Some remarkable Pseudo-Differential Operators with order -1, 0, 1

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Abstract

Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{\sqrt{2\pi}} \oint u(x) e^{-i\nu x} dx$$

enable the definitions of the norms (e.g. [ILi] 11.1.5, [BrK0])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 \ .$$

We consider the Pseudo-Differential model operators S_i $i = -1,0,1$, of Symm, Hilbert and Calderon-Zygmund type which are bounded and self-adjoint with respect to the energy inner norm

$$(u,u)_{i/2} = (S_i u, u)_0 \ .$$

Notations

Let $H = L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{\sqrt{2\pi}} \oint u(x) e^{i\nu x} dx =: \oint u(x) \psi_n(x) dx$$

enable the definitions of the norms (see e.g. [ILi] Remark 11.1.5, [BrK0])

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty} |\nu|^{2\beta} |u_\nu|^2 \quad .$$

Then H is the space of L_2-periodic function in R.

We note that the Fourier transform (denoted by $\hat{u}$) is an isomorphism between the Hilbert spaces $H_\beta := \left\{ u \middle| \|u\|_\beta^2 < \infty \right\}$ and its “dual” Hilbert space $H_{-\beta}$ (see appendix). For the Dirac function it holds

$$\delta(x) := \frac{1}{2\pi} \int_0^{2\pi} e^{ikx} dk = \frac{1}{\pi} \int_0^{\pi} \cos(kx) dk = \frac{1}{2} \operatorname{sgn}'(x) \in H_{-n/2-\varepsilon} \subset H_{-1} \quad .$$

Definition and remark: Let the kernel functions s_i ($i = -1,0,1$) with its corresponding Fourier transforms be given by:

$$s_{-1}(x) := \frac{1}{\sqrt{2\pi}} \ln \frac{1}{2 \sin \frac{x}{2}} \quad , \qquad \hat{s}_{-1}(\nu) = \frac{1}{\sqrt{2\pi}} \oint s_{-1}(x) e^{i\nu x} dx = \frac{1}{2\nu} \operatorname{sgn}(\nu)$$

$$s_0(x) := -\frac{1}{\sqrt{2\pi}} \frac{1}{2} \cot \frac{x}{2} \quad , \qquad \hat{s}_0(\nu) = \frac{1}{\sqrt{2\pi}} \oint s_0(x) e^{i\nu x} dx = i \frac{1}{2} \operatorname{sgn}(\nu)$$

$$s_1(x) := -\frac{1}{\sqrt{2\pi}} \frac{1}{\sin^2 \frac{x}{2}} = -s_0'(x) \quad , \qquad \hat{s}_1(\nu) = \frac{1}{\sqrt{2\pi}} \oint s_1(x) e^{i\nu x} dx = 2\nu \operatorname{sgn}(\nu) \quad .$$

The following kernel functions s_i ($i = -1,0,1$) with its symbols $\hat{s}_i$ define Pseudo-Differential model operators of Symm, Hilbert and Calderon-Zygmund, type (e.g. [LiI] (1.2.31)-(1.2.33), [IiI1]):

S(-1): $(S_{-1}u)(x) := \oint s_{-1}(x-y) u(y) dy$

S(0): $(S_0 u)(x) := \oint s_0(x-y) u(y) dy$

S(1): $(S_1 u)(x) := \oint s_1(x-y) u(y) dy$.

The Pseudo-Differential operators have remarkable properties. It holds

Lemma: i) The operator S_{-1} is symmetric in the form

$$(S_{-1}u,v)_{\alpha}=(u,v)_{\alpha-1/2} \quad .$$

ii) The operator S_0 is skew-symmetric in the space $L_2(0,2\pi)$ (e.g. [GaD], [PeB])

$$(S_0u,v)_{\alpha}=-(u,S_0v)_{\alpha}$$

and maps the space $H:=L_2(0,2\pi)-R$ isometric onto itself. It holds

$\|S_0u\|=\|u\|$ and $S_0^{\;2}=-I$, The constant Fourier term vanishes i.e. $(S_0u)_0=0$

$$S_0[xu](x)=xS_0[u](x)-\frac{1}{\pi}\int_{-\infty}^{\infty}u(y)dy$$

and therefore for u odd the commutator property: $S_0[xu](x)=xS_0[u](x)$

If $u,Hu\in L_2$ then u and S_0u are orthogonal, i.e.

