} = (x,x)_{(\tau)}.$$

The (τ)-norm is weaker than any α-norm, i.e.

$$\|x\|^2_{(\tau)} \leq c\|x\|^2_\alpha \quad \text{for any } \alpha\text{-norm}$$

with $c = c(\alpha,\tau)$ depending only on α and τ.

The counterpart of the related lemmata of the considered Hilbert scale is

Lemma: Let $\tau, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_\tau(x)$ according to

$$\|x-y\| \quad \leq \|x\|$$

$$\|y\|_1 \leq \delta^{-1}\|x\|$$

$$\|x-y\|_{(\tau)} \leq e^{-\tau/\delta}\|x\|.$$

Any Hilbert scale norm with negative index, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the 0-norm and the newly introduced (τ)-norm:

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|^2_{-\alpha} \leq \delta^{2\alpha}\|x\|^2_0 + e^{\tau/\delta}\|x\|^2_{(\tau)}$$

with $\delta > 0$ being arbitrary.

Proof: The inequality is a consequence of the following inequality

$$\lambda^{-\alpha} \leq \delta^{2\alpha} + e^{\tau(\delta^{-1}-\sqrt{\lambda})}, \text{ for any } \tau,\delta,\alpha > 0 \text{ and } \lambda \geq 1.$$

If $\lambda^{-1/2} \leq \delta$ then obviously $\lambda^{-\alpha} \leq \delta^{2\alpha}$, in case of $\lambda^{-1/2} \geq \delta$ it holds $e^{\tau(\delta^{-1}-\sqrt{\lambda})} \geq 1$, whereas $\lambda^{-\alpha} \leq 1$ is a consequence of $\alpha > 0$ and $\lambda \geq 1$.

Putting $\delta = \frac{1}{\theta}$ and $\lambda = \vartheta^2 \geq 1$ it follows from the lemma above the

Corollary: for any $\tau,\theta,\alpha > 0$ and $\vartheta \geq 1$ the following inequality is valid

$$\vartheta^{-2\alpha} \leq \theta^{-2\alpha} + e^{\tau(\theta-\vartheta)}.$$

Lemma: Because of $\int_0^\infty e^{-\sqrt{\lambda_i}\tau}d\tau = \frac{1}{\sqrt{\lambda_i}}$ it holds

$$\int_0^\infty \|x\|^2_{(\tau)}d\tau = \|x\|^2_{-1/2}.$$

Strong elliptic and hyperbolic PDO

By construction the Hilbert scales characterized by a polynomial decay in case of λ_i^α enables optimal shift theorem for the Laplacian operator in the form, (appendix I)

$$\|x\|_{\alpha+2}^2 = (Ax, Ax)_\alpha = \|Ax\|_\alpha^2.$$

The operator concerned with the time-harmonic Maxwell equation and the radiation problem is the D'Alembert (wave) operator related to the wave equation:

$$\Box w := \ddot{w} - \Delta w\ .$$

The Hilbert space defined by the inner product resp. norm

$$(x,y)_{(t)}^2 = \sum_{i=1}^\infty e^{-\sqrt{\lambda_i}t}(x,\phi_i)(y,\phi_i) \ \ t > 0$$

$$\|x\|_{(t)}^2 = (x,x)_{(t)}^2$$

provides „optimal" shift theorems for related strong hyperbolic operators.

Theorem: For the D'Alembert (wave) operator it holds

$$\int_0^T \|w\|_{k+2,(t)}^2 dt \le c \int_0^T \|f\|_{k,(t)}^2 dt\ .$$

Proof: Let $w_i := (w, \phi_i)$ resp. $f_i := (f, \phi_i)$ being the generalized Fourier coefficient related to the eigen-pairs $-w_i'' = \lambda_i w_i$ of the Laplacian operator. Th corresponding solution of ($\Box w = f$),

$$\ddot{w}_i(t) + \lambda_i w_i(t) = f_i(t) \ \text{ and } w_i(0) = \dot{w}_i(0) = 0\ .$$

is given by

$$w_i(t) = \frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\right) f_i(\tau) d\tau.$$

It holds for $\tau \le t$

$$\int_0^T \|w\|_{k+2,(t)}^2 dt = \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} w_i^2(t) dt$$

$$\le \sum \lambda_i^{k+2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\frac{1}{\sqrt{\lambda_i}} \int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\right) f_i(\tau) d\tau\right]^2 dt$$

$$\le \sum \lambda_i^{k+1} \int_0^T e^{-\sqrt{\lambda_i}t} \left(\int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\right) d\tau\right) \left[\int_0^t \sin\left(\sqrt{\lambda_i}(t-\tau)\right) d\tau\, f_i^2(\tau) d\tau\right] dt$$

$$\le \sum \lambda_i^{k+1/2} \int_0^T e^{-\sqrt{\lambda_i}t} \left[\int_0^t f_i^2(\tau) d\tau\right] dt\ .$$

Exchanging the order of integration gives

$$\int_0^T \int_0^t e^{-\sqrt{\lambda_i}t} f_i^2(\tau) d\tau dt = \int_0^T \int_t^T e^{-\sqrt{\lambda_i}t} f_i^2(\tau) dt d\tau$$

$$= \int_0^T f_i^2(\tau) dt \left[\int_t^T e^{-\sqrt{\lambda_i}t}\, d\tau\right]$$

$$\le \frac{1}{\sqrt{\lambda_i}} \int_0^T f_i^2(\tau) dt\ .$$

Theorem: In general there exists no „optimal" hyperbolic shift theorem in the standard Sobolev space framework in the form

$$\|w\|_{k+2}^2 \leq c\|f\|_k^2$$

Proof: the counter example is given by the function

$$\Phi(x,t) := e^{-(\frac{1}{2}-(x-t))^2}, u(x,t) := t^2\Phi(x,t), f(x,t) := 2\Phi(x,t) - 4t\Phi'(x,t)$$

fulfilling the relationships

$$\dot{\Phi}(x,t) = -\Phi'(x,t), \ddot{\Phi}(x,t) = \Phi''(x,t), \ddot{u}(x,t) - u''(x,t) = f(x,t)$$

and

$$\|u''\|_{L_2(L_2)} \sim \|\Phi''\|_{L_2(L_2)} \quad \text{but} \quad \|f\|_{L_2(L_2)} \sim \|\Phi'\|_{L_2(L_2)}.$$

Calculus of variations: the energy method

(VeW) p. 44

Let E denote a linear space, and U a linear subspace of E. We consider the boundary value problem as operator equation in the half-homogeous form

$$Au = f, u \in U$$

with a solution $\bar{u} \in U$. Additionally we assume

i) $(Au, v) = (u, Av), \quad \forall u, v \in U$

ii) $(Au, u) > 0, \qquad \forall u \in U, u \neq 0.$

This means, that the operator $A : U \to E$ is symmetric and positive. Then it follows that $Au = 0$ in U posseses only the solution $u = 0$, i.e., $\bar{u} \in U$ becomes the unique solution of $Au = f$.

Obviously, by

$$[u, v] := (Au, v), \; |||u||| := (Au, u)^{1/2}$$

There is an additonal inner product defined in U accompanied by an additional corresponding norm, which is denoted as „energy norm“ (in applications this norm often represents the physical notions „work“ or „energy“) Correspondingly, the inner product $[\cdot,\cdot]$ is called energetic inner product.

We now consider the so-called energy functional

$$I(u) = (Au, u) - 2(f, u).$$

As above, $\bar{u} \in U$ denotes the solution of $Au = f, u \in U$. Then it holds for all $u \in U$ it holds

$$I(u) = |||u - \bar{u}|||^2 - |||\bar{u}|||^2.$$

For the right side it holds

$$|||u - \bar{u}|||^2 - |||\bar{u}|||^2 = |||u|||^2 - 2[\bar{u}, u] = (Au, u) - 2(f, u) = I(u).$$

From $I(u) = |||u - \bar{u}|||^2 - |||\bar{u}|||^2$ it follows that $I(\bar{u}) = -|||\bar{u}|||^2$ and $I(u) = I(\bar{u}) + |||u - \bar{u}|||^2$.

Therefore, it holds $I(u) > I(\bar{u})$ für $u \neq \bar{u}$. In summary this means

Theorem: The operator equation $Au = f, u \in U$, is equivalent to the extrmal problem

$$I(u) \to min, u \in U.$$

The characterization of the solution $\bar{u}$ as a solution of the extremal problem defined by the energy functional is called the energy method.

