The Montgomery-Odlyzko law

The distribution of eigenvalue spacing
in a collection of Gaussian unitary operators
and the Hilbert-Polya (Berry-Keating) conjecture

The Montgomery-Odlyzko law in a nutshell
(DeJ)

(DeJ): p. 292: "*The distribution of the spacings between successive non-trivial zeros of the Riemann zeta function (suitable normalized) is statistically identical with the distribution of eigenvalue spacing in a Gaussian Unitary Ensemble (i.e. a collection of Gaussian unitary operators that share some common statistical properties)*".

(DeJ): p. 280 ff.: … „*The eigenvalues (of Gaussian-random Hermitian matrices)… are struggling to keep their distance from each other. … The statistical properties of spacings between long non-uniform string of numbers are encapsulated in a creature called „pair correlations function“ and a certain ratio associated with this function is called its „form factor“. … The form factor for the pair correlation of random Hermitian matrices is the conjectured distribution function for the differences between the non-trivial zeros of Riemann's zeta function. …*“

(DeJ): p. 285 ff.: „*The following points look pretty plausible on the basis of related comparing figures of „the eigenvalues of a 269-by-269-random matrix*“

(DeJ): p. 289: „*The first 269 values of „t“, where ½+it is a non trivial zero of the zeta function*“

> *1. neither the zeta zeros nor the eigenvalues look much like randomly scattered points*
> *2. they resemble each other*
> *3. in particular, they both show the repulsion (energy level) effect, trying to get as far as possible from each other, like a long standing line of antisocial people".*

A spectral interpretation of the zeta function
(CoB)

Hilbert and Polya are reputed to have suggested that the zeros of ζ(s) should be interpreted as eigenvalues of an appropriate operator.

Odlyzko wrote to Polya to ask about this.

Here is the text of Odlyzko's letter, dated Dec. 8, 1981.

Dear Professor Polya:
I have heard on several occasions that you and Hilbert had independently conjectured that the zeros of the Riemann zeta function correspond to the eigenvalues of a self-adjoint hermitian operator. Could you provide me with any references? Could you also tell me when this conjecture was made, and what was your reasoning behind this conjecture at that time?

The reason for my questions is that I am planning to write a survey paper on the distribution of zeros of the zeta function. In addition to some theoretical results, I have performed extensive computations of the zeros of the zeta function, comparing their distribution to that of random hermitian matrices, which have been studied very seriously by physicists. If a hermitian operator associated to the zeta function exists, then in some respects we might expect it to behave like a random hermitian operator, which in turn ught to resemble a random hermitian matrix. I have discovered that the distribution of zeros of the zeta function does indeed resemble the distribution of eigenvalues of random hermitian matrices of unitary type.

Any information or comments you might care to provide would be greatly appreciated.

Sincerely yours,
Andrew Odlyzko

and Polya's response, dated January 3, 1982.

Dear Mr. Odlyzko,
Many thanks for your letter of Dec. 8. I can only tell you what happened to me.
I spent two years in G¨ottingen ending around the beginning of 1914. I tried to learn analytic number theory from Landau. He asked me one day: "You know some physics. Do you know a physical reason that the Riemann Hypothesis should be true?"
This would be the case, I answered, if the non-trivial zeros of the ζ function were so connected with the physical problem that the Riemann Hypothesis would be equivalent to the fact that all the eigenvalues of the physical problem are real.
I never published this remark, but somehow it became known and it is still remembered.

With best regards.

Yours sincerely,
George, Polya

The conjecture

Conjecture. For fixed $0 < \alpha < \beta < \infty$,

$$\frac{|\{(\gamma, \gamma'): 0 < \gamma, \gamma' \leq T, 2\pi\alpha(\log T)^{-1} \leq \gamma - \gamma' \leq 2\pi\beta(\log T)^{-1}\}|}{\frac{T}{2\pi} \log T} \tag{1.4}$$

$$\sim \int_{\alpha}^{\beta} \left[1 - \left[\frac{\sin \pi u}{\pi u}\right]^2\right] du$$

as $T \to \infty$.

where $\frac{1}{2} + i\gamma$ and $\frac{1}{2} + i\gamma'$ denote nontrivial zeros of the zeta function, suggests some further topics for investigation. In the language of mathematical physics, this conjecture says that

$$1 - \left[\frac{\sin(\pi u)}{\pi u}\right]^2$$

is the pair correlation function of zeros of the zeta function. F. J. Dyson pointed out (MoH) that the gaussian unitary ensemble (GUE) has the same pair correlation function.

The possible connection between zeros of the zeta function and eigenvalues of random matrices is of interest in number theory because of the Hilbert and Polya conjectures, ((MoH) (OdA)), which say that the zeros of the zeta function correspond to eigenvalues of a positive linear operator. If true, the Hilbert and Polya conjectures would imply the RH, and some people feel that the most promising way to prove the RH is by finding the right operator and establishing the necessary results about it.

References

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(OdA) Odlyzko A. M., On the distribution of spacings between zeros of the zeta function, Math. Comp. Vol. 48, No. 177 (1987), 273-308