$$\int_{-\infty}^{\infty}u(y)(S_0u)(y)dy=0 \ .$$

The operator S_0 defines also a bijective mapping from the Hölder space $C^{0.\lambda}$ onto itself [NMu], §18, 19. We note

$$(S_0u)(x)=i\sum_{1}^{\infty}\left[u_{-\nu}e^{-i\nu x}-u_{\nu}e^{i\nu x}\right]\in L_2 \text{ for } u\in L_2 \ .$$

iii) The operator S_1 is symmetric in its domain and it holds

$$(S_1u,v)_{\alpha}=(u,v)_{\alpha+1/2} \quad .$$

From the lemma above it follows

Corollary: The operators S_i ($i=-1,0,1$) are bounded and self-adjoint with respect to the corresponding energy inner norm $(u,u)_{i/2}=(S_iu,u)_0$. It holds

$$(S_{-1}u')(s)=(S_0u)(s) \quad , \quad (S_0u')(s)=(S_1u)(s) \ .$$

Let $\bar{u}(x) := xu(x)$. Then, because of $\hat{\bar{u}}(\nu) = i \cdot \hat{u}'(\nu)$ and $2i\delta(\nu) = i \cdot \operatorname{sgn}'(\nu)$, it follows

$$\hat{\bar{s}}_0(\nu) = i \cdot \hat{s}'(\nu) = -\frac{1}{2}\operatorname{sgn}'(\nu) = \delta(\nu) \in H_{-1/2-\varepsilon} \;,$$

i.e. it holds

$$(S_0[xu], v')_{-1/2} = (S_0[xu], v)_0 = \hat{u}(o)\hat{v}(0) \quad .$$

Corollary: For each Hilbert transformed function $u_H(x) := Hu(x)$ it holds

$$(S_0[xu_H], v')_{-1/2} = (S_0[xu_H], v)_0 = \;.$$

Remark: The principle value of the not locally integrable function $1/x$ is the distribution g defined by ([PeB])

$$(g,\varphi) := \lim \int_{x \geq \varepsilon} \varphi(x)\frac{dx}{x} = \int_{-\infty}^{\infty} \log|x|\varphi'(x)dx \qquad \text{for each } \varphi \in C_c^\infty \;.$$

The corresponding Fourier transform is given by

$$\left[P.v.(\frac{1}{\pi x})\right]^{\wedge}(\nu) = -i\operatorname{sgn}(\nu) \quad .$$

Remark: From [StE1] IV 6.3, we note that a periodic function on R in the form

$$u(x) = \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \quad \text{with} \quad |u_\nu| \leq \frac{1}{\nu}$$

is an element of the function space of bounded mean oscillation, i.e. $u \in BMO(R)$.

Suppose

$$\sum_{-\infty}^{\infty} v_\nu e^{i\nu x} \in BMO(R) \;,\quad v_\nu \geq 0 \;\text{ then }\; \sum_{-\infty}^{\infty} u_\nu e^{i\nu x} \in BMO(R) \quad \text{whenever } |u_\nu| \leq v_\nu \;.$$

Appendix

Remark: From e.g. [GaD] pp.63, [GrI] 1.441, [Ili1], [MuN] chapter 3, §28, we recall

$$\frac{1}{2\pi}\oint_{0\to 2\pi} e^{in\varphi}\ln\frac{1}{2\sin\frac{\varphi-\vartheta}{2}}d\vartheta=\begin{cases}-\frac{1}{2n}e^{in\varphi} & n=1,2,3,...\\ 0 & n=0\\ \frac{1}{2n}e^{in\varphi} & n=-1,-2,..\end{cases}$$