Quadratic extremal problems for linear variational equations

(VeW) p. 48

Let E denote a linear space, and U a linear subspace of E. Additionally, let $l(\cdot) : E \rightarrow R$ denote a linear functional, and $a(\cdot,\cdot) : E \times E \rightarrow R$ a bilinear form with the following properties

1. $a(u,v) = a(v,u), \ \forall u,v \in E$
2. $a(u,u) \geq 0, \qquad \forall u \in E$
3. $a(u,u) > 0, \qquad \forall u \in U, u \neq 0$

Then, by $|||u||| := a(u,u)^{1/2}$ there is a half-norm defined in E, which is a norm in U (again called energy norm).

The extremal problem: For a given $u_0 \in E$ we look for a $u \in E$ as a solution of

$$J(u) = a(u,u) - 2l(u) \rightarrow min, \ u - u_0 \in U.$$

In order to enable the existence of such a solution it requires additional assumptions. However, the uniqueness of such a solution is guaranteed. Besides, the extremal problem is equivalent to the variational equation in the form

$$a(u,\varphi) = l(\varphi) \ \forall \varphi \in U, \ u - u_0 \in U.$$

The generalizations for physical relevant problems (Boltzmann equations, NSE equations, ..) is accompanied by the constructio of an operator-algebra consistent of integral and differential operators, leading to the concept of pseudo-differential operators. The counterpart of the symmetric and positive linear operator (accompanied by the energy norm) is Garding's inequality for strong elliptic pseudo-differential operators. In simple words, there is no conceptually difference regarding the application of the „energy method" for nonlinear strong elliptic or strong hyperbolic pseudo-differential operators. The non-linear terms of such operators may be interpreted as compact disturbances of the linear operator, defining the energy norm.

Non-linear minimization problems

Non-linear minimization problems can be analyzed as saddle point problems on convex manifolds in the following form (VeW):

$$(*) \quad J(u): a(u,u) - F(u) \rightarrow min \,, \quad u - u_0 \in U.$$

Let $a(\cdot,\cdot) : \ V \times V \rightarrow R$ a symmetric bilinear form with energy norm $\|u\|^2 := a(u,u)$. Let further $u_0 \in V$ and $F(\cdot): V \rightarrow R$ a functional with the following properties:

$F(\cdot): V \rightarrow R$ is convex on the linear manifold $u_0 + U$, i.e. for every $u,v \in u_0 + U$ it holds $F((1-t)u + tv) \leq (1-t)F(u) + tF(v)$ for every $t \in [0,1]$

$$F(u) \geq \alpha \text{ for every } u \in u_0 + U$$

$F(\cdot): V \rightarrow R$ is Gateaux differentiable, i.e. it exits a functional $F_u(\cdot): V \rightarrow R$ with

$$\lim_{t \to 0} \frac{F(u+tv) - F(v)}{t} = F_u(v).$$

Then the minimum problem (*) is equivalent to the variational equation

$$a(u,\phi) + F_u(\phi) = 0 \text{ for every } \phi \in U$$

and admits only an unique solution.
In case the sub-space U and therefore also the manifold $u_0 + U$ is closed with respect to the energy norm and the functional $F(\cdot): V \rightarrow R$ is continuous with respect to convergence in the energy norm, then there exists a solution. We note that the energy functional is even strongly convex in whole V.

The Hilbert transform operator & the mean ergotic theorem

Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. Then Hilbert spaces $\{H_\alpha | \alpha \in R\}$ and H_τ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \text{ , } \|x\|_{(\tau)}^2 = \sum_{n=1}^\infty e^{-\sqrt{\lambda_n}\tau} x_n^2,\ x_n := (x, \varphi_n).$$

The Hilbert transform of the orthogonal system $\Phi_n := \varphi_n^H := H[\varphi_n]$, where $(\Phi_n, \varphi_n) = 0$ provides an unitary operator U on those Hilbert spaces and theory Hilbert sub-space.

Mean ergotic theorem (HaP), (HoE): Let U be an isometry on a Hilbert space H; let P be the projection on the space of all x invariant under U, then

$$\frac{1}{n}\sum_{j=0}^{n-1} U^j x \to Px \text{ in a weak } L_2 \text{ sense for all } x \in H.$$

Note: If $x = y - Uy$ for some y, then $\frac{1}{n}\sum_{j=0}^{n-1} U^j x$ is a telescoping sum equal to $y - U^n y$ and $\left\|\frac{1}{n}\sum_{j=0}^{n-1} U^j x\right\| \le \frac{2}{n}\|y\| \to 0$.

Quadratic and complementary „least energy" Riesz-Galerkin methods

Hilbert-Krein space based least energy variational pseudo-differential equation representations enable the full power of quadratic and complementary „least energy" Riesz-Galerkin methods accompanied by FEM, BEM, and wavelet approximation methods, (BrK).

In (NiJ2) an extension of the stndard „inf-sup-condition" in the FEM is provided, in case applications where the underlying Banach spaces coincide and ar the cartesian product of two Hilbert spaces $X = Y = H \times H$.

The construction of an operator algebra consisting of integral and differential operators leads to the concept of pseudo-differential operators. The PDO theory provides the appropriate framework for affected physical differential and (singular) integral equations. In order to apply „least energy" Riesz-Galerkin methods it requires strong elliptic pseudo-differential operators, (BrK1). The hyperbolic wave equation operator (the D'Alembert operator) with domain in a $H_{(\tau)}$ framework defines a strong hyperbolic pseudo-differential operators (BrK1). This allows to revisit the current concept of „wave front sets" of the standard pseudo-differential operator theory, (PeB).

Complementary variational principles and the method of Noble

The method of Noble ((VeW) 6.2.4), (ArA) 4.2), is about two properly defined operator equations, to analyze (nonlinear) complementary extremal problems. The Noble method leads to a "Hamiltonian" function W(·,·) which combines the pair of underlying operator equations (based on the "Gateaux derivative" concept)

Let $(E, \langle , \rangle)$ and $(E', (,))$ be Hilbert spaces and $T: E \to E'$, $T^*: E' \to E$ linear operators fulfilling $(u', Tu) = \langle T^*u', u \rangle$ and let $W: E'xE \to R$ a functional fulfilling

$$T = \frac{\partial W(u',)}{\partial u'} \quad \text{and} \quad T^* = \frac{\partial W(.,u)}{\partial u}$$

i.e., the operators T and T^* are deviations from $W(.,.)$ in the sense of Gateaux, i.e.

$$lim \frac{F(u+tv)-F(v)}{t} = F_u(v) \text{ for all } v \in E \text{ .}$$

Putting $W(u', u) := \frac{1}{2}(u', u') - F(u)$ the minimization problem

$$(*) \qquad J(u) := (Tu, Tu) + 2F(u) \to min \quad , \quad u \in U \subset E$$

leads to $Tu = u'$ and $(T^*u', .) = -F_u(.)$ and therefore to

Lemma A.2 (method of Noble): If $F(.)$ is a convex functional it follows that $W(u', u)$ is convex concerning u' and concave concerning u. The minimization problem (*) is equivalent to the variational equation

$$(v', T\phi) + F_u(\phi) = 0 \text{ for all } \phi \in U \quad \text{resp.} \quad (T^*v', \phi) = -F_u(\phi) \text{ for all } \phi \in U.$$

i.e., there is a characterization of the solution of (*) as a saddle point.

The Hilbert spaces H_α, $H_{(\tau)}$, $H_\alpha \otimes H_{\alpha.(\tau)}$

For the technical details we refer to the appendix B. Let (λ_n, φ_n) be the orthogonal set of eigen-pairs of a linear self-adjoint & positive definite operator A, with A^{-1} compact. Then Hilbert spaces $\{H_\alpha | \alpha \in R\}$ are spanned by the finite norms

$$\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty \, , x_n := (x, \varphi_n).$$

In case of $\alpha = 0$ we skip the subscript.The bilinear form $a(x,y) := (Ax, y)$ defines an inner (kinetic energy) product in $D(A) = H_1$ and the operator equation $Ax = f$ is equivalent to, (BrK),

$$(x,y)_1 = (f,y), \forall y \in H_1.$$

For $\alpha < 0$ the Fourier coefficients x_n contribute to the α -norm with a polynomial decay. For $\tau > 0$ the inner product resp. norm in the form

$$(x,y)_{(\tau)} = \sum_{n=1}^\infty e^{-\sqrt{\lambda_n}\tau} x_n y_n \, , \|x\|_{(\tau)}^2 = (x,x)_{(\tau)}$$

spanning the Hilbert space $H_{(\tau)}$ have an exponential decay with

$$\|x\|_{(\tau)}^2 \le c(\alpha,\tau)\|x\|_\alpha^2, \forall x \in H_\alpha.$$

The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(\tau)}^2 \text{ with } \alpha, \delta > 0 \text{ being arbitrary.}$$

Especially for $\alpha = 1/2$ one get

$$\|x\|_{-1/2}^2 \le \delta\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(\tau)}^2 \text{ with } \delta > 0 \text{ being arbitrary.}$$

Putting

$$\|x\|_{\alpha.(\tau)}^2 := \sum_{n=1}^\infty \lambda_n^\alpha e^{-\sqrt{\lambda_n}\tau} x_n^2$$

one gets

i) $\int_0^\infty \|x\|_{(\tau)}^2 d\tau = \sum_{n=1}^\infty \lambda_n^{-1/2} x_n^2 = \|x\|_{-1/2}^2 \le \delta\|x\|_0^2 + e^{\tau/\delta}\|x\|_{(\tau)}^2$ for $\delta > 0$

ii) $(\ddot{x}, x)_{(\tau)} = \|\dot{x}\|_{(\tau)}^2 = \sum_{n=1}^\infty \lambda_n e^{-\sqrt{\lambda_n}\tau} x_n^2 = \|x\|_{1.(\tau).}^2$

iii) $\int_0^\infty \|\dot{x}\|_{(\tau)}^2 d\tau = \sum_{n=1}^\infty x_n^2 = \|x\|_0^2$.

Remark: We note that the D'Alembert operator with domain $L_2(H_{\alpha.(\tau)})$ is a strongly hyperbolic operator.

Let $\Phi_n := \varphi_n^H$ denote the Hilbert transform of φ_n with $(\varphi_n, \Phi_n) = 0$, (BrK1). The Hilbert space H_α of the composition $H_\alpha \otimes H_{\alpha.(\tau)}$ is built by the orthogonal system $\{\varphi_n\}$ while the Hilbert space $H_{(\tau)}$ is built by the orthogonal system $\{\Phi_n\}$ equipped with the related inner products resp. norms in the form

$$(x,y)_\alpha = \sum_{n=1}^{\infty} \lambda_n^\alpha x_n^{kin} y_n^{kin}\,, \qquad \|x\|_\alpha^2 = (x,x)_\alpha, \qquad x_n^{kin} := (x, \varphi_n),\ \ \alpha \in R$$

$$(x,y)_{\alpha.(\tau)} = \sum_{n=1}^{\infty} \lambda_n^\alpha e^{-\sqrt{\lambda_n}\tau} x_n^{pot} y_n^{pot}\,, \quad \|x\|_{\alpha.(\tau)}^2 = (x,x)_{\alpha.(\tau)}, \qquad x_n^{pot} := (x, \Phi_n),\ \ \tau > 0.$$

In the following we shall omit the Fourier coefficient indices refering to the related *kin*etic and *pot*ential energy norm case.

Then, the system $\left\{\psi_{n.\alpha.\tau}^{(1)}, \psi_{n.\alpha.\tau}^{(2)}\right\}$ with

$$\psi_{n.\alpha.\tau}^{(1)} := \lambda_n^{\alpha/2} \varphi_n - i\lambda_n^{\alpha/2} \Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}\,, \quad \psi_{n.\alpha.\tau}^{(2)} := \lambda_n^{\alpha/2} \varphi_n + i\lambda_n^{\alpha/2} \Phi_n e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}$$

defines an orthogonal system of the Hilbert space composition $H_\alpha \otimes H_{\alpha.(\tau)}$. For

$$x_{\alpha.\tau}^{(1)} := \sum_{n=1}^{\infty} x_n \psi_{n.\alpha.\tau}^{(1)}\,, \quad x_\tau^{(2)} := \sum_{n=1}^{\infty} x_n \psi_{n.\alpha.\tau}^{(2)}$$

the corresponding inner product of $H_\alpha \otimes H_{\alpha.(\tau)}$ is given by

$$\left(x_{\alpha.\tau}^{(1)}, x_{\alpha.\tau}^{(2)}\right) = (x,y)_\alpha + (x,y)_{\alpha.(\tau)}.$$

The relationship between the norms above and there relationship to the statistical L_2 norm is given by

$$\int_0^\infty \|x\|_{\alpha.(\tau)}^2 d\tau = \sum_{n=1}^{\infty} \lambda_n^\alpha \lambda_n^{-1/2} x_n^2 = \|x\|_{\alpha-1/2}^2 \le \delta^{2\alpha} \|x\|_0^2 + e^{\tau/\delta} \|x\|_{(\tau)}^2 \text{ for } \delta > 0$$

which is a consequence from

Lemma: Let $\alpha > 0$ be fixed. The α-norm of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \le \delta^{2\alpha} \|x\|_0^2 + e^{\tau/\delta} \|x\|_{(\tau)}^2$$

with $\delta > 0$ being arbitrary.

The Krein space $H_{(\tau)} = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$

The Hilbert space $H_{(\tau)}$ decomposition in the form

$$H_{(\tau)} = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$$

is supposed to be a quanta potential Hilbert-Krein space framework, where the parameter κ relates to correspondingly defined quantum number sequences in the form

$$\kappa^+_{\tau.n} := \frac{1}{2}\frac{e^{\kappa_n\tau}}{\cosh(\kappa_n\tau)}\,,\ \kappa^-_n := \frac{1}{2}\frac{e^{-\kappa_n\tau}}{\cosh(\kappa_n\tau)} \text{ with } \kappa_n \in R.$$

For

$$x_{(\tau)} := \sum_{n=1}^{\infty} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H_{(\tau)}$$

$$x^+_{\kappa.(\tau)} := \sum_{n=1}^{\infty} \kappa^+_{\tau.n} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H^+_{\kappa.(\tau)}$$

$$x^-_{\kappa.(\tau)} := \sum_{n=1}^{\infty} \kappa^-_{\tau.n} e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n \in H^-_{\kappa.(\tau)}$$

it follows (*)

$$x_{(\tau)} = x^+_{\kappa.(\tau)} + x^-_{\kappa.(\tau)}\ .$$

The Hilbert space decomposition $H_{(\tau)} = H^+_{\kappa.(\tau)} \otimes H^-_{\kappa.(\tau)}$ is accompanied by the indefinite inner products resp. metric

$$[x,y]_{\kappa.(\tau)} := \left(x^+_{\kappa.(\tau)}, y^+_{\kappa.(\tau)}\right) - \left(x^-_{\kappa(\tau)}, y^-_{\kappa.(\tau)}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} \frac{\sinh(2\kappa_n\tau)}{\cosh^2(\kappa_n\tau)} e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

$$= \sum_{n=1}^{\infty} \tanh(\kappa_n\tau) e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

where

$$\left(x^+_{\kappa.(\tau)}, y^+_{\kappa.(\tau)}\right) := \sum_{n=1}^{\infty} (\kappa^+_{\tau.n})^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n$$

$$\left(x^-_{\kappa(\tau)}, y^-_{\kappa.(\tau)}\right) := \sum_{n=1}^{\infty} (\kappa^-_{\tau.n})^2 e^{-\sqrt{\lambda_n}\tau} x_n y_n\ .$$

We note the corresponding relations in the form

$$x^+_{\kappa.(\tau)} - x^-_{\kappa.(\tau)} = \sum_{n=1}^{\infty} \tanh(\kappa_n\tau) e^{-\frac{1}{2}\sqrt{\lambda_n}\tau} x_n \Phi_n$$

$$\left\|x^+_{\kappa.(\tau)}\right\|^2 - \left\|x^-_{\kappa(\tau)}\right\|^2 = \sum_{n=1}^{\infty} \tanh(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau} x_n^2 = [x,x]_{\kappa.(\tau)}\ .$$

From the equivalent formulas

$$(x,y)_{(\tau)} = \left[x^+_{\kappa.(\tau)}, y^+_{\kappa.(\tau)}\right] - \left[x^-_{\kappa.(\tau)}, y^-_{\kappa.(\tau)}\right]$$

$$[x,y]_{\kappa.(\tau)} := \left(x^+_{\kappa.(\tau)}, y^+_{\kappa.(\tau)}\right) - \left(x^-_{\kappa(\tau)}, y^-_{\kappa.(\tau)}\right)$$

it follows the characterization of „positive", „negative", and „neutral" vectors $x \in H_{(\tau)}$ by the relations

$$\left\|x^+_{\kappa.(\tau)}\right\| > \left\|x^-_{\kappa(\tau)}\right\|\ ,\ \left\|x^+_{\kappa.(\tau)}\right\| < \left\|x^-_{\kappa.(\tau)}\right\|\ ,\ \left\|x^+_{\kappa.(\tau)}\right\| > \left\|x^-_{\kappa(\tau)}\right\|\ .$$

(*) appendix: $\kappa^+_{\tau.n} + \kappa^-_{\tau.n} = 1$, $\kappa^+_{\tau.n} - \kappa^-_{\tau.n} = \tanh(\kappa_n\tau)$, $(\kappa^+_{\tau.n})^2 - (\kappa^-_{\tau.n})^2 = \frac{\sinh(2\kappa_n\tau)}{\cosh^2(\kappa_n\tau)} = \tanh(\kappa_n\tau)$.