$$\frac{1}{\pi}\oint_{0\to 2\pi} e^{in\varphi}\frac{1}{2}\cot\frac{\varphi-\vartheta}{2}d\vartheta=\begin{cases}-ie^{in\varphi} & n=1,2,3,...\\ 0 & n=0\\ ie^{in\varphi} & n=-1,-2,..\end{cases}$$

$$\frac{1}{\pi}\oint_{0\to 2\pi} e^{in\varphi}\frac{1}{4\sin^2\frac{\varphi-\vartheta}{2}}d\vartheta=\begin{cases}-ne^{in\varphi} & n=1,2,3,...\\ 0 & n=0\\ ne^{in\varphi} & n=-1,-2,..\end{cases}$$

Remark:

$$S_0[xu](x)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{yu(y)}{x-y}dy=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(x-z)u(x-z)}{z}dz=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{xu(x-z)}{z}dz-\frac{1}{\pi}\int_{-\infty}^{\infty}u(x-z)dz=xS_0[u](x)-\frac{1}{\pi}\int_{-\infty}^{\infty}u(y)dy$$

Remark: For

$$f_n(y):=a_n\cos ny+b_n\sin ny$$

We note the identity ([LiI] (1.2.34):

$$-nf_n(x)=\frac{1}{\pi}\int_0^{2\pi}\frac{f(y)}{4\sin^2\frac{x-y}{2}}dy \quad .$$

For a relationship between conjugate functions on the open unit disk and the Hilbert transform S_0 above we recall from [GaD] Chapter II, §1:

Theorem 1.1.: *Let $f(z)=u(z)+iv(z)$ (u,v real) be regular in the open unit disk $|z|<1$ and continuous on the closed unit disk $|z|\le 1$. Then for $v(e^{i\varphi})$ the following integral representation of $u(e^{i\varphi})$ is valid (Cauchy integral):*

$$v(e^{i\varphi})=v(0)+\frac{1}{2\pi}\int_0^{2\pi}v(e^{i\varphi})\cot\frac{\varphi-\vartheta}{2})d\vartheta \ .$$

Remark: For a relationship between the differential calculus and its application in physics we recall from J. Plemelj ([PlJ] §8, also [AhJ], [NiJ]) his suggestion to replace the potential

(*) $v(s)=-\frac{1}{\pi}\oint\log|\zeta(s)-\zeta(t)|u(t)dt$ by (**) $v(s)=-\frac{1}{\pi}\oint\log|\zeta(s)-\zeta(t)|du(t)$

Plemelj's quote: "*Bisher war es üblich, für das Potential die Form (*) zu nehmen. Eine solche Einschränkung erweist sich aber als eine derart folgenschwere Einschränkung, dass dadurch dem Potentiale der grösste Teil seiner Leistungsfähigkeit hinweg genommen wird. Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form (**) verwendbar.*"

Remark: we note some relationships to the Euler's formula (see ([TiE] 2.1), [PeB]) and the sawtooth function: Let $[x]$ denote the largest integer not exceeding the real number x and let $\rho(x):=\{x\}:=x-[x]$ be the fractional part of x, then it holds:

i) $$\rho(x)=\{x\}=x-[x]=\frac{1}{2}-\sum_1^\infty\frac{\sin 2\pi\nu x}{2\pi|\nu|} \quad , \quad -i\pi sign(x)=-2i\int_0^\infty\frac{\sin(tx)dt}{t}=2\int_0^\infty\frac{\sinh(tx)dt}{t}=\left[P.v.(\frac{1}{x})\right]^\wedge .$$

ii) $$\sum_1^\infty\frac{\sin\nu x}{\nu}=\frac{\pi-x}{2} \quad , \quad \sum_1^\infty\frac{\cos\nu x}{\nu}=\frac{1}{2}\log\frac{1}{2(1-\cos x)} \quad , \quad 0<x<2\pi .$$

Remark: The Hilbert spaces $H_{-1/2},H_{-1}$ are characterized by

$$H_{-1/2}=\left\{\psi\,\middle|\,\|\psi\|_{-1/2}^2=(A\psi,\psi)_0<\infty\right\} , \; H_{-1}=\left\{\psi\,\middle|\,\|\psi\|_{-1}^2=(A\psi,A\psi)_0<\infty\right\}.$$