The potential operator of a Krein space

The canonical J-symmetric operator of a Krein space may be intepreted as a *„potential"* operator W (VaM) p. 90. In our case it is defined by

$$W_{\kappa.\tau}x := \frac{1}{2}grad\varphi_{\kappa.\tau}(x) := x^+_{\kappa.(\tau)} - x^-_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh(\kappa_n\tau)e^{-\frac{1}{2}\sqrt{\lambda_n}\tau}x_n\Phi_n \ .$$

It is complete, invertible, isometric ($W_{\kappa.\tau} = W^{-1}_{\kappa.\tau}$) and symmetric. Thus, the bilinear form

$$((x,y))_{\kappa.(\tau)} := [W_{\kappa.\tau}x, y]_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh^2(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau}x_n y_n$$

defines an inner product on all of the Hilbert space $H_{(\tau)}$ with related norm

$$|||x|||^2_{\kappa.(\tau)} := [W_{\kappa.\tau}x, x]_{\kappa.(\tau)} = \sum_{n=1}^{\infty} tanh^2(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau}x_n^2 \ .$$

The definition of the potential (canonical symmetry) operator enables a treatment of the results of its action as the „mirror reflection" of the space $H_{(\tau)}$ in the subspace $H^+_{\kappa.(\tau)}$. The sub-space $H^+_{\kappa.(\tau)}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = 1$. The sub-space $H^-_{\kappa.(\tau)}$ is an eigen-subspace of the operator $W_{\kappa.\tau}$ corresponding to the eigenvalue $\lambda = -1$. The whole spectrum of $W_{\kappa.\tau}$ lies on the join of the points $\lambda = \pm 1$.

We note that the operator norm of the potential operator with respect to the inner product $(x,y)_{(\tau)}$ is equivalent to the $|||x|||^2_{\kappa.(\tau)}$, i.e., it holds

$$(W_{\kappa.\tau}x, W_{\kappa.\tau}y)_{(\tau)} = \sum_{n=1}^{\infty} tanh^2(\kappa_n\tau)\, e^{-\sqrt{\lambda_n}\tau}x_n y_n = ((x,y))_{\kappa.(\tau)} \ .$$

The potential and hyperboloids of a Krein space

The indefinite metric (functional) of the considered Krein space

$$\varphi_{\kappa.\tau}(x) := [x,x]_{\kappa.(\tau)} = \left\|x^+_{\kappa.(\tau)}\right\|^2 - \left\|x^-_{\kappa(\tau)}\right\|^2 = \sum_{n=1}^{\infty} tanh\,(\kappa_n\tau)e^{-\sqrt{\lambda_n}\tau}x_n^2 \ .$$

in combination with the functional $\big((x)\big) := \sqrt{\varphi_{\kappa.\tau}(x)}$ generates hyperboloids H_c, hyperbolic regions V_c, and conical region V_0 in the form

$$H_c := \{x \in H_{(\tau)} | \varphi_{\kappa.\tau}(x) = c > 0\},\ V_c := \{x \in H_{(\tau)} | \big((x)\big) \geq c > 0\},\ V_0 := \{x \in H_{(\tau)} | \big((x)\big) \geq 0\}.$$

Evidently V_c is a subspace of V_0.

(VaM) p. 91: *„If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0,\infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0,\infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\left\|x^+_{\kappa.(\tau)}\right\| = \left\|x^-_{\kappa(\tau)}\right\|$ belong to K, and those points of S^1 for which $\left\|x^+_{\kappa.(\tau)}\right\| > \left\|x^-_{\kappa(\tau)}\right\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by*

$$t = c(\left\|x^+_{\kappa.(\tau)}\right\|^2 - \left\|x^-_{\kappa(\tau)}\right\|^2)^{-1/2}.$$

From this it is seen that $t \to \infty$ if $\left\|x^+_{\kappa.(\tau)}\right\|^2 - \left\|x^-_{\kappa(\tau)}\right\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$."

The angular and dissipative operators of a Krein space

The counterparts of W-norms $|||x|||_{\kappa.(\tau)}^2 := [W_{\kappa.\tau}x, x]_{\kappa.(\tau)}$ with respect to the H_α Hilbert spaces norms $\|x\|_\alpha^2 = \sum_1^\infty \lambda_n^\alpha x_n^2 < \infty$ are given by the norms

$$|||x|||_{\alpha.\kappa}^2 := \sum_{n=1}^\infty tanh^2(\kappa_n\tau)\, \lambda_n^\alpha x_n^2 \, .$$

Let $L := H_{\alpha.\kappa} \subset H_{(\tau)} = H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$ and $P^\pm$ be the canonical projectors. Then the bounded linear operator

$$K^+ = K_{\kappa.(\tau)}^+ \coloneqq \; P^-(P^+|H_{\alpha.\kappa})^{-1} \, : \, P^+|H_{\alpha.\kappa} \, \rightarrow \, H_{\kappa.(\tau)}^-$$

is called the angular operator for $H_{\alpha.\kappa}$ with respect to $H_{\kappa.(\tau)}^+$. Then, the set of vectors of the sub-space

$$L := H_{\alpha.\kappa} := \{x_{\alpha.\kappa}^+ + Kx_{\alpha.\kappa}^+\}_{x^+ \in H_{\kappa.\alpha}^+}$$

gives the general form of all $H_{\kappa.\alpha}^+ \subset H_{\kappa.(\tau)}^+$ of the Krein space $H = H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$.

Theorem 11.7 ((BoJ) p. 54, (PhR)): A subspace $L \subset H_{(\tau)} = H_{\kappa.(\tau)}^+ \otimes H_{\kappa.(\tau)}^-$ is positive if and only if the angular operator K^+ of L with respect to $H_{\kappa.(\tau)}^+$ exists and satisfies the condition

$$|||K_{\kappa.(\tau)}^+ x_{\kappa.(\tau)}^+|||_{\kappa.(\tau)}^2 \leq |||x_{\kappa.(\tau)}^+|||_{\kappa.(\tau)}^2 \, , \, x_{\kappa.(\tau)}^+ \in D(K_{\kappa.(\tau)}^+).$$

In particular, positive definite subspaces are characterized by the property

$$|||K_{\kappa.(\tau)}^+ x_{\kappa.(\tau)}^+|||_{\kappa.(\tau)}^2 < |||x_{\kappa.(\tau)}^+|||_{\kappa.(\tau)}^2, \, x_{\kappa.(\tau)}^+ \in D(K_{\kappa.(\tau)}^+), \, x_{\kappa.(\tau)}^+ \neq 0,$$

and neutral subspaces by

$$|||K_{\kappa.(\tau)}^+ x_{\kappa.(\tau)}^+|||_{\kappa.(\tau)}^2 = |||x_{\kappa.(\tau)}^+|||_{\kappa.(\tau)}^2, \, x_{\kappa.(\tau)}^+ \in D(K_{\kappa.(\tau)}^+).$$

The inclusion $H_{\kappa.\alpha}^+ \subset H_{\kappa.(\tau)}^+$ is accompanied by related inclusions $H_{\kappa.\alpha}^- \subset H_{\kappa.(\tau)}^-$. The related Krein space concept is called alternating (maximal) pairs and alternating extensions.

The physical application of maximal positive and negative sub-spaces is concerned with the concept of of maximal dissipative (and maximal accretive) operators accompanied with spectra of unitary and self-adjoint operators, (BoJ) p. 114 ff.