In ([ZyA], 5.28, 7.2, 13.11) the concept of "logarithmic", $\alpha-$capacity" of sets and convergence of Fourier series to functions with $\sum_1^\infty n\left[a_n^2+b_n^2\right]<\infty$ are provided. The following two examples are given:

$$\lambda(x)\approx\sum_1^\infty\frac{\cos 2\pi\nu x}{\nu}=-\log 2\sin(\pi x) \qquad \text{whereby} \qquad \left|\sum_1^N\frac{\cos\nu x}{\nu}\right|\le\log(\frac{1}{x})+C ,$$

$$\lambda(x)\approx\sum_1^\infty\frac{\cos\nu x}{\nu^{1-\alpha}}\cong c_\alpha|x|^{-\alpha} \quad , \quad (x\to 0;0<\alpha<1) .$$

The Eigenvalue problem for compact symmetric operators

In the following H denotes an (infinite dimensional) real Hilbert space with scalar product $(.,.)$ and the norm $\|..\|$. We will consider mappings $K : H \to H$. Unless otherwise noticed the standard assumptions on K are:

i) K is symmetric, i.e. for all $x, y \in H$ it holds $(x, Ky) = (x, Ky)$

ii) K is compact, i.e. for any (infinite) sequence $\{x_n\}$ bounded in H contains a subsequence $\{x_{n'}\}$ such that $\{Kx_{n'}\}$ is convergent,

iii) K is injective, i.e. $Kx = 0$ implies $x = 0$.

A first consequence is

Lemma: K is bounded, i.e.

$$\|K\| := \sup_{x \neq 0} \frac{\|Kx\|}{\|x\|} \quad .$$

Lemma: Let K be bounded, and fulfill condition i) from above, but not necessarily the two other condition ii) and iii). Then $\|K\|$ equals

$$N(K) = \sup_{x \neq 0} \frac{|(x, Kx)|}{\|x\|} \quad .$$

Theorem: There exists a countable sequence $\{\lambda_i, \varphi_i\}$ of eigenelements and eigenvalues

$K\varphi_i = \lambda_i \varphi_i$ with the properties

i) the eigenelements are pair-wise orthogonal, i.e. $(\varphi_i, \varphi_k) = \delta_{i,k}$

ii) the eigenvalues tend to zero, i.e. $\lim_{i \to \infty} \lambda_i$

iii) the generalized Fourier sums

$$S_n := \sum_{i=1}^{n} (x, \varphi_i)\varphi_i \to x \quad \text{with } n \to \infty \text{ for all } x \in H$$

iv) the Parseval equation

$$\|x\|^2 = \sum_{i}^{\infty} (x, \varphi_i)^2$$

holds for all $x \in H$.

Hilbert Scales

Let H be a (infinite dimensional) Hilbert space with scalar product $(.,.)$, the norm $\|...\|$ and A be a linear operator with the properties

i) A is self-adjoint, positive definite

ii) A^{-1} is compact.

Without loss of generality, possible by multiplying A with a constant, we may assume

$$(x, Ax) \geq \|x\| \qquad \text{for all } x \in D(A)$$

The operator $K = A^{-1}$ has the properties of the previous section. Any eigen-element of K is also an eigen-element of A to the eigenvalues being the inverse of the first. Now by replacing $\lambda_i \to \lambda_i^{-1}$ we have from the previous section

i) there is a countable sequence $\{\lambda_i, \varphi_i\}$ with

$$A\varphi_i = \lambda_i \varphi_i \quad , \quad (\varphi_i, \varphi_k) = \delta_{i,k} \text{ and } \lim_{i \to \infty} \lambda_i$$

ii) any $x \in H$ is represented by

$$(*) \qquad x = \sum_{i=1}^{\infty} (x, \varphi_i) \varphi_i \quad \text{and} \quad \|x\|^2 = \sum_1^{\infty} (x, \varphi_i)^2 .$$

Lemma: Let $x \in D(A)$, then

$$(**) \quad Ax = \sum_{i=1}^{\infty} \lambda_i (x, \varphi_i) \varphi_i \quad , \quad \|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2 ,$$

$$(Ax, Ay) = \sum_{i=1}^{\infty} \lambda_i^2 (x, \varphi_i)(y, \varphi_i) .$$

Because of (*) there is a one-to-one mapping I of H to the space $\hat{H}$ of infinite sequences of real numbers

$$\hat{H} := \{\hat{x} | \hat{x} = (x_1, x_2, ...)\}$$

defined by

$$\hat{x} = Ix \quad \text{with} \quad x_i = (x, \varphi_i) .$$

If we equip $\hat{H}$ with the norm

$$\|\hat{x}\|^2 = \sum_1^{\infty} (x, \varphi_i)^2$$

then I is an isometry.

By looking at (**) it is reasonable to introduce for non-negative α the weighted inner products

$$(\hat{x},\hat{y})_\alpha = \sum_i^\infty \lambda_i^\alpha (x,\varphi_i)(y,\varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and the norms

$$\|\hat{x}\|_\alpha^2 = (\hat{x},\hat{x})_\alpha$$

Let $\hat{H}_\alpha$ denote the set of all sequences with finite $\alpha -$ norm. then $\hat{H}_\alpha$ is a Hilbert space. The proof is the same as the standard one for the space l_2 .

Similarly one can define the spaces H_α : they consist of those elements $x \in H$ such that $Ix \in \hat{H}_\alpha$ with scalar product

$$(x,y)_\alpha = \sum_i^\infty \lambda_i^\alpha (x,\varphi_i)(y,\varphi_i) = \sum_i^\infty \lambda_i^\alpha x_i y_i$$

and norm

$$\|x\|_\alpha^2 = (x,x)_\alpha \cdot$$

Because of the Parseval identity we have especially

$$(x,y)_0 = (x,y)$$

and because of (**) it holds

$$\|x\|_2^2 = (Ax,Ax)_0 \;\;,\;\; H_2 = D(A) \cdot$$

The set $\{H_\alpha | \alpha \geq 0\}$ is called a Hilbert scale. The condition $\alpha \geq 0$ is in our context necessary for the following reasons:

Since the eigen-values λ_i tend to infinity we would have for $\alpha < 0$: $\lim \lambda_i^\alpha \to 0$. Then there exist sequences $\hat{x} = (x_1, x_2, ...)$ with

$$\|\hat{x}\|_2^2 < \infty \;\;,\;\; \|\hat{x}\|_0^2 = \infty \;\cdot$$

Because of Bessel's inequality there exists no $x \in H$ with $Ix = \hat{x}$. This difficulty could be overcome by duality arguments which we omit here.

There are certain relations between the spaces $\{H_\alpha | \alpha \geq 0\}$ for different indices:

Lemma: Let $\alpha < \beta$. Then

$$\|x\|_\alpha \leq \|x\|_\beta$$

and the embedding $H_\beta \to H_\alpha$ is compact.

Lemma: Let $\alpha < \beta < \chi$. Then

$$\|x\|_\beta \leq \|x\|_\alpha^\mu \|x\|_\gamma^\nu \text{ for } x \in H_\gamma$$

with $\mu = \frac{\gamma - \beta}{\gamma - \alpha}$ and $\nu = \frac{\beta - \alpha}{\gamma - \alpha}$.

Lemma: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\alpha \leq t^{\beta - \alpha} \|x\|_\beta$

ii) $\|x - y\|_\beta \leq \|x\|_\beta$, $\|y\|_\beta \leq \|x\|_\beta$

iii) $\|y\|_\gamma \leq t^{-(\gamma - \beta)} \|x\|_\beta$.

Corollary: Let $\alpha < \beta < \gamma$. To any $x \in H_\beta$ and $t > 0$ there is a $y = y_t(x)$ according to

i) $\|x - y\|_\rho \leq t^{\beta - \rho} \|x\|_\beta$ for $\alpha \leq \rho \leq \beta$

ii) $\|y\|_\sigma \leq t^{-(\sigma - \beta)} \|x\|_\beta$ for $\beta \leq \sigma \leq \gamma$.