The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1)$, $\mathrm{D}(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1x_1, x_1] + [x_1, T_1x_1] \leq 0, \, x_1 \in D(T_1)$$

$$[T_2x_2, x_2] + [x_2, T_2x_2] \leq 0, \, x_2 \in D(T_2).$$

Mean ergotic theorem: If U is an isometry on a complex Hilbert space and if P is a projection on the space of all vectors invariant under U, then $\frac{1}{n}\sum_{j=0}^{n-1} U^j x$ converges to Px for every x in the space.

Krein spaces, potentials and potential operators

(AzT), (BoJ)

A Krein space is a Hilbert space H with inner product (x,y), which can be written in the form $H = H^+ \otimes H^-$. There are two equivalent approaches defining Krein spaces based on

the concept of an indefinite metric (also called a Q-metric) $Q(x,y) \coloneqq [x,y]$, $\forall x,y \in H$

a self-adjoint operator B defined on all of the Hilbert space H inducing the decomposition of H.

A canonical decomposition of $H = H^+ + H^-$ enables the (positive definite) inner product of H according to

$$(*) \ \ (x,y) = [x^+,y^+] - [x^-,y^-], x = x^+ + x^-, y = y^+ + y^-.$$

For vectors $u,v \in H^+$ we have $(u,v) = [u,v]$; for vectors $u,v \in H^-$ we have $(u,v) = -[u,v]$. If $u \in H^+$ and $v \in H^-$, then it follows from (*) that $(u,v) = [u,,\theta] - [,\theta,v]$.

The formula (*) can be inverted in the following way

$$[x,y] = (x^+,y^+) - (x^-,y^-) \text{ resp. } [x,x] = (x^+,x^+) - (x^-,x^-)$$

from which it follows

„Positivity, negativity, and neutrality of a vector $x \in H$ are equivalent to the relations

$$\|x^+\| > \|x^-\|, \|x^+\| < \|x^-\|, \text{ or } \|x^+\| > \|x^-\| \text{ respectively.“}$$

In short, a Krein space can be looked on as an arbitrary Hilbert space decomposed into usual orthogonal sums of two subspaces, equipped in addition to the original Hilbert metric (i.e., the inner product (x,y)) with an additional indefinite metric $[x,y]$.

The decomposition of a Krein space generates two mutually complementary projectors P^+ and P^- mapping H on to H^+ and H^- respectively. Those orthogonal projection operators P^+ and P^- are linked to the indefinite metric by, (VaM) chapter IV,

$$\varphi(x) := [x,x] = \|P^+x\|^2 - \|P^-x\|^2 .$$

The indefinite metric $\varphi(x)$ can be interpreted as a „potential“. The related „potential operator“ (in mathematics it is called „the canonical symmetry“ J, (AzT) §3, (BoJ) p. 52) is then given by, (VaM) (10.7), (12.6)

$$\boldsymbol{W}(x) := \frac{1}{2} grad\varphi(x) = P^+x - P^-x = x^+ - x^-.$$

The fundamental properties of the potential operator $\boldsymbol{W}(x)$ are completeness, invertibility, ($\boldsymbol{W} = \boldsymbol{W}^{-1}$) isometry, and symmetry. Thus, the bilinear form $(x,y)_W \coloneqq (\boldsymbol{W}(x),y)$ defines an inner product, (BoJ) p. 52.

The sub-space H^+ is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = 1$.

The sub-space H^- is an eigen-subspace of the operator $\boldsymbol{W}$ corresponding to the eigenvalue $\lambda = -1$.

The whole spectrum of $\boldsymbol{W}$ lies on the join of the points $\lambda = \pm 1$.

The definition of the potential (canonical symmetry) operator enables a treatment of the results of its action as the „mirror reflection“ of the space H in the subspace H^+ .

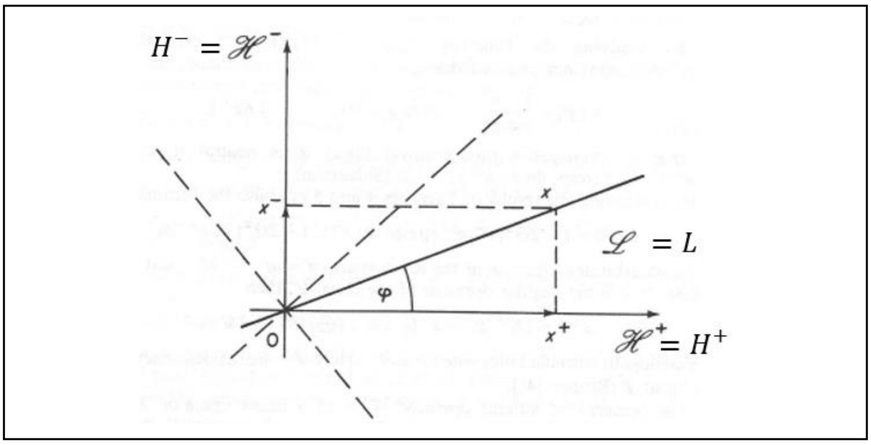
Krein spaces and angular (dissipative and accretive) operators
(AzT), (BoJ)

By the aid of J-norms a description of semi-definite subspaces L can be given enabling the definition of an angular operator $K^+ : H^+ \to H^-$ with domain $D(K^+) = P^+(L)$ and range $R(K^+) = P^-(L)$, (BoJ) p. 54. For the following we refer to (AzT) p. 48 ff. and (BoJ) p. 54.

Let $L \subset H$ in a Krein space $H = H^+ \otimes H^-$ and $P^{\pm}$ the canonical projectors (. Then the bounded linear operator

$$K^+ := P^-(P^+|L)^{-1} : P^+|L \to H^-$$

is called the angular operator for L with respect to H^+. The meaning of this nomenclature is explained by the following picture, (AzT) p. 61:



In the figure above a non-negative (even positive) subspace $L \subset H$ is shown. For any $x \in L$ we have $x = x^+ - x^-$, and $x^- = Kx^+$, where K is the operator of rotating the vector x^+ through an angle $\pi/2$ (in the positive direction), and then multiplying by a scalar $k = tan\varphi$ - the angular coefficient of the „line" L . …

If φ is always understood to be the *minimal* angle between L and „the axis" H^+, then $\tan(\varphi) = \|K\|$. In the general case too ($dimH \leq \infty$) for the angular operator K of a non-negative subspace L we have $tan\,(\varphi(L, H^+) = \|K\|$, if the (minimal) angle φ is defined by the equality $sin\,(\varphi(L, H^+) = sup_{e \in S(L)} \|e - ZP^+e\|$, where $S(L)$ is the unit sphere of the lineal L ($\|e\| = 1$).

Theorem 8.2 ((AzT) p. 49; see also Theorem 11.6, (BoJ) p. 54): The set of vectors

$$L = \{x^+ + Kx^+\}_{x^+ \in L^+}$$

in which L^+ is an arbitrary lineal from H^+, and $K : L^+ \to H^-$ is an arbitrary compression ($\|K\| \leq 1$), gives the general form of all $L \subset H$ of the Krein space $H = H^+ \otimes H^-$, and $L^+ = P^+(L)$ and K is the angular operator for L with respect to H^+.

Let $\|x\|_W^2 = \|x\|_J^2 = \|x^+\|^2 - \|x^-\|^2$ denote the $J = W$-inner product related (potential) norm.

Theorem 11.7 ((BoJ) p. 54): A subspace $L \subset H$ is positive if and only if the angular operator K^+ of L with respect to H^+ exists and satisfies the condition

$$\|K^+x^+\|_W^2 \leq \|x^+\|_W^2 \,,\, x^+ \in D(K^+).$$

In particular, positive definite subspaces are characterized by the property

$$\|K^+x^+\|_W^2 < \|x^+\|_W^2 \,,\, x^+ \in D(K^+),\, x^+ \neq 0,$$

and neutral subspaces by

$$\|K^+x^+\|_W^2 = \|x^+\|_W^2 \,,\, x^+ \in D(K^+).$$

For negative subspaces similar statements, involving K^- instead of K^+, are valid.

Theorem 8.2' ((AzT) p. 49): The set of vectors

$$L = \{Qx^- + x^-\}_{x^- \in L^-}$$

in which L^- is an arbitrary lineal from H^-, and $Q : L^- \to H^+$ is an arbitrary compression ($\|Q\| < 1$), gives the general form of all $L^+ \subset H$ of the Krein space $H = H^+ \otimes H^-$, and $L^- = P^-(L)$ and Q is the angular operator for L with respect to H^-.