Remark: Our construction of the Hilbert scale is based on the operator A with the two properties i) and ii). The domain $D(A)$ of A equipped with the norm

$$\|Ax\|^2 = \sum_{i=1} \lambda_i^2 (x, \varphi_i)^2$$

turned out to be the space H_2 which is densely and compactly embedded in $H = H_0$. It can be shown that on the contrary to any such pair of Hilbert spaces there is an operator A with the properties i) and ii) such that

$$D(A) = H_2 \quad R(A) = H_0 \text{ and } \|x\|_2 = \|Ax\|.$$

Extension and generalizations

For $t > 0$ we introduce an additional inner product resp. norm by

$$(x, y)_{(t)}^2 = \sum_{i=1} e^{-\sqrt{\lambda_i} t}(x, \varphi_i)(y, \varphi_i)$$

$$\|x\|_{(t)}^2 = (x, x)_{(t)}^2 \ .$$

Now the factor have exponential decay $e^{-\sqrt{\lambda_i} t}$ instead of a polynomial decay in case of λ_i^α.
Obviously we have

$$\|x\|_{(t)} \leq c(\alpha, t)\|x\|_\alpha \text{ for } x \in H_\alpha$$

with $c(\alpha, t)$ depending only from α and $t > 0$. Thus the $(t)-norm$ is weaker than any $\alpha - norm$. On the other hand any negative norm, i.e. $\|x\|_\alpha$ with $\alpha < 0$, is bounded by the $0 - norm$ and the newly introduced $(t) - norm$. It holds:

Lemma: Let $\alpha > 0$ be fixed. The $\alpha - norm$ of any $x \in H_0$ is bounded by

$$\|x\|_{-\alpha}^2 \leq \delta^{2\alpha}\|x\|_0^2 + e^{t/\delta}\|x\|_{(t)}^2$$

with $\delta > 0$ being arbitrary.

Remark: This inequality is in a certain sense the counterpart of the logarithmic convexity of the $\alpha - norm$, which can be reformulated in the form ($\mu, \nu > 0$, $\mu + \nu > 1$)

$$\|x\|_\beta^2 \leq \nu\varepsilon\|x\|_\gamma^2 + \mu e^{-\nu/\mu}\|x\|_\alpha^2$$

applying Young's inequality to

$$\|x\|_\beta^2 \leq (\|x\|_\alpha^2)^\mu (\|x\|_\gamma^2)^\nu \ .$$

The counterpart of lemma 4 above is

Lemma: Let $t, \delta > 0$ be fixed. To any $x \in H_0$ there is a $y = y_t(x)$ according to

i) $\|x - y\| \leq \|x\|$

ii) $\|y\|_1 \leq \delta^{-1}\|x\|$

iii) $\|x - y\|_{(t)} \leq e^{-t/\delta}\|x\|$.

Eigenfunctions and Eigendifferentials

Let H *be* a (infinite dimensional) Hilbert space with inner product $(.,.)$, the norm $\|...\|$ and A be a linear self-adjoint, positive definite operator, but we omit the additional assumption, that A^{-1} compact. Then the operator $K = A^{-1}$ does not fulfill the properties leading to a discrete spectrum.

We define a set of projections operators onto closed subspaces of H in the following way:

$$R \to L(H,H)$$

$$\lambda \to E_\lambda := \int_{\lambda_0}^{\lambda} \varphi_\mu(\varphi_\mu,*)d\mu \quad , \quad \mu \in [\lambda_0, \infty) \ ,$$

i.e.
$$dE_\lambda = \varphi_\lambda(\varphi_\lambda,*)d\lambda \ .$$

The spectrum $\sigma(A) \subset C$ of the operator A is the support of the spectral measure dE_λ.
The set E_λ fulfills the following properties:

i) E_λ is a projection operator for all $\lambda \in R$

ii) for $\lambda \le \mu$ it follows $E_\lambda \le E_\mu$ i.e. $E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$

iii) $\lim_{\lambda \to -\infty} E_\lambda = 0$ and $\lim_{\lambda \to \infty} E_\lambda = Id$

iv) $\lim_{\substack{\mu \to \lambda \\ \mu > \lambda}} E_\mu = E_\lambda$.

Proposition: Let E_λ be a set of projection operators with the properties i)-iv) having a compact support $[a,b]$. Let $f:[a,b] \to R$ be a continuous function. Then there exists exactly one Hermitian operator $A_f : H \to H$ with

$$(A_f x, x) = \int_{-\infty}^{\infty} f(\lambda) d(E_\lambda x, x) \ .$$

Symbolically one writes
$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda \ .$$

Using the abbreviation

$$\mu_{x,y}(\lambda) := (E_\lambda x, y) \quad , \quad d\mu_{x,y}(\lambda) := d(E_\lambda x, y)$$

one gets

$$(Ax,y) = \int_{-\infty}^{\infty} \lambda d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda) \qquad , \quad \|x\|_1^2 = \int_{-\infty}^{\infty} \lambda d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda d\mu_{x,x}(\lambda)$$

$$(A^2 x,y) = \int_{-\infty}^{\infty} \lambda^2 d(E_\lambda x, y) = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ , \quad \|Ax\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\|E_\lambda x\|^2 = \int_{-\infty}^{\infty} \lambda^2 d\mu_{x,x}(\lambda) \ .$$

The function $\sigma(\lambda) := \|E_\lambda x\|^2$ is called the spectral function of A for the vector x. It has the properties of a distribution function.

It hold the following eigen-pair relations

$$A\varphi_i = \lambda_i \varphi_i \qquad A\varphi_\lambda = \lambda \varphi_\lambda \qquad \|\varphi_\lambda\|^2 = \infty \text{ , } (\varphi_\lambda, \varphi_\mu) = \delta(\varphi_\lambda - \varphi_\mu) \text{ .}$$

The φ_λ are not elements of the Hilbert space. The so-called eigen-differentials, which play a key role in quantum mechanics, are built as superposition of such eigen-functions.

Let I be the interval covering the continuous spectrum of A. We note the following representations:

$$x = \sum_1^\infty (x,\varphi_i)\varphi_i + \int_I \varphi_\mu (\varphi_\mu, x) d\mu \text{ , } \quad Ax = \sum_1^\infty \lambda_i (x,\varphi_i)\varphi_i + \int_I \lambda \varphi_\mu (\varphi_\mu, x) d\mu$$

$$\|x\|^2 = \sum_1^\infty |(x,\varphi_i)|^2 + \int_I |(\varphi_\mu, x)|^2 d\mu \text{ ,}$$

$$\|x\|_1^2 = \sum_1^\infty \lambda_i |(x,\varphi_i)|^2 + \int_I \lambda |(\varphi_\mu, x)|^2 d\mu$$

$$\|x\|_2^2 = \|Ax\|^2 = \sum^\infty \lambda_i^2 |(x,\varphi_i)|^2 + \int_I \lambda^2 |(\varphi_\mu, x)|^2 d\mu \text{ .}$$

Example: The location operator Q_x and the momentum operator P_x both have only a continuous spectrum. For positive energies $\lambda \geq 0$ the Schrödinger equation

$$H\varphi_\lambda(x) = \lambda \varphi_\lambda(x)$$

delivers no element of the Hilbert space H, but linear, bounded functional with an underlying domain $M \subset H$ which is dense in H. Only if one builds wave packages out of $\varphi_\lambda(x)$ it results into elements of H. The practical way to find Eigen-differentials is looking for solutions of a distribution equation.