Alternating pairs and dissipative operators in Hilbert space

(BoJ)

(BoJ) p. 39: Let H_0 denote a Hilbert space with with inner product $(x, y)_0$, $x, y \in H_0$ and norm $\|x\|$ and let W be an arbitrary bounded self-adjoint operator ($W = W^*$) given on H_0. Then the Hermitian sesquilinear form $[x, y] = (Wx, y)_0 = Q(x, y)$ defines in H_0 an indefinite metric which we shall call the W-metric, and we shall call the space H_0 itself with the W-metric a W-space. W is called the Gram operator of the space H_0.

(BoJ) p. 91: A linear operator A with an arbitrary domain of definition $D(A)$, operating in a W-space H_0, is said to be W-dissipative if $Im[Ax, x] \geq 0$ for all $x \in D(A)$, and to be maximal W-dissipative if it is W-dissipative and coincides with any W -dissipative extension of it.

An ordered pair of subspaces $\{L_1, L_2\}$ of the Krein space H will be called an alternating pair provided L_1 is positive, L_2 is negative, and $L_1 \perp L_2$. If, in addition, L_1 is maximal positive and L_2 is maximal negative, the pair $\{L_1, L_2\}$ is called alternating maximal pair.

By an alternating extension of the alternating pair $\{L_1, L_2\}$ we mean an alternating pair $\{L_1' L_2'\}$ such that $L_1 \subset L_1'$, $L_2 \subset L_2'$.

Theorem 9.1 (BoJ) p. 115: Every alternating pair in the Krein space H can be extended to an alternating maximal pair.

The concept of alternating pairs can be applied to prove the existence of maximal dissipative operators $T_1^{(0)}, T_2^{(0)}$ of dissipative operators T_1, T_2 with dense domains $D(L_1)$, $D(L_2)$ in H_0 (i.e., dissipative operators having no dissipative proper extension) satsifying

$$[T_1 x_1, x_1] + [x_1, T_1 x_1] \leq 0,\ x_1 \in D(T_1)$$

$$[T_2 x_2, x_2] + [x_2, T_2 x_2] \leq 0,\ x_2 \in D(T_2).$$

Theorem (BoJ) p. 118: If $\left\{L_1^{(0)}, L_1^{(0)}\right\}$ is an alternating maximal pair extending $\{D(-T_1), D(-T_2)\}$, then the operators $T_1^{(0)}, T_2^{(0)}$ defined by the relations $L_1^{(0)} = D(-T_1^{(0)})$, $L_2^{(0)} = D(-T_2^{(0)})$ are maximal dissipative operators of the dissipative operators T_1, T_2, and every solution can be obtained in this way.

Krein spaces and hyperboloids
accompanied by hyperbolic and conical regions
(VaM) p. 89 ff.

Putting $x^+ := P^+x$, $x^- := P^-x$ the corresponding potential $\varphi(x^+ + x^-)$ defined by

$$\varphi(x^+ + x^-) = \|x^+\|^2 - \|x^-\|^2 = c > 0$$

generates hyperboloids in the form

$$H_c := \{x \in H | (x^+ + x^-) = \|x^+\|^2 - \|x^-\|^2 = c > 0\}.$$

A hyperbolic region V_c is defined by

$$\big((x)\big) = \sqrt{\|x^+\|^2 - \|x^-\|^2} \geq c > 0$$

A conical region V_0 is defined by

$$\big((x)\big) = \sqrt{\|x^+\|^2 - \|x^-\|^2} \geq 0 .$$

Evidently V_c is a subspace of V_0.

If x is an exterior point of the conical region V_0, then those points of the ray $tx, t \in [0, \infty)$ for which $t \geq c/a$ belong to the hyperbolic region V_c, and those for which $0 \leq t < c/a$ do not belong to V_c. If x is not an element of V_0, then the ray $tx, t \in [0, \infty)$ does not have any point in common with V_c. Thus, every interior ray of the conical region V_0 intersects the hyperbolid $\big((x)\big) = c > 0$ in a single point. We denote by K the boundary of the conical region V_0. The manifold K is defined by the condition $\big((x)\big) = 0$. If we look at the unit sphere S^1 ($\|x\|^2 = 1$), then those points of S^1 for which $\|P^+x\| = \|P^-x\|$ belong to K, and those points of S^1 for which $\|P^+x\| > \|P^-x\|$ intersect the hyperboloid $\big((x)\big) = c > 0$ at the point whose distance from θ is given by

$$t = \mathrm{c}(\|x^+\|^2 - \|x^-\|^2)^{-1/2}.$$

From this it is seen that $t \to \infty$ if $\|x^+\|^2 - \|x^-\|^2 \to 0$, i.e. the manifold K is an asymptotic conical manifold for the hyperboloid $\big((x)\big) = c > 0$.

Lemma: If the (proper) subspace $H_1 \subset H$ is finite dimensional, then the region V_c ($c \geq 0$) is weakly closed.

Remark: Ellipsoids are defined by the condition $\frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} = 1$. The related elliptical region is defined by

$$E_c := \left\{x \in H | \frac{\|x^+\|^2}{a_+^2} + \frac{\|x^-\|^2}{a_-^2} \leq c, c > 0\right\}.$$

Theorem (ZaC) p. 291: Let H denote a Hilbert space with inner product $(\cdot,\cdot)$ and $K \subset H$ be a closed convex cone. For every $x \in H$ let $P^K x$ (which is uniqly defined) denote the projection of x on K. Putting $K^- := -K^+ := \{y \in H] \, (x, y) \leq 0, \forall x \in H\}$ it holds $\forall x \in H \; x = P^K x + P^{K^-} x$ and $(P^K x, P^{K^-} x) = 0$. Conversely, if $x = x_1 + x_2$ with $x_1 \in K$, $x_2 \in K^-$ and $(x_1, x_2) = 0$ then $x_1 = P^K x$ and $x_2 = P^{K^-} x$.

Hyperboloids generated by operators

(VaM) p. 92

Let B be self-adjoint operator defined on all of the Hilbert space H. Since it follows that B is bounded, then

$$inf\{\,(Bx,x) = a \mid \|x\| = 1\} > \infty\,,\; sup\{\,(Bx,x) = b \mid \|x\| = 1\} < \infty\ .$$

We shall assume that $a < 0$, $b > 0$. Further, let E_t be the resolution of the odentity corresponding to B; then $E_b - E_0 = P_1$ is a projection operator onto subspace $H_1 \subset H$ which reduces B. Thus, the operator induces a decomposition of into the direct sum of subspaces H_1 and H_2 ($H = H_1 \otimes H_2$) and thereby generated a hyperboloid

$$\varphi(x) = \varphi(x^+ + x^-) = \sqrt{\|P_1\|^2 - \|P_2\|^2} = c > 0\,,$$

where P_2 is the projection onto H_2.

In this case where the positive part of the spectrum of B lies in an interval $[m,b]$, where $m > 0$, then the inequality

$$\|Bx\| \geq \frac{m}{\sqrt{2}}\sqrt{\varphi^2(x) + \|x\|^2} \geq \frac{m}{\sqrt{2}}\sqrt{c^2 + \|x\|^2}$$

holds for every x in the hyperbolic region V_c defined by

$$\varphi(x) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq c > 0\,,$$

as well as in the conical region V_0 defined by

$$\varphi(x) = \sqrt{\|P^+x\|^2 - \|P^-x\|^2} \geq 0\ .$$

Remark: It should be remarked that in some cases the operator B leaves invariant the hyperbolic regions V_c, which it generates. This is the case, for example, when the positive part of the spectrum of B lies in the interval $[1,b]$ and the negative part lies in $[-1,0]$. In fact, we then have

$$\big((Bx)\big) = \|P^+Bx\|^2 - \|P^-Bx\|^2 = \|BP^+x\|^2 - \|BP^-x\|^2$$

$$= \int_1^b t^2 d(E_t\,P^+x, P^+x) - \int_{-1}^0 t^2 d(E_t\,P^-x, P^-x)$$

$$\geq \|P^+x\|^2 - \|P^-x\|^2 \geq c^2.$$

The telegraph equation
(CoR) p. 192 ff.