References

[AhJ] J. F. Ahner, Some spectral properties of an integral operator in potential theory, Proc. Edinburgh Math. Soc. (1986) 29, 405-411

[ArA] A. M. Arthurs, Complementary Variational Principles, Clarendon Press, Oxford, 1970

[BeM] M. V. Berry, J. P. Keating, $H = xp$ and the Riemann zeros, in Supersymmetry and Trace Formulae: Chaos and Disorder (Ed. I.V. Lerner, J.P. Keating, D.E.Khmelnitski), Kluver, New York, 355-367 (1999)

[BrR] R. Bracewell, The Fourier Transform and Its Applications, McGraw-Hill, New York, 1999

[BrK0] K. Braun, Interior Error Estimates of the Ritz Method for Pseudo-Differential Equations, Jap. Journal of Applied Mathematics, 3, 1, 59-72, (1986)

[BrK1] K. Braun, Two proofs of the Riemann hypothesis, www.riemann-hypothesis.de

[BrK2] K. Braun, A new ground state energy model, www.riemann-hypothesis.de

[BuW] W. Bulla, F. Gesztesy, Deficiency indices and singular boundary conditions in quantum mechanics, J. Math. Phys. 26, 2520-2528 (1985)

[DuR] R. J. Duffin, Hilbert Transforms in Yukawan Potential Theory, Proc. Nat. Acad. Sci. USA, 69, No 12, pp. 3677-3679 (1972)

[EsG] G. I. Eskin, Boundary Value Problems for Elliptic Pseudodifferential Equations, Amarican Math, society, Providence, Rhode Island (Russian original edition: 1973)

[FeR] R.P. Feyman, A.R. Hibbs, Quantum Mechanics and Path Integrals, Dover Publications, Inc., Mineola, New York, 1965

[GaD] D. Gaier, Konstruktive Methoden der konformalen Abbildung, Springer-Verlag, Berlin, 1964

[GlH] h. Glauert, Grundlagen der Tragflügel- und Luftschraubentheorie, Verlag von Julius Springer, Berlin, 1929

[GrI] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals Series and Products, Fourth Edition, Academic Press, New York, San Francisco, London, 1965

[JaM] M. A. Jaswon, G. T. Symm, Integral Equation Methods in Potential Theory and electrostatics, Academic Press, London, New York, San Francisco, 1977

[LiI] I. K. Lifanov, L. N. Poltavskii, G. M. Vainikko, Hypersingular Integral Equations and their Applications, Chapman & Hall/CRC, Boca Rato, London, New York, Washington, D. C., 2004

[liI1] I. K. Lifanov, A. S. Nenashev, Generalized functions on Hilbert spaces, singular integral equations, and problems of aerodynamics and electrodynamics, Differential Equations, 2007, Band: 43, Heft 6, 862-872

[MoC] C. B. Morrey, Multiple Integrals in the Calculus of Variations, Springer-Verlag, New York Inc., 1966

[MuN] N. I. Muskhelishvili, Singular Integral Equations, P. Noordhoff N. V., Groningen, Holland, 1953

[NiJ] J.A. Nitsche, Finite Element Methods for Conformal Mapping, SIAM J. Num. Anal., 26, 6, 1525-1533 (1989)

[PeB] B. E. Petersen, Introduction to the Fourier Transform & Pseudo-Differential Operators, Pitman Publishing Limited, Boston, London, Melbourne

[PlJ] J. Plemelj, Potentialtheoretische Untersuchungen, B.G. Teubner, Leipzig, 1911

[ShM] M. A. Shubin, Pseudodifferential Operators and spectral Theory, Springer Verlag, Berlin, Heidelberg, New York, 2001 (Russian original edition: 1978)

[StE] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, New Jersey, 1970

[StE1] E. M. Stein, Harmonic Analysis, Princeton University Press, Princeton, New Jersey, 1993

[TrE] E. Trefftz, Zur Behandlung der Prandtlschen Tragflächentheorie, Math. Anal. 82, Issue 3-4, 306-319, 1921

[ZeA0] A. H. Zemanian, Generalized Integral Transformation, Dover Publications, New York, 1987

[ZeA1] A. Zemanian, Distribution Theory and Transform Analysis, Dover Publications, Inc., New York, 1987

[ZyA] A. Zygmund, Trigonometric series, Vol. I, Cambridge University Press, 1968