For the wave equation

$$\frac{1}{c^2}\ddot{u} - \Delta u = 0\,,$$

progressing undistorted plane waves with speed c and the arbitrary form

$$\Phi(\sum_{i=1}^{n} \alpha_i x_i - ct), \sum_{i=1}^{n} \alpha_i^2 = 1$$

are possible in every direction. A more general example is given by the telegraph equation

$$\ddot{u} - c^2 u'' + (\alpha + \beta)\dot{u} + \alpha\beta u = 0,$$

satisfied by the voltage or the current u as a function of the time t and the position x along a cable; here x measures the length of the cable from an initial point. Unless this equation represents dispersion. If we introduce $v := e^{\frac{1}{2}(\alpha+\beta)t} u$, we obtain the simpler equation

$$\ddot{v} - c^2 v'' + (\frac{\alpha-\beta}{2})^2 v = 0$$

for the function v. This new equation represents the dispersionless case if and only if $\alpha = \beta$. In this case the original telegraph equation, of course, possesses no absolutely undistorted wave solutions of arbitrarily prescribed form. However, our result may be stated in the following way:

> If condition $\alpha = \beta$ holds, the telegraph equation posses damped, yet „relatively“ undistorted, progressing wave solutions of the form $u = e^{-\frac{1}{2}(\alpha+\beta)t} f(x \pm ct)$, with arbitrary f, progressing in both directions of the cable.

The telegraph equation

$$\ddot{u} - c^2 u'' + (\alpha + \beta)\dot{u} + \alpha\beta u = 0,$$

is derived by elimination of one of the unknown functions from the following system of two differential equations of first order for the current $i = i(x,t)$ and the voltage $u = u(x,t)$ as functions of x and t:

$$C\dot{u} + Gu + i' = 0$$

$$L\dot{i_t} + Ri + u' = 0.$$

Here L is the inductance of the cable, R its resistence, C its shunt capacity, and, finally, G, its shunt conductance (loss of current divided by voltage). The constants in the telegraph equation, which arise in the elimination process, have the meaning

$$\frac{1}{c^2} = LC, \alpha = \frac{G}{C}, \beta = \frac{R}{L}$$

where c is the speed of light and α the capacitive and β the inductive damping factor.

Global boundedness of the 3D-Navier-Stokes equations
in a variational $H_{-1/2}$ based Hilbert space framework
(BrK1)
www.navier-stokes-equations.com

It turned out that based on the physical modelling assumption of a variational representation of the 3D NSE in a $H_{-1/2}$ Hilbert space framework (interpreted as a fluid element test space) the 3D NSE enjoy global solutions. Its a consequence of the well-known Sobolevskii-estimates for the 3D case. Those estimates fail in case of a H_0 test space.

Lemma (Sobolevskii): For $0 \leq \delta < 1/2 + n \cdot (1 - 1/p)/2$ it holds

$$\left|A^{-\delta} P(u, grad)v\right|_p \leq M \cdot \left|A^{\theta} u\right|_p \cdot |A^{\rho} u|_p$$

with a constant $M := M(\delta, \theta, \rho, p)$ if $\delta + \theta + \rho \geq n/2p + 1/2$, $\theta, \rho > 0$, $\theta + \rho > 1/2$.

The NSE initial-boundary equation is given by

$$\frac{du}{dt} + Au + Bu = Pf \, , u(0) = u_0$$

where $B(u) := P(u, grad)u)$ and $Pu_0 = u_0$. Multiplying this homogeneous equation with $A^{-1/2}u$ leads to

$$(\dot{u}, u)_\alpha + (Au, u)_\alpha + (Bu, u)_\alpha = 0, \; (u(0), v)_\alpha = (u_0, v)_\alpha \text{ for all } v \in H_{-1/2}.$$

For $\alpha := -1/2$ one gets the generalized "energy" inequality in the form

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq \left|(Bu, u)_{-1/2}\right| \leq \|u\|_{-1/2}\|Bu\|_{-1/2} \cong \|u\|_{-1/2}\left\|A^{-1/4}Bu\right\|_0.$$

Putting $p := 2$, $\delta := 1/4$,$\theta := \rho := 1/2$ fulfilling $\theta + \rho \geq \frac{1}{4}(n+1) = 1$ one gets from the Sobolevskii-lemma above

$$\left\|A^{-\delta} P(u, grad)u\right\| \leq c\left\|A^{\theta} u\right\| \cdot \|A^{\rho} u\| = c\|u\|_{2\theta} \cdot \|u\|_{2\rho} = c\|u\|_1^2$$

and therefore

$$\frac{1}{2}\frac{d}{dt}\|u\|_{-1/2}^2 + \|u\|_{1/2}^2 \leq \left|(Bu, u)_{-1/2}\right| \leq c \cdot \|u\|_{-1/2}\|u\|_1^2.$$

Putting $y(t) := \|u\|_{-1/2}^2$ one gets $y'(t) \leq c \cdot \|u\|_1^2 \cdot y^{1/2}(t)$, resulting into the a priori estimate

$$\|u(t)\|_{-1/2} \leq \|u(0)\|_{-1/2} + \int_0^t \|u\|_1^2(s)ds \leq c\left\{\|u_0\|_{-1/2} + \|u_0\|_0^2\right\},$$

which ensures global boundedness by the a priori energy estimate provided that $u_0 \in H_0$.

Remark: We note that the pressure p in the variational representation

$$(Au, v)_{-\frac{1}{2}} := (\nabla u, \nabla v)_{-\frac{1}{2}} + (\nabla p, v)_{-\frac{1}{2}} = (u, v)_{\frac{1}{2}} + (p, v)_0 \qquad \text{for all } v \in H_{-1/2}$$

$$(u(0), v)_{-1/2} = (u_0, v)_{-1/2}$$

can be expressed in terms of the velocity by the formula

$$p = -\sum_{j,k=1}^{3} R_j R_k (u_j u_k)$$

with (R_1, R_2, R_3) is the Riesz transform.

The real & complex Lorentz groups
(StR)

The Lorentz transformation in special relativity is a simple type of rotation in hyperbolic space.

The Lorentz group L has four components, each of which is connected in the sense that any point can be connected to any other, but no Lorentz transformation in one component can be connected to another in another component. One of this components is the restricted Lorentz group, which is the group of $2x2$ complex matrices of determinant one, $SL(2,C)$. It is isomophic to the symmetry group $SU(2) \cong SL(2,C)$, containing as elements the complex-valued rotations, which can be written as a complex-valued matrix of type

$$\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix} \text{ with determinant one.}$$

It is important in describing the transformation properties of spinors. In SMEP the group $SU(2) \cong SL(2,C)$ describes the weak force interaction with 3 bosons W^+, W^-, Z.

Another group associated with the Lorentz group L is the complex Lorentz group $L(C)$, (*). It has just two connected components, $L_+(C)$ and $L_-(C)$. The transformations 1 and -1, which are disconnected in L are connected in $L(C)$. In other words, the complex Lorentz transformation connects

- the two components containing the 1-transformation and space-time inversion
- the two components containing the space inversion and the time inversion.

Just as the restricted Lorentz group is associated with $SL(2,C)$, the complex Lorentz group is associated with $SL(2,C) \otimes SL(2,C) \cong SU(2) \otimes SU(2)$. There is also a two-to-one homomorphism from $SL(2,C) \times SL(2,C)$ onto $L(+,C)$.

The spin of an elementary particle is its eigen-rotation with exactly two rotation axes, one parallel and one anti-parallel axis to a magnetic field. This is the 2×2 complex number scheme, where every „normal“ rotation is contained twice. Consequently, an electron has a charge only half of the Planck's quantum of action.

(*) The complex Lorentz group is composed of all complex matrices satisfying

$$\Lambda^{\kappa}{}_{\mu}\Lambda_{\kappa\nu} = g_{\mu\nu} \text{ or } \Lambda^T G \Lambda = G, \qquad (1\text{-}5).$$

It has just two connected components, $L_+(C)$ and $L_-(C)$ according to the sign of det(Λ). The transformations 1 and -1, which are disconnected in L are connected in $L(C)$. In other words, the complex Lorentz transformation connects

- the two components containing the 1-transformation and space-time inversion, i.e. the pair

$$\{\det(\Lambda) = +1, \det(\Lambda^0{}_0 = +1)\}, \{\det(\Lambda) = +1, \det(\Lambda^0{}_0 = -1)\},$$

- the two components containing the space inversion and the time inversion, i.e. the pair

$$\{\det(\Lambda) = -1, \det(\Lambda^0{}_0 = +1)\}, \{\det(\Lambda) = -1, \det(\Lambda^0{}_0 = -1)\}.$$

Summary:

While two (real) Lorentz transformations need to be connected to one another by an appropriately defined continuous curve of Lorentz transformations, there are two pairs of components of the complex Lorentz transform, which are both already connected by definition.

Just as the restricted Lorentz group is associated with $SL(2,C)$, the complex Lorentz group is associated with $SL(2,C) \otimes SL(2,C)$. The latter group is the set of all pairs of $2x2$ matrices of determinants one with the multiplication law

$$\{A_1, B_1\} \cdot \{A_2, B_2\} = \{A_1A_2, B_1B_2\}.$$

Is is easy to see that only matrix pairs which yield a given $\Lambda(\mathrm{A},\mathrm{B})$ are $(\pm\mathrm{A}, \pm\mathrm{B})$. In particular,

$$\Lambda(-1,1) = \Lambda(1,-1) = -1.$$

The corresponding complex Poincare group admits complex translation but also the multiplication law

$$\{a_1, \Lambda_1\} \cdot \{a_2, \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2\}.$$

It has two components $\mathrm{P}_{\pm}(C)$, which are distinguished by $\det(\Lambda)$ and a corresponding inhomogeneous group to $SL(2,C)$.

Wavelets
(HoM), (LoA), (MeY)

The decomposition of the quantum element space $H_{-1/2} = H_0 \otimes H_0^{\perp}$ resp. its related quantum element energy space decomposition $H_{1/2} = H_1 \otimes H_1^{\perp} = H_{-1/2}^{*}$ is very much related to the Calderón wavelet tool. In contrst to the one-parameter depending Fourier wave the Calderón wavelet depends from two parameters. It may be interpreted as a mathematical microscope analysing Fourier wave behavior beyong their statistical L_2 domain:

> (HoM) 1.2: „The idea of wavelet analysis is to look at the details are added if one goes from scale a to scale $a - da$ with $da > 0$ but infinitesimal small. … Therefore, the wavelet transform allows us to unfold a function over the one-dimensional space R into a function over the two-dimensional half-plane $\boldsymbol{H}$ of positions and details (where is which details generated?). … Therefore, the parameter space $\boldsymbol{H}$ of the wavelet analysis may also be called the position-scale half-plane since if g localized around zero with width Δ then $g_{b,a}$ is localized around the position b with width $a\Delta$. The wavelet transform itself may now be interpreted as a mathematical microscope where we identify
>
> b ↔ position; $(a\Delta)^{-1}$ ↔ enlargement; g ↔ optics. "

While the Fourier waves enable an analysis of the test space H_0, wavelets enable an alternative analysis tool for a specific densely embedded subspace of H_0, as the (wavelet) admissibility condition for a $\psi \in H_0$ is a weak one, as for each $\psi, \hat{\psi} \in H_0$: it holds $\|\psi_\varepsilon - \psi\|_{L_2}^2 \to 0$ for

$$\hat{\psi}_\varepsilon := \begin{cases} \hat{\psi}(\omega), & |\omega| \geq \varepsilon \\ 0, & else \end{cases}$$

There are at least two approaches to wavelet analysis, both are addressing the somehow contradiction by itself, that a function over the one-dimensional space R can be unfolded into a function over the two-dimensional half-plane. The Fourier transform of a wavelet transformed function f is given by, (LoA), (MeY),

$$\widehat{W_\vartheta[f]}(a,\omega) := (2\pi|a|)^{\frac{1}{2}} c_\vartheta^{-\frac{1}{2}} \hat{\vartheta}(-a\omega)\hat{f}(\omega) \,.$$

For $\varphi, \vartheta \in L_2(R)$, $f_1, f_2 \in L_2(R)$,

$$0 < |c_{\vartheta\varphi}| := 2\pi \left| \int_R \frac{\hat{\vartheta}(\omega)\bar{\hat{\varphi}}(\omega)}{|\omega|} d\omega \right| < \infty$$

and $|c_{\vartheta\varphi}| \leq c_\vartheta c_\varphi$ one gets the duality relationship, (LoA)

$$(W_\vartheta\ f_1, W_\varphi^* f_2)_{L_2(R^2, \frac{dadb}{a^2})} = c_{\vartheta\varphi}(f_1, f_2)_{L_2}$$

i.e.

$$W_\varphi^* W_\vartheta\ [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{ −sense.}$$

For $\varphi, \vartheta \in L_2(R)$, $f_1, f_2 \in L_2(R)$,

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i.e.

$$W_\varphi^* W_\vartheta\ [f] = c_{\vartheta\varphi} f \quad \text{in a } L_2 \text{ −sense.}$$

This identity provides an additional degree of freedom to apply wavelet analysis with appropriately (problem specific) defined wavelets in a (distributional) Hilbert scale framework where the "microscope observations" of two wavelet (optics) functions ϑ, φ can be compared with each other by the above "reproducing" ("duality") formula.

Some formulas
(Grl)

i) $cosh(x) \pm sinh(x) = e^{\pm x}$

ii) $cosh^2(x) - sinh^2(x) = 1$

iii) $tanh(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2kx}, x > 0$ (Grl) 1.232

iv) $tanh(x) = \sum_{k=1}^{\infty}\frac{2^{2k}(2^{2k}-1)}{(2k)!}B_{2k}x^{2k-1}$

v) $e^{ax} - e^{bx} = (a-b)xe^{\frac{1}{2}(a+b)x}\prod_{k=1}^{\infty}(1 + \frac{(a-b)^2x^2}{4k^2\pi^2}$ (Grl) 1.223

vi) $sinh(2x) = 2\,sinh(x)\,cosh(x), cosh(2x) = 2cosh^2(x) - 1$ (Grl) 1.334

vii) $tanh(x)\frac{sinh\,(2x)}{cosh^2(x)} = tanh(x)\frac{2\,sinh(x)cosh(x)}{cosh^2(x)} = 2\,tanh^2(x)$

viii) $\int sinh(ax)dx = \frac{1}{a}cosh(ax)$, $\int cosh(ax)dx = \frac{1}{a}sinh(ax)$ (Grl) 2.414

ix) $\int\frac{dx}{cosh^2(x)} = tanh(x)$, $\int\frac{dx}{sinh^2(x)} = -\,coth(x)$ (Grl) 2.422

x) $\int tanh(x)dx = ln(cosh(x))$, $\int coth(x)dx = ln(sinh(x))$ (Grl) 2.423

xi) $\int\frac{\sinh\,(2nx)}{\cosh\,(x)}dx = 2\sum_{k=0}^{n-1}(-1)^k\frac{\cosh((2n-2k-1)x)}{2n-2k-1}$ (Grl) 2.433

xii) $\int\frac{sinh\,(2x)}{cosh\,(x)}dx = 2cosh(x)$

xiii) $\int_0^{\infty} e^{-\alpha x}\,tanh(x)\,dx = \beta\left(\frac{\alpha}{2}\right) - \frac{1}{\alpha}$, $Re(\alpha) > 0$ (Grl) 3.541

xiv) $a^2 \neq b^2$, (Grl) 2.481

$$\int e^{ax}sinh\,(bx+c)dx = \frac{e^{ax}}{a^2-b^2}[a\cdot sinh\,(bx+c) - b\cdot cosh(bx+c)]$$

$$\int e^{ax}cosh\,(bx+c)dx = \frac{e^{ax}}{a^2-b^2}[a\cdot cosh\,(bx+c) - b\cdot sinh(bx+c)]$$

xv) $a^2 \neq b^2$, (Grl) 2.484

$$\int e^{ax}sinh\,(bx)\frac{dx}{x} = \frac{1}{2}\{[Ei(a+b)x] - [Ei(a-b)x]\}$$

$$\int e^{ax}cosh\,(bx)\frac{dx}{x} = \frac{1}{2}\{[Ei(a+b)x] + [Ei(a-b)x]\}$$

xvi) $tanh(x) = 1 + 2\sum_{k=1}^{\infty}(-1)^k e^{-2kx}$ (Grl) 1.232

xvii) $e^{ax} - e^{bx} = (a-b)xe^{\frac{1}{2}(a+b)x}\prod_{k=1}^{\infty}(1 + \frac{(a-b)^2x^2}{4k^2\pi^2}$ (Grl) 1.223

xviii) $\int_0^{\infty} e^{-zx}\,\tanh(x)\,dx = \beta\left(\frac{z}{2}\right) - \frac{1}{z}, \beta(1) = log2, \beta\left(\frac{1}{2}\right) = \frac{\pi}{2}, \ Re(z) > 0$ (Grl) 3.541

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